Positivity
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Positivity

Spectral analysis of a generalized buckling problem on a ball

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Abstract In this paper, the spectrum of the following fourth order problem

$$\begin{cases} \Delta^2 u + \nu u = -\lambda \Delta u & \text{in } D_1, \\ u = \partial_r u = 0 & \text{on } \partial D_1, \end{cases}$$

where D_1 is the unit ball in $\mathbb{R}^N$, is determined for $\nu < 0$ as well as the nodal properties of the corresponding eigenfunctions. In particular, we show that the first eigenvalue is simple and that the corresponding eigenfunction is radial and (up to a multiplicative factor) positive and decreasing with respect to the radius. This completes earlier results obtained for $\nu \geqslant 0$ (see Coster et al. in Positivity 19:843–875, 2015) and for $\nu < 0$ (see Laurençot and Walker in J Anal Math 127:69–89, 2014).

Keywords Buckling problem · Spectral analysis · Properties of eigenfunctions

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1 Introduction

In this paper we look for any pairs of real numbers (ν, λ) such that a nontrivial solution u of

$$\begin{cases} \Delta^2 u + \nu u = -\lambda \Delta u & \text{in } D_1, \\ u = \partial_r u = 0 & \text{on } \partial D_1, \end{cases} \tag{1}$$

exists, where D_1 is the unit ball of $\mathbb{R}^N$, $N \geqslant 2$. This problem can be viewed as an eigenvalue problem in λ, once ν is fixed, or as an eigenvalue problem in ν, once λ is fixed; both cases being points of view on finding pairs (ν, λ) in the plane.

There are few references concerning the general problem (1) and they concern either the eigenvalue problem in λ ($\geqslant 0$) for $\nu \geqslant 0$ fixed or the eigenvalue problem in ν ($\leqslant 0$) for $\lambda \leqslant 0$ fixed i.e., only the situation in the first or the third quadrant of the plane is considered (see in particular [11,19,25] and the references therein). These papers are mainly concerned with the behaviour of functions $\lambda : \mathbb{R}^+ \to \mathbb{R} : \nu \mapsto \lambda(\nu)$ (resp. $\nu : \mathbb{R}^+ \to \mathbb{R} : \lambda \mapsto -\nu(-\lambda)$). In particular, in [19], the authors prove that the first eigenvalue $\lambda_1 : \mathbb{R}^+ \to \mathbb{R}^+ : \nu \mapsto \lambda_1(\nu)$, as a function of ν, is strictly concave, increasing, and satisfies

$$\forall \nu \in \,]0, +\infty[, \qquad \max\left\{\lambda_1(0), 2\sqrt{\nu}\right\} < \lambda_1(\nu) < \lambda_1(0) + \frac{\nu}{\xi_1}$$

with ξ_1 being the first eigenvalue of the Laplacien in $H_0^1(D_1)$ with corresponding eigenfunction ψ_1. Similar results are obtained for $-\nu_1(-\lambda)$. Moreover, they prove that $-\nu_1(-\lambda)/\lambda \to \xi_1$ as λ tends to zero and that the corresponding eigenfunction converges to ψ_1.

More recently, we find in the literature results on the structure of the eigenfunctions and in particular the positivity of the first eigenfunction. There is a vast literature on this last question about the case $\nu = 0$ or $\lambda = 0$, see [4,7–9,12,13,15,16,20,27,28] which corresponds to the situations on the axes.

In [22], the authors consider the problem

$$\begin{cases} \Delta^2 u - \tau \Delta u = \omega u & \text{in } D_1, \\ u = \partial_r u = 0 & \text{on } \partial D_1, \end{cases} \tag{2}$$

where the term $\Delta^2 u$ accounts for the bending, while the term $-\tau \Delta u$, with $\tau > 0$ for stretching; ω being the (positive) eigenvalues they are looking for. Observe that reversing the point of view, problem (2) is related to (1) with $\nu = -\omega$ and $\lambda = -\tau$. In particular, for $\tau \geqslant 0$, the authors of [22] prove the existence of ω_1 such that the problem (2) has a positive radially symmetric eigenfunction and that ω_1 is the only such eigenvalue. However, the authors observe that there could exist non-radially symmetric positive eigenfunctions i.e., ω_1 may not be the only eigenvalue with positive eigenfunctions nor the smallest eigenvalue. Our result proves that, in fact, it is not the case. We can also refer to [5,6] where other boundary conditions are considered.

In [10], motivated by the study of clamped thin elastic membranes supported on a fluid substrate, we considered the case $\nu \geqslant 0$ and, in particular, we gave a complete description of the smallest eigenvalue λ_1 and its eigenfunction u. In this work, we want to extend the analysis started in [10] to any $\nu \in \mathbb{R}$. More precisely, we determine pairs (ν, λ) with $\nu < 0$ such that (1) has a nontrivial solution u as well as the shape of the corresponding solution u. In particular, we obtain precise information about the smallest eigenvalue $\lambda_1(\nu)$ and its associated eigenfunction as a function of ν. Putting together the results of this paper with those of [10], we obtain the following theorem concerning the case where D_1 is the unit ball of $\mathbb{R}^2$. In this result and throughout the paper, $(j_{k,\ell})_{\ell \geqslant 1}$ denote the roots of J_k, the Bessel function of the first kind of order k.

Theorem 1 *If D is the unit ball of $\mathbb{R}^2$, the first eigenvalue $\lambda_1 : \mathbb{R} \to \mathbb{R} : \nu \mapsto \lambda_1(\nu)$ of (1) is a continuous, increasing function of ν such that*

$$\lim_{\nu \to -\infty} \lambda_1(\nu) = -\infty, \quad \lim_{\nu \to +\infty} \lambda_1(\nu) = +\infty \quad \text{and} \quad \lambda_1(0) = j_{1,1}^2.$$

Hence it is a bijection from $\mathbb{R}$ into itself. Moreover,

- *If $\nu \in \,]-\infty, (j_{0,1} j_{0,2})^2[$, the first eigenvalue is simple and the eigenfunctions φ_1 are radial, one-signed and $|\varphi_1|$ is decreasing with respect to the radius r.*
- *If $\nu \in \,](j_{1,n} j_{1,n+1})^2, (j_{0,n+1} j_{0,n+2})^2[$, for some $n \geqslant 1$, the first eigenvalue is simple and the eigenfunctions are radial and have $n+1$ nodal regions.*
- *If $\nu \in \,](j_{0,n+1} j_{0,n+2})^2, (j_{1,n+1} j_{1,n+2})^2[$, for some $n \geqslant 0$, the eigenfunctions φ_1 have the form*

  $$R_{1,1}(r)(c_1 \cos\theta + c_2 \sin\theta), \qquad c_1, c_2 \in \mathbb{R}.$$

  Moreover the function $R_{1,1}$ has n simple zeros in $]0, 1[$, i.e., φ_1 has $2(n+1)$ nodal regions.

Information on the eigenspaces at the countably many $\nu > 0$ not considered in the previous theorem is also provided in [10]. For these ν, the eigenspaces have even larger dimensions (see [10, Theorem 4.18]).

For $\nu < 0$, we can also give a characterization of higher eigenvalues. In particular, the nodal properties of their eigenfunctions are completely determined.

Theorem 2 *If D is the unit ball of $\mathbb{R}^2$, there exist increasing, differentiable functions $\lambda_{k,\ell} : \,]-\infty, 0[\, \to \mathbb{R} : \nu \mapsto \lambda_{k,\ell}(\nu)$ for $k \in \mathbb{N}$ and $\ell \in \mathbb{N}^+$ such that the spectrum of (1) for a given $\nu \in \,]-\infty, 0[$ is exactly $\{\lambda_{k,\ell}(\nu) \mid k \in \mathbb{N},\ \ell \in \mathbb{N}^+\}$. Moreover*

$$\lim_{\nu \to -\infty} \lambda_{k,\ell}(\nu) = -\infty \quad \text{and} \quad \lim_{\nu \to 0} \lambda_{k,\ell}(\nu) = j_{k+1,\ell}^2.$$

Associated to $\lambda_{k,\ell}$ is a space of eigenfunctions in spherical coordinates of the form $R_{k,\ell}(r)\big(c_1 \cos(k\theta) + c_2 \sin(k\theta)\big)$ where $c_1, c_2 \in \mathbb{R}$ and

$$R_{k,\ell}(r) := c J_k(\alpha_{k,\ell}\, r) + d I_k\Big(\frac{\kappa}{\alpha_{k,\ell}}\, r\Big),$$

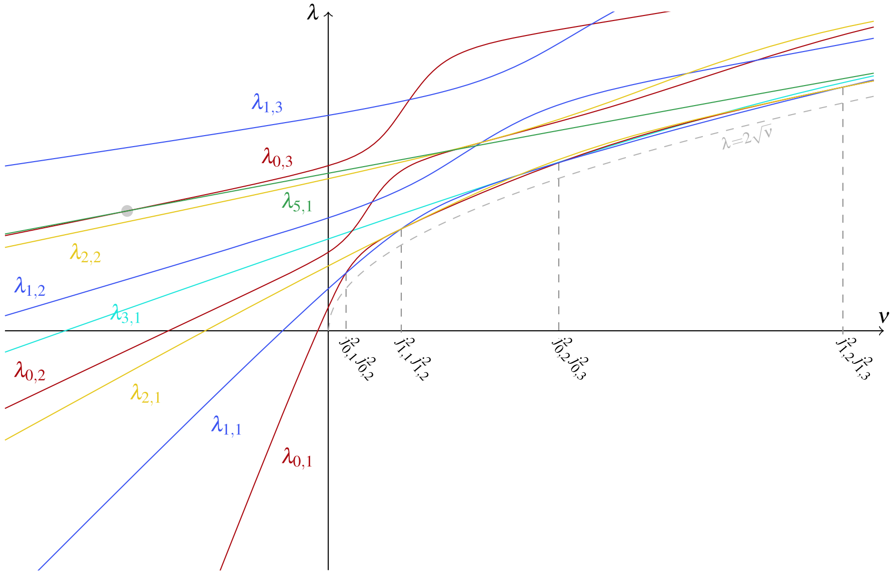

Fig. 1 Curves of (ν, λ) along which (1) possesses a nontrivial solution. The *dot* indicates where the graphs of $\lambda_{0,3}$ (*red*) and $\lambda_{5,1}$ (*green*) cross

for some $(c, d) \neq (0, 0)$ *suitably chosen (depending on* κ*,* k*, and* ℓ*). Here* I_k *(resp.* J_k*) denotes the modified Bessel function (resp. the Bessel function) of the first kind of order* k*. In addition* $R_{k,\ell}$ *possesses* $\ell - 1$ *roots in* $]0, 1[$*, all of which are simple.*

Here we write $\mathbb{N}$ for the set $\{0, 1, 2, \dots\}$ and $\mathbb{N}^+$ for $\{1, 2, \dots\}$. Figure 1 shows the graph of a few of the functions $\lambda_{k,\ell}$. It shows (and we prove) that

$$\lambda_1(\nu) = \min\bigl\{\lambda_{0,1}(\nu), \lambda_{1,1}(\nu)\bigr\}$$

where the minimum coincides with $\lambda_{0,1}$ for $\nu \leqslant (j_{0,1} j_{0,2})^2$ and alternates between $\lambda_{0,1}$ and $\lambda_{1,1}$ for larger ν, which explains the results of Theorem 1. The nodal properties of the eigenfunctions are illustrated by the graphs of the first six eigenfunctions for $\nu = -1$ which are drawn in Fig. 2.

The paper is organized as follows and concerns the case $\nu < 0$. In Sect. 2, we explain how to find solutions to (1) despite the fact that the method of separation of variables is not directly applicable. In Sect. 3, we show (see Theorem 8) that, for all $k \in \mathbb{N}$, there exists an increasing sequence $\alpha_{k,\ell} = \alpha_{k,\ell}(\kappa) \in \,]j_{k,\ell},\ j_{k+1,\ell}[$, $\ell \geqslant 1$, with $\kappa = \sqrt{-\nu}$, such that $\lambda_{k,\ell}(\nu) = \alpha_{k,\ell}^2 - \kappa^2/\alpha_{k,\ell}^2$ is an eigenvalue of (1) with corresponding eigenfunctions of the form $R_{k,\ell}(r)\,\mathrm{e}^{\pm \mathbf{i}k\theta}$. The minimal value of the spectrum $\lambda_1 = \lambda_1(\kappa) := \min_{k,\ell} \lambda_{k,\ell}$ corresponds to the minimum of $\{\alpha_{k,\ell} \mid k \in \mathbb{N},\ \ell \geqslant 1\}$ which is given by $\alpha_{0,1}$.

In Sect. 4, we show that $\lambda_1(\nu)$ is simple and its eigenfunction φ_1 is radial, one-signed and $|\varphi_1|$ is decreasing with respect to r (see Theorem 13). We further give precise statements about the nodal properties of the other eigenfunctions in Theorem 18.

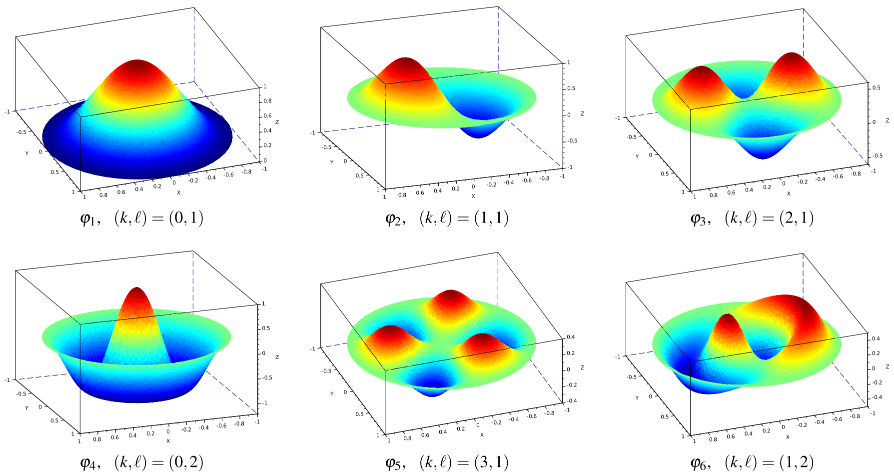

Fig. 2 Graphs of the first eigenfunctions for $\nu = -1$, mentioning the value (k, ℓ) of Theorem 2 to which they correspond

Section 5 explains how Theorems 1 ans 2 are obtained from the results in this paper and in [10].

Finally, the case $D_1 \subseteq \mathbb{R}^N$ with $N > 2$ is also considered in Sect. 6.

2 Preliminaries

Let us first consider the case $N = 2$ (see Sect. 6 for the case $N > 2$).

As the case $\nu \geqslant 0$ in (1) was fully treated in [10], we can restrict ourselves to the case $\nu < 0$, that we rewrite as

$$\begin{cases} \Delta^2 u - \kappa^2 u = -\lambda \Delta u & \text{in } D_1, \\ u = \partial_r u = 0 & \text{on } \partial D_1, \end{cases} \tag{3}$$

with $\kappa > 0$. Similarly to [10], we factorize

$$\Delta^2 u + \lambda \Delta u - \kappa^2 u = (\Delta + \alpha^2)(\Delta + \beta^2) u = 0 \tag{4}$$

with $\alpha, \beta \in \mathbb{C}$ such that $\alpha^2 \beta^2 = -\kappa^2$ and $\alpha^2 + \beta^2 = \lambda$. As $\kappa > 0$, α and β are both non zero and we can choose α as a positive real number and $\beta = \mathbf{i}\kappa/\alpha$ that is clearly different from α.

As we work in two dimensions, we use the ansatz $u(r, \theta) = R(r)\, \mathrm{e}^{\mathbf{i}k\theta}$ with $k \in \mathbb{Z}$, where (r, θ) are the polar coordinates, and notice that (4) is equivalent to a fourth order ordinary differential equation (in ∂_r)

$$L(\partial_r, r, \alpha, \beta, |k|) R = 0. \tag{5}$$

Hence by the theory of ordinary differential equations, L has four linearly independent solutions. To find them it suffices to notice that

$$(\Delta + \alpha^2)u = 0 \quad \Rightarrow \quad (\Delta + \alpha^2)(\Delta + \beta^2)u = 0.$$

Thus, if

$$(\Delta + \alpha^2)\big(R(r)\,\mathrm{e}^{\mathbf{i}k\theta}\big) = 0, \tag{6}$$

then R is a solution to (5). But a solution to (6) is simpler to find. Indeed such R satisfy the Bessel equation

$$(r\partial_r)^2 R + \alpha^2 r^2 R = k^2 R.$$

As $\alpha \neq 0$, R is then a linear combination of $J_{|k|}(\alpha r)$ and of $Y_{|k|}(\alpha r)$.

Similarly, if

$$\Big(\Delta - \frac{\kappa^2}{\alpha^2}\Big)\big(R(r)\,\mathrm{e}^{\mathbf{i}k\theta}\big) = 0, \tag{7}$$

then R is a solution to (5). But R is solution to (7) if and only if R satisfies the modified Bessel equation

$$(r\partial_r)^2 R - \frac{\kappa^2}{\alpha^2} r^2 R = k^2 R.$$

Therefore, R is a linear combination of $I_{|k|}(\frac{\kappa r}{\alpha})$ and of $K_{|k|}(\frac{\kappa r}{\alpha})$.

Summing up, we have proved the following result.

Lemma 3 *Let $k \in \mathbb{Z}$. If $\alpha \in \,]0, \infty[$ and $\beta = \mathbf{i}\kappa/\alpha$, then the four linearly independent solutions to (5) are $J_{|k|}(\alpha r)$, $Y_{|k|}(\alpha r)$, $I_{|k|}(\frac{\kappa r}{\alpha})$ and $K_{|k|}(\frac{\kappa r}{\alpha})$.*

3 Eigenvalue problem

Here we want to characterize the full spectrum of the buckling problem on the unit disk. In other words, we look for a $u \neq 0$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \Delta^2 u - \kappa^2 u = -\lambda \Delta u & \text{in } D_1, \\ u = \partial_r u = 0 & \text{on } \partial D_1, \end{cases} \tag{8}$$

where D_1 is the unit ball of $\mathbb{R}^2$.

Proposition 4 *The eigenfunctions of the boundary value problem (8) are of the form $u = R(r)\,\mathrm{e}^{\mathbf{i}k\theta}$ with $k \in \mathbb{Z}$ and R given by*

$$R(r) = c J_{|k|}(\alpha r) + d I_{|k|}\Big(\frac{\kappa}{\alpha} r\Big), \tag{9}$$

for some $c, d \in \mathbb{R}$, where α is a positive solution to

$$F_k(\alpha) := \frac{\kappa}{\alpha} J_{|k|}(\alpha) I'_{|k|}\Big(\frac{\kappa}{\alpha}\Big) - \alpha I_{|k|}\Big(\frac{\kappa}{\alpha}\Big) J'_{|k|}(\alpha) = 0. \tag{10}$$

The corresponding eigenvalue is $\lambda = \alpha^2 - \kappa^2/\alpha^2$.

Proof According to the previous section, we look for solutions u to (3) in the form

$$u = R(r)\,\mathrm{e}^{\mathbf{i}k\theta}, \qquad \text{with } k \in \mathbb{Z}.$$

From Lemma 3, we see that

$$R(r) = c\,J_{|k|}(\alpha r) + d\,I_{|k|}\Big(\frac{\kappa}{\alpha} r\Big), \tag{11}$$

for some $c, d \in \mathbb{R}$ (since R and R' are bounded near $r = 0$). Hence the boundary conditions at $r = 1$ lead to the system

$$\begin{cases} c\,J_{|k|}(\alpha) + d\,I_{|k|}\big(\frac{\kappa}{\alpha}\big) = 0, \\ c\,\alpha\,J'_{|k|}(\alpha) + d\,\frac{\kappa}{\alpha} I'_{|k|}\big(\frac{\kappa}{\alpha}\big) = 0. \end{cases} \tag{12}$$

This 2×2 system has a non-trivial solution (c, d) if and only if its determinant is equal to zero, namely if and only if (10) is satisfied.

The fact that $(\mathrm{e}^{\mathbf{i}k\theta})_{k\in\mathbb{Z}}$ form a basis of $L^2(]0, 2\pi[)$ allows to conclude that no other eigenvalues exist. For the reader's convenience, let us quickly recall the argument given in [10, Theorem 3.2] to prove this fact. Let u be a solution to (8) with eigenvalue λ. Let us express u in polar coordinates (r, θ) and write

$$u = \sum_{k\in\mathbb{Z}} u_k(r)\,\mathrm{e}^{\mathbf{i}k\theta}, \qquad \text{where } u_k(r) := \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\,\mathrm{e}^{-\mathbf{i}k\theta}\,\mathrm{d}\theta$$

(the integral makes sense because u is smooth). Multiplying both sides of equation (8) by $\mathrm{e}^{-\mathbf{i}k\theta}$ and integrating by parts in θ shows that

$$\forall k \in \mathbb{Z}, \quad L(\partial_r, r, \alpha, \mathbf{i}\kappa/\alpha, |k|)u_k = 0.$$

The discussion following (5) then shows that u_k is of the form (11). The boundary conditions on u will give boundary conditions on u_k which in turn imply that (12) is satisfied. In conclusion u can be expressed as a series of functions $R(r)\,\mathrm{e}^{\mathbf{i}k\theta}$ where R has the form (11) with $F_k(\alpha) = 0$. □

Let us start with a technical result.

Lemma 5 *Let $k \in \mathbb{N}$. Then the function*

$$G_k :]0, +\infty[\to \mathbb{R} : z \mapsto \frac{z I'_k(z)}{I_k(z)}$$

has a positive derivative and thus is increasing on $]0, +\infty[$.

Remark 6 Observe that, by (45),

$$G_k(z) = \frac{z I'_k(z)}{I_k(z)} = \frac{z I_{k+1}(z)}{I_k(z)} + k. \tag{13}$$

Hence by [21, Theorem 1.1] we have that

$$G_k(z) > -1 + \sqrt{(k+1)^2 + z^2}.$$

This implies in particular that

$$\lim_{z \to +\infty} G_k(z) = +\infty.$$

Proof Direct calculations using the differential equation satisfied by modified Bessel functions (47) yield

$$G_k'(z) = \frac{(z^2 + k^2) I_k^2(z) - z^2 (I_k'(z))^2}{z I_k^2(z)}.$$

Then using the recurrence relation (45), we deduce that

$$G_k'(z) = \frac{z^2 (I_k^2(z) - I_{k+1}^2(z)) - 2kz I_k(z) I_{k+1}(z)}{z I_k^2(z)}.$$

Hence with the notation $u = \frac{I_{k+1}(z)}{I_k(z)}$ from Lemma 21 (in the Appendix), we have

$$G_k'(z) = -z\Big(u^2 + \frac{2k}{z} u - 1\Big), \tag{14}$$

and then Lemma 21 implies that $G_k'(z) > 0$. □

Remark 7 Due to the relation (1.5) of [2] (see also [1, p. 256]),

$$\forall z > 0, \qquad z\left(I_k^2(z) - I_{k-1}(z) I_{k+1}(z)\right) = (I_k(z))^2 G_k'(z),$$

Lemma 5 is in fact equivalent to the Turán type inequality (53). Let us further mention that this Lemma was also proved by Gronwall [14, p. 277].

Theorem 8 *For all $k \in \mathbb{N}$ and $\kappa > 0$, the roots of F_k (defined by (10)) are simple and can be ordered as an increasing sequence $\alpha_{k,\ell} = \alpha_{k,\ell}(\kappa) > 0$, with $\ell \in \mathbb{N}^+$, such that*

$$\forall \ell > 0, \quad j_{k,\ell} < \alpha_{k,\ell} < j_{k+1,\ell}.$$

Each $\ell > 0$ gives rise to the eigenvalue

$$\lambda_{k,\ell} = \alpha_{k,\ell}^2 - \frac{\kappa^2}{\alpha_{k,\ell}^2}, \tag{15}$$

of (8) *and to corresponding eigenfunctions of the form* $R_{k,\ell}(r)\,\mathrm{e}^{\pm \mathbf{i}\,k\theta}$ *with*

$$R_{k,\ell}(r) = c\,J_k(\alpha_{k,\ell}\,r) + d\,I_k\Big(\frac{\kappa}{\alpha_{k,\ell}}\,r\Big),$$

where (c, d) *is a solution to* (12) *with* $\alpha = \alpha_{k,\ell}$.

Proof Since I_k is positive on $]0, +\infty[$ and the positive roots of J_k are simple, the positive roots of F_k never coincide with those of J_k. Hence, we can write

$$F_k(\alpha) = J_k(\alpha)\,I_k\Big(\frac{\kappa}{\alpha}\Big)\,\tilde{F}_k(\alpha)$$

with

$$\tilde{F}_k(\alpha) := G_k\Big(\frac{\kappa}{\alpha}\Big) - H_k(\alpha)$$

where G_k was defined in Lemma 5 and $H_k(z) := \frac{z J_k'(z)}{J_k(z)}$ (see [10, Lemma 4.1]) and we have

$$F_k(\alpha) = 0 \qquad \Leftrightarrow \qquad \tilde{F}_k(\alpha) = 0.$$

Using formulas (40) and (45), $\tilde{F}_k$ may also be written as

$$\tilde{F}_k(\alpha) = \frac{\kappa}{\alpha}\,\frac{I_{k+1}\big(\frac{\kappa}{\alpha}\big)}{I_k\big(\frac{\kappa}{\alpha}\big)} + \alpha\,\frac{J_{k+1}(\alpha)}{J_k(\alpha)}. \tag{16}$$

Observe first that if $\alpha \in\,]0, j_{k,1}[$, then $J_k(\alpha) > 0$ and $J_{k+1}(\alpha) > 0$ and so $\tilde{F}_k(\alpha) > 0$. Thus the roots of $\tilde{F}_k$ lie in $]j_{k,1}, +\infty[$. Easy computations show that, for all $\ell \geqslant 1$,

$$\lim_{\alpha \overset{>}{\longrightarrow} j_{k,\ell}} \tilde{F}_k(\alpha) = -\infty \qquad \text{and} \qquad \tilde{F}_k(j_{k+1,\ell}) = \frac{\kappa}{j_{k+1,\ell}}\,\frac{I_{k+1}\big(\frac{\kappa}{j_{k+1,\ell}}\big)}{I_k\big(\frac{\kappa}{j_{k+1,\ell}}\big)} > 0$$

This implies that $\tilde{F}_k$ possesses a root between $j_{k,\ell}$ and $j_{k,\ell+1}$.

To establish that it has a unique root in this interval, it is enough to prove that, for every root α^*, we have $\tilde{F}_k'(\alpha^*) > 0$. Direct calculations yield

$$\tilde{F}_k'(\alpha) = -\frac{\kappa}{\alpha^2}\,G_k'\Big(\frac{\kappa}{\alpha}\Big) - H_k'(\alpha). \tag{17}$$

First, let us use [10, Lemma 4.1] which states that

$$H_k'(z) = \frac{k^2 - z^2}{z} - z\,\frac{(J_k'(z))^2}{J_k(z)^2},$$

and, using (40), we obtain

$$H_k'(z) = -z + 2k\,\frac{J_{k+1}(z)}{J_k(z)} - z\,\frac{J_{k+1}^2(z)}{J_k^2(z)}.$$

Now using that α^* is a root of $\tilde{F}_k$ and remembering (16), we have

$$\frac{J_{k+1}(\alpha^*)}{J_k(\alpha^*)} = -\frac{\kappa}{(\alpha^*)^2}u,$$

where we have set $u := \frac{I_{k+1}(z)}{I_k(z)}$ with $z = \frac{\kappa}{\alpha^*}$. These two identities show that

$$H_k'(\alpha^*) = -\alpha^* - 2k\frac{\kappa}{(\alpha^*)^2}u - \frac{\kappa^2}{(\alpha^*)^3}u^2.$$

Using this identity together with (14) in (17), we get

$$\begin{aligned}
\tilde{F}_k'(\alpha^*) &= \frac{\kappa^2}{(\alpha^*)^3}\Big(u^2 + \frac{2k\alpha^*}{\kappa}u - 1\Big) + \alpha^* + 2k\frac{\kappa}{(\alpha^*)^2}u + \frac{\kappa^2}{(\alpha^*)^3}u^2 \\
&= \frac{2\kappa^2}{(\alpha^*)^3}\Big(u^2 + \frac{2k\alpha^*}{\kappa}u + \frac{(\alpha^*)^4 - \kappa^2}{2\kappa^2}\Big) \\
&= \frac{2\kappa^2}{(\alpha^*)^3}\Big(u^2 + \frac{2ku}{z} + \frac{(\alpha^*)^2}{2z^2} - \frac{1}{2}\Big) \qquad \text{with } z = \kappa/\alpha^*.
\end{aligned}$$

Hence to show that $\tilde{F}_k'(\alpha^*)$ is positive, it remains to prove that

$$u^2 + \frac{2ku}{z} + \frac{(\alpha^*)^2}{2z^2} - \frac{1}{2} > 0. \tag{18}$$

But the estimate (2.4) of [21] says that

$$u^2 + \frac{2(k+1)}{z}u - 1 > 0,$$

therefore

$$u^2 + \frac{2ku}{z} + \frac{(\alpha^*)^2}{2z^2} - \frac{1}{2} > \frac{1}{2} - \frac{2u}{z} + \frac{(\alpha^*)^2}{2z^2} = \frac{1}{2z^2}\big(z^2 - 4uz + (\alpha^*)^2\big).$$

Since $\alpha^* > j_{k,1} > j_{0,1} \geqslant \frac{3\pi}{4} > 2$ (see [17, Theorem 3] for the estimate on $j_{0,1}$) and, by Lemma 21, $u < 1$ for $k \geqslant 0$ and $z > 0$, we deduce that

$$u^2 + \frac{2ku}{z} + \frac{(\alpha^*)^2}{2z^2} - \frac{1}{2} > \frac{1}{2z^2}(z-2)^2 \geqslant 0.$$

This implies that, for every root α^*, we have $\tilde{F}_k'(\alpha^*) > 0$ and prove the uniqueness of the root of $\tilde{F}_k$ in $]j_{k,\ell}, j_{k,\ell+1}[$. This concludes the proof. □

We now give some further informations on the functions $\alpha_{k,\ell}$.

Lemma 9 *For all $k \in \mathbb{N}$ and $\ell \in \mathbb{N}^+$, the function $\alpha_{k,\ell} :]0, +\infty[\to \mathbb{R} : \kappa \mapsto \alpha_{k,\ell}(\kappa)$ is of class $\mathscr{C}^1$ and $\partial_\kappa \alpha_{k,\ell} < 0$.*

Proof Let us note $F_k(\alpha, \kappa)$ the function $F_k(\alpha)$ defined by (10) where we have explicited the dependence on κ. The assertion will result from the Implicit Function Theorem. Let us fix $k \in \mathbb{N}$, $\kappa^* > 0$ and $\alpha^* = \alpha_{k,\ell}(\kappa^*) > 0$.

Again as in the proof of Theorem 8, instead of working on $F_k(\alpha, \kappa)$, we consider the function $\tilde{F}_k(\alpha, \kappa)$. In the proof of Theorem 8, we already observed that

$$\partial_\alpha \tilde{F}_k(\alpha^*, \kappa^*) > 0.$$

Hence we can apply the Implicit Function Theorem to $\tilde{F}_k$ and there exists a $\mathscr{C}^1$ curve β_ℓ defined around κ^* such that, in a neighbourhood V of (α^*, κ^*),

$$F_k(\alpha, \kappa) = 0 \text{ if and only if } \alpha = \beta_\ell(\kappa).$$

Moreover, using Lemma 5, it is easily seen that

$$\partial_\kappa \tilde{F}_k(\alpha^*, \kappa^*) > 0$$

and so

$$\partial_\kappa \beta_\ell(\kappa^*) = -\frac{\partial_\kappa \tilde{F}_k(\alpha^*, \kappa^*)}{\partial_\alpha \tilde{F}_k(\alpha^*, \kappa^*)} < 0.$$

As, for all κ, $\alpha_{k,\ell}(\kappa)$ is the only root of $\tilde{F}_k$ in $]j_{k,\ell}, j_{k,\ell+1}[$, we have, for all $\kappa \in V$,

$$\beta_\ell(\kappa) = \alpha_{k,\ell}(\kappa),$$

whence the desired result. □

Lemma 10 *Let $k \in \mathbb{N}$ and $\ell > 0$. Then we have*

$$\lim_{\kappa \to 0} \alpha_{k,\ell}(\kappa) = j_{k+1,\ell} \quad \textit{while} \quad \lim_{\kappa \to \infty} \alpha_{k,\ell}(\kappa) = j_{k,\ell}.$$

Proof As $\alpha_{k,\ell}$ is decreasing and bounded (for all $\kappa > 0$, $j_{k,\ell} < \alpha_{k,\ell}(\kappa) < j_{k+1,\ell}$), these two limits exist. Let us denote

$$\alpha_0 := \lim_{\kappa \to 0} \alpha_{k,\ell}(\kappa) \quad \text{and} \quad \alpha_\infty := \lim_{\kappa \to \infty} \alpha_{k,\ell}(\kappa).$$

It remains to find their value. Recall that, for all $z > 0$, we have $0 < \frac{I_{k+1}(z)}{I_k(z)} < 1$ and hence, formula (16) implies

$$\begin{aligned}
0 &= \lim_{\kappa \to 0} \tilde{F}_k\big(\alpha_{k,\ell}(\kappa)\big) \\
&= \lim_{\kappa \to 0} \left[\frac{\kappa}{\alpha_{k,\ell}(\kappa)} \frac{I_{k+1}\big(\frac{\kappa}{\alpha_{k,\ell}(\kappa)}\big)}{I_k\big(\frac{\kappa}{\alpha_{k,\ell}(\kappa)}\big)} + \alpha_{k,\ell}(\kappa) \frac{J_{k+1}(\alpha_{k,\ell}(\kappa))}{J_k(\alpha_{k,\ell}(\kappa))} \right] \\
&= \alpha_0 \frac{J_{k+1}(\alpha_0)}{J_k(\alpha_0)}.
\end{aligned}$$

This implies that $\alpha_0 = j_{k+1,\ell}$.

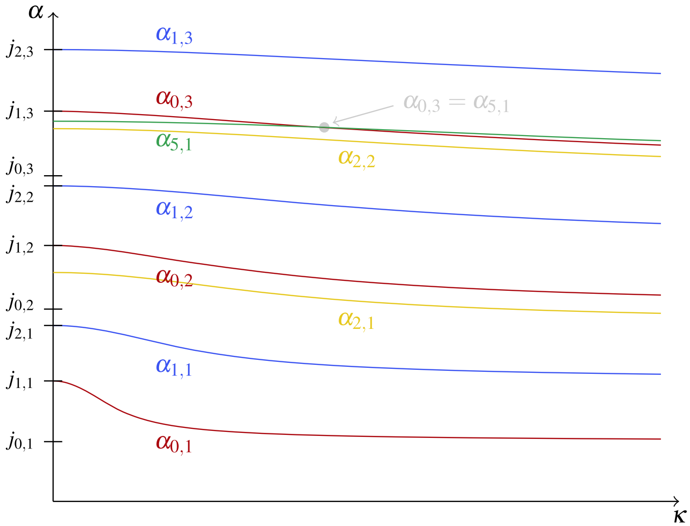

Fig. 3 Graph of some $\alpha_{k,\ell}$

On the other hand, Remark 6 and $\alpha_{k,\ell} > j_{k,\ell}$ imply

$$\lim_{\kappa\to\infty} G_k\left(\frac{\kappa}{\alpha_{k,\ell}(\kappa)}\right) = +\infty.$$

Given that $0 = \tilde{F}_k(\alpha_{k,\ell}) = G_k(\kappa/\alpha_{k,\ell}) - H_k(\alpha_{k,\ell})$, one has

$$\lim_{\kappa\to\infty} H_k(\alpha_{k,\ell}) = +\infty$$

and hence $J_k(\alpha_\infty) = 0$. This shows that $\alpha_\infty = j_{k,\ell}$ and concludes the proof. □

Remark 11 Computer generated graphs of the first $\alpha_{k,\ell}$ are drawn on Fig. 3. From this, one would for instance naturally conjecture[1] that

$$\forall\kappa > 0, \quad \alpha_{0,1}(\kappa) < \alpha_{1,1}(\kappa) < \alpha_{2,1}(\kappa) < \alpha_{0,2}(\kappa) < \alpha_{1,2}(\kappa) < \alpha_{2,2}(\kappa).$$

However, the picture does not stay that simple once $\alpha_{k,\ell}$ with higher indices are drawn: many crossings appear. As an example, we have drawn graph of $\alpha_{5,1}$ which crosses the one of $\alpha_{0,3}$ for $\kappa \approx 44.616$, thereby generating an eigenspace of higher dimension. Nevertheless we can prove the following result concerning the first eigenvalue.

Theorem 12 *The first eigenvalue of* (8) *is simple and given by*

$$\lambda_1 = \alpha_{0,1}^2 - \frac{\kappa^2}{\alpha_{0,1}^2} \tag{19}$$

[1] Some of these inequalities are obvious from the bounds on $\alpha_{k,\ell}$ that were obtained in Theorem 8.

with corresponding eigenfunctions given by

$$R_{0,1}(r) = c\, J_0(\alpha_{0,1}\, r) + d\, I_0\Big(\frac{\kappa}{\alpha_{0,1}}\, r\Big), \tag{20}$$

where (c, d) *is a nontrivial solution to* (12) *with* $\alpha = \alpha_{0,\ell}$. *Moreover* λ_1 *is a decreasing function of* κ,

$$\lim_{\kappa \to 0} \lambda_1(\kappa) = j_{1,1}^2 \qquad \textit{and} \qquad \lim_{\kappa \to \infty} \lambda_1(\kappa) = -\infty.$$

Proof First note that the function $]0, +\infty[\, \to \mathbb{R} : \alpha \mapsto \alpha^2 - \kappa^2/\alpha^2$ is increasing. The localization of $\alpha_{k,\ell}$ given in Theorem 8 readily shows that $\alpha_{0,1}$ is the smaller of all $\alpha_{k,\ell}$. Formula (19) and (20) immediately follow. The monotonicity comes from Lemma 9 and the asymptotic values from Lemma 10. □

4 Nodal properties of eigenfunctions

In this section, we prove that the eigenfunction $R_{0,1}$ is positive and decreasing, and give precise statements about the change of sign of the other eigenfunctions.

Theorem 13 *Let* $\kappa > 0$ *and* $R_{0,1}$ *be the function associated with* $\alpha_{0,1}$ *defined by* (20) *in Theorem 12. Then* $r \mapsto |R_{0,1}(r)|$ *is positive in* $[0, 1[$ *and decreasing.*

Proof As $R_{0,1}$ is given by (20) where (c, d) is a nontrivial solution of (12) with $\alpha = \alpha_{0,1}$, we may choose $c = I_0\big(\frac{\kappa}{\alpha_{0,1}}\big)$ and $d = -J_0(\alpha_{0,1})$. Since $\alpha_{0,1} \in\,]j_{0,1}, j_{1,1}[$, we deduce that c and d are positive.

Now we want to show that $v(r) := \partial_r R_{0,1}(r) < 0$ for all $r \in\,]0, 1[$. As $R_{0,1}(1) = 0$, we then obtain also $R_{0,1} > 0$ on $[0, 1[$. First observe that v is given by

$$v(r) = -c\, \alpha_{0,1} J_1(\alpha_{0,1} r) + d\, \frac{\kappa}{\alpha_{0,1}} I_1\Big(\frac{\kappa}{\alpha_{0,1}} r\Big). \tag{21}$$

A simple computation using (42) and (47) shows that v solves

$$\begin{aligned} &-\partial_r^2 v - \frac{1}{r}\partial_r v + \Big(\frac{1}{r^2} + \frac{\kappa^2}{\alpha_{0,1}^2}\Big) v = -c\alpha_{0,1}\Big(\alpha_{0,1}^2 + \frac{\kappa^2}{\alpha_{0,1}^2}\Big) J_1(\alpha_{0,1} r), \\ &v(0) = 0, \quad v(1) = 0. \end{aligned} \tag{22}$$

This problem can be rewritten under the form

$$\begin{aligned} &-\partial_r(r\, \partial_r v) + r\Big(\frac{1}{r^2} + \frac{\kappa^2}{\alpha_{0,1}^2}\Big) v = -c\alpha_{0,1}\Big(\alpha_{0,1}^2 + \frac{\kappa^2}{\alpha_{0,1}^2}\Big) r\, J_1(\alpha_{0,1} r), \\ &v(0) = 0, \quad v(1) = 0. \end{aligned} \tag{23}$$

Since $-c\alpha_{0,1}\Big(\alpha_{0,1}^2 + \frac{\kappa^2}{\alpha_{0,1}^2}\Big) r\, J_1(\alpha_{0,1} r)$ is negative in $]0, 1[$, multiplying (23) by v^+ and integrating we obtain

$$0 \leqslant \int_0^1 r|\partial_r(v^+)(r)|^2 \,\mathrm{d}r \leqslant \int_0^1 -c\alpha_{0,1}\Big(\alpha_{0,1}^2 + \frac{\kappa^2}{\alpha_{0,1}^2}\Big) r\, J_1(\alpha_{0,1} r) v^+ \,\mathrm{d}r \leqslant 0.$$

This implies that $v^+ \equiv 0$ i.e., $v \leqslant 0$. On the other hand, if there exists $r_0 \in]0, 1[$ such that $v(r_0) = \max_{[0,1]} v = 0$ then $\partial_r v(r_0) = 0$, $\partial_r^2 v(r_0) \leqslant 0$ which gives a contradiction with (22).

This implies that $v < 0$ on $]0, 1[$ and concludes the proof.

Theorem 14 *Let $\kappa > 0$, $k \in \mathbb{N}$, and $R_{k,1}$ be the function defined by*

$$R_{k,1}(r) = c\, J_k(\alpha_{k,1} r) + d\, I_k\Big(\frac{\kappa}{\alpha_{k,1}} r\Big) \tag{24}$$

with (c, d) a nontrivial solution to (12). Then $|R_{k,1}| > 0$ in $[0, 1[$.

Proof As (c, d) is a nontrivial solution of (12) with $\alpha = \alpha_{k,1}$, we may choose $c = I_k\big(\frac{\kappa}{\alpha_{k,1}}\big)$ and $d = -J_k(\alpha_{k,1})$ in (24). Since $\alpha_{k,1} \in]j_{k,1}, j_{k+1,1}[$, we deduce that c and d are positive.

For $r \in]0, j_{k,1}/\alpha_{k,1}]$, $R_{k,1}(r)$ is clearly positive as the sum of a non-negative and a positive term.

To prove that $R_{k,1}$ is positive on $]j_{k,1}/\alpha_{k,1}, 1[$, suppose on the contrary the existence of $r^* \in]j_{k,1}/\alpha_{k,1}, 1[$ such that $R_{k,1}(r^*) = 0$. A simple computation using (42) and (47) shows that $u := R_{k,1}$ is a solution to

$$\begin{aligned} &-\partial_r(r\,\partial_r u) + \Big(\frac{k^2}{r^2} + \frac{\kappa^2}{\alpha_{k,1}^2}\Big) r\, u = c\Big(\alpha_{k,1}^2 + \frac{\kappa^2}{\alpha_{k,1}^2}\Big) r\, J_k(\alpha_{k,1} r) \\ &u(r^*) = 0, \quad u(1) = 0. \end{aligned} \tag{25}$$

Note that, for any $r \in]r^*, 1[$, $\alpha_{k,1} r \in]j_{k,1}, \alpha_{k,1}[\subseteq]j_{k,1}, j_{k+1,1}[\subseteq]j_{k,1}, j_{k,2}[$ and so the right hand side of (25) is negative. Multiplying the equation by u^+ and integrating yields

$$0 \leqslant \int_{r^*}^1 r|\partial_r(u^+)|^2 \,\mathrm{d}r \leqslant \int_{r^*}^1 c\Big(\alpha_{k,1}^2 + \frac{\kappa^2}{\alpha_{k,1}^2}\Big) r\, J_k(\alpha_{k,1} r) u^+ \,\mathrm{d}r \leqslant 0,$$

and so $u^+ \equiv 0$ i.e., $u \leqslant 0$ on $[r^*, 1]$. Now, evaluating (25) at $r = 1$ and taking into account the clamped boundary conditions yields

$$\partial_r^2 R_{k,1}(1) = -c\Big(\alpha_{k,1}^2 + \frac{\kappa^2}{\alpha_{k,1}^2}\Big) J_k(\alpha_{k,1}) > 0.$$

This contradicts the fact that $R_{k,1}$ is nonpositive on $[r^*, 1]$ and concludes the proof.

Lemma 15 *Let $\kappa > 0$, $k \in \mathbb{N}$, $\ell \in \mathbb{N}^+$ and $R_{k,\ell}$ be the function defined by (9) with $\alpha = \alpha_{k,\ell}$ and $(c, d) \neq (0, 0)$ a solution to (12). As it is customary, let $(j'_{k,n})_{n\geqslant 1}$ be the positive zeros of J'_k in increasing order, except for $k = 0$ for which we set $j'_{0,1} = 0$. Then*

$$\forall n = 1, \ldots, \ell, \qquad \operatorname{sign} R_{k,\ell}\Big(\frac{j'_{k,n}}{\alpha_{k,\ell}}\Big) = \operatorname{sign}(c)\,(-1)^{n+1}. \tag{26}$$

Proof Up to a multiplicative constant, the function $R_{k,\ell}$ can be written as

$$R_{k,\ell}(r) := c J_k(\alpha_{k,\ell} r) + d I_k\Big(\frac{\kappa}{\alpha_{k,\ell}} r\Big) \tag{27}$$

with $c = I_k\big(\frac{\kappa}{\alpha_{k,\ell}}\big) > 0$ and $d = -J_k(\alpha_{k,\ell})$. To fix the ideas, the proof will be carried out for ℓ even; the case of ℓ odd being similar. Because $\alpha_{k,\ell} \in \,]j_{k,\ell}, j_{k+1,\ell}[$ (see Theorem 8), $d < 0$. Note also that the lower bound on $\alpha_{k,\ell}$ implies that $j'_{k,n}/\alpha_{k,\ell} < 1$ for all $n = 1, \ldots, \ell$. If n is even, then $J_k(j'_{k,n}) < 0$ which immediately implies that $R_{k,\ell}(j'_{k,n}/\alpha_{k,\ell}) < 0$. To conclude the proof, it remains to show that $R_{k,\ell}(j'_{k,n}/\alpha_{k,\ell}) > 0$ when n is odd. It is well known [26, p. 37] that $|J_k(j'_{k,n})|$ decreases with respect to n. Thus, for odd $n \in \{1, \ldots, \ell\}$, $J_k(j'_{k,n}) > J_k(j'_{k,\ell+1}) \geqslant J_k(\alpha_{k,\ell}) = -d$ where the last inequality results from the fact that $j'_{k,\ell+1}$ is the point of maximum of J_k over the interval $[j_{k,\ell}, j_{k,\ell+1}]$ and $\alpha_{k,\ell} \in \,]j_{k,\ell}, j_{k+1,\ell}[$. Moreover, as I_k is increasing, $I_k\Big(\frac{\kappa}{\alpha_{k,\ell}} \frac{j'_{k,n}}{\alpha_{k,\ell}}\Big) < I_k\big(\frac{\kappa}{\alpha_{k,\ell}}\big) = c$. Putting the last two inequalities together proves that $R_{k,\ell}(j'_{k,n}/\alpha_{k,\ell}) > 0$ for odd n.

Lemma 16 *Let $k \in \mathbb{N}$ and $\ell \in \mathbb{N}^+$. Then*

$$\forall \kappa \in \,]0, +\infty[, \qquad I_k\Big(\frac{\kappa}{\alpha_{k,\ell}(\kappa)}\Big)\alpha^k_{k,\ell}(\kappa) - J_k\big(\alpha_{k,\ell}(\kappa)\big)\Big(\frac{\kappa}{\alpha_{k,\ell}(\kappa)}\Big)^k > 0.$$

Remark 17 For odd values of ℓ, the estimate $\alpha_{k,\ell} \in \,]j_{k,\ell}, j_{k+1,\ell}[$ implies $J_k(\alpha_{k,\ell}) < 0$ and the statement thus clearly holds. The following proof shows that the statement is true for all values of ℓ.

Proof Let us rewrite the statement as

$$g\Big(\frac{\kappa}{\alpha_{k,\ell}}\Big) - h(\alpha_{k,\ell}) > 0, \quad \text{where } g(z) := \frac{I_k(z)}{z^k} \text{ and } h(z) = \frac{J_k(z)}{z^k}.$$

A simple computation using (45) yields $\partial_z g(z) = I_{k+1}(z)/z^k > 0$. So g is increasing and, using the expansion (49) of I_k around 0, one gets:

$$\forall z > 0, \qquad g(z) > \lim_{z\to 0} g(z) = \frac{1}{2^k\, k!}.$$

The proof will be complete if we show

$$\forall z > 0, \qquad \frac{1}{2^k\, k!} = \lim_{z\to 0} h(z) > h(z). \tag{28}$$

The equality directly follows from the expansion (48). Formula (40) implies that $\partial_z h(z) = -J_{k+1}(z)/z^k$. Thus h is decreasing on $]0, j_{k+1,1}] \supseteq]0, j_{k,1}]$.

Because $|J_k(j'_{k,n})|$ decreases with respect to n (see [26, p. 37]), one deduces that, for all $z \geqslant j_{k,1}$, $J_k(j'_{k,1}) > |J_k(z)|$ and so $h(j'_{k,1}) > |h(z)|$. The fact that $j'_{k,1} \in]0, j_{k,1}]$ where h is decreasing establishes the inequality (28).

Theorem 18 *Let $\kappa > 0$, $k \in \mathbb{N}$, $\ell \in \mathbb{N}^+$ and $R_{k,\ell}$ be the function defined by* (9) *with $\alpha = \alpha_{k,\ell}$ and $(c, d) \neq (0, 0)$ a solution to* (12). *The function $R_{k,\ell}$ possesses $\ell - 1$ roots in $]0, 1[$, all of which are simple.*

Proof Lemma 15 says that $R_{k,\ell}(j'_{k,n}/\alpha_{k,\ell})$ takes alternate signs when n runs from 1 to ℓ. Thus $R_{k,\ell}$ possesses at least $\ell - 1$ zeros. It remains to show that there is only one root in each interval $]j'_{k,n}/\alpha_{k,\ell}, j'_{k,n+1}/\alpha_{k,\ell}[$, $n = 1, \ldots, \ell - 1$, and that there are no roots in $]0, j'_{k,1}/\alpha_{k,\ell}[$ and $]j'_{k,\ell}/\alpha_{k,\ell}, 1[$.

A direct computation using the definition (27) of $R_{k,\ell}$ as well as (42) and (47) shows that $u = R_{k,\ell}$ is a solution to:

$$-\partial_r(r\,\partial_r u) + \Big(\frac{k^2}{r^2} + \frac{\kappa^2}{\alpha_{k,\ell}^2}\Big) r\, u = c\Big(\alpha_{k,\ell}^2 + \frac{\kappa^2}{\alpha_{k,\ell}^2}\Big) r\, J_k(\alpha_{k,\ell} r). \tag{29}$$

Note also that, for $n = 1, \ldots, \ell$,

$$\operatorname{sign} R_{k,\ell}\Big(\frac{j_{k,n}}{\alpha_{k,\ell}}\Big) = \operatorname{sign}\Big(d\, I_k\Big(\frac{\kappa}{\alpha_{k,\ell}} r\Big)\Big) = \operatorname{sign}(d). \tag{30}$$

The expansions (48) and (49) yield

$$R_{k,\ell}(r) = \gamma \frac{r^k}{2^k\, k!}\big(1 + o(1)\big)$$

where $\gamma := c\alpha_{k,\ell}^k + d(\kappa/\alpha_{k,\ell})^k$ is positive because of the choice of c and d in (27) and Lemma 16. Thus, there exists $\epsilon > 0$ such that $R_{k,\ell}(r) > 0$ for all $r \in]0, \epsilon]$. Without loss of generality, we can assume that $\epsilon < j'_{k,1}/\alpha_{k,\ell}$.

On $]\epsilon, j_{k,1}/\alpha_{k,\ell}[$, the right hand side of (29) is positive. Because $R_{k,\ell}(\epsilon)$ and $R_{k,\ell}(j'_{k,1}/\alpha_{k,\ell})$ are both positive, the maximum principle implies that $R_{k,\ell} > 0$ on $[\epsilon, j'_{k,1}/\alpha_{k,\ell}]$. Thus $R_{k,\ell}$ has no root in $]0, j'_{k,1}/\alpha_{k,\ell}[$.

If $d > 0$, by (30), we have $R_{k,\ell}(j_{k,1}/\alpha_{k,\ell}) > 0$ and the same reasoning shows that $R_{k,\ell} > 0$ on $]j'_{k,1}/\alpha_{k,\ell}, j_{k,1}/\alpha_{k,\ell}[$. If $d < 0$, $R_{k,\ell}$ must have a root in the previous interval. That root is unique because, if $r_1 < r_2$ were two roots, applying the maximum principle on $]j'_{k,1}/\alpha_{k,\ell}, r_2[$ would imply that $R_{k,\ell}(r_1) > 0$, a contradiction. Moreover, the Hopf boundary Lemma implies that this root is simple.

On the interval $]j_{k,1}/\alpha_{k,\ell}, j_{k,2}/\alpha_{k,\ell}[$, the right hand side of (29) is negative. If $d < 0$, $R_{k,\ell}$ is negative at both endpoints of $]j_{k,1}/\alpha_{k,\ell}, j'_{k,2}/\alpha_{k,\ell}[$ and applying the maximum principle shows that $R_{k,\ell} < 0$ on the whole interval. If $d > 0$, $R_{k,\ell}$ must have a root in the previous interval. As before that root must be unique (if $r_1 < r_2$ are

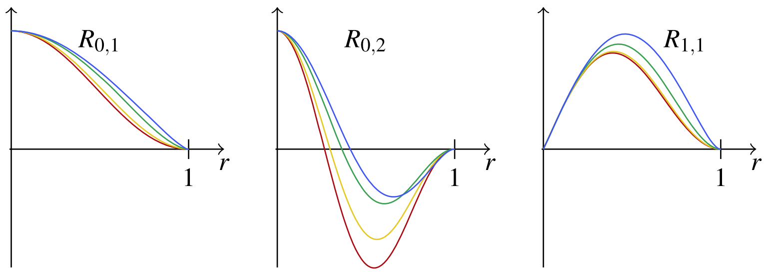

Fig. 4 Graphs of $R_{0,1}$, $R_{0,2}$ and $R_{1,1}$ for $\kappa = 0.1$ (*red*), $\kappa = 10$ (*orange*), $\kappa = 30$ (*green*) and $\kappa = 100$ (*blue*)

two roots, apply the maximum principle on $]r_1, j'_{k,2}/\alpha_{k,\ell}[$ to get a contradiction) and simple.

The above arguments show that in all cases, $R_{k,\ell}$ possesses a single root in the interval $]j'_{k,1}/\alpha_{k,\ell}, j'_{k,2}/\alpha_{k,\ell}[$ and that root is simple. The same reasoning applies to all subsequent intervals $]j'_{k,n}/\alpha_{k,\ell}, j'_{k,n+1}/\alpha_{k,\ell}[$.

To conclude, let us now prove that there is no root in the last interval $]j'_{k,\ell}/\alpha_{k,\ell}, 1[$. Choosing $c > 0$ and d as above, by Lemma 15 for the first equality, (30) and the bounds on $\alpha_{k,\ell}$ of Theorem 8 for the second one and evaluating (29) at $r = 1$ and taking into account the clamped boundary conditions for the last one, we obtain

$$\begin{aligned}
\operatorname{sign} R_{k,\ell}\Big(\frac{j'_{k,\ell}}{\alpha_{k,\ell}}\Big) &= (-1)^{\ell+1},\\
\operatorname{sign} R_{k,\ell}\Big(\frac{j_{k,\ell}}{\alpha_{k,\ell}}\Big) &= \operatorname{sign}(d) = \operatorname{sign}\big(-J_k(\alpha_{k,\ell})\big) = (-1)^{\ell+1},\\
\operatorname{sign} \partial_r^2 R_{k,\ell}(1) &= -\operatorname{sign} J_k(\alpha_{k,\ell}) = (-1)^{\ell+1}.
\end{aligned} \tag{31}$$

Because the sign of the right hand side of (29) on $]j_{k,\ell-1}/\alpha_{k,\ell}, j_{k,\ell}/\alpha_{k,\ell}[$ is also $(-1)^{\ell+1}$ and that interval contains $]j'_{k,\ell}/\alpha_{k,\ell}, j_{k,\ell}/\alpha_{k,\ell}[$, the maximum principle implies that $R_{k,\ell}$ has the same sign on the whole interval $]j'_{k,\ell}/\alpha_{k,\ell}, j_{k,\ell}/\alpha_{k,\ell}[$. In particular, it has no root there.

On $]j_{k,\ell}/\alpha_{k,\ell}, 1[$, the sign of the right hand side of (29) is $(-1)^{\ell}$. Thus, if there was a root r^* in that interval, the maximum principle applied to $]r^*, 1[$ would imply that $R_{k,\ell}$ has sign $(-1)^{\ell}$ over that interval. This contradicts (31) and shows that there is no root in that interval either.

Remark 19 The function $R_{0,1}$ is pictured on Fig. 4 for “small” and “large” values of κ. The different graphs of $R_{0,2}$ illustrate the nodal properties proved in Theorem 18. In view of Fig. 3, the second eigenfunction space is spanned by $R_{1,1}(r)\,\mathrm{e}^{\pm \mathrm{i}\theta}$ and thus necessarily changes sign due to its angular part. However, its radial part $R_{1,1}$ does not change sign as established in Theorem 14. It is no longer monotone though.

5 Proof of Theorems 1 and 2

Theorem 1 speaks both of non-negative and negative ν. For non-negative ν, the fact that the map $\nu \to \lambda_1(\nu)$ is smooth and increasing is a consequence of Theorem 4.3, Lemma 4.4, Lemma 4.5 and Theorem 3.2 in [10]. Theorem 3.2 of [10] also implies that $\lambda_1(0) = j_{1,1}^2$. The fact that $\lambda_1(\nu) \to +\infty$ as $\nu \to +\infty$ results from [10, Lemma 4.6]. The following three claims (with $\nu \in [0, (j_{0,1} j_{0,2})^2[$ for the first one) are stated in [10, Theorem 1.1] (recalling that, in that paper, $\nu = \kappa^2$). For $\nu \in]-\infty, 0[$, the fact that $\lambda_1(\nu)$ is simple, increasing, continuous at $\nu = 0$ and its behavior when $\nu \to -\infty$ are direct consequences of Theorem 12. The claims about $|\varphi_1|$ are stated in Theorem 13.

The existence of the functions $\nu \mapsto \lambda_{k,\ell}(\nu)$ and the form of the corresponding eigenfunctions given in Theorem 2 is stated in Theorem 8. The monotonicity and smoothness of $\lambda_{k,\ell}$ results from Lemma 9 and formula (15) while the value of the limits as $\nu \to 0$ and $\nu \to -\infty$ are immediate consequences of Lemma 10. Finally, the claims about the roots of $R_{k,\ell}$ are established in Theorems 14 and 18.

6 Extension to any dimension

In this section, we show how the previous results may be extended to any dimension $N \geqslant 2$. This generalization is straightforward so we only sketch the modifications to be made.

To find the eigenvalues of (1), we use spherical coordinates $r = |x| \in [0, +\infty[$ and $\theta = \frac{x}{|x|} \in \mathbb{S}^{N-1}$ and the ansatz $u = R(r)\mathscr{Y}_k(\theta)$ where $\mathscr{Y}_k$ is a spherical harmonic of degree $k \in \mathbb{N}$, i.e., an harmonic homogeneous polynomial of degree k. Expressing Δ in spherical coordinates, equations (6) and (7) become, respectively,

$$\partial_r^2 R + \frac{N-1}{r}\partial_r R + \Big(\alpha^2 - \frac{k(k+N-2)}{r^2}\Big) = 0, \tag{32}$$

$$\partial_r^2 R + \frac{N-1}{r}\partial_r R - \Big(\frac{\kappa^2}{\alpha^2} + \frac{k(k+N-2)}{r^2}\Big) = 0. \tag{33}$$

Performing the change of variables $R(r) = r^{(N-2)/2}B(r)$, one respectively gets the following equations

$$\partial_r^2 B + \frac{1}{r}\partial_r B + \Big(\alpha^2 - \frac{\nu_k^2}{r^2}\Big)B = 0, \tag{34}$$

$$\partial_r^2 B + \frac{1}{r}\partial_r B - \Big(\frac{\kappa^2}{\alpha^2} + \frac{\nu_k^2}{r^2}\Big)B = 0. \tag{35}$$

where $\nu_k := k + \frac{N-2}{2}$. Following the same approach as in Proposition 4, one deduces that the eigenfunctions have the form $R(r)\mathscr{Y}_k(\theta)$ with

$$R(r) = cJ_{\nu_k}(\alpha r) + dI_{\nu_k}\Big(\frac{\kappa}{\alpha}r\Big) \tag{36}$$

and α being a positive solution to

$$F_k(\alpha) := \frac{\kappa}{\alpha} J_{\nu_k}(\alpha) I'_{\nu_k}\Big(\frac{\kappa}{\alpha}\Big) - \alpha I_{\nu_k}\Big(\frac{\kappa}{\alpha}\Big) J'_{\nu_k}(\alpha) = 0. \tag{37}$$

Because $\nu_k \geqslant 0$, $\nu_{k+1} = \nu_k + 1$ and the proofs do not use the fact that k is an integer, Lemma 5, Theorem 8, Lemma 9, Lemma 10,... remain valid with k replaced by ν_k.

Theorem 20 *Let $\kappa > 0$. Denote $(\alpha_{k,\ell})_{\ell\geqslant 1}$ the infinitely many simple roots of F_k ordered in increasing order. The eigenvalues of (3) are given by*

$$\lambda_{k,\ell} = \alpha_{k,\ell}^2 - \frac{\kappa^2}{\alpha_{k,\ell}^2}, \qquad k \in \mathbb{N},\ \ell \in \mathbb{N}^+,$$

and, in spherical coordinates (r, θ), the corresponding eigenfunctions are $R_{k,\ell}(r)\mathscr{Y}_k(\theta)$ where $R_{k,\ell}$ is defined by (36) with $\alpha = \alpha_{k,\ell}$ and $(c, d) \neq (0, 0)$ chosen to satisfy the boundary conditions, and $\mathscr{Y}_k$ is a spherical harmonic of degree k. In addition the following holds.

- *The first eigenvalue is given by $\lambda_{0,1}$ and the corresponding eigenspace is spanned by $x \mapsto R_{0,1}(|x|)$. Moreover $r \mapsto |R_{0,1}(r)|$ is positive and decreasing on $[0, 1[$.*
- *For any k and ℓ, the function $R_{k,\ell}$ possesses $\ell - 1$ roots in $]0, 1[$, all of which are simple.*

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Appendix: Bessel and modified Bessel functions

Standard properties

As convenience to the reader, we gather in this section various properties of Bessel and modified Bessel functions (see for instance [24]) that are used in this paper.

Recurrence relations and derivatives

The Bessel functions J_ν satisfy

$$\nu J_\nu(z) = \frac{z}{2}\big(J_{\nu-1}(z) + J_{\nu+1}(z)\big), \tag{38}$$

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z), \tag{39}$$

$$J'_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z), \tag{40}$$

$$J'_0(z) = -J_1(z), \tag{41}$$

$$z^2 J''_\nu(z) + z J'_\nu(z) + (z^2 - \nu^2) J_\nu(z) = 0. \tag{42}$$

The modified Bessel functions I_ν satisfy

$$\nu I_\nu(z) = \frac{z}{2}\big(I_{\nu-1}(z) - I_{\nu+1}(z)\big), \tag{43}$$

$$I'_\nu(z) = I_{\nu-1}(z) - \frac{\nu}{z} I_\nu(z), \tag{44}$$

$$I'_\nu(z) = I_{\nu+1}(z) + \frac{\nu}{z} I_\nu(z), \tag{45}$$

$$I'_0(z) = I_1(z), \tag{46}$$

$$z^2 I''_\nu(z) + z I'_\nu(z) - (z^2 + \nu^2) I_\nu(z) = 0. \tag{47}$$

Asymptotic behaviour

For any given $\nu \neq -1, -2, -3, \ldots$, when $z \to 0$,

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)}(1 + o(1)), \tag{48}$$

$$I_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)}(1 + o(1)). \tag{49}$$

Zeros

When $\nu \geqslant 0$, the positive zeros of J_ν are simple and interlace according to the inequalities

$$j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \cdots \tag{50}$$

On the other hand, for all $z > 0$, $I_k(z) > 0$ and is increasing.

Some inequalities

Lemma 21 *For all real number $\nu > -1$, we have*

$$\forall z > 0, \qquad \left(\frac{I_{\nu+1}(z)}{I_\nu(z)}\right)^2 + \frac{2\nu}{z}\frac{I_{\nu+1}(z)}{I_\nu(z)} - 1 < 0 \tag{51}$$

whence

$$\forall z > 0, \qquad \frac{I_{\nu+1}(z)}{I_\nu(z)} < \frac{-\nu + \sqrt{\nu^2 + z^2}}{z}. \tag{52}$$

Proof We start with the Turán type inequality [1] (see also [18,21,23])

$$\forall z > 0, \qquad I_{\nu-1}(z) I_{\nu+1}(z) < I_\nu^2(z), \tag{53}$$

valid for all $\nu > -1$. Then using the recurrence relation (43) this can be rewritten under the form

$$\Big(\frac{2\nu}{z} I_\nu(z) + I_{\nu+1}(z)\Big) I_{\nu+1}(z) < I_\nu^2(z).$$

Setting $u := \frac{I_{\nu+1}(z)}{I_\nu(z)}$, the previous inequality is equivalent to (51). The estimate (52) directly follows as $z > 0$. □

Remark 22 Note that the estimate (52) follows from [3, estimate (14)] that says that for $\nu > -1$

$$\forall z > 0, \qquad \frac{I_{\nu+1}(z)}{I_\nu(z)} < \frac{-\nu - 1 + \sqrt{\nu^2 + \frac{\nu+1}{\nu} z^2}}{\frac{\nu+1}{\nu} z},$$

since the right-hand side of this estimate is smaller than the right-hand side of (52). Actually our proof uses the estimate (53), while the previous estimate uses the converse Turán type inequality

$$\forall z > 0, \qquad I_\nu^2(z) - I_{\nu-1}(z) I_{\nu+1}(z) \leqslant \frac{I_\nu^2(z)}{\nu + 1},$$

valid for all $\nu > -1$.

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