Twisted semilocal strings in the MSSM

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Abstract

The standard electroweak model is extended by means of a second Brout-Englert-Higgs-doublet. The symmetry breaking potential is chosen in such a way that (i) the Lagrangian possesses a custodial symmetry, (ii) a stationary, axially symmetric ansatz of the bosonic fields consistently reduces the Euler-Lagrange equations to a set of differential equations. The potential involves, in particular, a direct interaction between the two doublets. Stationary, axially-symmetric solutions of the classical equations are constructed. Some of them can be assimilated to embedded Nielsen-Olesen strings. From these solutions there are bifurcations and new solutions appear which exhibit the characteristics of the recently constructed twisted semilocal strings. A special emphasis is set on "doubly-twisted" solutions for which the two doublets present different time-dependent phase factors. They are regular and have a finite energy which can be lower than the energy of the embedded twisted solution. Electric-type solutions, such that the fields oscillate asymptotically far from the symmetry-axis, are also reported.

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I. INTRODUCTION

Since the discovery of topological string solutions in the so called "Abelian-Higgs-model" by Nielsen-Olesen [1], more than thirty years ago, there have been several investigations for applying these solutions in high energy physics and cosmology. The existence of these solutions is closely connected to the symmetry breaking of the local U(1)-symmetry by a Brout-Englert-Higgs mechanism. In the last years it has been shown that string-like configurations also appear as classical solutions in several other models within several pattern of symmetry breaking. One of the most famous generalisation of the string is the "semi-local" string [2]. It appears in models involving both local and global symmetry breaking, see [3] for a review. One of the most striking properties of these solutions is that they constitute stable vortices without being supported by a standard topological argument related to the homotopy group of the vacuum manifold [4]. The semi-local string appears naturally in models involving N charged complex scalar fields coupled to electromagnetism and presenting an SU(N) global symmetry suitably broken by a Brout-Englert-Higgs potential. Along with the basic Nielsen-Olesen (NO) strings they possess a cylindrical symmetry, they do not depend on the coordinate associated to the symmetry axis (usually chosen as the oz-axis) and have a finite energy per unit length of the z coordinate.

Very recently, the classical equations of this model (in the case N=2) were reinvestigated [5, 6] and new families of solutions were shown to exist. These remarkable solutions are called "twisted-semilocal strings". The crucial difference between the twisted semilocal and the conventional ones [2, 3] resides in the fact that the two scalar fields have different phase factors depending linearly on time as well as on the coordinate corresponding to their axis of symmetry. This needs an axially symmetric ansatz but, e.g. does not invalidate the cylindrical symmetry of the energy density of these solutions.

In this paper, we consider the semi-local strings (twisted or not) in the framework of a model involving two Brout-Englert-Higgs doublets. This is motivated by the Minimal Supersymmetric Standard Model (MSSM). Along with [5, 6], we consider the classical equations in the limit where the Weinberg angle θ_W is set to $\theta_W = \pi/2$. The most general potential used in the context of MSSM is very involved and depends roughly on ten parameters. Here we limit our investigation to the subfamily of parameters allowing for a custodial symmetry. This restricts the number of parameters to five. We give numerical evidence that, in a specific but rather large region of the space of the parameters (involving to a large extent the physical region determined by the experimental lower bounds for the Brout-Englert-Higgs bosons) solutions exist for which the two doublets present independent twists in the phases associated with specific components of them. We call this type of solutions "doubly-twisted", in general they coexist with the embedded simply-twisted solutions [5, 6] where one of the doublets does not depend on t and z. For some region of the parameters, doubly twisted solutions have the lowest energy. The parameters labeling the twist can, in particular, be tuned in such a way that the associated Noether currents have opposite signs. Another feature of doubly twisted solutions is that, for generic values of the potential, the components A_0 and A_z of the Maxwell field turn out to be not proportional to each other. This contrasts with the case of one doublet [6].

The description of the model is presented in Sect. II, where a special emphasis is set to the parameters of the potential and their usual relations to the masses of the Higgs-particles. The ansatz for the fields and the corresponding equations are given is Sect. III. In Sect. IV, we characterize the constraints of the potential that allow embedded NO strings. This is useful since it leads to a family of semi-analytical solutions (in a sense that they are parametrized in terms of the profile of the NO solution only). Several features of the solutions constructed numerically for generic values of the potential are discussed in Sect. V. Due to the possibilities allowed by the twist with respect to the two doublets, the solutions depend on several independent parameters. We present a few plots which hopefully capture a large part of qualititive properties of the twisted semi-local string available in the MSSM. Some conclusions are given in Sect. VI.

II. THE MODEL

The lagrangian that we consider in this paper reads

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + (D_\mu\Phi_{(1)})^\dagger(D^\mu\Phi_{(1)}) + (D_\mu\Phi_{(2)})^\dagger(D^\mu\Phi_{(2)}) - V(\Phi_{(1)},\Phi_{(2)}) \tag{1}$$

Where $\Phi_{(1)}, \Phi_{(2)}$ denote the two BEH-doublets and the standard definitions are used for the covariant derivative and gauge field strengths :

$$F^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g\epsilon^{abc}V^b_\mu V^c_\nu \tag{2}$$

$$D_\mu\Phi = \Big(\partial_\mu - \frac{i}{2}g\tau^a V^a_\mu - \frac{i}{2}g' A_\mu\Big)\Phi \tag{3}$$

(the case $\theta_W = \pi/2$, i.e. $g = 0$ will be used later in this paper).

The most general gauge invariant potential constructed with two BEH-doublets is presented namely in [7, 8] and depends on nine coupling constants. Here we consider the family of potentials of the form

$$V(\Phi_{(1)},\Phi_{(2)}) = \lambda_1(\Phi^\dagger_{(1)}\Phi_{(1)} - \frac{v_1^2}{2})^2 + \lambda_2(\Phi^\dagger_{(2)}\Phi_{(2)} - \frac{v_2^2}{2})^2 + \lambda_3(\Phi^\dagger_{(1)}\Phi_{(1)} - \frac{v_1^2}{2})(\Phi^\dagger_{(2)}\Phi_{(2)} - \frac{v_2^2}{2}) \tag{4}$$

depending on five parameters. The direct coupling between the two doublets is parametrized by the constant λ_3 and one of the main characteristic about this potential resides in the fact that it imposes a symmetry breaking mechanism to each of the BEH-doublets. Several classical solutions in the models have been studied in the past. Spherically symmetric sphaleron solutions were studied at length in the case $\lambda_3 = 0$ in [9, 10, 11] and a similar study for $\lambda_3 \neq 0$ was reported in [12].

The lagrangian (1) is of course invariant under SU(2) gauge transformations but it further possesses a larger global symmetry under SU(2) $\times$ SU(2) $\times$ SU(2). In fact, the part of the lagrangian (1) involving the scalar fields can be written in terms of 2×2 matrices defined by

$$M_{1,2} \equiv \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_+^* & \phi_0 \end{pmatrix}_{1,2} \quad \text{for} \quad \Phi_{1,2} = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}_{1,2} \tag{5}$$

When written in terms of the matrices M_1 and M_2, the lagrangian (1) becomes manifestly invariant under the transformation

$$V'_\mu = AV_\mu A^\dagger \quad , \quad M'_1 = AM_1B \quad , \quad M'_2 = AM_2C \tag{6}$$

with $A, B, C \in \mathrm{SU}(2)$; this is the custodial symmetry. The double symmetry breaking mechanism imposed by the potential (4) leads to a mass M_W for two of the three gauge vector bosons and, namely, to two neutral BEH-particles with masses M_h, M_H. In terms of the parameters of the Lagrangian, these masses are given by [8]

$$M_W = \frac{g}{2}\sqrt{v_1^2 + v_2^2} \quad , \quad M_{H,h}^2 = \frac{1}{2}[A_1 + A_2 \pm \sqrt{(A_1 - A_2)^2 + 4B^2}] \tag{7}$$

with

$$A_1 = 2v_1^2(\lambda_1 + \lambda_3) \quad , \quad A_2 = 2v_2^2(\lambda_2 + \lambda_3) \quad , \quad B = 2\lambda_3 v_1 v_2 \tag{8}$$

For later convenience we also define

$$\tan\beta = \frac{v_2}{v_1} \quad , \quad \rho_{H,h} = \frac{M_{H,h}}{M_W} \quad , \quad \epsilon_p = 4\frac{\lambda_p}{g^2} \quad , \quad p = 1, 2, 3 \tag{9}$$

Note that the mass ratios $\rho_{1,2}$ used in [10] are related to $\rho_{H,h}$ by $\rho_H = \max\{\rho_1, \rho_2\}$, $\rho_h = \min\{\rho_1, \rho_2\}$. For physical reasons, we consider only $v_1 \geq 0$, $v_2 \geq 0$ so that $0 \leq \beta \leq \pi/2$. Interestingly, the parameter ϵ_3 can be negative but cannot take arbitrary values. The following relations are useful to determine the physical region :

$$\begin{aligned} \epsilon_1 \cos^2\beta + \epsilon_2 \sin^2\beta &= \frac{1}{2}(\rho_H^2 + \rho_h^2) - \epsilon_3 \\ \epsilon_1 \cos^2\beta - \epsilon_2 \sin^2\beta &= \frac{1}{2}\sqrt{(\rho_H^2 - \rho_h^2)^2 - 4\epsilon_3^2 \sin^2(2\beta)} - \epsilon_3 \cos(2\beta) \end{aligned} \tag{10}$$

The physical domain is then determined by the conditions

$$\frac{\rho_h^2 - \rho_H^2}{2\sin(2\beta)} \leq \epsilon_3 \leq \frac{\rho_H^2 - \rho_h^2}{2\sin(2\beta)} \quad , \quad 0 \leq \rho_h^2 \leq \rho_H^2 \tag{11}$$

III. CYLINDRICAL SYMMETRY

In this paper, we will assume that only the U(1)-subgroup of the gauge group is really gauged (that is to say that we assume $V_\mu^1 = V_\mu^2 = 0$) and look for solutions presenting a cylindrical symmetry. For this purpose, we impose a form for the fields inspired from [5, 6]. After some algebra, it turns out that the form

$$\phi_{1(1)} = f_1(\rho)\exp^{in_1\phi} \quad , \quad \phi_{2(1)} = f_2(\rho)\exp^{im_1\phi}\exp^{i(\omega_0 t + \omega_3 z)} \tag{12}$$

$$\phi_{1(2)} = g_1(\rho) \exp^{in_2\phi} \exp^{i(\sigma'_0 t + \sigma'_3 z)} \quad , \quad \phi_{2(2)} = g_2(\rho) \exp^{im_2\phi} \exp^{i(\sigma_0 t + \sigma_3 z)} \tag{13}$$

$$A_0 = \omega_0 a_0(\rho) \quad , \quad A_\rho = 0 \quad , \quad A_\phi = na(\rho) \quad , \quad A_3 = \omega_3 a_3(\rho) \quad , \tag{14}$$

n_1, n_2 and m_1, m_2 being integers is consistent with the classical equations.As pointed out in [5, 6], solutions exist for $0 \le m_1 \le n_1$ and $0 \le m_2 \le n_2$. Inserting this ansatz into the energy functional, it turns out that finite energy configurations are possible only for $n_1 = n_2 = n$ and $\sigma'_0 = \sigma'_3 = 0$. In fact we parametrize the U(1) field as close as possible to [5, 6] (although without losing generality) in order to crosscheck our solutions with the solutions constructed of [5, 6] in the following two limits : (i) the second doublet $\Phi_{(2)}$ becomes trivial, (ii) the parameters $\sigma_{0,3}$ appearing in this doublet vanish.

The ansatz above leads to a reduced lagrangian density of the form :

$$\begin{aligned}
\mathcal{L} \;=\; & \rho \, \frac{1}{2} \left[\omega_0^2 (a'_0)^2 - \omega_3^2 (a'_3)^2 - \frac{1}{\rho^2} n^2 (a')^2 \right] \\
- \; & \rho (\cos\beta)^2 [(f'_1)^2 + (f'_2)^2 + (g'_1)^2 + (g'_2)^2 + (-\omega_0^2 a_0^2 + \omega_3^2 a_3^2 + \frac{n^2(1-a)^2}{\rho^2}) f_1^2 \\
+ \; & (-\omega_0^2 (1-a_0)^2 + \omega_3^2 (1-a_3)^2 + \frac{(m_1 - na)^2}{\rho^2}) f_2^2 + (-\omega_0^2 a_0^2 + \omega_3^2 a_3^2 + \frac{n^2(1-a)^2}{\rho^2}) g_1^2 \\
+ \; & (-(\sigma_0 - \omega_0 a_0)^2 + (\sigma_3 - \omega_3 a_3)^2 + \frac{(m_2 - na)^2}{\rho^2}) g_2^2] \\
- \; & \rho (\cos\beta)^4 \left[\epsilon_1 (|f|^2 - 1)^2 + \epsilon_2 (|g|^2 - \mathrm{tg}^2\beta)^2 + \epsilon_3 (|f|^2 - 1)(|g|^2 - \mathrm{tg}^2\beta) \right]
\end{aligned} \tag{15}$$

where the prime means the derivative with respect to the axial variable ρ. The quantity Q defined according to

$$Q \equiv 2\pi \int d\rho \rho a_3 (f_1^2 + g_1^2) = 2\pi \int d\rho \rho (1 - a_3)(f_2^2 + g_2^2) \tag{16}$$

is also of great interest. It determines the vortex worldsheet currents and generalizes Eqs.(5.4)-(5.6) of [6]. They are associated with the conserved quantities related to the Noether current of the residual U(1) symmetry. In passing we note that the equality in (16) provides a useful test of accuracy for our numerical solutions.

In order to write the corresponding Euler-Lagrange equations, we find it convenient to define $K = (a_0 + a_3)/2$, $\Delta = (a_0 - a_3)/2$. It is straightforward to establish the Euler-Lagrange equations :

$$\begin{aligned}
\rho (\frac{a'}{\rho})' \;&=\; 2(\cos\beta)^2 \left\{ f_1^2 (a-1) + f_2^2 (a - \frac{m_1}{n}) + g_1^2 (a-1) + g_2^2 (a - \frac{m_2}{n}) \right\} \\
\frac{1}{\rho} (\rho f'_1)' \;&=\; f_1 \left\{ -\omega_0^2 a_0^2 + \omega_3^2 a_3^2 + \frac{n^2(1-a)^2}{\rho^2} \right\} \\
&\qquad + (\cos\beta)^2 \left\{ 2\epsilon_1 f_1 (|f|^2 - 1) + \epsilon_3 f_1 (|g|^2 - \mathrm{tg}^2\beta) \right\} \\
\frac{1}{\rho} (\rho f'_2)' \;&=\; f_2 \left\{ -\omega_0^2 (1-a_0)^2 + \omega_3^2 (1-a_3)^2 + \frac{(m_1 - na)^2}{\rho^2} \right\}
\end{aligned}$$

$$+(\cos\beta)^2\{2\epsilon_1 f_2(|f|^2-1)+\epsilon_3 f_2(|g|^2-\mathrm{tg}^2\beta)\}$$

$$\frac{1}{\rho}(\rho g_1')' = g_1\left\{-\omega_0^2 a_0^2+\omega_3^2 a_3^2+\frac{n^2(1-a)^2}{\rho^2}\right\}$$

$$+(\cos\beta)^2\left\{2\epsilon_2 g_1(|g|^2-\mathrm{tg}^2\beta)+\epsilon_3 g_1(|f|^2-1)\right\}$$

$$\frac{1}{\rho}(\rho g_2')' = g_2\left\{-(\sigma_0-\omega_0 a_0)^2+(\sigma_3-\omega_3 a_3)^2+\frac{(m_2-na)^2}{\rho^2}\right\}$$

$$+(\cos\beta)^2\left\{2\epsilon_2 g_2(|g|^2-\mathrm{tg}^2\beta)+\epsilon_3 g_2(|f|^2-1)\right\}$$

$$\frac{1}{\rho}(\rho K')' = (\cos\beta)^2\left\{2K|f|^2-2f_2^2+2K|g|^2-g_2^2(\frac{\sigma_0}{\omega_0}+\frac{\sigma_3}{\omega_3})\right\}$$

$$\frac{1}{\rho}(\rho\Delta')' = (\cos\beta)^2\left\{2\Delta|f|^2+2\Delta|g|^2-g_2^2(\frac{\sigma_0}{\omega_0}-\frac{\sigma_3}{\omega_3})\right\} \qquad (17)$$

where $|f|^2 \equiv f_1^2+f_2^2$ and $|g|^2 \equiv g_1^2+g_2^2$. The interest for using Δ appears on the last equation; defining $R_0 \equiv (\sigma_0/\omega_0)$, $R_3 \equiv (\sigma_3/\omega_3)$ and $\delta \equiv R_0 - R_3$, we see that $\delta = 0$ (corresponding to parallel twists in the two doublets) implies $\Delta = 0$ by the positivity argument. If however solutions exist such that $\delta \neq 0$, they will have $\Delta \neq 0$; this is a new feature with respect to the case with one doublet [5, 6].

In order to obtain regular and finite energy solutions, the above system has to be solved with the following boundary conditions respectively at $\rho = 0$ and $\rho = \infty$:

$$a_0'(0) = a_3'(0) = a(0) = f_1(0) = f_2'(0) = g_1(0) = g_2'(0) = 0 \qquad (18)$$

$$a_0(\infty) = a_3(\infty) = f_2(\infty) = g_2(\infty) = 0 \quad , a(\infty) = 1,\ f_1(\infty) = 1,\ g_1(\infty) = \tan(\beta) \qquad (19)$$

Before discussing the solutions for generic values of the potential's parameters, it is useful to study a few specific limits of these equations.

IV. PARTICULAR SOLUTIONS

A. Nielsen-Olesen strings

Setting $\epsilon_2 = \epsilon_3 = \beta = 0$ in the potential and $a_3 = a_0 = f_2 = g_2 = g_1 = 0$, the system of equations above naturally reduces to the two equations :

$$a'' - \frac{1}{\rho}a' = 2f^2(a-1) \quad , \quad f'' + \frac{1}{\rho}f' = f\frac{n^2}{\rho^2}(1-a)^2 + 2\epsilon f(f^2-1) \ , \ \epsilon = \epsilon_1 \qquad (20)$$

whose solution is the NO string. For later convenience, we denote this particular solution by $\bar{a}_\epsilon$, $\bar{f}_\epsilon$.

B. Embedded Nielsen-Olesen string

The system of seven equations above admits embedded NO solutions of the form

$$a = \bar{a}_\epsilon \quad , \quad f_1 = \bar{f}_\epsilon \quad , \quad g_1 = \tan(\beta)\bar{f}_\epsilon \quad , \quad a_0 = a_3 = f_2 = g_2 = 0 \qquad (21)$$

provided the following relations between the parameters of the potential hold

$$2\epsilon = 2\epsilon_1 \cos^2\beta + \epsilon_3 \sin^2\beta \quad , \quad 2\epsilon = 2\epsilon_2 \sin^2\beta + \epsilon_3 \cos^2\beta \quad . \tag{22}$$

Accordingly, embedded semilocal strings will exist if $\epsilon > 1/2$. For coupling constants away from these constraints, we expect the solutions to be deformed progressively from the embedded NO-solutions. The linearized equations [4] associated with the fields f_2 and g_2 corresponding to an embedded NO-solution lead to a Schroedinger equation

$$-\frac{1}{\rho}(\rho f_2')' + [\frac{1}{\rho^2}(n\bar{a} - m)^2 - 2\epsilon(1 - \bar{f}^2)]f_2 = -\omega^2 f_2 \quad , \quad m = m_1 = m_2 \tag{23}$$

and an identical equation for g_2 with the spectral parameter ω^2 replaced by σ^2. Using the reasoning of [5, 6], we conclude that superconducting solutions with both fields $\Phi_{(1)}$ and $\Phi_{(2)}$ twisted will exist as soon as the potential parameters are such that $2\epsilon > 1$. These will exist for

$$\omega^2 \in]0, \omega_b^2] \quad , \quad \sigma^2 \in]0, \omega_b^2] \tag{24}$$

where ω_b^2 are computed in [6] (see e.g. Tables I,II and Fig. 3 in [6]). As a consequence, we expect solutions to exist with arbitrary relative phases associated with the two Higgs doublets. This is indeed confirmed by our numerical analysis. See e.g. Fig. 4 where the energy, the charge Q and the parameters $\Delta(0), f_2(0)$ are reported as functions of the ratio σ_3/ω_3, for $\omega_3 = 0.017$.

It is a natural question to ask whether the bifurcation point corresponding to $2\epsilon = 1$ in (22) belongs to the physical region of the parameter of the MSSM. In terms of the mass ratios ρ_H, ρ_h, it turns out that embedded NO-solutions exist if

$$\rho_H^2 + \rho_h^2 = 2 + \epsilon_3 \quad , \quad |\epsilon_3|/(4 + 5\cos(2\beta)^2)^{1/2} = |\rho_H^2 - \rho_h^2| \tag{25}$$

If we fix ρ_h, the conditions above define a point

$$\rho_H = \sqrt{2 + \epsilon_3 - \rho_h^2} \quad , \quad \epsilon_3 = \frac{2\sigma(1 - \rho_h^2)}{(4 + 5\cos^2(2\beta))^{1/2} - \sigma} \quad , \quad \sigma \equiv \text{sign}(\epsilon_3) \tag{26}$$

in the ρ_H, ϵ_3 plane. After some algebra, one can check that, if $\rho_h < 1$, this point is inside the physical region for generic values of β and *on* the curve determining the physical region for $\beta = \pi/4$. Since the experimental results show that the Higgs mass is larger than the W-boson mass, no NO-embeded solution is available for realistic values of the Higgs mass parameter.

C. Embedded and mixed semi-local strings

The twisted semi-local strings of [5, 6] are characterized by the winding number n and form n families of solutions labelled by m with $0 \le m \le n-1$. Around $\rho = 0$, the two components of the Higgs field behave according to $f_1 \sim f_1^{(n)}\rho^n$ and $f_2 \sim f_2^{(m)}\rho^m$.

When the condition (22) holds, these solutions can also be embedded in the system above provided $g_1 = f_1 \tan\beta$ and $g_2 = f_2 \tan\beta$ and $\delta = 0$, implying $\Delta(\rho) = 0$ as already mentionned. These embeddings holds for any n and $m_1 = m_2$. However, it is likely that solutions can be constructed with $m_1 \neq m_2$ and $\delta = 0$, we will see that this is indeed the case and call these solutions 'mixed'.

D. Simply and doubly twisted solutions

For generic values of the potential, the system of seven equations above admits solutions where all the functions are non trivial and the parameters R_0, R_3 are independant. We will denote these solutions doubly twisted ones. However, as can be seen from the equations and the boundary conditions, solutions possibly exist which have $g_2 = 0$ and $f_2 \neq 0$ (or $f_2 = 0$ and $g_2 \neq 0$). In the following chapter, we will denote such solutions simply twisted.

V. NUMERICAL RESULTS

The system above cannot be solved analytically, we relied on numerical method to construct solutions; the numerical routine used is based on the collocation method [20]. Because both twisted and untwisted solutions obey the same set of boundary conditions, for instance (18), (19), some trick has to be implemented in order to enforce the numerical routine to produce the twisted strings. For this purpose we supplement the system by three trivial equations, namely $d\omega_3/d\rho = d\sigma_0/d\rho = d\sigma_3/d\rho = 0$ and we take advantage of these supplementary equations to impose by hand values for $f_2(0), g_2(0)$ and $a_3(0)$ as supplementary boundary conditions. If we start from an appropriate initial configuration, the values of $\omega_3, \sigma_0, \sigma_3$ for which these supplementary boundary conditions are fullfilled can be reconstructed numerically. Once such a solution is available, the trick allows to "navigate" very efficiently into the manifold of simply or doubly twisted solutions.

A. One doublet

In order to test the efficiency of our technique, we first reconsidered the one-Higgs equations in the case $\epsilon_1 = 2$, $\epsilon_{2,3} = 0$ (corresponding to $\beta = 4$ in the notation of [6]). The second doublet then vanishes identically and the equations reduce to the ones of [5, 6]). The numerical values that we obtain are in full agreement with those of [6]. Several parameters characterizing this family of solutions are represented on Fig.1.

B. Embedded solutions

Assuming $R_3 = R_0$, we manage to embed the solutions of [6] in the MSSM model discussed above i.e. solutions with $f_1 = g_1, f_2 = g_2$. On Fig. 2, we present several

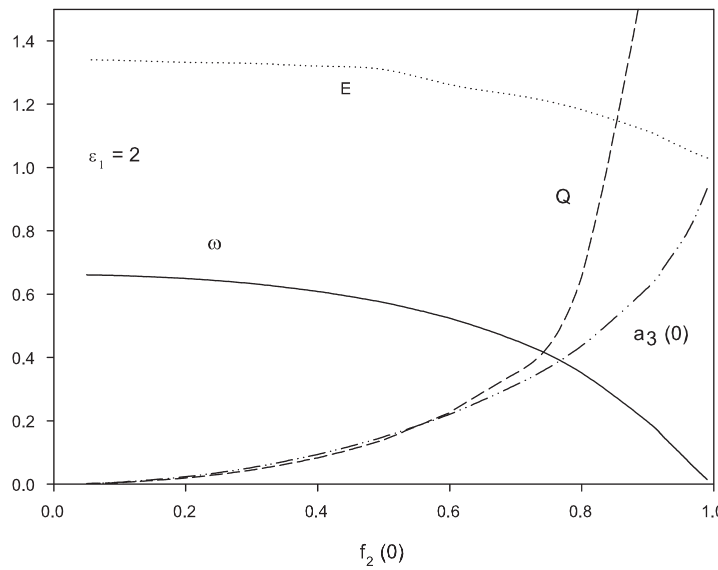

FIG. 1: The energy, the charge Q and parameters $a_3(0), \omega$ are plotted as functions of the value $f_2(0)$ for the solution with one doublet and $\epsilon_1 = 2$.

parameters characterizing these embedded solutions for $\epsilon_1 = \epsilon_2 = 2$ $\epsilon_3 = 0$ and $\beta = \pi/4$ and for the solutions corresponding to $n = 1,\ m_1 = m_2 = 0$; $n = 2,\ m_1 = m_2 = 0$ and $n = 2,\ m_1 = m_2 = 1$. On these plots, the different parameters are represented as functions of ω. The plots confirm that the embedded solutions (like the original ones) exist only for a finite interval of ω and that the solution bifurcates into an embedded NO solution (with $f_2 = g_2 = 0 = a_3$) in the limit $\omega \to \omega_b$. In the limit $\omega \to 0$, a so called Skyrme type solution is approached but we did not study this limit in details. We just note that the charge Q becomes infinite in this limit, as a consequence of the fact that the function a_3 reaches its asymptotic value $a_3(\infty) = 0$ more and more slowly (see [6] for more details).

C. Mixed solutions

The profile of a mixed a solution is presented on Fig. 3 corresponding to $n = 2, m_1 = 0, m_2 = 1,\ \delta = 0$ and $f_2(0) = 0.8$; a branch of solutions of this type can be constructed by varying $f_2(0)$ A few parameters characterizing this branch with respect to ω are represented in Fig. 2. For the parameters choosen, it exists for $\omega < 0.32$. In the limit $\omega \to 0.32$, we have $g_2'(0) \to 0$ and the function g_2 tends uniformly to zero while a_3, f_2 remain non zero, this feature is specific to mixed solutions.

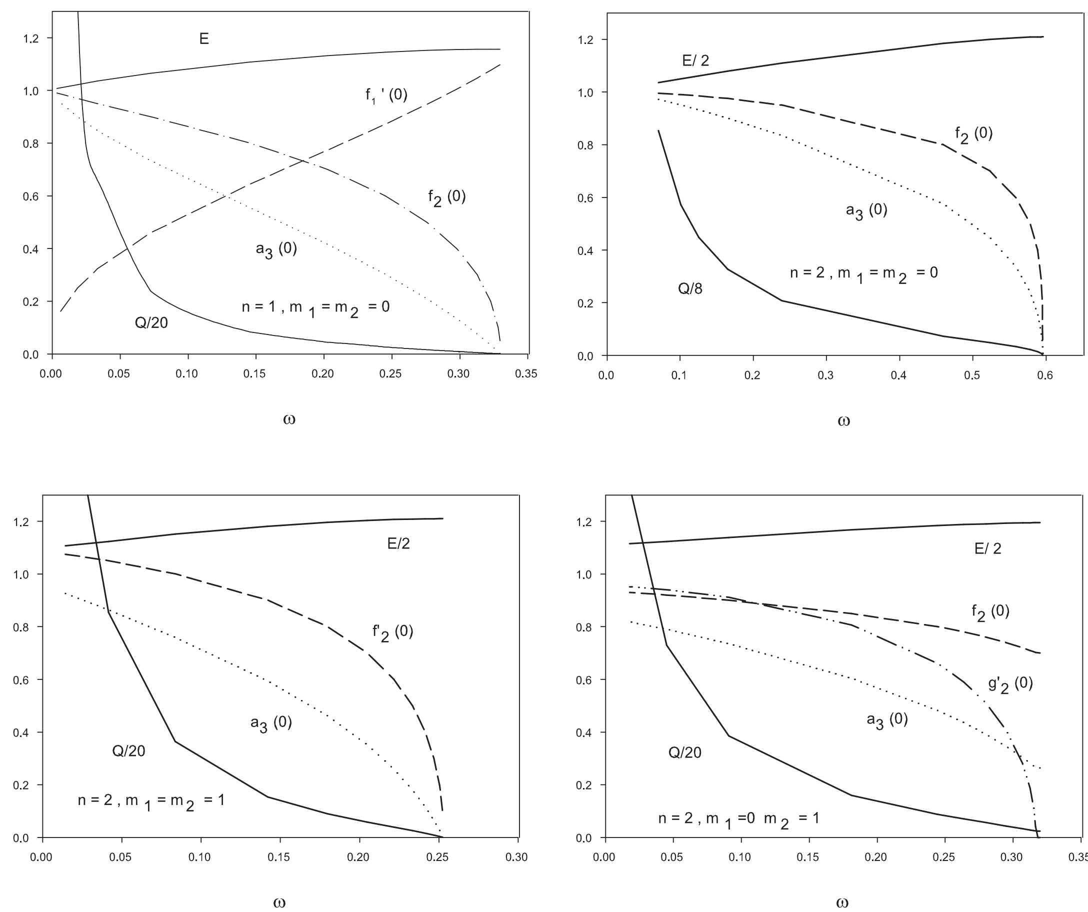

FIG. 2: The energy, the charge Q and parameters $a_3(0), f_1(0), f_2'(0)$, etc... are plotted as functions of ω for the embedded solutions corresponding to a few values of n, m_1, m_2 .

D. Doubly twisted solutions

We first investigate the effects of a doubly twisted solution of the NO-string type corresponding to $\epsilon_1 = \epsilon_2 = 2$, $\epsilon_3 = 0$, $\beta = \pi/4$ and assuming $\omega_3 = 0.017$, $R_0 = 1$. The effect of the parameter R_3 on the solutions is reported on Fig. 4. Remark that for $R_3 = 1$, we have $\Delta(0) = 0$; in fact we have $\Delta(\rho) = 0$ as a consequence of the fact that the positivity argument applies in the equation for the function Δ if $R_0 = R_3$.

We manage to construct doubly-twisted solutions for generic values of the potential, i.e. not NO-embedded solutions. From now on we will use the parametrisation of the coupling constants formulated in terms of ρ_H, ρ_h and we will concentrate on the case $\rho_h = 1, \rho_H = 2$, $\beta = \pi/4$; corresponding to the physical region for $-3/2 < \epsilon_3 < 3/2$. The profile of a doubly twisted solution is reported on Fig. 5. Let us stress that the functions Δ and a_3 are *not* proportional to each other for generic doubly twisted solutions. The non vanishing components of the electromagnetic fields

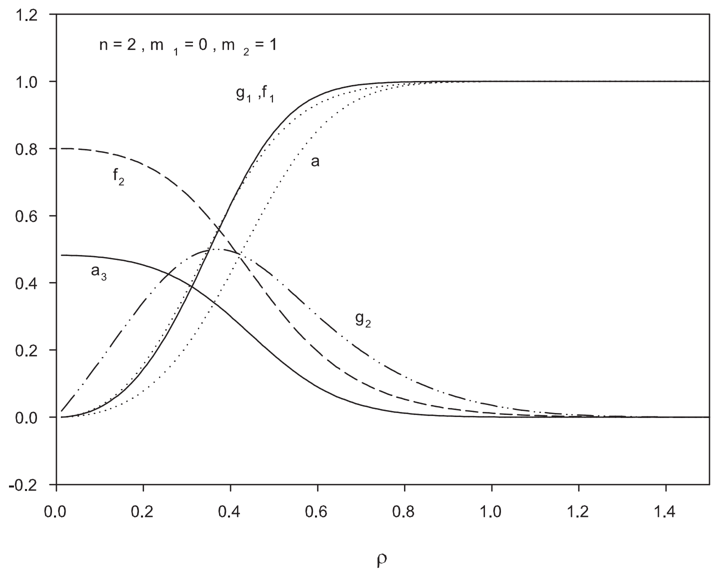

FIG. 3: The profile of a mixed solution corresponding to $n = 2, m_1 = 0, m_2 = 1$.

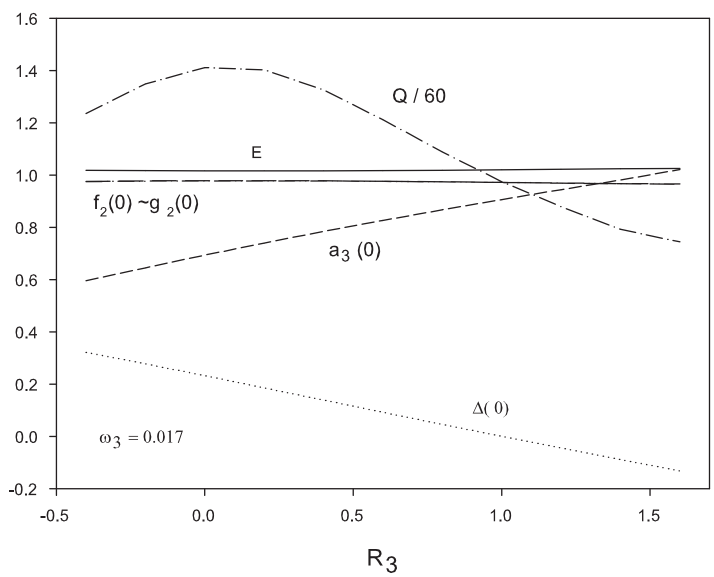

FIG. 4: The energy, the charge Q and parameters $\Delta(0), a_3(0), f_2(0)$ are plotted as functions of the ratio R_3 for $R_0 = 1$ and $\omega_3 = 0.017$.

$\vec{E}$ and $\vec{B}$ read

$$E_\rho = -\frac{dA_0}{d\rho} \quad , \quad B_\phi = -\frac{dA_z}{d\rho} \quad , \quad B_z = \frac{dA_\phi}{d\rho} + \frac{A_\phi}{\rho} \tag{27}$$

These functions are represented on Fig.6 for an embedded solution (solid lines) and for the doubly twisted solution of Fig. 5. Note that the embedded solution has $E_\rho = B_\phi$ since $\Delta(\rho) = 0$.

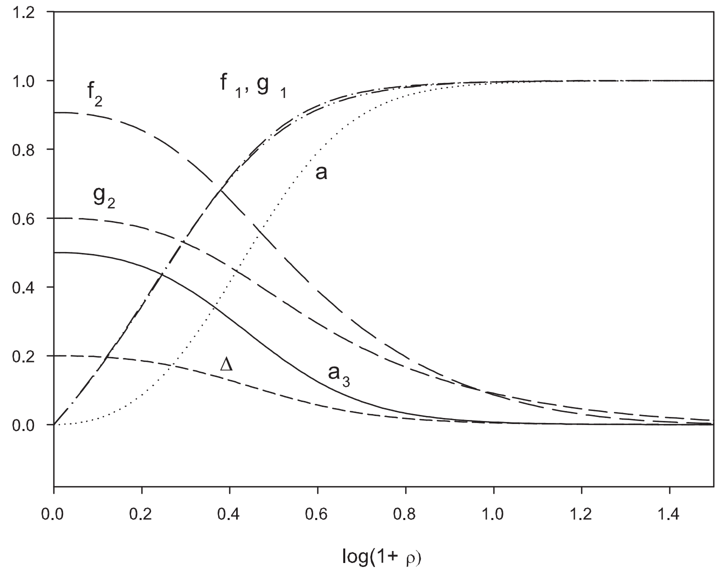

FIG. 5: The profile of a generic solution corresponding to $\rho_h = 1$, $\rho_H = 2$, $\epsilon = 0.1$, $\beta = \pi/4$.

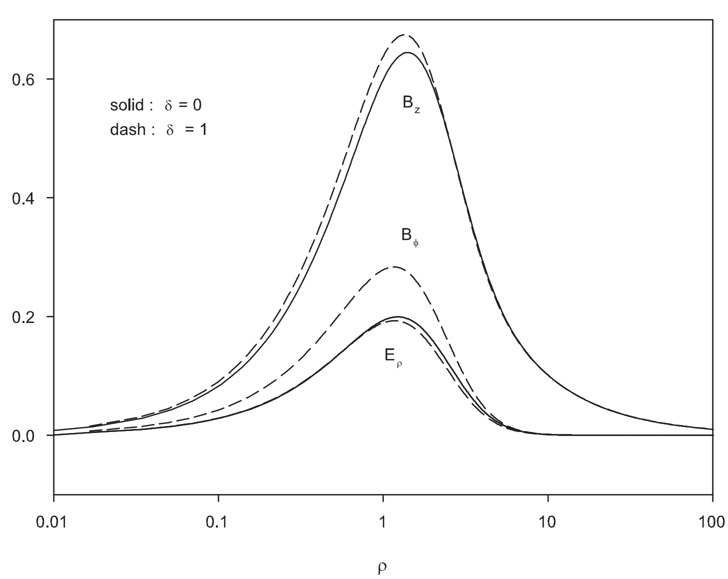

FIG. 6: The profile of the electromagnetic fields associated with an embedded solution (solid) and a generic doubly twisted solution (dashed) are compared

Several features of the stationary twisted solutions are summarized on Fig.7 and Fig. 8. On Fig. 7 some parameters characterising the evolution of a doubly-twisted solution ($f_2(0) \neq 0$, $g_2(0) \neq 0$) into a simply-twisted solution (in the limit $g_2(0) \to 0$) are presented. Although the effect on the energy is rather small, we have checked numerically that the energy slightly decreases while $g_2(0)$ increases. Finally, on Fig. 8, the evolution of the limiting solution of Fig.7 for $g_2(0) = 0$ is reported for decreasing $f_2(0)$. As expected, the energy of the twisted solution is lower than the one of the untwisted one. The critical value ω_b can be read on the picture. Although it would need more checks, these results suggest that doubly twisted string can have lower classical energy than the simply-twisted and the untwisted ones. A systematic check of this statement would need more numerical and/or analytical work.

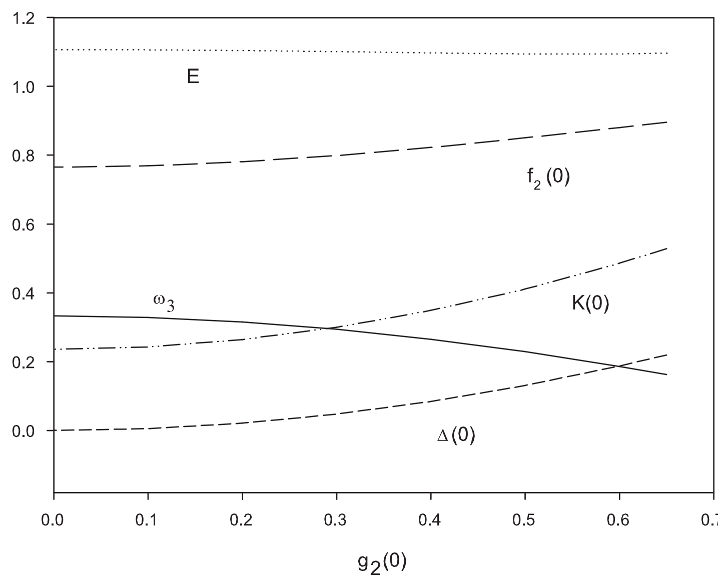

FIG. 7: The evolution of some data for a doubly twisted solution with $R_0 = 2, R_3 = -0.2$ and for $\rho_H = 2, \rho_h = 1$, and $\epsilon = 0.1$, $\beta = \pi/4$.

Further aspects of the doubly twisted solutions is presented on Fig. 9. Here some features of the solutions are superposed for two values of the parameter R_3 with all other parameters fixed (namely $R_0 = 1$, $\omega_0 = 0.01$) and $g_2(0)$ labels the branch of solutions. The solid and dashed curves correspond to $R_3 = -0.5$ and $R_3 = 1.5$ respectively.

In the limit $g_2(0) \to 0$ the branch bifurcates into a simply twisted solution. In the limit $g_2(0) \to 1$ it ends up into a non localized solution. This figure also confirms the property that doubly twisted solutions have a lower energy than the simply twisted ones at least close to the bifurcating point where $g_2(0) = 0$. In fact the decrease of the enegy is very small for the dashed curve but more significant for $R_3 < 0$ corresponding to the solid curve. After reaching a minimal value, the energy increases with $g_2(0)$.

So far, all figures are presented with fixed value of $\epsilon_3 = 0$. However, as suggested

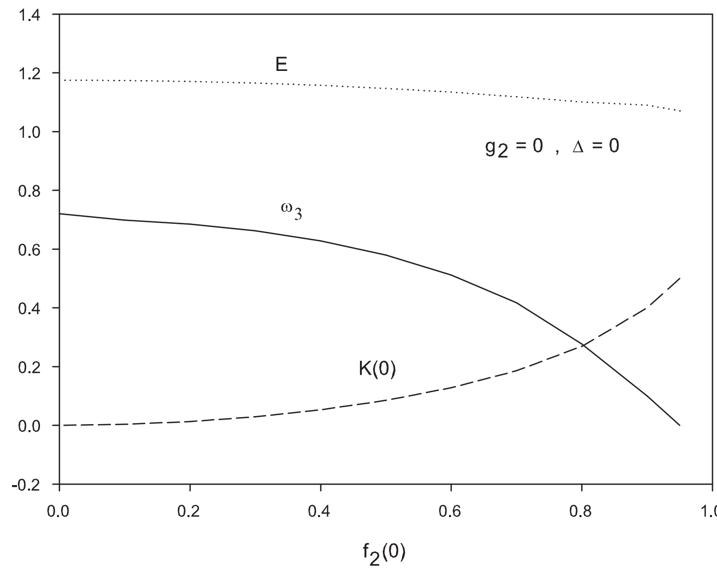

FIG. 8: The evolution of some data for a simply twisted solution with $R_0 = 2, R_3 = -0.2$ and for $\rho_H = 2, \rho_h = 1$, and $\epsilon_3 = 0.1$, $\beta = \pi/4$.

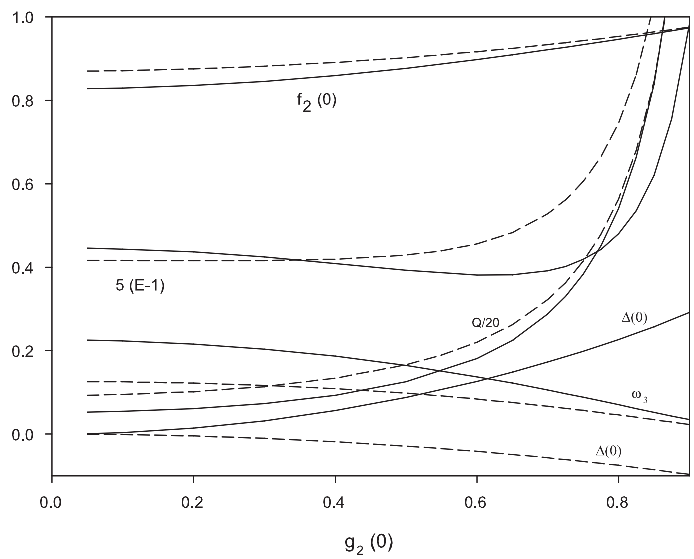

FIG. 9: The evolution of some parameters as functions of the value $g_2(0)$ for $R_0 = 1$ and $R_3 = -0.5$ (solid) and $R_3 = 1.5$ (dashed)

in [12], this coupling constant plays a role in the study of bifurcations. The analysis of the full bifurcation pattern for varying ϵ_3 (see Fig. 1 of [12]) in the present context is out of the scope of this paper due to the richness of the solution's manifold. Some

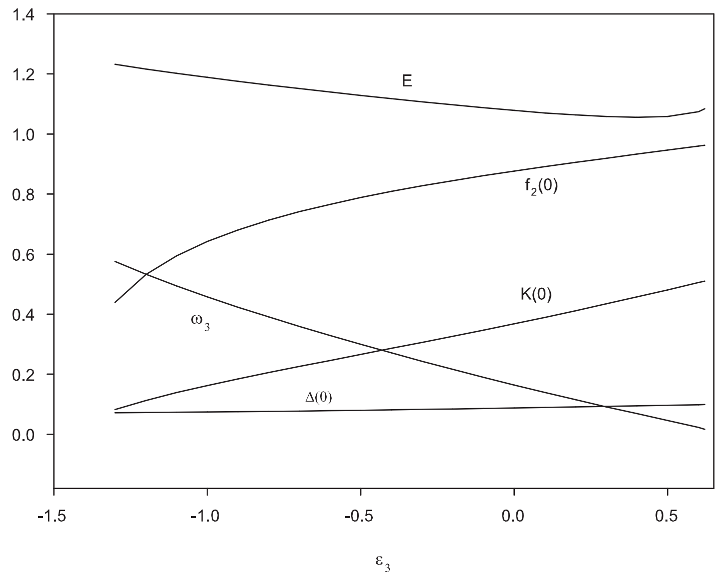

FIG. 10: The evolution of some parameters as functions of the coupling constant ϵ_3 for $R_0 = 1$ and $R_3 = -0.5$

information about the response of doubly-twisted solutions to ϵ_3 is presented on Fig. 10. This graphic is obtained for $R_0 = 1, R_3 = -0.5$, $g_2(0) = 0.5$ and $\rho_h = 1, \rho_H = 2$, $\beta = \pi/4$ (corresponding to the physical domain $-3/2 < \epsilon_3 < 3/2$). For increasing ϵ_3, finite energy solutions stop to exist for $\epsilon_3 \sim 0.62$. In this limit we observe that $\omega^2 = 0$; as a consequence the various functions stop to be localized. For negative values of ϵ_3, the numerical analysis becomes very involved for $\epsilon_3 \to -3/2$. Our results strongly suggest that, in this limit, the solution ends up into a simply twisted solution with respect to the doublet $\phi_{(2)}$, i.e. it has $f_2(\rho) = 0$ and $g_2(\rho) \neq 0$. The fact that $\Delta(\rho) \neq 0$ in this limit is related to the parametrisation (14) adopted for this field in terms of $\vec{\omega}$.

E. Oscillating solution

In [6] it was argued that timelike (or electric, i.e. with $\omega^2 < 0$), finite energy solutions do not exist but no such solutions was reported. We failed to construct such solution in the case of one doublet but in the case of two doublets, the numerical techniques allowed us to construct such solutions. Refering to the asymptotic expansion of [6] it can be expected that several functions associated with timelike solutions oscillate asymptotically. We obtained indeed several such solutions. The profile for one of them is presented on Fig.11. The parameters are such that $\sigma^2 \equiv \sigma_0^2 - \sigma_3^2 < 0$, explaining in particular the oscillations of the function g_2. Because this function behaves asymptotically according to $g_2 \sim \sin(i\sigma\rho)/\sqrt{\rho}$, the term $\rho(g_2')^2$ in the energy density leads to an infinite energy.

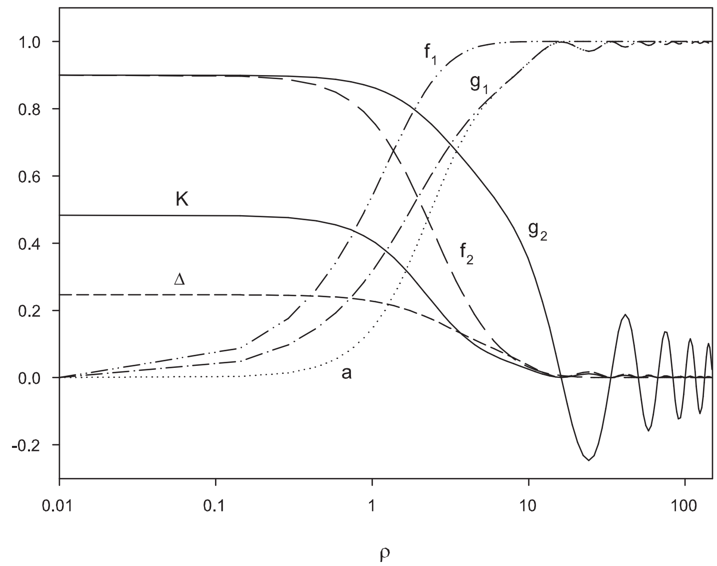

FIG. 11: An example of oscillating solution with $R_0 = 1, R_3 = -0.19, \omega_0 = 0.2, \omega_3 = 0.3009$ and for $\rho_H = 2, \rho_h = 1$, and $\epsilon = 0.1$, $\beta = \pi/4$.

VI. CONCLUSIONS

The lagrangian considered in this paper admits numerous types of physically relevant classical solutions. The most studied ones so far are the sphaleron [13, 14] constructed by using spherically symmetric solutions in the limit $g' = 0$ and with an axial symmetry [21] for the physical value of the Weinberg angle. It is believed that they are related to baryon number violations processes which likely occurred copiously at some stages of the evolution of the Universe [15, 16, 17]. It has been known for some time that new solutions: the bisphaleron [18], bifurcates from the standard sphaleron for a sufficiently large value of the Brout-Englert-Higgs-boson mass [18, 19].

The recently discovered twisted semilocal strings [5, 6] and the superconducting electroweak strings [22] open new ways of investigations of classical solutions in the MSSM. These solutions have applications in cosmology. The cylindrically symmetric ansatz used in this paper transforms the formidable system of Euler-Lagrange equations into a system of seven differential equations with boundary conditions and depending effectively on the five parameters characterizing the potential. The solutions are characterized by several integers labelling the winding numbers of the different components of the scalar fields relative to the axis of the symmetry. The solutions present a rich pattern of bifurcations occuring at specific values of the coupling constants which parametrize the underlying BEH-boson masses. Solutions with $\Delta = a_3 = f_2 = g_2 = 0$ (NO-type) likely exist for all values of the coupling constants. However for sufficiently high values of the BEH-boson masses, solutions bifurcate from the NO-type solution and new solutions exist. They depend on time through a non trivial phase factor and can be made z-depending (denoting z as the coordinate associated with the symmetry axes) by a Lorentz boost in this direction. Very reminiscent of the bisphaleron-sphaleron bifurcation, the static twisted strings have lower energy than the untwisted one. In contrast to the bisphaleron, which exist for $M_H \sim 12 M_W$, twisted semilocal strings arise in regions of the parameter space completely compatible with the experimental lower bounds of the BEH-boson masses.

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