ON THE EXPANSION $(\mathbb{N}, +, 2^x)$ OF PRESBURGER ARITHMETIC

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1. Introduction.

This is based on a preprint ([9]) which appeared in the *Proceedings of the fourth Easter Conference on model theory, Gross Köris, 1986, 17-34, Seminarberichte 86, Humboldt University, Berlin*, where, with G. Cherlin, we gave a detailed proof of a result of Alexei L. Semenov that the theory of $(\mathbb{N}, +, 2^x)$ is decidable and admits quantifier elimination in an expansion of the language containing the Presburger congruence predicates and a logarithmic function.

Expansions of Presburger arithmetic have been (and are still) extensively studied (see, for instance [5]). Let us give a quick review on the expansions of $(\mathbb{N}, +, P_2)$, where P_2 is the set of powers of 2. J. Richard Büchi showed that this expansion is decidable using the fact that the definable subsets are recognizable by a finite 2-automaton (and Kleene's theorem that the empty problem for finite automata is decidable). (In his article, a stronger result is claimed, namely that $Th_\omega(\mathbb{N}, S)$, the weak monadic second-order theory of $\mathbb{N}$ with the successor function S, is bi-interpretable with $Th(\mathbb{N}, +, P_2)$, which is incorrect, as later pointed out by R. McNaughton ([19])).

In his review, McNaughton suggested to replace the predicate P_2 by the binary predicate $\epsilon_2(x, y)$ interpreted by *"x is a power of 2 and appears in the binary expansion of y"*. It is easily seen that this predicate is inter-definable with the unary function $V_2(y)$ sending y to the highest power of 2 dividing it. Since then, several proofs of the fact that $Th(\mathbb{N}, +, V_2)$ is bi-interpretable with $Th_\omega(\mathbb{N}, S)$ and that $Def(\mathbb{N}, +, V_2)$ are exactly the 2-recognizable sets (in powers of $\mathbb{N}$) appeared (see [6], [7]), where $Def(\mathbb{N}, +, V_2)$ are the definable sets in the structure $(\mathbb{N}, +, V_2)$.

A.L. Semenov exhibited a family of 2-recognizable subsets which are not definable in $(\mathbb{N}, +, P_2)$ (see [24] Corollary 4 page 418). Another way to show that this last theory has less expressive power than $Th(\mathbb{N}, +, V_2)$ is to use a result of C. Elgot and M. Rabin ([16] Theorem 2) that if g is a function from P_2 to P_2 with the property that *g skips at least one value*, namely that $\forall n > 1\ \forall m\ (m > n\ \rightarrow\ (\exists y \in P_2\ g(m) > y > g(n)))$, then $Th(\mathbb{N}, +, V_2, n \rightarrow g(n))$ is undecidable and so $Th(\mathbb{N}, +, V_2, 2^x)$ is undecidable (another proof was given by G. Cherlin (see [9]). Consequences are that neither the graph of 2^x is definable in $(\mathbb{N}, +, V_2)$, nor the graph of V_2 in $(\mathbb{N}, +, 2^x)$ and that $Th(\mathbb{N}, +, P_2)$ has less expressive power than $Th(\mathbb{N}, +, 2^x)$.

Which unary predicate can we add to the structure $(\mathbb{N}, +, V_2)$ and retain decidability? Let us mention two kinds of results. On one hand, R. Villemaire showed that

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$Th(\mathbb{N},+,V_2,V_3)$ is undecidable ([26]) and this has been strengthened by A. Bès who proved that $Th(\mathbb{N},+,V_2,P_3)$ is undecidable ([4]), and on the other hand, P. Bateman, C. Jockusch and A. Woods showed that, under the linear Schintzel hypothesis, $Th(\mathbb{N},+,V_2,(2^n)_{n\in\mathcal{P}})$, where $\mathcal{P}$ denotes the set of prime numbers, is decidable ([2]).

Then, one may ask what are the complexities of the definable sets of these structures? In an analogous way to his proof of the model-completeness of the theory of $(\mathbb{R},+,.,\lambda_2)$ ([13]), where λ_2 is a unary function sending a strictly positive real number r to the biggest power of 2 smaller than r, L. van den Dries ([14]) gave a universal axiomatisation T of $Th(\mathbb{N},+,P_2)$ in the language $\{+,\dot{-},\leq,0,1,\frac{\div}{n};n\in\omega,\lambda_2,P_2\}$ and showed that T was model-complete (and so it admits quantifier elimination). Both proofs were model-theoretic, using a description of 1-extensions. Recently, J. Avigad and Y. Yin gave an effective proof of van den Dries' quantifier elimination result for $(\mathbb{R},+,.,\lambda_2)$ (see [1]).

R. Villemaire ([26]) showed that the quantifier complexity of $Def(\mathbb{N},+,V_2)$ has no more than three alternations of quantifiers : $\exists\forall\exists$, by showing that any subset of $\mathbb{N}^n$ which is recognizable by a finite 2-automaton is definable in $(\mathbb{N},+,V_2)$.

Finally, let us say a word on other expansions of Presburger arithmetic. H. Putnam showed that $(\mathbb{N},+,C)$, where C is a unary predicate for the set of squares, is undecidable, then J.R. Büchi extended that result to $(\mathbb{N},+,R)$, where R is a unary predicate which is ultimately the image of a polynomial of degree bigger than or equal to 2 ([8]). On the other hand he showed that $Th(\mathbb{N},+,R)$ is decidable whenever it is ultimately periodic.

A.L. Semënov ([25]) described a class of unary predicates R (*effectively*) *sparse* and a class of functions $f:\mathbb{N}\to\mathbb{N}$, (*effectively*)*compatible with addition* for which one gets analogous (decidability and) definability results as for $(\mathbb{N},+,P_2)$ and $(\mathbb{N},+,2^x)$ (see section 4).

Revisiting a result of A. Muchnik ([21]), C. Michaux and R. Villemaire showed that whenever one has a subset R' of $\mathbb{N}^m$, for some m, which is not already definable in Presburger arithmetic, then one can find a subset R of $\mathbb{N}$ such that $R\in Def(\mathbb{N},+,R')-Def(\mathbb{N},+)$ (see [20]). A. Bès showed showed that there exists $\tilde{R}:=(r_n)_{n\in\omega}\subset\mathbb{N}$ with $\tilde{R}\in Def(\mathbb{N},+,R)$ and such that $r_{n+1}-r_n\geq n$ ([4]).

To a subset of $\mathbb{N}$, one can associate a numeration system. In [23], we gave necessary conditions on the numeration system under which $R=(r_n)_{n\in\mathbb{N}}$ and $f_R:n\to r_n$, fulfilled Semënov requirements and so for which $Th(\mathbb{N},+,R)$ (respectively $Th(\mathbb{N},+,f_R)$) was model-complete and/or decidable.

Note that there are unary predicates R such that $Th(\mathbb{N},+,R)$ is decidable and model-complete, but $Th(\mathbb{N},+,V_R)$ is undecidable. (For instance, take $R:=(2^n+n)_{n\in\omega}$.)

2. Axiomatisation of $(N,+,2^x)$

Let $\mathbb{N}^*:=\mathbb{N}-\{0\}$.

Definition 2.1. Let $\mathcal{L}:=\{+,\dot{-},\leq,0,1,\frac{\div}{n};n\in\mathbb{N}^*,2^x,\ell_2(x)\}$.

Let x, y, $z\in\mathbb{N}$ and $n\in\mathbb{N}^*$, let us define

(1) $x\dot{-}y=0$ if $y>x$, and $x\dot{-}y=x-y$ if $y\leq x$.

(2) $\frac{x}{n} = y$ iff $x = n.y + z$, where $0 \leq z < n$.
(3) If $x \neq 0,\ y \neq 0$, then $(\ell_2(x) = y$ iff $2^y \leq x < 2^{y+1})$ and $\ell_2(0) = 0$.
(4) If $x \neq 0$, then $(P_2(x)$ iff $x = 2^{\ell_2(x)})$.
(5) If $x \neq 0,\ y \neq 0$, then $(\lambda_2(x) = y$ iff $(y \leq x < 2.y$ and $P_2(y)))$ and $\lambda_2(0) = 0)$.

We abbreviate $x \dot{-} \frac{x}{d}.d = m$, where $0 \leq m \leq d-1$ and $d \in \mathbb{N}^*$, by the congruence condition: $x \equiv_d m$. Let $\mathcal{L}_P := \{+, \dot{-}, \leq, 0, 1, \frac{\cdot}{n}; n \in \mathbb{N}^*\}$.

Let T_{Pres} be the $\mathcal{L}_P$-theory of Presburger arithmetic:

(1) $\forall x\ \forall y\ \forall z\quad ((x+y)+z = x+(y+z))$,
(2) $\forall x\ (x+0 = 0+x = x)$,
(3) $\forall x\ \forall y\ \forall z\ (x+z = y+z \to x = y)$,
(4) $\forall x\ \forall y\ (x+y = y+x)$,
(5) $\forall x\ \forall y\ (x \leq y \leftrightarrow \exists u\ (x+u = y))$,
(6) $\forall x\ \forall y\ (x \leq y$ or $x \leq y)$,
(7) $\forall x\ (x \geq 0\ \&\ x \neq 0 \to x \geq 1)\ \&\ 0 \neq 1$,
(8) $\forall x\ \exists y\ (\bigvee_{0 \leq k < n}\ x = n.y + k)$, for each $n \in \mathbb{N}^*$,
(9) $\forall x\ \forall y (\frac{x}{n} = y\ \leftrightarrow\ \bigvee_{0 \leq k < n}\ x = n.y + k)$, for each $n \in \mathbb{N}^*$,
(10) $\forall x\ \forall y\ \forall z\ (x \dot{-} y = z \leftrightarrow ((x \geq y\ \&\ x = y+z)$ or $(x \leq y\ \&\ z = 0)))$.

Let ϕ be the Euler function, namely $\phi(m)$ for natural number m is the number of natural numbers coprime to m and less than or equal to m. We will include in our axioms a special case of Euler's theorem, namely that if m is odd, then $2^{\phi(m)} \equiv_m 1$. In the following, we will use $2.x$ as an abbreviation for $x+x$.

Let T_{exp} be the following $\mathcal{L} \cup \{\lambda_2\}$-theory

(1) T_{Pres}
(2) $\forall x\ (x \neq 0 \to (\lambda_2(x) \leq x < 2.\lambda_2(x)))$, $\lambda_2(0) = 0$,
(3) $\forall x\ \forall y\ (0 < x \leq y\ \to\ \ell_2(x) \leq \ell_2(y))$,
(4) $\ell_2(0) = 0$, $\ell_2(1) = 0$,
(5) $\forall x\ (x \geq 1\ \to\ \ell_2(2.x) = \ell_2(x) + 1)$,
(6) $\forall x\ (x \geq 1\ \to 2^{\ell_2(x)} = \lambda_2(x))$,
(7) $\forall x\ (\ell_2(2^x) = x)$,
(8) $\forall x\ (2^{x+1} = 2^x + 2^x)$,
(9) $\forall\ (x \geq 1 \to 2^{x \dot{-} 1} \geq x)$,
(10) $\forall x\ (x \equiv_{\phi(m)} 0\ \to\ 2^x \equiv_m 1)$ for every odd natural number $m \in \mathbb{N}^*$.

We will show that T_{exp} axiomatizes the theory of $(\mathbb{N}, +, 2^x)$; this will be a consequence of the quantifier elimination (q.e.) result for this theory. Indeed, $(\mathbb{N}, +, 2^x)$ is a model of T_{exp} and it embeds in any model of that theory, so in particular it will be a prime model of T_{exp}.

Before proving the q.e. result, we list a series of properties that hold in any model of T_{exp}.

(1) $\lambda_2(1) = 1$ since $\lambda_2(1) \leq 1$ and $1 < 2.\lambda_2(1)$ (axiom (2)).
(2) $2^0 = 1$ since $2^{\ell_2(1)} = \lambda_2(1) = 1$ (axioms (6), (4) and property (1) above).
(3) If $x \geq 1$, $\lambda_2(2^x) = 2^x$ and $\lambda_2(2^0) = \lambda_2(1) = 1 = 2^0$.
(4) $\lambda_2(2.x) = 2.\lambda_2(x)$ [if $x \geq 1$, $\lambda_2(2.x) = 2^{\ell_2(2.x)} = 2^{\ell_2(x)+1} = 2.2^{\ell_2(x)} = 2.\lambda_2(x)$, if $x = 0$, $\lambda_2(0) = 2.\lambda_2(0) = 0$.

(5) $\forall x \, \forall y \ (2^{\ell_2(x)} < y < 2^{\ell_2(x)+1} \ \rightarrow \ y \neq 2^{\ell_2(y)})$ [since ℓ_2 is an increasing function, we have that $\ell_2(x) \leq \ell_2(y) \leq \ell_2(x)+1$. So either, $\ell_2(x) = \ell_2(y)$ and so $2^{\ell_2(x)} = 2^{\ell_2(y)}$ and $2^{\ell_2(y)} \neq y$, or $\ell_2(x)+1 = \ell_2(y)$, so $2^{\ell_2(y)} = 2^{\ell_2(x)+1}$ which implies that $2^{\ell_2(y)} > y$, a contradiction with axioms (2) and (6).
(6) $\lambda_2(\lambda_2(x)) = \lambda_2(x)$ [$\lambda_2(x) = 2^{\ell_2(x)}$ and $\lambda_2(\lambda_2(x)) = \lambda_2(2^{\ell_2(x)}) = 2^{\ell_2(x)}$ (property (3)).
(7) $\ell_2(x) = \ell_2(\lambda_2(x))$ [$\lambda_2(x) = \lambda_2(\lambda_2(x))$, namely $2^{\ell_2(x)} = 2^{\ell_2(\lambda_2(x))}$; so $\ell_2(x) = \ell(\lambda_2(x))$.
(8) Let $m, n, N \in \mathbb{N}^*$ with $m \leq n$ and $N > \ell_2(n) - \ell_2(m) + 1$. Then,

$$\forall x \ (x \geq 2N \rightarrow nx \leq m.2^x).$$

First, we prove that if $x \geq 2.N$, then $2^x \geq 2^N.x$. By axiom (9), $x \geq 1 \rightarrow 2^{x-1} \geq x$. So, $x \geq N+1$ implies that $2^{(x-N)-1} \geq x - N$. So, if $2.(x-N) \geq x$ i.e. $x \geq 2.N$, then we get the result.
Then, $2.\lambda_2(n) \leq 2^N.\lambda_2(m)$ and $n.x \leq 2.\lambda_2(n).x$, so $n.x \leq 2.\lambda_2(n).x \leq 2^N.\lambda_2(m).x \leq \lambda_2(m).2^x \leq m.2^x$.

Finally, we will show that axiom (8) of T_{exp} follows from properties (3) and (4) above.
First we prove that $\forall x \ \ell_2(2^{x+1}) = \ell_2(2.2^x)$. [$\ell_2(2^{x+1}) = x+1$ and $\ell_2(2.2^x) = \ell_2(2^x) + 1 = x+1$.]
Then, $2^{\ell_2(2^{x+1})} = 2^{\ell_2(2.2^x)}$ and $2^{(\ell_2(2^x+1))} = \lambda_2(2.2^x) = 2.\lambda_2(2^x) = 2.2^x$.

Therefore T^*_{exp}, where in T_{exp}, we replace axiom (8) by $\forall x \ (\lambda_2(2.x) = 2.\lambda_2(x))$ (property (4)), is equivalent to T_{exp}.

3. Quantifier Elimination.

The following results were proven by A.L. Semenov in [25] and a more detailed proof was written in [9].

Theorem 3.1. *The theory T_{exp} admits quantifier elimination in $\mathcal{L}$ and $\mathbb{N}_{exp}$ is a prime model of T_{exp}.*

Corollary 3.2. *The theory T_{exp} is complete and decidable.* □

As noted by K. Compton and C.W. Henson ([11] Remark 8.9, p.187) , one can define a pairing function in models of T_{exp}, namely $p(x,y) := 2^{2^x} + 2^{2y+1}$, which implies that T_{exp} has a hereditary $exp_\infty(cn)$ lower bound.

As a byproduct of the proof of the above theorem, we obtain the following result.

Theorem 3.3. *Given any $\mathcal{L}$-formula $\theta(x, \bar{y})$, there exists a term $t(\bar{y})$ that one can built from θ such that*

$$T_{exp} \models \forall \bar{y}[\exists x \ \theta(x, \bar{y}) \leftrightarrow \exists x \leq t(\bar{y}) \ \theta(x, \bar{y})].$$

Proof of theorems 3.1 and 3.3:
We will show that any 1-existential formula $\exists x \ \theta(x, \bar{y})$, where $\theta(x, \bar{x})$ is a conjunction of basic formulas, is equivalent to an open formula. We will assume that all the basic

formulas are of the form $t_1 \leq t_2$ where t_1 and t_2 are $\mathcal{L}$-terms. (Indeed, $t_1 < t_2$ is equivalent to $t_1 + 1 \leq t_2$ and $t_1 = t_2$ is equivalent to $(t_1 \leq t_2 \;\&\; t_2 \leq t_1)$).

Tracing through the proof, we will show that we can bound the variable x by a term in $\bar{y}$. We will proceed as follows.

We will make the following convention. If x is a variable, $n.x$, where $n \in \mathbb{N}^*$, means $x + \cdots + x$ (n times), we will introduce terms of the form $z.x$, $z \in \mathbb{Z} - \{0\}$, in atomic formulas of the form $t_1 + z.x \leq (=, \geq) t_2$ and this will mean, if $z < 0$ that $t_1 \leq (=, \geq) t_2 + (-z).x$.

First step:
By adding possibly more quantified variables, we transform the formula $\exists x\, \theta(x, \bar{y})$ into a formula $\exists \bar{x}\ \theta_0(\bar{x}, \bar{y})$ where now $\theta_0(\bar{x}, \bar{y})$ is a disjunction of conjunction of inequations between terms of the following forms:
(i) $\sum_i a_i.2^{c.x_i} + \sum_j b_j.x_j + d$ with $a_i, b_j, d \in \mathbb{Z}$, $c \in \mathbb{N}$ [we will call such terms *S-terms*],
(ii) $\mathcal{L}$-terms in $\bar{y}$.

Note that the coefficient c of the $x_i's$ does not depend on i (and we will use later that it is allowed to be unequal to 1.

To achieve this, we replace in θ:-assuming that x occurs non trivially in $t(x, \bar{y})$.

(1) any term of the form $2^{t(x,\bar{y})}$, where $t(x, \bar{y})$ is not the variable x, by a new term 2^{x_j}, where x_j is a new variable, and we add the atomic formula $x_j = t(x, \bar{y})$,
(2) any term of the form $\ell_2(t'(x, \bar{y}))$ by a new variable x_i and we add the formula $2^{x_i} \leq t'(x, \bar{y}) < 2^{x_i+1}$,
(3) any term of the form $\frac{t(x,\bar{y})}{n}$ by a new variable x_k and we add the disjunction $\bigvee_{0 \leq m \leq n} t(x, \bar{y}) = n.x + m$,
(4) any term of the form $\frac{t_1(x,\bar{y}) \dot{-} t_2(x,\bar{y})}{n}$ by either $\frac{t_1(x,\bar{y}) - t_2(x,\bar{y})}{n}$ or 0, adding the disjunction of the corresponding two cases whether $t_1(x, \bar{y}) > t_2(x, \bar{y})$ or $t_1(x, \bar{y}) \leq t_2(x, \bar{y})$,
(5) any inequation of the form $t_1(x, \bar{y}) + s_1(\bar{y}) \leq t_2(x, \bar{y}) + s_2(\bar{y})$ by $s_1(\bar{y}) - s_2(\bar{y}) \leq t_2(x, \bar{y}) - t_1(x, \bar{y})$.

Second step:
Rename x, x_0 and assume we have introduced n new variables $\bar{x} := (x_1, \cdots, x_n)$ in the above process.

Let S_{n+1} be the group of permutations on $\{0, 1, \cdots, n\}$ and $\sigma \in S_{n+1}$. Let $\chi_\sigma(x, \bar{x}) := x_{\sigma(0)} \leq \cdots \leq x_{\sigma(n)}$. Set $\theta_{0,\sigma}(x, \bar{x}) := \chi_\sigma(x, \bar{x}) \;\&\; \theta_0(x, \bar{x}, \bar{y})$.

We have that

$$\theta_0(x, \bar{x}, \bar{y}) \leftrightarrow \bigvee_{\sigma \in S_{n+1}} \theta_{0,\sigma}(x, \bar{x}, \bar{y})$$

and

$$\exists x\ \theta(x, \bar{y}) \leftrightarrow \exists x\ \exists \bar{x}\ \theta_0(x, \bar{x}, \bar{y}) \leftrightarrow \bigvee_{\sigma \in S_{n+1}} \exists x_{\sigma(0)} \cdots \exists x_{\sigma(n)}\ \theta_{0,\sigma}(x, \bar{x}).$$

From now on, we will deal with the 1-existential formula $\exists x_{\sigma(n)}\ \theta_{0,\sigma}(x, \bar{x}, \bar{y})$ and we will show how to eliminate this existential quantifier. In order that the process terminates, we have to obtain a formula where we don't have to use again processes (1) up to (3).

We will show that we can bound x by a multiple of $2^{2^{\cdot^{\cdot^{2^{t(\bar{y})}}}}}$, where $t(\bar{y})$ is a subterm of θ, the number of iterations of the exponential function is equal to n and this multiple only depends on the coefficients of the variable x and on the constant terms appearing in θ.

Put $\theta_{0,\sigma}$ in disjunctive normal form, say $\bigvee_{i\in I}\theta_{i,0,\sigma}$; rename $x_{\sigma(n)}$ by x_0, $\theta_{i,0,\sigma}$ by θ_0 and $(x_{\sigma(0)},\cdots,x_{\sigma(n-1)})$ by $\bar{x}$.

Third step:
We distinguish between two different ways x_0 can occur in the formula θ_0.
A) x_0 occurs linearly in every inequation occurring in θ_0.

We may assume that the system of inequations is of the form:

$$\bigwedge_{1\leq i\leq p,1\leq j\leq q,1\leq k\leq s} f_j(\bar{x})+g_j(\bar{y})\leq d_k x_0\leq f_i(\bar{x})+g_i(\bar{y}),\quad (\star)$$

where $f_i(\bar{x}),f_j(\bar{x})$ are S-terms and $g_i(\bar{y}),g_i(\bar{y})$ are $\mathcal{L}$-terms, $d_k\in\mathbb{Z}$ and depends on both i,j. [Indeed, $f_j(\bar{x})+d_k.x_0\leq g_j(\bar{y})$ is equivalent to $f_j\leq g_j$ & $d_k.x_0\leq g_j-f_j$ or $f_j>g_j$ & $f_j-g_j\leq(-d_k).x_0$.]

Let d be the least common positive multiple of the d_k's and decompose d as follows: $d=2^r.d_0$, where $d_0,r\in\mathbb{N}$ and d_0 is an odd natural number. We multiply out each inequation occurring in $(\star)$ in order to get d as the coefficient of x_0.

For each x_i, $1\leq i\leq n$, the following disjunct holds:

$$\bigvee_{0\leq k_i<d.\phi(d_0)}(x_i\geq r\ \&\ x_i\dot{-}r=k_i+d.\phi(d_0).x_i')\text{ or }\bigvee_{0\leq z<r}x_i=z.$$

We replace each inequation in $(\star)$ by a disjunction of inequations which are obtained by either replacing x_i by $r+k_i+d.\phi(d_0).x_i'$ or by z with $0\leq z<r$.

Let us consider a S-term $f(\bar{x})$ of the form $\sum_{i=1}^n a_i.2^{c.x_i}+\sum_t b_t.x_t+e$ and let us do one of the above substitutions. Assume that $x_i>r$ (otherwise we replace x_i by a constant in the interval $[0;r]$), we get

$$f'(\bar{x}'):=\sum_{i=1}^n a_i.2^{c.(k_i+r+d.\phi(d_0).x_i')}+\sum_t b_t.(r+k_i+d.\phi(d_0).x_t')+e.$$

By axiom scheme (10), we have $2^{\phi(d_0).d.x_i'}\equiv_{d_0}1$, so $2^r.2^{\phi(d_0).d.x_i'}\equiv_d 2^r$ and $2^{x_i}=2^{r+k_i}.2^{\phi(d_0).d.x_i'}\equiv_d 2^{r+k_i}$. Therefore, setting $u_i=x_i$ if $x_i\leq r$, and $u_i=r+k_i$, if $x_i>r$, we get that $f(\bar{x})\equiv_d f(\bar{u})$ and $x_i\equiv_{d.\phi(d_0)}u_i$.

So in replacing the tuple of variables $\bar{x}$ by $\bar{x}'$, we again obtain S-terms since the coefficients of the x_i''s are all equal to $c.d.\phi(d_0)$ and we have for each S-term $f(\bar{x})$ that $f'(\bar{x}')\equiv_d f(\bar{u})$. In particular, we obtain a disjunction of systems of inequations each of the form $(\star)$, over the possible values for the tuple $\bar{u}$.

Consider the existential formula

$$\exists x_0\bigwedge_{1\leq i\leq p,1\leq j\leq q} f_j'(\bar{x}')+g_j(\bar{y})\leq dx_0\leq f_i'(\bar{x}')+g_i(\bar{y}).$$

It is equivalent to the open formula:

$$\bigvee_{\rho\in S_p}\bigvee_{\tau\in S_q}[f'_{\tau(1)}(\bar{x}')+g_{\tau(1)}(\bar{y})\geq\cdots\geq f'_{\tau(q)}(\bar{x}')+g_{\tau(q)}(\bar{y})\ \&\ f'_{\tau(1)}(\bar{x}')+g_{\tau(1)}(\bar{y})\leq f'_{\rho(1)}(\bar{x}')+g_{\rho(1)}(\bar{y})\ \&$$

$$f'_{\rho(1)}(\bar{x}')+g_{\rho(1)}(\bar{y})\leq\cdots\leq f'_{\rho(p)}(\bar{x}')+g_{\rho(p)}(\bar{y})]\ \&[(f'_{\rho(1)}(\bar{x}')+g_{\rho(1)}(\bar{y}))\dot{-}(f'_{\tau(1)}(\bar{x}')+g_{\tau(1)}(\bar{y}))\geq d \text{ or}$$

$$[\bigvee_{0\leq c_{\tau,\rho}<d}[(f'_{\rho(1)}(\bar{x}')+g_{\rho(1)}(\bar{y}))\dot{-}(f'_{\tau(1)}(\bar{x}')+g_{\tau(1)}(\bar{y}))=c_{\tau,\rho}\ \&\ (\bigvee_{0\leq c'\leq c_{\tau,\rho}}g_{\tau(1)}(\bar{y})+f_{\tau(1)}(\bar{u})\equiv_d d-c')]]].$$

Note that we can bound the variable $x_{\sigma(n)}$ by $\frac{1}{d}.max_\rho\{f_{\rho(1)}(x_{\sigma(0)},\cdots,x_{\sigma(n-1)})+g_{\rho(1)}(\bar{y})\}$. Then, we iterate the procedure considering the next largest variable and applying either A) or B) below. We will show that in the case B) below that we can bound the variable by either a term of the form $\ell_2(t(\bar{y}))+1$ where $t(\bar{y})$ is a sub-term occurring in θ, or by $max\{x_i+\delta'',N''\}$, for some $1\leq i\leq n$, where δ'' and N'' are some explicit constants depending on the coefficients appearing in the formula θ_0. So, at the end, we will obtain, in an explicit way, a term in $\bar{y}$ bounding x.

B) There is an inequation where x_0 occurs in an exponential term. Let

$$a_0.2^{d.x_0}+\sum_{i=1}^n a_i.2^{d.x_i}+\sum_{j=0}^n b_j.x_j+c\leq t(\bar{y})$$

be such inequation, where $t(\bar{y})$ is an $\mathcal{L}$-term, $d\in\mathbb{N}^*$, $a_i,b_j,c\in\mathbb{Z}$, $a_0\neq 0$. We denote such inequation by $\tau(x_0,\bar{x},\bar{y})$.

We are going to replace τ by a boolean combination of inequations between S-terms in $x_0,\cdots,x_n$, where now x_0 occurs linearly, and $\mathcal{L}$-terms in $\bar{y}$. We will assume that $d=1$. Let $J:=\{0,\cdots,n\}$. Let $J_1:=\{j\in J: b_j\geq 0\}$.

If $J_1\neq\emptyset$, let $b_+:=2.(\ell_2(\sum_{j\in J_1}b_j)+3)$ and otherwise set $b_+=0$.

If $J-J_1\neq\emptyset$, let $b_-:=2.(\ell_2(\sum_{j\in J_1}(-b_j))+4)$, otherwise set $b_-:=0$.

Let $c_+:=\ell_2(c)+3$ and $c_-:=0$, if $c>0$ and let $c_+=0$ and $c_-:=\ell_2(-c)+4$, otherwise.

Case: $a_0>0$. We will distinguish four subcases.

(1) $2.\lambda_2(a_0).2^{x_0}\leq\lambda_2(t(\bar{y}))$,
(2) $\lambda_2(a_0).2^{x_0}=\lambda_2(t(\bar{y}))$ (equivalently, $x_0=\ell_2(t(\bar{y}))-\ell_2(a_0)$),
(3) $\lambda_2(a_0).2^{x_0}=2.\lambda_2(t(\bar{y}))$ (equivalently, $x_0=\ell_2(t(\bar{y}))+1-\ell_2(a_0)$),
(4) $\lambda_2(a_0).2^{x_0}>2.\lambda_2(t(\bar{y}))$,

Note that in subcase (2) (respectively subcase (3)), we may substitute x_0 by a $\mathcal{L}$-term in $\bar{y}$, namely $\ell_2(t(\bar{y}))-\ell_2(a_0)$ (respectively $\ell_2(t(\bar{y}))+1-\ell_2(a_0)$).

In the remaining cases, we will estimate the S-term

$$a_0.2^{x_0}+\sum_{i=1}^n a_i.2^{x_i}+\sum_{j=0}^n b_j.x_j+c$$

as follows. Let $\delta:=\ell_2(\sum_i|a_i|)+3$.

Claim: In subcase (1), if $x_0 \geq max\{b_+, c_+\}$ and if for all $1 \leq i \leq n$, $x_i \leq x_0 - \delta$, then $\tau(x_0, x, \bar{y})$ holds.
This can be expressed as follows.

$$x_0 \leq \ell_2(t(\bar{y})) - \ell_2(a_0) - 1 \ \& \quad [(x_0 \leq b_+) \text{ or } (x_0 \leq c_+) \text{ or}$$

$$[(x_0 \geq b_+ \ \& \ x_0 \geq c_+) \ \& \ [(\bigwedge_{i=1}^{n} x_i + \delta \leq x_0) \text{ or } \bigvee_{i=1}^{n} \bigvee_{0 \leq k \leq \delta} (x_0 = x_i + k \ \& \ \tau(x_i + k, \bar{x}, \bar{y})]]].$$

Proof of the Claim:
(a) First, assume that $\sum_j b_j.x_j \geq 0$.
We have $\sum_{j \in J} b_j.x_j \leq \sum_{j \in J_1} b_j.x_j \leq (\sum_{j \in J_1} b_j).x_0 \leq 2^{x_0 - 2}$. To see that this last inequality (5) holds, we use property (9) and the fact that $x_0 \geq b_+ := 2.(\ell_2(\sum_{j \in J_1} b_j) + 3)$. Now,

$$a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + \sum_{j \in J} b_j.x_j + c \leq a_0.2^{x_0} + \sum_{i \neq 0} |a_i|.2^{x_i} + \sum_{j \in J_1} b_j.x_j + c.$$

Then,

$$\begin{aligned} a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + \sum_{j \in J} b_j.x_j + c &\leq 2^{x_0}.(a_0 + \sum_{i \neq 0} |a_i|.2^{-\delta} + 2^{x_0}.2^{-2}) + c \\ &\leq 2^{x_0}.(a_0 + \frac{\sum_{i \neq 0} |a_i|}{2^2.2.\lambda_2(\sum_{i \neq 0} |a_i|)} + 2^{x_0 - 2}) + c \\ &\leq 2^{x_0}.(a_0 + \frac{1}{4} + \frac{1}{4}) + c \quad (6) \end{aligned}$$

(a.1) Assume that $\sum_j b_j.x_j \geq 0$ and $c \geq 0$.
So, $c_+ := \ell_2(c) + 3$ and since $x_0 \geq c_+$, we get $2^{x_0} \geq 2.\lambda_2(c).2^2 > 4.c \quad (7)$.
Using inequation (6), we get:

$$\begin{aligned} a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + \sum_{j} b_j.x_j + c &\leq 2^{x_0}.(a_0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}) \\ &< 2^{x_0}.(a_0 + 1) \leq 2^{x_0}.(2.\lambda_2(a_0)) \leq \lambda_2(t(\bar{y})) \quad (8) \\ &\leq t(\bar{y}). \end{aligned}$$

(a.2) Assume that $\sum_j b_j.x_j \geq 0$ and $c \leq 0$. Using inequations (6) and (8), we get:

$$\begin{aligned} a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + \sum_{j \in J} b_j.x_j + c &\leq 2^{x_0}.(a_0 + \frac{1}{4} + \frac{1}{4}) \\ &< 2^{x_0}.(a_0 + \frac{1}{2}) \\ &\leq t(\bar{y}). \end{aligned}$$

(b) Second, assume that $\sum_{j\in J} b_j.x_j \leq 0$. Again we will use the fact that in the case $c \geq 0$, since $x_0 \geq c_+$, then $c \leq 2^{x_0-2}$ (see inequation (7)).

$$\begin{aligned} a_0.2^{x_0} + \sum_{i\neq 0} a_i.2^{x_i} + \sum_{j\in J} b_j.x_j + c &\leq a_0.2^{x_0} + \sum_{i\neq 0} a_i.2^{x_i} + c \\ &\leq 2^{x_0}.(a_0 + \sum_{i\neq 0} |a_i|.2^{-\delta}) + c \\ &\leq 2^{x_0}.(a_0 + \frac{\sum_{i\neq 0}|a_i|}{2^2.2.\lambda_2(\sum_{i\neq 0}|a_i|)}) + c \\ &\leq 2^{x_0}.(a_0 + \frac{1}{2}) \\ &\leq t(\bar{y}) \end{aligned}$$

Claim: In subcase (4), if $x_0 \geq max\{b_-, c_-\}$ and if for all $1 \leq i \leq n$, $x_i \leq x_0 - \delta$, then $\tau(x_0, x, \bar{y})$ does not hold.
This can be expressed as follows.

$$x_0 \geq \ell_2(t(\bar{y})) - \ell_2(a_0) + 2) \ \& \ [(x_0 \leq b_-) \text{ or } (x_0 \leq c_-) \text{ or}$$

$$[(x_0 \geq b_- \ \& \ x_0 \geq c_-) \ \& \ [\bigvee_{i=1}^{n} \bigvee_{0\leq k\leq \delta} (x_0 = x_i + k \ \& \ \tau(x_i + k, \bar{x}, \bar{y}))]].$$

Proof of the Claim:
(a) Assume that $\sum_{j\in J} b_j.x_j \leq 0$.
So, $x_0 \geq 2.((\ell_2(-\sum_{J-J_1} b_j)+4)$. We have $\sum_{j\in J} b_j.x_j \geq \sum_{j\in J-J_1} b_j.x_j \geq (\sum_{j\in J-J_1} b_j).x_0$; using property (9), we obtain $(\sum_{j\in J-J_1} b_j).x_0 \geq -2^{x_0-3}$ (9). Using inequation (9) and the following inequation (10)

$$\sum_{i\neq 0} a_i.2^{x_i} \geq -\sum_{i\neq 0} |a_i|.2^{x_0-\delta} \geq -2^{x_0}.\frac{\sum_{i\neq 0}|a_i|}{2^2.2.\lambda_2(\sum_{i\neq 0}|a_i|)}) \geq -2^{x_0}.2^{-2},$$

we get

$$\begin{aligned} a_0.2^{x_0} + \sum_{i\neq 0} a_i.2^{x_i} + \sum_{j\in J} b_j.x_j + c &\geq a_0.2^{x_0} + \sum_{i\neq 0} a_i.2^{x_i} + \sum_{j\in J-J_1} b_j.x_j + c \\ &\geq (a_0.2^{x_0} - \sum_{i\neq 0} |a_i|.2^{x_0-\delta} + \sum_{j\in J-J_1} b_j.x_j + c) \\ &\geq 2^{x_0}.(a_0 - \frac{\sum_{i\neq 0}|a_i|}{2^2.2.\lambda_2(\sum_{i\neq 0}|a_i|)}) - 2^{x_0-3} + c \\ &\geq 2^{x_0}.(a_0 - 2^{-2} - 2^{-3}) + c. \ (11) \end{aligned}$$

If $c \geq 0$, using inequation (11), we get:

$$\begin{aligned} a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + \sum_{j \in J} b_j.x_j + c &\geq 2^{x_0}.(a_0 - 2^{-1}) \\ &\geq 2^{x_0}.2^{-1}.\lambda_2(a_0) \geq 2.\lambda_2(t(\bar{y})) \\ &> t(\bar{y}). \end{aligned}$$

If $c \leq 0$, we have $c_- = \ell_2(-c) + 4$. Since $x_0 \geq c_-$, $2^{x_0} \geq \lambda_2(-c).2^4$, so we get $-2^{x_0} < c.2^3$ (12).

Using inequations (11) and (12), we get:

$$\begin{aligned} a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + \sum_{j \in J} b_j.x_j + c &\geq 2^{x_0}.(a_0 - 2^{-2} - 2^{-3} - 2^{-3}) \\ &\geq 2^{x_0}.(a_0 - 2^{-1}) \\ &\geq 2^{x_0}.\frac{\lambda_2(a_0)}{2} \geq 2.\lambda_2(t(\bar{y})) \\ &> t(\bar{y}). \end{aligned}$$

(b) Assume now that $\sum_j b_j.x_j \geq 0$.
Using inequation (10) and inequation (11) in case $c < 0$, we get:

$$\begin{aligned} a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + \sum_{j \in J} b_j.x_j + c &\geq a_0.2^{x_0} + \sum_{i \neq 0} a_i.2^{x_i} + c \\ &\geq 2^{x_0}.(a_0 - 2^{-2}) - 2^{-3}.2^{x_0} \\ &\geq 2^{x_0}.(a_0 - 2^{-2} - 2^{-3}) \\ &> 2^{x_0}.\frac{\lambda_2(a_0)}{2} \geq 2.\lambda_2(t(\bar{y})) \\ &> t(\bar{y}). \end{aligned}$$

Set $N := max\{b_+, c_+, b_-, c_-\}$. In the case where $a_0 > 0$, we get the following equivalence.

$$\begin{aligned}
&\tau(x_0,\bar{x},\bar{y}) \leftrightarrow \\
&\quad [(x_0 = \ell_2(t(\bar{y})) - \ell_2(a_0) \;\&\; \tau(\ell_2(t(\bar{y})) - \ell_2(a_0), \bar{x}, \bar{y})) \text{ or} \\
&\quad (x_0 = \ell_2(t(\bar{y})) - \ell_2(a_0) + 1 \;\&\; \tau(\ell_2(t(\bar{y})) - \ell_2(a_0) + 1, \bar{x}, \bar{y})) \text{ or} \\
&\quad \bigvee_{0\le k\le N} (x_0 = k \;\&\; \tau(k,\bar{x},\bar{y})) \text{ or} \\
&\quad [x_0 \ge N \;\&\; (x_0 \le \ell_2(t(\bar{y})) - \ell_2(a_0) - 1 \;\&\; [(\bigwedge_{i=1}^{n} x_i + \delta \le x_0) \text{ or} \\
&\quad \bigvee_{i=1}^{n} \bigvee_{0\le k\le\delta} (x_0 = x_i + k \;\&\; \tau(x_i+k,\bar{x},\bar{y})]) \text{ or} \\
&\quad (x_0 \ge \ell_2(t(\bar{y})) - \ell_2(a_0) + 2 \;\&\; \bigvee_{i=1}^{n} \bigvee_{0\le k\le\delta} (x_0 = x_i + k \;\&\; \tau(x_i+k,\bar{x},\bar{y}))))]].
\end{aligned}$$

So, we can bound the largest variable x_0 either by $max\{N, \ell_2(t(\bar{y})) + 1 - \ell_2(a_0)\}$, where $t(\bar{y})$ occurs as a subterm of θ and so all the variables are bounded by that term, or by $x_i + \delta$, for some $1 \le i \le n$.

Case: $a_0 < 0$.
Let $N' := max\{b_+, c_+\}$, let $\delta' := \ell_2(\sum_i |a_i|) + 2 - \ell(-a_0)$.
Claim: If $x_0 \ge N'$ and if for all $1 \le i \le n$, $x_i \le x_0 - \delta'$, then $\tau(x_0, x, \bar{y})$ holds.
Proof of Claim:
First, we note the following. If $\sum_{j\in J} b_j.x_j \ge 0$, then using $x_0 \ge b_+$, we have that $\sum_{j\in J_1} b_j.x_0 \le 2^{x_0-2}$ (inequation (5)) and if $c \ge 0$, then using that $x_0 \ge c_+$, we get $c < 2^{x_0-2}$ (inequation (7)).
So, either $\sum_{j\in J} b_j.x_j \le 0$ and we will replace it by 0 in the above inequation, or $\sum_{j\in J} b_j.x_j \ge 0$, and since $\sum_{j\in J} b_j.x_j \le \sum_{j\in J_1} b_j.x_j \le \sum_{j\in J_1} b_j.x_0$, we have $\sum_{j\in J} b_j.x_j \le 2^{x_0-2}$. Likewise, either $c < 0$ and we replace it by 0 in the inequation below, or $c \ge 0$ and then $c < 2^{x_0-2}$. So, we get:

$$\begin{aligned}
a_0.2^{x_0} + \sum_{i\ne 0} a_i.2^{x_i} + \sum_j b_j.x_j + c &\le a_0.2^{x_0} + \sum_{i\ne 0} |a_i|.2^{x_0-\delta'} + 2^{x_0-2} + 2^{x_0-2} \\
&< 2^{x_0}.(a_0 + \sum_{i\ne 0} |a_i|.2^{-\delta'} + 2^{-1}) \\
&< 2^{x_0}.(a_0 + \frac{\lambda_2(-a_0)}{2}.\frac{\sum_{i\ne 0}|a_i|}{2.\lambda_2(\sum_{i\ne 0}|a_i|)} + \frac{1}{2}) \\
&< 2^{x_0}.(a_0 + \frac{\lambda_2(-a_0)}{2} + \frac{1}{2}) \le 0 \\
&< t(\bar{y}).
\end{aligned}$$

Therefore, if $a_0 < 0$, we get the following equivalence.

$$\tau(x_0,\bar{x},\bar{y}) \leftrightarrow$$
$$[\bigvee_{0\leq k\leq N'} (x_0 = k \;\&\; \tau(k,\bar{x},\bar{y})) \text{ or}$$
$$(x_0 \geq N' \;\&\; [(\bigwedge_{i=1}^{n} x_i + \delta' \leq x_0) \text{ or} (\bigvee_{i=1}^{n} \bigvee_{0\leq k\leq\delta'} (x_0 = x_i + k \;\&\; \tau(x_i + k,\bar{x},\bar{y}))))])].$$

So, we can bound the largest variable x_0 either by N', and so all the variables are bounded by that term, or by $x_i + \delta'$, for some $1 \leq i \leq n$.

4. Generalisation to $(\mathbb{N},+,f)$.

Actually, A.L. Semënov directly considered expansions of the form $(\mathbb{N},+,f)$ where f is (effectively) compatible with addition ([25] paragraph 2), with the exponential function as a special case, proving a quantifier elimination result when one expands the language with the congruence predicates and a new symbol for the integral part of the inverse of the function ([25] Theorem 2). In [24], A.L. Semënov described a family of (*effectively*) *sparse* predicates and proved model-completeness and (decidability) for the expansions of the form $(\mathbb{N},+,R)$. [Examples of sparse predicates is P_2, the Fibbonacci sequence, $(n!)_{n\in\mathbb{N}^*}$. (See [24] paragraph 3).] An example of a non-sparse one is $(2^n + n)_{n\in\mathbb{N}}$ ([23] p.1354).

Below, we will axiomatize the theory T_f of such expansions $(\mathbb{N},+,f)$. For all finite tuples $\bar{a} := (a_i)_{i\in I}$, $\bar{b} := (b_i)_{i\in I}$ of integers, denote by $A_{(\bar{a},\bar{b})}(n)$ the term $\sum_{i\in I} a_i.f(n+b_i)$, where we make the following abuse of notation: if z is negative $n+z$ means $n\dot{-}(-z)$.

Definition 4.1. ([25][1]) The function f is *compatible with addition* if
(0) for every $m \in \mathbb{N}^*$, the values of f are eventually periodic modulo m (namely there exists n_0 such that the function f is periodic in $\mathbb{Z}/m\mathbb{Z}$ when restricted to the natural numbers x bigger than n_0),
and if for every term $A_{\bar{a},\bar{b}}(n)$ one of the following holds:
(i) $A_{\bar{a},\bar{b}}(n)$ is bounded, (we denote by c_A such bound) or
(ii) there exists a constant Δ_A such that $\forall x\ A_{\bar{a},\bar{b}}(x+\Delta_A) \geq f(x)$ ($A_{\bar{a},\bar{b}}$ is positive definite),
(iii) there exists a constant Δ_A such that $\forall x\ -A_{\bar{a},\bar{b}}(x+\Delta_A) \geq f(x)$ ($A_{\bar{a},\bar{b}}$ is negative definite).

f is *effectively compatible with addition* if there is an algorithm which first determines for each m, the period of f modulo m and the natural number n_0 and second which tells in which of the above cases (i), (ii) or (iii) we are and produces the corresponding constants c_A, Δ_A.

Let $\mathcal{L}_f := \{+,\dot{-},\leq,0,1,\frac{\div}{n}; n\in\omega, f, f^{-1}\}$.
let T_f be the $\mathcal{L}_f$-theory corresponding to a function compatible with addition:

[1] In [25] page 616, Semënov requires that the values of f are periodic modulo m, for every $m \in \mathbb{N}^*$

(1) T_{Pres},
(2) $\forall x \forall y\ (x < y \rightarrow f(x) < f(y))$,
(3) $\forall x \forall y (f^{-1}(x) = y \rightarrow f(x) \leq x < f(y+1)$,
(4) $\forall x\ A_{\bar{a},\bar{b}}(x) \leq c_A$, for all finite tuples $(a_i)_{i \in I}$, $(b_i)_{i \in I}$ of integers such that the corresponding term is bounded,
(5) $\forall x\ A_{\bar{a},\bar{b}}(x + \Delta_A) \geq f(x)$, for all finite tuples $(a_i)_{i \in I}$, $(b_i)_{i \in I}$ of integers such that the corresponding term is positive definite,
(6) $\forall x\ -A_{\bar{a},\bar{b}}(x + \Delta_A) \geq f(x)$, for all finite tuples $(a_i)_{i \in I}$, $(b_i)_{i \in I}$ of integers such that the corresponding term is negative definite,
(7) the values of f are eventually periodic modulo m, for every $m \in \mathbb{N}^*$.

Theorem 4.1. ([25] Theorem 2) *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function compatible with addition. Then, the theory T_f admits q.e. in $\mathcal{L}_f$. If moreover, f is effectively compatible with addition, then T_f is decidable.*

In [23] paragraph 5, we gave a proof of that quantifier elimination result, along the same lines as the above corresponding result for the exponential function, under the hypothesis that the values of f are periodic modulo m, for every $m \in \mathbb{N}^*$, but there is no harm in replacing it by the hypothesis that f is eventually periodic.

Let us make a few remarks. In [9], we noted that if f is compatible with addition and if $f(x) - x$ is unbounded, then f has the following two properties:
$\forall c \exists \Delta \forall x\ (f(x + \Delta) \geq c.f(x) + c.x)$ and
$\forall x\ (x \geq 1 \rightarrow f(x) \geq n.f(x-1) \geq x)$, for all $n > 1$.
Denote the corresponding scheme of axioms, $n \in \mathbb{N}^*$,
$(\star)_{n,\Delta(n)}$: $\forall x\ (f(x + \Delta(n)) \geq n.f(x) + n.x)$, and
$(\star\star)_n$: $\forall x\ (x \geq 1 \rightarrow f(x) \geq n.f(x-1) \geq x)$.
Note that $(\star\star)_n$ implies that $f(x) \geq n^k.f(x-k)$ and that for k such that $n^k - m > 0$, $(x \geq \frac{n^k.k}{n^k-m} \rightarrow f(x) \geq m.x)$.
Moreover, a theorem analogous to Theorem 4.1 holds for theories T'_f and T''_f where we replace the schemes of axioms (4) up to (6) by respectively the schemes $(\star)_{n,\Delta(n)}$ and $(\star\star)_n$ below, $n \in \mathbb{N}^*$.

In [23], we showed that if there is an real number $\theta > 1$ such that $lim_{n \rightarrow +\infty} f(n)/\theta^n$ exists and is non-zero, then the function f satisfies the scheme $(\star)_{n,\Delta(n)}$. Moreover if $(f(n))_{n \in \omega}$ is an A. Bertrand sequence, then $\Delta(n)$ can be found effectively in terms of n.

5. Comments

The results of van den Dries ([13], [14]) bring out the question to what extent can we draw a parallel between the expansions of the theory of $(\mathbb{Z}, +, <, 0, 1)$ and the theory of $(\mathbb{R}, +, ., <, 0, 1)$? Of course for both structures, we do have a notion of minimality ($Th(\mathbb{Z}, +, <, 0, 1)$ is coset-minimal, $Th(\mathbb{R}, +, ., 0, 1)$ is o-minimal) and definable subsets can be endowed with a dimension function. Both structures have uniform elimination of imaginaries and the non independence property (NIP). (See [10], [18], [17]).

Expansions of $(\mathbb{Z}, +, 0)$	Expansions of $(\mathbb{R}, +, ., 0, 1)$
$(\mathbb{Z}, +, <, 2^x)$ model-complete and decidable	$(\mathbb{R}, +, ., 0, 1, <, exp)$ model-complete ([27])
$(\mathbb{Z}, +, 0, <, \lambda_2)$ model-complete and decidable	$(\mathbb{R}, +, ., 0, 1, <, \lambda_2)$ model-complete, decidable and has definable Skolem functions ([13])
$(\mathbb{Z}, +, P_2, P_3)$??	$Def(\mathbb{R}, +, ., 0, 1, <, 2^{\mathbb{Z}}.3^{\mathbb{Z}})$ are a boolean combination of existentially definable subsets ([15])

Note that the theory of $(\mathbb{Z}, +, P_2, V_3)$ is undecidable ([4]) and that the question whether one can extend the results for $(\mathbb{R}, +, ., 0, 1, <, 2^{\mathbb{Z}}.3^{\mathbb{Z}})$ to $(\mathbb{R}, +, ., 0, 1, <, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$ is still open ([15] page 76, paragraph 7). Recently, O. Belegradek and B. Zil'ber also considered non trivial expansions of the field of real numbers ([3]).

One may also consider the expansions of the ordered additive group $(\mathbb{Q}, +, 0, <)$ and in some respects it behaves similarly to expansions of $(\mathbb{Z}, +, 0)$. For instance, one can also prove that the theory of $(\mathbb{Q}, +, P_2)$ is model-complete ([14]) and for generalisations with a unary predicate R (see [23] paragraph 7). However, the following result shows that this is not always the case. Consider the expansion $(\mathbb{Q}, P_2, <, +, 0, f)$, where $f : P_2 \times \mathbb{Q} \to \mathbb{Q} : (2^z, q) \to 2^z.q$, then its theory is decidable ([12]), whereas the theory of $(\mathbb{Z}, P_2, <, +, 0, f)$ is undecidable [22].

References

[1] Avigad J., Yin Y., Quantifier elimination for the reals with a predicate for the powers of two, Theoretical Computer Science 370, 2007, 48-59.

[2] Bateman, P. T., Jockusch, C. G., Woods, A. R., Decidability and undecidability of theories with a predicate for the primes. J. Symbolic Logic 58 (1993), no. 2, 672-687.

[3] Belegradek O., Zil'ber B., The model theory of the field of reals with a subgroup of the unit circle, J. London Math. Soc. (2) 78 (2008) 563579.

[4] Bès A., Undecidable extensions of Büchi arithmetic and Cobham-Semënov Theorem, Journal of Symb. Logic, volume 62, number 4, 1997, 1280-1296.

[5] Bès A., A survey of arithmetical definability, *A tribute to Maurice Boffa*, Bull. Belg. Math. Soc. Simon Stevin 2001, suppl., 1-54.

[6] Bruyère V., Entiers et automates finis, Mémoire de fin détudes, Université de Mons, 1985.

[7] Bruyère V., Hansel G., Michaux C., Villemaire R., Logic and p-recognizable sets of integers, Bull. Belg. Math. Soc. 1, 1994, 191-238.

[8] Büchi, J.R., Weak second-order arithmetic and finite automata, Z. Math. Logik Grundlag. Math. 6 (1960) 66-92.

[9] Cherlin G., Point F., On extensions of Presburger arithmetic, Proceedings of the fourth Easter Conference on model theory, Gross Köris, 1986, 17-34, Seminarberichte 86, Humboldt University, Berlin. (See http://www.logique.jussieu.fr/ point/index.html).

[10] Cluckers R., Presburger sets and p-minimal fields. J. Symbolic Logic 68, 2003, no. 1, 153-162.

[11] Compton K., Henson W., A uniform method for proving lower bounds on the computational complexity of logical theories, in Handbook of Logic in Computer Science, volume 5, edited by S. Abramsky, Dov M. Gabbay and T.S.E. Maibaum, Clarendon Press, Oxford, 2000.

[12] Delon F., $\mathbb{Q}$ muni de l'arithmétique faible de Penzin est décidable. ($\mathbb{Q}$ equipped with Penzin's weak arithmetic is decidable), Proc. Am. Math. Soc. 125, No.9, 1997, 2711-2717.

[13] van den Dries L., The field of reals with a predicate for powers of two, Manuscripta Mathematica 54, 1985, 187-195.
[14] van den Dries L., manuscript, 1985.
[15] van den Dries, L., Günaydın, A., The fields of real and complex numbers with a small multiplicative group, Proc. London Math. Soc. (3), 93, 2006, no. 1, 43-81.
[16] Elgot, C.C., Rabin M.O., Decidability and undecidability of extensions of second (first) order theory of generalized) successor, Journal of Symbolic Logic 31, no. 2, 1966, 169-181.
[17] Gurevich Y., Schmitt P.H., The Theory of Ordered Abelian Groups does not have the Independence Property, Trans. Amer. Math. Soc. 284, no. 1, 1984, 171-182.
[18] Hodges W., Model Theory, Encyclopedia of Mathematics and its applications 42, Cambridge University Press, 1993.
[19] McNaughton R., Review, Journal Symbolic Logic 29, no. 1, 1963, 100-102.
[20] Michaux, C., Villemaire, R., Presburger arithmetic and recognizability of sets of natural numbers by automata: new proofs of Cobham's and Semënov's theorems, Annals of Pure and Applied Logic 77 (1996), 251-277.
[21] Muchnik, A., Definable criterion for definability in Presburger Arithmetic and its application, preprint, Institute of New Technologies (1991).
[22] Penzin Ju.G., Decidability of the theory of integers with addition, oder and multiplication by an arbitrary number, Mat. Zametki 13, 1973, 667-675.
[23] Point F., On decidable extensions of Presburger arithmetic: from A. Bertrand numeration systems to Pisot numbers, Journal of Symbolic Logic 65, no. 3, 2000, 1347-1374.
[24] Semënov A.L., On certain extensions of the arithmetic of addition of natural numbers, Izv. Akad. Nauk. SSSR ser. Mat. 43 (1979) 1175-1195, English translation, Math. USSR-Izv. 15 (1980) 401-418.
[25] Semënov A.L., Logical theories of one-place functions on the set of natural numbers (in Russian), Izv. Akad. Nauk. SSSR ser. Mat. 47 (1983) 623-658, English translation, Math. USSR-Izv. 22 (1984) 587-618.
[26] Villemaire R., The theory of $\langle N, +, V_k, V_l \rangle$ is undecidable, Theoret. Comput. Sci. 106 (1992), no. 2, 337-349.
[27] Wilkie A.J., Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, J. Amer. Math. Soc. 9 (1996), no. 4, 1051-1094.

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