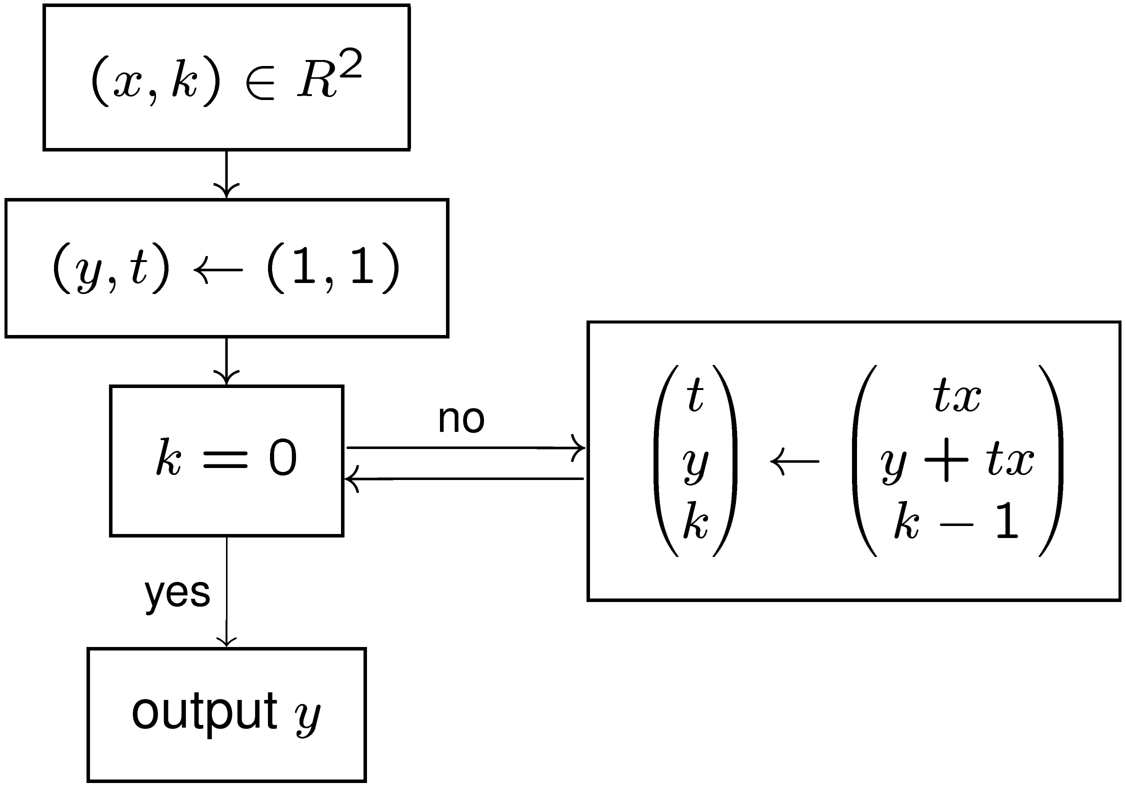
BSS machines — briefly

Let R be a totally ordered ring.
Let $f : R \times R \circ\!\!\rightarrow R$, with domain $R \times \mathbb{N}$, be defined by

$$f(x,k) := \sum_{i=0}^{k} x^i.$$

f is *computable* by the machine on the right whose variables are $x, k, y, t \in R$.



Special operations for infinite dimensional variables — that is belonging to $R^\infty := \left\{ (x_i)_{i=1}^{\infty} : \text{all } x_i \in R \text{ and all but a finite number of them are null} \right\}$.

Turing machines correspond to the case $R = \mathbb{Z}$.

Recursive functions

➢ *Basic functions*

- the polynomial maps $R^k \to R$ (or the rational maps $R^k \circ\!\!\to R$ if R is a field) where $k < \infty$;
- the characteristic function $\chi : R \to R$ defined by

$$\chi(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0; \end{cases}$$

- the inclusions $R^k \hookrightarrow R^\infty : (x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, 0, \dots)$ and the projections $\mathrm{pr}_k : R^\infty \to R : (x_i)_{i=1}^\infty \mapsto x_k$;
- the function $\sigma : R \times R \times R^\infty \circ\!\!\to R^\infty$ with domain $\mathbb{N} \times \mathbb{N} \times R^\infty$ defined by $\sigma(n, m, x) = y = (y_i)_{i=1}^\infty$ with $y_i = x_i$ if $i \neq m$ and $y_m = x_n$;
- the length function $R^\infty \to R : x \mapsto \mathtt{length}(x)$;

➢ *Generation rules*

- *composition:* if $f : R^k \circ\!\!\rightarrow R^\ell$ and $g : R^\ell \circ\!\!\rightarrow R^m$ are recursive functions, so is $g \circ f : R^k \circ\!\!\rightarrow R^m : x \mapsto g(f(x))$ with domain $f^{-1}(\mathrm{Dom}\, g)$;
- *juxtaposition:* if $f : R^k \circ\!\!\rightarrow R^\ell$ and $g : R^k \circ\!\!\rightarrow R^m$ are recursive functions, so is $(f, g) : R^k \circ\!\!\rightarrow R^\ell \times R^m : x \mapsto \big(f(x), g(x)\big)$ with domain $\mathrm{Dom}\, f \cap \mathrm{Dom}\, g$;
- *recursion:* given two recursive maps $f_0 : R^k \circ\!\!\rightarrow R^\ell$ and $F : R \times R^k \times R^\ell \circ\!\!\rightarrow R^\ell$, recursion says that the following map $f : R \times R^k \circ\!\!\rightarrow R^\ell$ is recursive where f is uniquely defined by

$$\begin{cases} f(0, x) = f_0(x) \\ f(n+1, x) = F\big(n, x, f(n, x)\big) \end{cases}$$

with domain being the set $\bigcup_{n=0}^{\infty} D_n \subset \mathbb{N} \times R^k$ with $D_0 := \{0\} \times \mathrm{Dom}\, f_0$ and $D_{n+1} := \Big\{(n+1, x) : \big(n, x, f(n, x)\big) \in \mathrm{Dom}\, F\Big\}$.

- *minimalization:* given a recursive map $F : R \times R^k \circ\!\!\rightarrow R$ defined at least on $\mathbb{N} \times R^k$, minimalization defines a function $f : R^k \circ\!\!\rightarrow R$ by

$$f(x) = \boldsymbol{\mu}_n F(n, x) := \min\{n \in \mathbb{N} : F(n, x) = 0\}$$

whose domain is the set of $x \in R^k$ for which there exists a $n \in \mathbb{N}$ with $F(n, x) = 0$.

The set of recursive functions will be denoted $\mathscr{R}$. It coincide with the set of BSS-computable functions on R.

$\mathscr{R}(R^k, R^\ell)$:= the set of partial maps from R^k to R^ℓ that are BSS-recursive.

Scalar- v.s. vector-recursion

The recursion

$$\begin{cases} f(0,x) = f_0(x) \\ f(n+1,x) = F\Big(n, x, f(n,x)\Big) \end{cases}$$

where $F : R \times R^k \times R^\ell \circ\!\!\to R^\ell$, is said to be *scalar* iff $\ell = 1$.

$\mathscr{R}^1$:= set of scalar-recursive functions.
$\mathscr{R}^1(R^k, R^\ell) := \mathscr{R}^1 \cap \mathscr{R}(R^k, R^\ell)$.

$$\mathscr{R}^1 = \mathscr{R}\,?$$

Motivation:

- basic property of BSS-recursive functions;
- Grzegorczyk complexity classes:
  Muchnick result ('74): $\mathscr{E}^n_{\mathbb{Z}} = \Big\{ f : \mathbb{Z}^k \to \mathbb{Z} : f \in \overrightarrow{\mathscr{E}}^n_{\mathbb{Z}}, k \in \mathbb{N} \Big\}$ for $n \geqslant 2$.

Answer:

- for $k \in \mathbb{N} \cup \{\infty\}$ and $\ell \in \mathbb{N}$:

$$\mathscr{R}^1(R^k, R^\ell) = \mathscr{R}(R^k, R^\ell)$$

- for $k \in \mathbb{N} \cup \{\infty\}$and $\ell = \infty$:

$$\mathscr{R}^1(R^k, R^\infty) \subsetneqq \mathscr{R}(R^k, R^\infty)$$

REMARK: $\mathbb{R}^k \ncong \mathbb{R}$ by a bi-computable bijection ($k \neq 1$).

The equality case

Given $f \in \mathscr{R}(R^k, R^\ell)$ ($k \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$), one has to show it is computable using only scalar-recursion. Assume f is computed by a BSS-machine with input variable x and output variable y.

➢ *Reduction to R-valued functions*

One can assume $f \in \mathscr{R}(R^k, R)$ — consider the components f_i of f and use the juxtaposition rule.

➢ *Get rid of the constants*

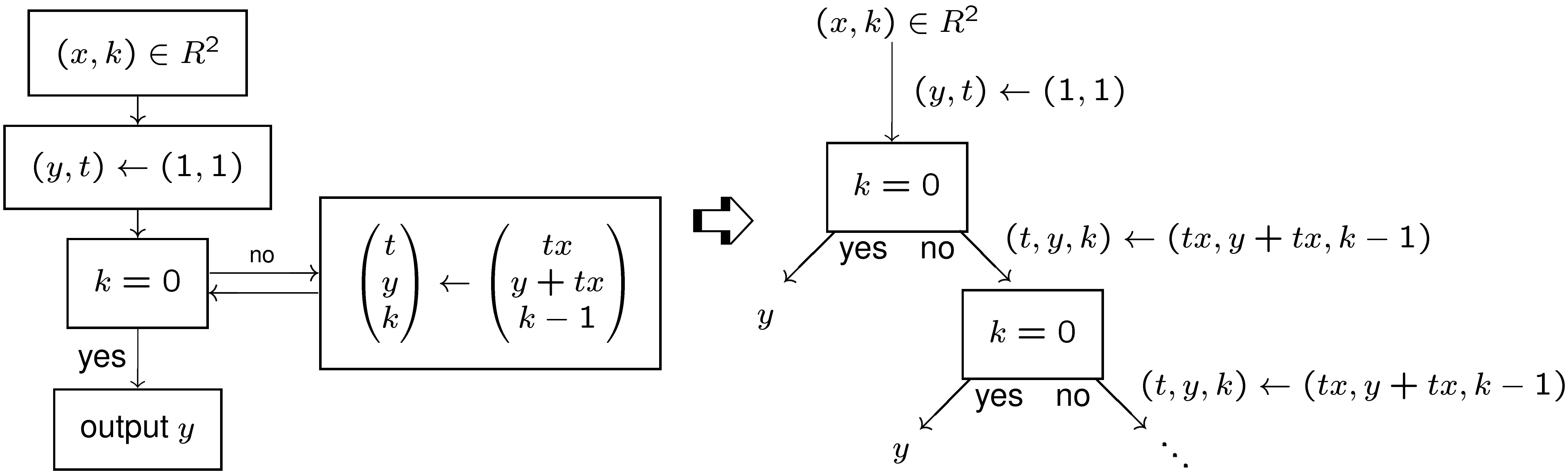
$\mathscr{R}_{\mathbb{Z}}$:= set of functions that are computable by a machine with coefficients in $\mathbb{Z}$.

$f(x) = \tilde{f}(a_1, \ldots, a_k, x)$ where $a_1, \ldots, a_k$ are the constants of the machine computing f and $\tilde{f} \in \mathscr{R}_{\mathbb{Z}}$. From now on, one assumes $f \in \mathscr{R}_{\mathbb{Z}}$.

➢ *Encode the machine structure*

The machine computing $\tilde{f}$ can be encoded into an integer $\boldsymbol{m} \in \mathbb{N}$.

➢ *Unroll the loops*



Possible computational paths (answers 'yes' and 'no' to the tests) can be coded as integers $\gamma \in \Gamma \subset \mathbb{N}$.

- $P(\gamma, z_i) \in \mathbb{Z}[x]$ denotes (a coding of) the polynomial giving the value of the ith component of z as a function of the input x at the ending node of γ.

$$(\gamma, z, i) \mapsto P(\gamma, z_i) \text{ is computable}$$

- $T(\gamma) \in \mathsf{Bool}(\mathbb{Z}[x])$ denotes (a coding of) the conjunction of all tests along the path γ.

$$\gamma \mapsto T(\gamma) \text{ is computable}$$

➢ *Evaluation*

Polynomial evaluation. The function ev : $\mathbb{Z}[x] \times R^k \subset R \times R^k \circ\!\!\rightarrow R$: $(P, x) \mapsto$ ev$(P, x) := P(x)$ is scalar-BSS-recursive.

Corollary. The map ev : $\mathsf{Bool}(\mathbb{Z}[x]) \times R^k \to \mathsf{Bool}$: $(\varphi, x) \mapsto$ ev(φ, x), where ev$(\varphi, x) = 1$ if φ is true when evaluated with data x and 0 otherwise, is scalar-BSS-recursive.

➢ *Conclusion*

Given an input x, on can compute the path $\gamma(x) \in \Gamma$ followed by x — check ev$(T(\gamma), x) = 1$ for all γ.

One has $f(x) =$ ev$\Big(P(\gamma(x), y), x\Big)$. Therefore f is scalar-BSS-recursive.

The non-equality case

Lemma. If $f : R^k \circ\!\!\!\longrightarrow R^\ell$ belongs to $\mathscr{R}^1$, there exists a finite number of recursive functions $a_1, \ldots, a_N : R^k \circ\!\!\!\longrightarrow R$ such that

- $\mathsf{Dom}\, f \subset \bigcap_{n=1}^{N} \mathsf{Dom}\, a_i$,
- $\forall x \in \mathsf{Dom}\, f,\ \forall j, \quad f_j(x) \in \{x_i\}_i \cup \{a_1(x), \ldots, a_N(x)\}$

PROOF. Satisfied by the basic functions and preserved by the generation rules. (Trivial if $\ell < \infty$.) $\square$

Example of $f \in \mathscr{R}(R, R^\infty) \setminus \mathscr{R}^1(R, R^\infty)$:

$$f : R \circ\!\!\!\longrightarrow R^\infty : x \mapsto (1, 2, 3, \ldots, x, 0, 0, \ldots) \text{ if } x \in \mathbb{N}.$$

Therefore $\mathscr{R}^1(R, R^\infty) \subsetneq \mathscr{R}(R, R^\infty)$.

➢ *Vectorization*

If $F : R \times R^k \circ\!\!\rightarrow R$ and $L : R^k \circ\!\!\rightarrow \mathbb{N} \subset R$ are recursive functions such that

$$D := \{x \in R^k : \forall n \in \mathbb{N}_0,\ (n, x) \in \operatorname{Dom} F\} \subset \operatorname{Dom} L,$$
$$\forall x \in D,\ \forall n > L(x),\ F(n, x) = 0,$$

then the function $\operatorname{vect} F : R^k \circ\!\!\rightarrow R$ is recursive, where $\operatorname{vect} F$ is defined by

$$(\operatorname{vect} F)(x) = \Big(F(n, x)\Big)_{n=1}^{\infty}$$

with domain $\operatorname{Dom}(\operatorname{vect} F) = D$.

PROOF. Use recursion on nodes of "the fifth type" — given by the function σ. □

Result. The set $\mathscr{R}(R^k, R^\infty)$ coincide with $\mathscr{R}^1(R^k, R^\infty)$ if and only if we allow vectorization rule to be used to construct $\mathscr{R}^1$.