JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
Volume **6**, Number **4**, November **2010**

doi:10.3934/jimo.2010.6.811

pp. **811–823**

A METHOD FOR OPTIMIZING OVER THE INTEGER EFFICIENT SET

Chaabane Djamal

USTHB - Faculty of Mathematics - Operations Research Department
Bp 32 El Alia, BEZ
Algiers, 16121, Algeria

Pirlot Marc

UMONS, Faculté Polytechnique
Rue de Houdain,
7000 Mons, Belgium

(Communicated by Xiaojun Chen)

Abstract. In this paper, we are interested in optimizing a linear function on the set of efficient solutions of a Multiple Objective Integer Linear Programming problem ($MOILP$). We propose an exact algorithm for maximizing a linear function denoted ϕ on the set of efficient solutions of a $MOILP$ problem without having to enumerate explicitly all the elements of this set. Two techniques are used: the first is to reduce progressively the admissible domain by adding more constraints eliminating all the dominated points by the current solution; the second, when the new solution obtained by maximizing the function ϕ in the reduced area is not efficient, an exploration procedure is applied over the edges incident to this solution in order to find new alternative efficient solutions if they exist. The algorithm produces not only an optimal value of the linear function but also a subset of non-dominated solutions in the direction of ϕ that can be helpful in the practice.

1. **Introduction.** The problem of maximizing a linear objective function over the set of efficient solutions of a multicriteria linear program is considered. This is a linear program with nonconvex constraints. The difficulty of this problem is mainly due to the nonconvexity of this set (see [15]) and hence delicate algorithms are necessary. In the case where the decision variables are continuous, the developed algorithms are classified into several groups : adjacent vertex search algorithm, non-adjacent vertex search algorithm, branch-and-bound based algorithm, Lagrangian relaxation based algorithm, dual approach and bisection algorithm ([21]). When the decision variables are integers, few methods exist in the literature and cuts or branch and bound techniques are unavoidable.

An illustration can be solved by our proposed method is a traveling salesman who must conduct a tour by visiting only once n cities while minimizing various fees under some scenarios (certain weather conditions for example). Consider a list of n cities $\aleph = \{1, 2, \dots, n\}$, $A = \{(i, j); i, j \in \aleph, i \neq j\}$ and c_{ij}^k is the corresponding cost under the meteorological conditions k, $(k = 1, \dots, p)$. Let x_{ij} be a decision

2000 *Mathematics Subject Classification.* Primary: 90C29, 90C26; Secondary: 90C10.

Key words and phrases. Multiple Objective Optimization, Discrete Optimization, Linear Programming, Efficient Set.

variable that is equal to 1 if $(i,j) \in \aleph$ and 0 if not. The problem is then formulated by

$$\left\{\begin{array}{ll} \text{“min”} z^k = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} & k = 1,\dots,p \\ s.t. & \\ \sum_{i=1}^{n} x_{ij} = 1 & j = 1,\dots,n \\ \sum_{j=1}^{n} x_{ij} = 1 & i = 1,\dots,n \\ \sum_{i\in T}\sum_{j\in \overline{T}} x_{ij} \geq 1 & \forall T \subset \aleph \text{ and } \overline{T} = \aleph \backslash T \\ x_{i,j} \in \{0,1\} & \end{array}\right. \tag{1}$$

Suppose that the decision-maker (D.M.) has got a utility function $U(x)$ expressed in decision variables. For instance, the D.M. prefers toll highway which implies a cost of d_{ij}, the utility function can be written as $\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_{ij}$. However, optimization over the efficient solutions is needed in order to choose a robust solution. Similar examples of robustness in multiple objective combinatorial optimization problems can be formulated in the same manner. In this paper we focus on the problem of optimizing a linear function, denoted by ϕ, over the efficient set of a Multiple Objective Integer Linear Programming problem ($MOILP$). We address the general case where ϕ is any linear function and not necessarily a linear combination of the objectives of the $MOILP$ problem. In such a case a formulation of criterion ϕ as a linear combination of the other objectives may be impossible. It may be the case for instance when the number of variables is larger than the number of objectives. We propose an implicit technique that avoids searching for all efficient solutions but guarantees finding one that optimizes ϕ. We take advantage of Sylva and Crema's idea in reducing progressively the feasible set (eliminating the solutions dominated by the current non dominated ones) (see [7]), then we perform an exploration process of the edges incident to the current optimal solution.(see [2]).
Consider the $MOILP$ problem

$$(P)\left\{\begin{array}{l} \text{“max”} \quad Z_i = c^i x, \quad i = 1,2,...,p \\ \text{s.t.} \quad x \in D \end{array}\right.$$

where $D = S \cap \mathbb{Z}^n$, $S = \{x \in \mathbb{R}^n |\ Ax \leq b,\ x \geq 0\}$, $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $p \geq 2$; $c^1, c^2, ..., c^p \in \mathbb{Z}^n$ are row vectors, $\mathbb{Z}$ is the set of integers. We assume throughout the paper that D is not empty and S is a bounded convex polyhedron. The set of all integer efficient solutions of (P) is denoted by $E\,(P)$.

Definition 1.1. The following spaces are defined by :

- The decision space is the set S of the actions in $\mathbb{R}^n$.
- The criteria space is the image of S in $\mathbb{R}^p$ by linear mapping ψ given by

$$\psi : \mathbb{R}^n \mapsto \mathbb{R}^p : \quad x \mapsto (Z_1(x), \cdots, Z_p(x))$$

Efficiency and non-dominance are defined as follows :

Definition 1.2 ([19])**.** A point $\overline{x} \in D$ is an *efficient* solution if and only if there is no $x \in D$ such that $Z_i(x) \geq Z_i(\overline{x})$ for all $i \in \Im = \{1,2,...,p\}$ and $Z_i(x) > Z_i(\overline{x})$

for at least one $i \in \Im$. Otherwise, $\overline{x}$ is not efficient and the corresponding vector $(Z_1(\overline{x}),\ Z_2(\overline{x}), ..., Z_p(\overline{x}))$ is said to be *dominated*.

The problem that we are studying is

$$(P_E) \begin{cases} \max & \phi(x) = dx \\ s.t. & x \in E(P) \end{cases} \tag{2}$$

where d denotes an n dimensional integer row vector.

Let the relaxed problem be:

$$(P_R) \begin{cases} \max & \phi(x) = dx \\ \text{s.t.} & x \in D; \end{cases} \tag{3}$$

The problem (P_R) can have several optimal solutions. In this case we define the notion of an alternate solution as follows :

Definition 1.3. Let x^* be an optimal solution of (P_R), a feasible solution $\tilde{x} \in D$ is said to be an *alternate solution* to x^* if $\phi(x^*) = \phi(\tilde{x})$ and $x^* \neq \tilde{x}$.

2. Necessary results.

In this section, we shall present some theoretical results that characterize efficient solutions. We state first recall a well-known theorem that gives necessary and sufficient conditions characterizing efficient solutions of a multiple objective linear programming problem.

Theorem 2.1 ([10])**.** *If $\overline{x}$ is an optimal solution of the problem*

$$(P_\lambda) \begin{cases} \max Z_\lambda = \sum\limits_{k=1}^{p} \lambda_k z_k\,(x) \\ s.t.\ x \in S, \end{cases} \tag{4}$$

where $\lambda_k > 0, \forall k \in \{1, \ldots, p\}$, then $\overline{x}$ is an efficient solution of (P_S), i.e. problem (P) defined on S instead of D.
If $\overline{x}$ is an efficient solution of (P_S) and $\psi\,(S)$ is convex, then there exists a vector λ with components $\lambda_k > 0$ such that $\overline{x}$ is an optimal solution of the problem (P_λ).

When the decision variables are integers, the reciprocal of this theorem does not hold (see [5]); consequently, the set of efficient solutions is partitioned into two incompatible sub-sets : supported efficient solutions set that are optimal solutions of the problem (P_λ) and non supported efficient solutions that are not.

The following theorem provides another characterization of an efficient solution integrated in the developed algorithm as a test-procedure.

2.1. Testing efficiency.

The proof of this theorem is omitted and it can be found in [3]. Its utilization guarantees that the feasible solution is either efficient or otherwise, provides an efficient solution for problem (P).

Theorem 2.2. *Let x^0 be an arbitrary element of the region D. $x^0 \in E\,(P)$ if and only if the optimal value of the objective function $\Theta(\psi, x)$ is null in the following* *integer linear programming problem:*

$$\left(P\left(x^0\right)\right)\left\{\begin{array}{ll}\max & \Theta(\psi_1,\ldots,\psi_p,x_1,\ldots,x_n)=\sum\limits_{i=1}^{p}\psi_i\\ s.t. & \left\{\begin{array}{l}\displaystyle\sum_{j=1}^{n}c_j^i x_j-\psi_i=\sum_{j=1}^{n}c_j^i x_j^0 \quad \forall i\in\{1,\ldots,p\}\\ x=(x_1,\ldots,x_n)\in D\\ \psi_i:\ \textit{are real nonnegative integer variables for all } i\\ c_j^i:\ \textit{is the } j^{th} \textit{ component of row vector}\\ \qquad C^i \textit{ in problem } (P)\end{array}\right.\end{array}\right. \tag{5}$$

2.2. **Exploring edges.** In the proposed algorithm we may have to explore edges incident to a feasible solution, looking for efficient solutions. We use a type of cuts, as defined below. The current admissible domain is being reduced by estimating all feasible solutions on those edges.

Let D_k be the current feasible region at iteration k,

- x^k is an optimal integer solution of problem (P_λ) obtained in D_k,
- B_k is the basis associated with solution x^k,
- $a_{k,j}$ is the activity vector of x_j^k with respect to the current region D_k,
- $I_k = \{j|$ the vector $a_{k,j}$ is a column of the basis $B_k\}$ (indices of basic variables),
- $N_k = \{j|$ the vector $a_{k,j}$ is not a column of the basis $B_k\}$ (indices of non-basic variables),
- $y_{k,j} = (y_{k,ij}) = (B_k)^{-1}\,a_{k,j}$,
- $\phi_j = d_{B_k}y_{k,ij}$; d_{B_k} is the vector of cost-coefficients of basic variables associated with B_k in vector d of problem (P_R) (equation 3).

Definition 2.3. Assume that $j_k\in N_k$. An edge E_{j_k} incident to a solution x^k is defined as the set

$$E_{j_k}=\left\{(x_1,\cdots,x_n)\in D_k\;\middle|\;\begin{array}{l}x_i=x_i^k-\theta_{j_k}y_{k,ij_k}\ \text{for } i\in I_k\\ x_{j_k}=\theta_{j_k}\\ x_i=0 \text{ for } i\in N_k\backslash\{j_k\}\end{array}\right\}$$

where $0<\theta_{j_k}\le\min\limits_{i\in I_k}\left\{\frac{x_i^k}{y_{k,ij_k}};\,y_{k,ij_k}>0\right\}$, θ_{j_k} is a positive integer and $\theta_{j_k}\times y_{k,ij_k}$ are integers for all $i\in I_k$ if such integer values exist.

We present some results that will be useful for proving the finiteness of the procedure we propose in section 3. The following theorem addresses the case in which the optimal solution of $(P_R(D_k))$, i.e. problem (P_R) relative to the current feasible solution set D_k, is not unique.

Note that a sufficient condition for the uniqueness of the optimal solution x^1 of $(P_R(D_k))$ is that the set $J_1=\{j\in N_1\,|\,\phi_j-d_j=0\}$ is empty.

Let x^1 be an optimal solution of (P_R). $A_1x^1=\sum\limits_{i\in I_1}a_{1,i}x_i^1=b_1$.

Let $j_1\in J_1$; we have $\sum\limits_{i\in I_1}a_{1,i}x_i^1-\theta_{j_1}a_{1,j_1}+\theta_{j_1}a_{1,j_1}=b_1$, where θ_{j_1} is a non-zero positive scalar. Trivially, $a_{1,j_1}=\sum\limits_{i\in I_1}a_{1,i}y_{1,ij_1}$; hence :

$$\sum_{i\in I_1}a_{1,i}x_i^1-\theta_{j_1}\left(\sum_{i\in I_1}a_{1,i}y_{1,ij_1}\right)+\theta_{j_1}a_{1,j_1}=b_1\ ;$$

$\sum\limits_{i\in I_1} a_{1,i}(x_i^1 - \theta_{j_1} y_{1,ij_1}) + \theta_{j_1} a_{1,j_1} = b_1.$

For $0 < \theta_{j_1} \leq \min\limits_{i\in I_1} \left\{ \frac{x_i^1}{y_{1,ij_1}} \; ; \; y_{1,ij_1} > 0 \right\}$, we define x^2 as follows :

$$x^2 = \begin{cases} x_i^2 = x_i^1 - \theta_{j_1} \times y_{1,ij_1} & i \in I_1 \\ x_{j_1}^2 = \theta_{j_1} \\ x_i^2 = 0 \text{ for } i \in N_1 \backslash \{j_1\}, \end{cases}$$

which is a new integer feasible solution of (P_R), provided θ_{j_1} is a positive integer and $\theta_{j_1} \times y_{1,ij_1}$ are integers for all $i \in I_1$.

We now show that $\phi(x^2) = \phi(x^1)$.

$$\begin{aligned} \phi(x^2) &= d'x^2 = \sum_{i\in I_1} d_i x_i^2 + d_{j_1} x_{j_1}^2 + \sum_{i\in N_1\backslash\{j_1\}} d_i x_i^2 \\ &= \sum_{i\in I_1} d_i(x_i^1 - \theta_{j_1} y_{1,ij_1}) + d_{j_1}\theta_{j_1} = \sum_{i\in I_1} d_i x_i^1 - \sum_{i\in I_1} d_i \theta_{j_1} y_{1,ij_1} \\ &+ d_{j_1}\theta_{j_1} \\ &= \sum_{i\in I_1} dix_i^1 - \theta_{j_1}\left(\sum_{i\in I_1} d_i y_{1,ij_1} - d_{j_1} \right) \\ &= \phi(x^1) - \theta_{j_1}\left(\phi_j - d_{j_1}\right). \end{aligned}$$

As $j_1 \in J_1$, then $\phi_j - d_{j_1} = 0$. Then $\phi(x^2) = \phi(x^1)$.
x^2 is an integer feasible solution of (P_R), alternate to x^1, lying on an edge

$$E_{j_1} = \left\{ (x_i) \in \mathbb{R}^{(|I_1|+|N_1|)} \;\middle|\; \begin{array}{l} x_i^2 = x_i^1 - \theta_{j_1} \times y_{1,ij_1}, \; i \in I_1 \\ x_{j_1}^2 = \theta_{j_1} \\ x_i^2 = 0 \;\text{ for }\; i \in N_1 \backslash \{j_1\} \end{array} \right\} \tag{6}$$

We have $\sum\limits_{j\in N_1\backslash\{j_1\}} x_j^2 < 1$, since $x_j^2 = 0$ for all $j \in N_1 \backslash \{j_1\}$. Thus, the point x^2 lies in the open half space $\sum\limits_{j\in N_1\backslash\{j_1\}} x_j < 1$.

Equation (6) enables us to compute the integer feasible alternate solutions when the optimal solution obtained by solving (P_λ) is not unique.

The following theorem suggests a cut that can be viewed as a generalization of Dantzig's cut, see [8]; it truncates a whole edge while the latter truncates only a point. Obviously, it leads to a reduction of the feasible set that is more drastic than the classical Dantzig cut. For the proof see [2].

Theorem 2.4. *An integer feasible solution of problem $(P_\lambda(D_k))$ that is distinct from x_k and not on an edge E_{j_k} of the truncated region S_k (or region S) through an integer optimal solution x_k of $(P_\lambda(D))$ lies in the closed half space*

$$\sum_{j\in N_k\backslash\{j_k\}} x_j \geq 1 \tag{7}$$

3. **Description of the procedure.** This section describes the algorithm that will be further detailed and tested in section 4. We prove that the algorithm yields an optimal solution of (P_E) in a finite number of steps.

The procedure starts from an initial efficient solution x^0 of problem (P), obtained by solving problem $(P\,(x^*))$ see equation (5), where x^* is an optimal solution of the relaxed problem (P_R).

x^0 is then used to optimize the main criterion on solutions with the same criterion vector by solving the problem (T_l) defined by

$$(T_l) : \max\{dx|Cx = Cx^0, x \in D\} \tag{8}$$

where C is the criteria matrix. The optimal solution $\overline{x}^l$ of this problem is considered as a first efficient solution; we initialize $X_{opt} := \overline{x}^l$ and $\phi_{opt} := d\overline{x}$.

Afterwards, at each iteration, say l, the research area for efficient solutions is reduced gradually by eliminating all dominated solutions by $\overline{x}^l$ using Sylva and Crema's idea, see [7]. The resolution of the following problem enables us to perform this elimination.

$$(P_l) : \max\left\{\phi(x) = dx|x \in D\backslash \bigcup_{s=1}^{l} D_s\right\} \tag{9}$$

where $D_s = \{x \in \mathbb{Z}^n \mid Cx \leq C\bar{x}^s\}$, $s \in \{1, \cdots, l\}$; with $\bar{x}^1, \bar{x}^2, ..., \bar{x}^{l-1}$ efficient solutions of (P) obtained at iterations $1, 2, ..., l-1$ respectively.

The feasible region of problem (P_l) will be iteratively defined by the following constraints

$$\begin{cases} H^0 = D \\ H^k = H^{k-1} \cap \Delta_k, \qquad \forall k \in \{1, \ldots, l-1\} \end{cases} \tag{10}$$

where

$$\Delta_k = \left\{ \begin{array}{c|l} x \in D & Z_i(x) \geq (Z_i(X_{opt}) + 1)y_i^k - M_i(1 - y_i^k) \quad (*) \\ & \forall i \in \{1, 2, ..., p\} \\ & \displaystyle\sum_{i=1}^{p} y_i^k \geq 1, y_i^k \in \{0, 1\} \; \forall i \in \{1, 2, ..., p\} \end{array} \right\} \tag{11}$$

$-M_i$ is a lower bound to the i^{th} objective function for all $x \in D$ and we associate a binary variable $y_i^k \in \{0, 1\}$ to the criterion $Z_i = c^i x$.

If $y_i^k = 0$, the constraint $(*)$ gives $Z_i(x) \geq -M_i$, which is not restrictive and the constraint

$\displaystyle\sum_{i=1}^{p} y_i^k \geq 1$ means that at least one criterion is improved.

The optimal solution of the problem (P_l), x^l, produces a maximum value of the criterion ϕ in the reduced domain. If it is efficient, the procedure terminates with x^l an optimal solution of the main problem (P_E); otherwise, we use the final optimal simplex tableau to explore all incident edges to this optimal solution searching for an eventual alternative efficient point. If no such efficient point is found, the feasible region is reduced and the process continues improving the value of $\phi(x)$ and reducing the domain of admissibility until no pivot operation can be done indicating that the current feasible region is empty.

A technical description of the algorithm is presented below. It contains the following procedures:

- *Pb_relaxed*: is used to solve the relaxed problem (P_R),
- *test_efficiency*: solves problem $P(x^*)$,
- *opt*: solves problem (T_l),
- *solve_P_l*: searches for a new efficient solution and reduces the domain of admissibility; this procedure produces among other outputs, the final optimal tableau *table_opt*
- *Exploring*: seeks eventual efficient solutions on the edges emanating from an efficient existing solution.

4. **The Algorithm.** The technical description of the method is given below, Dom_k denotes the current feasible region of (P_ℓ)

Algorithm 1: Optimizing a Linear Function over Integer Efficient Set

```
Input
$\downarrow A_{(m\times n)}$: matrix of constraints;
$\downarrow b_{(m\times 1)}$: RHS vector;
$\downarrow c_{(1\times n)}$: main criterion vector;
$\downarrow Cr_{(p\times n)}$ : matrix of criteria;
Output
$\uparrow X_{opt}$: optimal solution of the problem $(P_E)$.
$\uparrow \phi_{opt}$: optimal value of criterion $\phi$.
Initialization $\phi_{opt} \leftarrow -\infty$, $l \leftarrow 0$, research $\leftarrow$ $true$.
Solve the relaxed problem $(P_R)$.
$[\uparrow x_0, \uparrow z_0]$ = Pb_relaxed $(\downarrow c, \downarrow A, \downarrow b)$.
if the problem does not have a feasible solution then problem $(P_E)$ is not feasible;
else
    $[\uparrow x_t, \uparrow z_t] = test\_efficiency\ (\downarrow Cr, \downarrow A, \downarrow b, \downarrow x_0)$;
    if $z_t = 0$ then $X_{opt} \leftarrow x_0$, $\phi_{opt} \leftarrow z_0$;
    else
        put $x_{ef} \leftarrow x_t,\ (P_0) \leftarrow (P_R)$;
        while $research = true$ do
            Solve problem $(T_l)$: $[\uparrow x_{eq}, \uparrow z_{eq}] = opt\ (\downarrow Cr, \downarrow A, \downarrow b, \downarrow c, \downarrow x_{ef})$;
            if $z_{eq} > \phi_{opt}$ then $X_{opt} \leftarrow x_{eq},\ \phi_{opt} \leftarrow z_{eq}$;
            l:=l+1;
            Solve problem $(P_l)$: $[\uparrow x_l, \uparrow z_l, \uparrow Dom_l, \uparrow table_{opt}] = solve\_P_l\ (\downarrow\downarrow A, \downarrow b, \downarrow Cr, \Delta_l, \downarrow d, \downarrow x_{ef})$;
            if $Dom_l = \emptyset$ Or $z_l < \phi_{opt}$ then $research \leftarrow false$ ;
            else
                $[\uparrow x_t^l, \uparrow z_t^l] = test\_efficiency(\downarrow Cr, \downarrow A, \downarrow b, \downarrow x_l)$;
                if $z_t^l = 0$ then $X_{opt} \leftarrow x_l,\ \phi_{opt} \leftarrow z_l,\ research \leftarrow false$ ;
                else
                    Construct the set $J_l = \{j \in N_l | \phi_j - c_j = 0\}$, where $N_l$ is the set of out of basis variable indices of $x_l$;
                    if $J_l \neq \emptyset$ then $[\uparrow x_{exp}, \uparrow z_{exp}, exist] = Exploring(\downarrow Cr, \downarrow A, \downarrow b, \downarrow c, \downarrow table_{opt}, J_l)$;
                    else
                        if $exist = true$ then
                        $X_{opt} \leftarrow\ x_{exp},\ \phi_{opt} \leftarrow\ z_{exp},\ research \leftarrow false$
```

Proposition 1. *The algorithm above converges in a finite number of iterations.*

Proof. The convergence of the procedure to problem (P_E) solution is assured by the theorems above, and since the feasible region S is assumed to be bounded (there is a limited number of integer solutions), it is reduced at each step until infeasibility. Thus the procedure converges to the optimal solution of P_E in a finite number of iterations. □

5. **A didactic example.** Consider the following $MOILP$ problem:

$$(P(D))\begin{cases} \max & Z_1 = 2x_1 - x_2 \\ \max & Z_2 = -x_1 + 2x_2 \\ s.t. & \\ & x_1 \leq 5, \\ & x_2 \leq 7, \\ & x_1 + x_2 \leq 10, \\ & x_1, x_2 \in \mathbb{N} \end{cases}$$

Let (P_E) be the main problem $(P_E)\begin{cases} \max & \phi = -x_1 - 3x_2 \\ s.t. & x_1, x_2 \in E(P). \end{cases}$

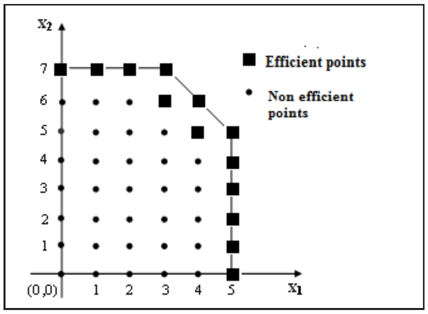

FIGURE 1. The feasible region D

Step 0 : Initialization The relaxed problem (P_R) is solved.

$$(P_R)\begin{cases} \max & \phi = -x_1 - 3x_2 \\ s.t. & \\ & x_1 \leq 5, \\ & x_2 \leq 7, \\ & x_1 + x_2 \leq 10, \\ & x_1, x_2 \in \mathbb{N} \end{cases}$$

The optimal solution of (P_R) is $x^0 = (0,0)$, $-M_1 = -7, -M_2 = -5$.
Step 1 $l = 0, \phi_{opt} = -\infty$, $H^0 = D$, the solution x^0 is not efficient, $\hat{x}^0 = (5,5)$ is the solution obtained when solving the efficiency test problem $P(x^0)$; $Z(\hat{x}^0) = (5,5)$, see Figure 2.
Step 2 We solve the problem (T_0)

$$(T_0)\begin{cases} \max & \phi = -x_1 - 3x_2 \\ s.t. & (x_1, x_2) \in D \cap \{(x_1,x_2) \in \mathbb{N}^2 | 2x_1 - x_2 = 5, -x_1 + 2x_2 = 5\}. \end{cases}$$

An optimal solution of this problem is $\bar{x}^0 = (5,5)$, $\phi(\bar{x}^0) = -20 > \phi_{opt}$, let $X_{opt} := (5,5), \phi_{opt} = -20$.
Step 3 Let $l := l + 1 = 1$ and solve problem (P_1)

$$(P_1)\left\{\begin{array}{lll} \max & \phi = -x_1 - 3x_2 & \\ s.t. & (x_1, x_2) \in D & \\ & 2x_1 - x_2 \geq 6y_1^1 - 7(1 - y_1^1) & (1) \\ & -x_1 + 2x_2 \geq 6y_2^1 - 5(1 - y_2^1) & (2) \\ & y_1^1 + y_2^1 \geq 1 & \\ & (x_1, x_2) \in \mathbb{N}^2, (y_1^1, y_2^1) \in \{0, 1\}^2. & \end{array}\right.$$

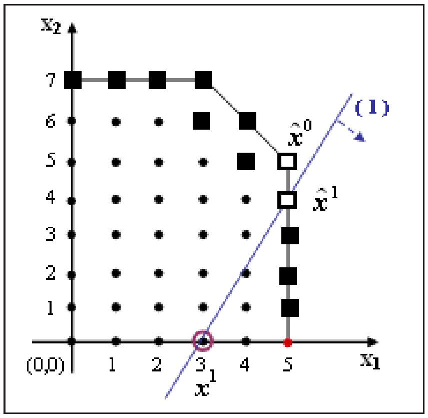

FIGURE 2. The reduced feasible region D^1

An optimal solution of this problem is $x^1 = (3, 0), \phi(x^1) = -3$, $Z(x^1) = (6, -3)$ and $y = (1, 0)$.

As $\phi(x^1) > \phi_{opt}$ we test the efficiency of this solution by solving the problem

$$(P(x^1))\left\{\begin{array}{ll} \max & \Theta = \psi_1 + \psi_2 \\ s.t. & x_1 \leq 5, \\ & x_2 \leq 7, \\ & x_1 + x_2 \leq 10, \\ & 2x_1 - x_2 - \psi_1 = 6, \\ & -x_1 + 2x_2 - \psi_2 = -3, \\ & (x_1, x_2) \in \mathbb{N}^2, \psi_i \in \mathbb{N}, i = 1, 2. \end{array}\right.$$

We obtain $\Theta^* \neq 0$ and $\hat{x}^1 = (5, 4)$ an optimal solution of $(P(x^1))$; the corresponding vector of the criteria is $Z(\hat{x}^1) = (6, 3)$, see Figure 2.

Step 4 The set $J_1 = \{j \in N_1 | c_j - d_j = 0\} = \emptyset$ (we use the optimal tableau of x^1). We return to step 2.

iteration 2

Step 2 We solve the problem (T_1)

$$(T_1)\left\{\begin{array}{ll} \max & \phi = -x_1 - 3x_2 \\ s.t. & (x_1, x_2) \in D. \\ & 2x_1 - x_2 = 6 \\ & -x_1 + 2x_2 = 3 \end{array}\right.$$

$\bar{x}^1 = (5, 4)$, $\phi(\bar{x}^1) = -17 > \phi_{opt}$, let $X_{opt} := (5, 4), \phi_{opt} := -17$.

Step 3 We put $l := l + 1 = 2$ and solve the problem (P_2)

$$(P_2)\left\{\begin{array}{lll} \max & \phi = -x_1 - 3x_2 & \\ s.t. & (x_1, x_2) \in D^1 & \\ & 2x_1 - x_2 \geq 7y_1^2 - 7(1 - y_1^2) & (3) \\ & -x_1 + 2x_2 \geq 4y_2^2 - 5(1 - y_2^2) & (4) \\ & y_1^2 + y_2^2 \geq 1 & \\ & (x_1, x_2) \in \mathbb{N}, (y_1^2, y_2^2) \in \{0, 1\}^2. & \end{array}\right.$$

An optimal solution is $x^2 = (4, 0)'$ which is not efficient; $\phi(x^2) = -4$, $Z(x^2) = (8, -4)$ and $Y = (1, 0, 1, 0)$. Let $\hat{x}^2 = (5, 2)$ as shown in Figure 3, $Z(\hat{x}^2) = (8, -1)$ be an optimal solution of efficiency test problem of x^2.

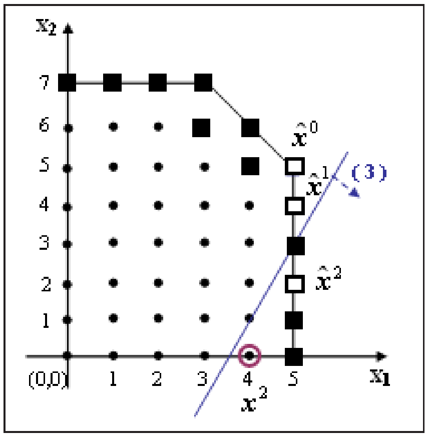

FIGURE 3. The reduced feasible region D^2

Step 4 Set $J_2 = \{j \in N_2 | c_j - d_j = 0\} = \emptyset$. We return again to step 2

iteration 3

Step 2 We solve problem (T_2)

$$(T_2)\left\{\begin{array}{ll} \max & \phi = -x_1 - 3x_2 \\ s.t. & (x_1, x_2) \in D. \\ & 2x_1 - x_2 = 8 \\ & -x_1 + 2x_2 = -1 \end{array}\right.$$

$\bar{x}^2 = (5, 2), \phi(\bar{x}^2) = -11 > \phi_{opt}$, let $X_{opt} := (5, 2), \phi_{opt} := -11$.

Step 3 We put $l := l + 1 = 3$ and we solve problem (P_3)

$$(P_3)\left\{\begin{array}{lll} \max & \phi = -x_1 - 3x_2 & \\ s.t. & (x_1, x_2) \in D^2 & \\ & 2x_1 - x_2 \geq 9y_1^3 - 7(1 - y_1^3) & (5) \\ & -x_1 + 2x_2 \geq -5(1 - y_2^3) & (6) \\ & y_1^3 + y_2^3 \geq 1 & \\ & (x_1, x_2) \in \mathbb{N}, (y_1^3, y_2^3) \in \{0, 1\}^2. & \end{array}\right.$$

$x^3 = (5, 0), \phi(x^3) = -5 > \phi_{opt}$, the efficiency test is positive. The algorithm terminates with $X_{opt} = (5, 0)$ and $\phi_{opt} = -5$.

The set of all efficient solutions of the MOILP problem is

$$E = \{(0, 7), (1, 7), (2, 7), (3, 7), (3, 6), (4, 6), (4, 5), (5, 5), (5, 4), (5, 3), (5, 2), (5, 1), (5, 0)\}$$

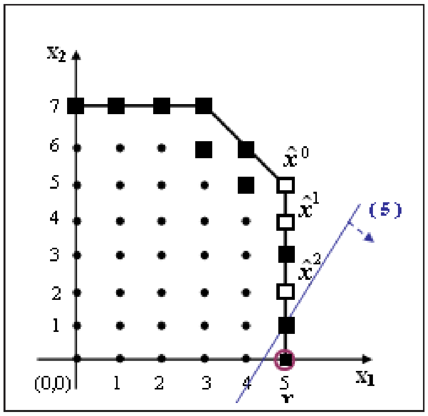

FIGURE 4. The reduced feasible region D^3

and the list of efficient solutions that our procedure goes over is $L = \{(5,5),(5,4),(5,2),(5,0)\}$, see Figure 4.

6. **Implementation.** The procedure presented above was implemented in the MATLAB 7.0 environment. We used the optimization predefined functions to solve integer linear programming problems. First, we checked the efficiency of our exact algorithm by applying it to some didactic examples used by several authors and researchers. Table 1 below shows the obtained results.

Instances	n	m	p	#iter	CPU (sec)	# eff-sol	$\lvert List\rvert$
Jesus [12]	2	3	2	3	0.062	7	3
Ecker and Song [9]	2	3	2	2	0.031	5	2
Philip [16]	2	3	2	4	0.125	13	4
Bowman [5]	3	4	2	1	0.000	3	1
Crema & Sylva [7]	3	4	2	1	0.000	4	1
Sylva & Crema [17]	4	6	2	2	0.030	4	2
Gupta& Malhotra [11]	2	3	3	5	0.149	9	5
Karaivanova & Narula [13]	3	3	3	5	0.547	14	4
Klein & Hannan [14]	4	6	3	2	0.034	6	2

TABLE 1. Examples treated in the literature

where

- $n, m, p (p \geq 2) \in \mathbb{N}$: are respectively the number of variables, the number of constraints and the number of criteria.
- #iter: is the number of iterations performed.
- CPU : is the execution time of the code expressed in seconds.
- # eff-sol: is the number of efficient solutions in the corresponding example.
- $|List|$: is the number of efficient solutions gone over.

According to table 1 an efficient solution of problem (P_E) is found without enumerating explicitly all the efficient solutions of the relative MOILP problem, which undoubtedly reduces the CPU time. The algorithm is also tested with instances randomly generated from discrete uniform distribution. Elements of matrix A are drawn randomly in the set $\{-30, \ldots, +30\}$; the criteria coefficients in the set $\{-20, \ldots, +20\}$ and the right hand side and the coefficients of the main criterion in the set $\{-10, \ldots, +10\}$. We use a procedure that produces regions containing at least three feasible solutions.

We present some results for several generated examples taking for each sample type the mean result of 25 instances (see table 2). For each example we provide also lower and upper bounds for the execution time. We notice that for these sizes the method gives in a reasonable time the answer about the best efficient solution in the direction of some preference ϕ. This procedure avoids the explicit generation of all efficient solutions which may prove costly.

p	$n\times\ m$	5×5	10×10	20×15	30×15	40×15	50×15
3	cpu (sec)	0.89 $[0.005; 3]$	2.14 $[0.02; 6.5]$	18.23 $[0.2; 97]$	25.85 $[0.5; 119]$	104.04 $[0.225; 352.5]$	87 $[0.105; 707]$
	#iter	3.39 $[1; 5]$	3.23 $[1; 6]$	3.37 $[1; 6]$	4.23 $[2; 7]$	3.69 $[2; 7]$	4.7 $[1; 10]$
5	cpu (sec)	6.3 $[0.015; 43]$	9.67 $[0.1; 71.5]$	49.82 $[0.012; 559]$	142.75 $[0.02; 885.5]$	86.38 $[0.125; 759]$	273.18 $[0.4; 1500]$
	#iter	3.9 $[1; 7]$	3.58 $[2; 6]$	4.38 $[2; 9]$	4.16 $[2; 11]$	3.41 $[1; 8]$	4.8 $[1; 10]$
8	cpu (sec)	1.545 $[0.02; 8]$	11.28 $[0.075; 35.6]$	52.56 $[0.02; 225]$	219 $[0.5; 1550]$	196 $[0.5; 982]$	147 $[2.02; 658]$
	#iter	2.37 $[1; 5]$	2.76 $[1; 6]$	3.66 $[2; 7]$	4.22 $[1; 10]$	4.34 $[1; 11]$	3.9 $[2; 9]$

Table 2. Samples generated randomly

7. Conclusion.

We have presented an exact method that optimizes a linear function over an integer efficient solutions set. We achieve this objective by combining two ideas : one uses Sylva and Crema cuts to reduce the admissible region and the second explores incident edges to a current solution in order to find a new efficient solution increasing the linear function $\phi(x)$; then the domain is being reduced using generalized Dantzig cut. In this method we bring together resolution in criteria space and exploration within decision variable space just when needed.

We found in our experimental study that the proposed algorithm is very efficient in terms of the number of iterations performed (the number of efficient points considered). The number of iterations as well as computing time do not grow very fast with the size of the problem.

Acknowledgments. The authors would like to thank the referees for the many helpful suggestions.

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Received September 2009; 1st revision February 2010; final revision May 2010.

E-mail address: chaabane_dj@yahoo.fr
E-mail address: Marc.Pirlot@umons.ac.be