Quantum N-body problem with a minimal length

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The quantum N-body problem is studied in the context of nonrelativistic quantum mechanics with a one-dimensional deformed Heisenberg algebra of the form $[\hat{x},\hat{p}] = i(1+\beta\hat{p}^2)$, leading to the existence of a minimal observable length $\sqrt{\beta}$. For a generic pairwise interaction potential, analytical formulas are obtained that allow estimation of the ground-state energy of the N-body system by finding the ground-state energy of a corresponding two-body problem. It is first shown that in the harmonic oscillator case, the β-dependent term grows faster with increasing N than the β-independent term. Then, it is argued that such a behavior should also be observed with generic potentials and for D-dimensional systems. Consequently, quantum N-body bound states might be interesting places to look at nontrivial manifestations of a minimal length, since the more particles that are present, the more the system deviates from standard quantum-mechanical predictions.

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I. INTRODUCTION

The existence of a minimal observable length in nature is an appealing suggestion of string theory and quantum gravity (see, e.g., [1–6]). For this reason, and also because of their intrinsic interest, the study of quantum theories characterized by a minimal length has become an active area in theoretical physics. An economical way of introducing such a minimal length is to modify the canonical commutation relations between the position and momentum operators in quantum mechanics, i.e., to use a modified Heisenberg algebra [7–9]. As discussed in detail in [7–9], in one dimension, an algebra of the form

$$[\hat{x},\hat{p}] = i\,\Theta(\hat{p}) \tag{1}$$

(in units where $\hbar = c = 1$) is able to yield a minimal uncertainty on $\hat{x}$. The function $\Theta(\hat{p})$ can be expanded in powers of $\hat{p}$. Assuming an isotropic situation and demanding to recover the standard Heisenberg algebra at the lowest order, at order $\hat{p}^2$, one has

$$\Theta(\hat{p}) = 1 + \beta\hat{p}^2. \tag{2}$$

The ansatz (2) is the simplest way of generating a minimal length. Indeed, the uncertainty relation

$$\Delta\hat{x} \geqslant \frac{1}{2}\left(\frac{1}{\Delta\hat{p}} + \beta\Delta\hat{p}\right) \tag{3}$$

imposes a nonzero minimal uncertainty on $\Delta\hat{x}$ given by $\sqrt{\beta}$. One- or two-body problems have been studied thoroughly using the modified algebra defined by Eqs. (1) and (2), especially the harmonic oscillator [10–13], the hydrogen atom [14–17], and the gravitational quantum well [18,19]. Note that the parameter β should be such that $\beta\langle\hat{p}^2\rangle \ll 1$, otherwise such a modification would already have been detected experimentally. The most stringent upper bound on the minimal length scale obtained so far is the one coming from the hydrogen atom and is equal to 3.3×10^{-18} m leading to $\beta \leqslant 4\times10^{-6}\,(\text{fm}/\hbar)^2$, or 10^{-4} GeV^{-2}, in units where $\hbar = c = 1$ (which will be used in the rest of this paper). Notice that the minimal length could be system dependent; thus this bound is *stricto sensu* valid for electrons.

We propose to focus on the following straightforward generalization of the algebra (1) to an N-body system:

$$[\hat{x}_j,\hat{p}_k] = i\delta_{jk}\big(1+\beta\hat{p}_k^2\big),\qquad [\hat{x}_j,\hat{x}_k] = [\hat{p}_j,\hat{p}_k] = 0, \tag{4}$$

where $j,k = 1,\dots,N$. Notice that no summation is meant in the first line: The commutativity between the coordinates of different particles has been kept, as in standard quantum mechanics. Moreover, the inequality (3) holds separately for each particle. The N-body Hamiltonian that we are interested in reads

$$\hat{H}^{(N)} = \sum_{j=1}^{N}\frac{\hat{p}_j^2}{2m} + \sum_{j<k=1}^{N} V(\hat{x}_j - \hat{x}_k), \tag{5}$$

which is the Hamiltonian describing a one-dimensional system of N particles with mass m interacting via the pairwise potential V. The N-body problem with a minimal length has been studied in Ref. [20], where macroscopic (classical) systems are considered. In particular, the analysis of Mercury's perihelion precession leads to the upper bound 0.024 fm for the minimal length for quarks. In the limit of very large N, it is worth mentioning that the modifications of statistical physics due to a nonzero value of β have also been discussed in [21,22]. Thus, to our knowledge, no solution for the quantum N-body problem with a minimal length is currently known.

This paper is organized as follows: In Sec. II A, it is shown that a lower bound for the ground-state energy of the Hamiltonian (5) can be obtained in terms of the ground state of a corresponding two-body problem. Then, the scaling in N of the β-dependent corrections at first order in β is studied in Sec. II B. Those general results are particularized to the case of a harmonic interaction potential in Sec. III. Finally, the results are summarized in Sec. IV, with comments concerning their validity in D dimensions and for generic interaction potentials.

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II. GENERAL FORMALISM

A. Lower bound

In standard quantum mechanics, it is quite natural to work with the relative positions $r_{jk} = x_j - x_k$ when dealing with systems whose potential is of the form $V(x_j - x_k)$. Notice that the symbols without carets denote the standard operators used with the unmodified Heisenberg algebra. The relative momenta are then defined as $\pi_{jk} = (p_j - p_k)/2$ so that the Heisenberg algebra $[r_{jk},\pi_{jk}] = i$ is recovered along with the relative coordinates. The shape of the Hamiltonian (5) therefore suggests the introduction of the modified relative positions

$$\hat{r}_{jk} = \hat{x}_j - \hat{x}_k, \tag{6}$$

and, by analogy with the standard case, the definition of the corresponding relative momenta $\hat{\pi}_{jk}$, such that the modified Heisenberg algebra

$$[\hat{r}_{jk},\hat{\pi}_{jk}] = i\left(1 + \beta\hat{\pi}_{jk}^2\right) \tag{7}$$

is obtained. The general form of the commutator $[\hat{r}_{jk},\hat{\pi}_{lm}]$ with $j \neq l$ and $k \neq m$ is not needed, since the final expressions we will find are actually separable with respect to the relative coordinates. To find the explicit form of $\hat{\pi}_{jk}$, we recall that, as suggested by Eq. (2), the modified Heisenberg algebra we consider comes from an expansion in $\hat{p}$. It is thus sufficient for our formulas to be valid at order $\hat{p}^2$. One can then check that the commutation relation (7), in which

$$\hat{\pi}_{jk} = \left(\frac{\hat{p}_j - \hat{p}_k}{2}\right)\left(1 - \frac{\beta}{4}(\hat{p}_j + \hat{p}_k)^2\right), \tag{8}$$

is satisfied at the second order in the momenta $\hat{p}_j$ as required. It is antisymmetric in j and k, and reduces to the standard relative momentum for $\beta = 0$, as expected.

Still, at the second order in the momenta $\hat{p}_j$, one has

$$\frac{4}{N}\sum_{j<k=1}^{N}\hat{\pi}_{jk}^2 = \sum_{j=1}^{N}\hat{p}_j^2 - \frac{1}{N}\left(\sum_{j=1}^{N}\hat{p}_j\right)^2 \leqslant \sum_{j=1}^{N}\hat{p}_j^2. \tag{9}$$

The above inequality yields the following lower bound of the Hamiltonian (5):

$$\hat{H}^{(N)} \geqslant \sum_{j<k=1}^{N}\left[\frac{\hat{\pi}_{jk}^2}{2\mu} + V(\hat{r}_{jk})\right], \tag{10}$$

with

$$\mu = \frac{mN}{4}. \tag{11}$$

Since the lower-bound Hamiltonian (10) is separable, it can be shown that a lower bound on the ground-state energy $\mathcal{E}^{(N)}$ of $\hat{H}^{(N)}$ is given by [23]

$$\mathcal{E}^{(N)} \geqslant E^{(N)} = \frac{N(N-1)}{2}E^{(2)}, \tag{12}$$

where $E^{(2)}$ is the ground-state energy of the two-body Hamiltonian

$$\hat{H}^{(2)} = \frac{\hat{\pi}^2}{2\mu} + V(\hat{r}). \tag{13}$$

Note that in $\hat{H}^{(2)}$, $\hat{r}$ and $\hat{\pi}$ satisfy $[\hat{r},\hat{\pi}] = i(1 + \beta\hat{\pi}^2)$. The lower bound (12) implicitly assumes that the spatial wave function of the bound state is totally symmetric. Consequently, it is valid for either bosons or fermions provided that extra degrees of freedom (spin, isospin, color, and so on) bring an antisymmetric wave function.

Following Eq. (12), any two-body problem with modified Heisenberg algebra in which the ground-state energy is known can be used to bound from below the ground-state energy of a corresponding N-body problem. It should be stressed that the mass μ appearing in $H^{(2)}$ is proportional to N, as shown by the definition (11). The fact that the β-dependent terms do not necessarily have the same dependence on μ as the β-independent terms is a first indication that their corresponding dependence on N might be different also.

B. $O(\beta)$ approach

The smallness of β with respect to typical quantum-mechanical energy scales suggests that working at the first order in β should be relevant. In that case, it is convenient to work with the representation of Ref. [14] that can be generalized to the N-body case as follows:

$$\hat{x}_j = x_j, \quad \hat{p}_k = \left(1 + \frac{\beta}{3}p_k^2\right)p_k, \tag{14}$$

with the standard position and momentum operators satisfying $[x_j,p_k] = i\delta_{jk}$. Using the representation (14), the two-body Hamiltonian (13) at the first order in β reads

$$\hat{H}^{(2)} = \frac{\pi^2}{2\mu} + V(r) + \frac{\beta}{3\mu}\pi^4. \tag{15}$$

Let us now choose the case of a power-law potential,

$$V(x) = \Lambda\,\text{sgn}(a)|x|^a. \tag{16}$$

Applied to the one-dimensional case, the virial theorem [14,24] leads to

$$\langle\pi^2\rangle = \frac{2\mu a}{a+2}E^{(2)} = \frac{a}{2(a+2)}NmE^{(2)}, \tag{17}$$

and thus to

$$\frac{\langle\pi^4\rangle}{\mu} \propto \mu(E^{(2)})^2 \propto Nm(E^{(2)})^2. \tag{18}$$

Scaling arguments impose that $E^{(2)} = \Lambda^{\frac{2}{a+2}}\mu^{-\frac{a}{a+2}}e(a,n) \equiv \Lambda^{\frac{2}{a+2}}(mN)^{-\frac{a}{a+2}}e_0(a,n)$, where e and e_0 are dimensionless functions of a and of a quantum number n [25]. One finally gets from Eq. (12) that the lower bound of the ground-state energy is schematically given at large N by

$$E^{(N)} \approx N^{\frac{a+4}{a+2}}\Lambda^{\frac{2}{a+2}}m^{-\frac{a}{a+2}}e_0\left[1 + \beta(m\Lambda N)^{\frac{2}{a+2}}e_1\right], \tag{19}$$

where e_0 and e_1 are dimensionless functions of a. This last relation suggests that the β-dependent term of the ground-state energy increases with N faster than the β-independent term, whose dependence on N agrees with the recent analytical calculation in [26]. The smaller a is (especially when a is negative), the more this effect is significant. For the Coulomb case, for example, one would have schematically at large N: $E^{(N\gg1)} \sim mN^3 + \beta\, m^3N^5$.

III. THE HARMONIC OSCILLATOR

Let us now particularize the results obtained so far to an N-body harmonic oscillator, i.e., to a potential of the form

$$V(\hat{x}_i - \hat{x}_j) = \Omega(\hat{x}_i - \hat{x}_j)^2. \quad (20)$$

In that case, the ground-state energy can also be easily bounded from above. Indeed,

$$\hat{H}^{(N)} = \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2m} + \Omega N \hat{x}_j^2\right] - \Omega\left(\sum_{j=1}^{N}\hat{x}_j\right)^2 \leqslant \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2m} + \Omega N \hat{x}_j^2\right], \quad (21)$$

and the upper bound of the ground-state energy reads

$$\mathcal{E}^{(N)} \leqslant N\left\langle \frac{\hat{p}^2}{2m} + \Omega N \hat{x}^2 \right\rangle, \quad (22)$$

where $[\hat{x},\hat{p}] = i(1+\beta\hat{p}^2)$ and where the average is computed with the ground-state wave function.

The exact spectrum of the two-body harmonic oscillator with a minimal length has been exactly computed in [8,10]. It can be deduced from those results that

$$\left\langle \frac{\hat{p}^2}{2\nu} + \theta\hat{x}^2 \right\rangle = \sqrt{\frac{\theta}{2\nu} + \frac{\beta^2\theta^2}{4}} + \frac{\beta\theta}{2}. \quad (23)$$

Combining this last result with the lower and upper bounds (12) and (22) leads to the conclusion that the ground-state energy of a one-dimensional N-body harmonic oscillator is bounded by

$$(N-1)\left[\sqrt{\frac{\Omega N}{2m} + \left(\frac{\beta\Omega N}{4}\right)^2} + \frac{\beta\Omega N}{4}\right] \leqslant \mathcal{E}^{(N)} \leqslant N\left[\sqrt{\frac{\Omega N}{2m} + \left(\frac{\beta\Omega N}{2}\right)^2} + \frac{\beta\Omega N}{2}\right]. \quad (24)$$

The lower bound is actually exact for $N=2$ and arbitrary β, as well as for $\beta=0$ and arbitrary N. Furthermore, the upper bound is exact for $N=1$ and arbitrary β. At the first order in β, the above inequalities become

$$(N-1)\sqrt{\frac{\Omega N}{2m}} + \frac{\beta\Omega N(N-1)}{4} \leqslant \mathcal{E}^{(N)} \leqslant N\sqrt{\frac{\Omega N}{2m}} + \frac{\beta\Omega N^2}{2}. \quad (25)$$

At large N, the exact ground-state energy is thus of the form

$$\mathcal{E}^{(N\gg 1)} = \sqrt{\frac{\Omega}{2m}}\,N^{\frac{3}{2}} + \beta A\Omega N^2, \quad (26)$$

where $A \in [1/4, 1/2]$ is a coefficient independent of m. This formula is in qualitative agreement with the lower bound estimate (19) for $a=2$, which thus seems to provide a reliable estimation of the behavior in N of the exact ground-state energy.

IV. SUMMARY AND DISCUSSION

The one-dimensional quantum N-body problem has been studied within the framework of a modified Heisenberg algebra leading to a minimal length. The system under study is composed of N nonrelativistic particles with a mass m, interacting via a pairwise potential. We have shown that the ground-state energy can be bounded from below by a convenient formula that only requires knowledge of the ground-state energy of a corresponding two-body system. It is also possible to work at the first order in β; the correction term is then simply related to the averaged fourth power of the two-body relative momentum. The formalism developed has been explicitly applied to the case of harmonic interactions to check that the several formulas obtained are coherent with each other once applied to a common case. In particular, we can conclude that the ground-state energy of the one-dimensional N-body harmonic oscillator is of the form $bN^{\frac{3}{2}} + \beta c N^2$ at large N. As illustrated by this last relation, it appears from this study that for power-law potentials, the β-dependent term grows faster with N than the β-independent term, with the effect being especially important for singular attractive potentials of the form $-1/|x|^k$.

Some comments can be made concerning the extension of the present results to more realistic potentials and higher-dimensional systems. First, provided that it is not located too close to the continuum, the ground state will be mostly sensitive to the short-range behavior of the potential $V(x)$ that can be approximated by its Taylor expansion near $x=0$. One can thus say that in a first approximation, the faster increase with N of the β-dependent term obtained for power-law potentials will qualitatively be a feature of generic potentials if the ground-state binding energy is significant. Second, D-dimensional generalizations of the modified N-body algebra considered here generally depend on two parameters, denoted β and β' [20]. In analogy with Sec. II A, let us assume that we know the relative coordinates $\hat{\boldsymbol{r}}_{jk} = \hat{\boldsymbol{x}}_j - \hat{\boldsymbol{x}}_k$, $\hat{\boldsymbol{\pi}}_{jk} = (\hat{\boldsymbol{p}}_j - \hat{\boldsymbol{p}}_k)[1 + s(\beta,\beta',\hat{\boldsymbol{p}}_j,\hat{\boldsymbol{p}}_k)]/2$, where the bold symbols denote vectors, such that the relative coordinates satisfy the same algebra as the particle coordinates. The function s should be such that $s(0,0,\hat{\boldsymbol{p}}_j,\hat{\boldsymbol{p}}_k) = 0$ and $s(\beta,\beta',\hat{\boldsymbol{p}}_j,\hat{\boldsymbol{p}}_k) = s(\beta,\beta',\hat{\boldsymbol{p}}_k,\hat{\boldsymbol{p}}_j)$. Assuming that it has a nontrivial dependence on the momenta, one would again find the lower bound (12) and (13) at second order in the momenta. A case of interest is the algebra in which $\beta' = 2\beta$, keeping commutative positions at $O(\beta)$, and admitting at this order the representation $\hat{\boldsymbol{x}}_j = \boldsymbol{x}_j$, $\hat{\boldsymbol{p}}_k = (1+\beta\boldsymbol{p}_k^2)\boldsymbol{p}_k$ [14]. One would finally find the term $\beta\boldsymbol{\pi}^4/\mu$ in the D-dimensional generalization of Eq. (15), leading again to the estimation (19) for the ground-state energy in the case of radial power-law potentials.

In conclusion, it has been shown that there exists a nontrivial interplay between the quantum N-body dynamics and the existence of a minimal length, whose manifestation is an enhancement of the minimal length effects at large numbers of particles. Although explicit examples are out of the scope of this work, the results suggest that the comparison between high-precision models and measurements related to quantum N-body systems (atomic, molecular, etc.) might eventually be an interesting and new way of constraining the value of a minimal length in nature.

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