HSGRA: Off-shell formulation and AKSZ quantisation

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Plan

Higher-spin gauge theories : some motivations

- Gauge Principle : HS theories contain gravity ; ∞-dim gauge algebra ;
- Vasiliev's unfolding : a geometric approach to field theory ;
- AdS/CFT dualities between Vasiliev's theory and free CFT's [Sezgin-Sundell, Klebanov-Polyakov] for AdS_4/CFT_3 and [Gaberdiel-Gopakumar] for AdS_3/CFT_2 . Relations with statistical physics, non-commutative field theory, strings.

The gauge principle [H. Weyl, 1929]

In Classical Field Theory : remarkable achievement by M. A. Vasiliev with formulation of *fully nonlinear field equations* for higher-spin gauge fields in 4D [Vasiliev, 1990 – 1992] and in D space-time dimensions [hep-th/0304049]. Some salient features are

- Manifest diffeomorphism invariance, no explicit reference to a metric ;
- Manifest Cartan integrability $\Rightarrow$ *gauge invariance* under infinite-dimensional HS algebra ;
- Formulation in terms of two infinite-dimensional modules of $\mathfrak{so}(2, D-1)$: The *adjoint* and *twisted-adjoint* representations $\rightsquigarrow$ master 1-form and master zero-form. Uses unfolding in terms of FDA.

Unfolded equations and FDA

A free (graded commutative, associative) differential algebra $\mathfrak{R}$ is set $\{X^\alpha\}$ of *a priori* independent variables, locally-defined differential forms obeying first-order equations of motion

$$\mathscr{R}^\alpha = dX^\alpha + Q^\alpha(X) \approx 0 \,, \quad Q^\alpha(X) = \sum_n f^\alpha_{\beta_1\dots\beta_n} X^{\beta_1}\cdots X^{\beta_n} \;.$$

Nilpotency of d and integrability condition $d\mathscr{R}^\alpha \approx 0$ require

$$Q^\beta \, \frac{\partial^L Q^\alpha}{\partial X^\beta} \equiv 0 \quad .$$

For $X^\alpha_{[p_\alpha]}$ with $p_\alpha > 0$, gauge transformation preserving $\mathscr{R}^\alpha \approx 0$:

$$\delta_\epsilon X^\alpha = d\epsilon^\alpha - \epsilon^\beta \frac{\partial^L}{\partial X^\beta} Q^\alpha \quad .$$

- The concepts of spacetime, dynamics and observables are *derived* from infinite-dimensional FDA's.
- Unfolded dynamics is an inclusion of local d.o.f. into field theories described *on-shell* by flatness conditions on generalized curvatures.
- Spin-2 couplings arise in the limit in which the $\mathfrak{so}(2, D-1)$-valued part of the higher-spin connection one-form is treated exactly while its remaining spin $s > 2$ components become weak fields together with all curvature (Weyl) zero-forms.

  $\hookrightarrow$ Lorentz-covariant derivative, minimal coupling.

Action principle with QP-structure

Want an action principle reproducing non-linear and background-independent Vasiliev equations in four spacetime dimensions. These equation possess

- an algebraic structure that enables one to construct a *Hamiltonian action* with nontrivial QP-structures in a manifold $\mathscr{B}$ with boundary $\partial\mathscr{B}$;
- a geometric structure which allows to construct additional boundary deformations.

Manifold : bulk with non-empty boundary

- Like for the nonlinear Poisson sigma-model, introduce bulk $\mathscr{B}$ with non-empty boundary, and add extra momentum-like variables $\{P_\alpha\}$.

- Impose boundary conditions compatible with a *globally* well-defined action principle
  $\hookrightarrow$ the action $S = \int_{\mathscr{B}} L$ should be gauge invariant, and $\delta_\varepsilon L = \mathrm{d}K_\varepsilon$;
  $\hookrightarrow$ compatibility between gauge transformations of field configurations and transition functions between charts.

- The action has two pieces : a bulk part plus various classically marginal deformations on boundary $\rightarrow$ amplitudes.

Relation with Fronsdal's programme

Unlike the original Fronsdal programme [formulate higher-spin gauge theory off shell in a perturbative expansion around constantly curved spacetime], *background-independent* formulation in terms of master fields living in the correspondence space, *i.e.* the local product of a non-commutative phase-spacetime containing the commutative spacetime as a Lagrangian submanifold and a non-commutative twistor space.

Vasiliev's system has a huge classical solution space that admits many different perturbative expansions of which *only some* reduce to Fronsdal systems (with Λ).

Some elements of Vasiliev's 4D equations (1)

The master fields are locally-defined (chart index ξ) **operators**

$$O_\xi(x^M_\xi, \mathrm{d}x^M_\xi; Z^{\underline{\alpha}}, \mathrm{d}Z^{\underline{\alpha}}; Y^{\underline{\alpha}}; K) \ ,$$

where

$$[Y^{\underline{\alpha}}, Y^{\underline{\beta}}] \ = \ 2iC^{\underline{\alpha\beta}} \ , \quad [Z^{\underline{\alpha}}, Z^{\underline{\beta}}] \ = \ -2iC^{\underline{\alpha\beta}} \ , \qquad \underline{\alpha}, \underline{\beta} = 1,2,3,4,$$

with charge conjugation matrix $C^{\alpha\beta} = \epsilon^{\alpha\beta}$, $C^{\dot\alpha\dot\beta} = \epsilon^{\dot\alpha\dot\beta}$, $\underline{\alpha} = (\alpha, \dot\alpha)$, and where $K = (k, \bar{k})$, are two outer Kleinian operators.

The operators are represented by **symbols** $f[O_\xi]$ obtained by going to **specific bases** for the operator algebra $\rightsquigarrow$ **ordering prescriptions**.

Some elements of Vasiliev's 4D equations (2)

One may think of the symbols as functions $f(x, Z; \mathrm{d}x, \mathrm{d}Z; Y)$ on a **correspondence space** $\mathfrak{C}$

$$\mathfrak{C} = \bigcup_{\xi} \mathfrak{C}_{\xi} \ , \qquad \mathfrak{C}_{\xi} = \mathscr{B}_{\xi} \times \mathscr{Y} \ , \qquad \mathscr{B}_{\xi} = \mathscr{M}_{\xi} \times \mathscr{Z}$$

equipped with a suitable `associative` star-product operation $\star$ which reproduces, in the space of symbols, the composition rule for operators.

↬ The exterior derivative on $\mathscr{B}_{\xi}$ is given by

$$\mathrm{d} = \mathrm{d}x^{M} \partial_{M} + \mathrm{d}Z^{\underline{\alpha}} \, \partial_{\underline{\alpha}} \quad .$$

Some elements of Vasiliev's 4D equations (3)

The master fields of the *minimal bosonic model* are an adjoint one-form

$$A \;=\; W + V \;,$$

$$W \;=\; \mathrm{d}x^M\, W_M(x, Z; Y) \;, \qquad\qquad V \;=\; \mathrm{d}Z^{\underline{\alpha}}\, V_{\underline{\alpha}}(x, Z; Y) \;,$$

and a twisted-adjoint zero-form

$$\Phi \;=\; \Phi(x, Z; Y) \;.$$

Generically, start with locally-defined differential forms of *total degree* p

$$f \;=\; \sum_{p=0}^{\infty} f_{[p]}(x^M, \mathrm{d}x^M; Z^{\underline{\alpha}}, \mathrm{d}Z^{\underline{\alpha}}; Y^{\underline{\alpha}}; k, \bar{k}) \;,$$

$$f_{[p]}(\lambda\, \mathrm{d}x^M; \lambda\, \mathrm{d}Z^{\underline{\alpha}}) \;=\; \lambda^p\; f_{[p]}(\mathrm{d}x^M; dZ^{\underline{\alpha}}) \;, \quad \lambda \in \mathbb{C} \;.$$

Some elements of Vasiliev's 4D equations (4)

The x^M's are commuting coordinates, while $(Y^{\underline{\alpha}}, Z^{\underline{\alpha}}) = (y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}})$ are non-commutative , k , $\bar{k}$ are outer Kleinians :

$$k \star f \;=\; \pi(f) \star k \;, \quad \bar{k} \star f \;=\; \bar{\pi}(f) \star \bar{k} \;, \quad k \star k \;=\; 1 \;=\; \bar{k} \star \bar{k} \;,$$

with automorphisms π and $\bar{\pi}$ defined by $\pi \mathrm{d} = \mathrm{d}\pi \,, \bar{\pi}\mathrm{d} = \mathrm{d}\bar{\pi}$ and

$$\pi[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] \;=\; f(-z^\alpha, \bar{z}^{\dot{\alpha}}; -y^\alpha, \bar{y}^{\dot{\alpha}}) \;,$$

$$\bar{\pi}[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] \;=\; f(z^\alpha, -\bar{z}^{\dot{\alpha}}; y^\alpha, -\bar{y}^{\dot{\alpha}}) \;.$$

Bosonic and irreducibility projections : $\pi\bar{\pi}(f) = f = P_+ \star f$, $P_+ = \frac{1}{2}(1 + k \star \bar{k})$,

$$\hookrightarrow \; f = \left[f^{(+)}(x, \mathrm{d}x; Z, \mathrm{d}Z; Y) + f^{(-)}(x, \mathrm{d}x; Z, \mathrm{d}Z; Y) \star \frac{(k + \bar{k})}{2} \right] \star P_+ \;.$$

Some elements of Vasiliev's 4D equations (5)

- **Bosonic projection** : removes component fields $\rightsquigarrow$ spacetime spinors.
- **Irreducible *minimal* bosonic models : by imposing** reality conditions and discrete symmetries that remove all **odd** spins.

$\hookrightarrow$ $\dagger$ and anti-automorphism τ defined by $\mathrm{d}[(\cdot)^\dagger] = [\mathrm{d}(\cdot)]^\dagger\,,\quad \mathrm{d}\,\tau = \tau\,\mathrm{d}\,,$

$$
\begin{aligned}
[f(z^\alpha,\bar z^{\dot\alpha};y^\alpha,\bar y^{\dot\alpha};k,\bar k)]^\dagger &= \bar f(\bar z^{\dot\alpha},z^\alpha;\bar y^{\dot\alpha},y^\alpha;\bar k,k)\ ,\\
\tau[f(z^\alpha,\bar z^{\dot\alpha};y^\alpha,\bar y^{\dot\alpha};k,\bar k)] &= f(-iz^\alpha,-i\bar z^{\dot\alpha};iy^\alpha,i\bar y^{\dot\alpha};k,\bar k)\ ,\\
[f_{[p]}\star f'_{[p']}]^\dagger &= (-1)^{pp'}(f'_{[p']})^\dagger\star(f_{[p]})^\dagger\ ,\\
\tau(f_{[p]}\star f'_{[p']}) &= (-1)^{pp'}\tau(f'_{[p']})\star\tau(f_{[p]})\ .
\end{aligned}
$$

Some elements of Vasiliev's 4D equations (6)

Back to Vasiliev's A and Φ, the minimal models are imposed by the following projection and reality conditions :

$$\tau(A,\Phi) \;=\; (-A,\pi(\Phi))\;, \qquad (A,\Phi)^\dagger \;=\; (-A,\pi(\Phi))\;.$$

Full equations of motion of the minimal bosonic model with fixed interaction ambiguity : $F+\Phi\star J=0$, with two-form J defined globally on correspondence space, obeying $\tau(J)=-J=J^\dagger$ and

$$\mathrm{d}J \;=\; 0\;, \quad [f,J]^\pi_\star \;:=\; f\star J - J\star\pi(f) \;=\; 0 \quad \forall\, f \quad \text{s.t.} \quad \pi\bar{\pi}(f)=f\;. \qquad (1)$$

In the minimal model,

$$J \;=\; -\frac{i}{4}(b\,dz^2\;\kappa+\bar{b}\,d\bar{z}^2\;\bar{\kappa})\;,$$

Brief review of Vasiliev's 4D equations (7)

... where the chiral inner Kleinians

$$\kappa \;=\; \exp(i y^\alpha z_\alpha)\;, \qquad \bar\kappa \;=\; \kappa^\dagger = \exp(-i\bar y^{\dot\alpha}\bar z_{\dot\alpha})\;.$$

By making use of field redefinitions $\Phi \to \lambda\Phi$ with $\lambda \in \mathbb{R}\,,\ \lambda \neq 0\,,$ the complex parameter b in J can be taken to obey

$$|b| \;=\; 1\;, \qquad \arg(b) \;\in\; [0,\pi]\;.$$

The phase breaks parity P $\quad [P\,\mathrm{d} = \mathrm{d}\,P]$

$$P\left[f(x^M; z^\alpha, \bar z^{\dot\alpha}; y^\alpha, \bar y^{\dot\alpha}; k, \bar k)\right] \;=\; (Pf)(x^M; -\bar z^{\dot\alpha}, -z^\alpha; \bar y^{\dot\alpha}, y^\alpha; \bar k, k)\;,$$

except in the following two cases :

Type-A model (parity-even physical scalar) : $\quad b = 1\;,$

Type-B model (parity-odd physical scalar) : $\quad b = i\;.$

Some elements of Vasiliev's 4D equations (8)

[The integrability of $F + \Phi \star J = 0$ implies that $D\Phi \star J = 0$, that is, $D\Phi = 0$, where the twisted-adjoint covariant derivative $D\Phi = d\Phi + A \star \Phi - \Phi \star \pi(A)$. This constraints is integrable since $D^2\Phi = F \star \Phi - \Phi \star \pi(F) = -\Phi \star J \star \Phi + \Phi \star \pi(\Phi) \star J$ gives zero, using the constraint on F and (1).]

$\hookrightarrow$ Summary : minimal higher-spin gravity given by

$$F + \Phi \star J = 0 \,, \qquad D\,\Phi = 0 \,, \qquad \mathrm{d}J = 0 \,,$$

$$F := \mathrm{d}A + A \star A \,, \qquad D\Phi := \mathrm{d}\Phi + [A, \Phi]_\pi \,,$$

$$\tau(A, \Phi) = (-A, \pi(\Phi)) \,, \qquad (A, \Phi)^\dagger = (-A, \pi(\Phi)) \,,$$

$$\hookrightarrow [A, J]_\pi = 0 = [\Phi, J]_\pi \,.$$

Some elements of Vasiliev's 4D equations (9)

↪ Integrability implies invariance under Cartan gauge transformations

$$\delta_\epsilon A \ = D\epsilon \ , \qquad \delta_\epsilon \Phi \ = \ -[\epsilon, \Phi]_\star^\pi \ ,$$

for zero-form gauge parameters $\epsilon(x, Z; Y)$ obeying the same kinematic constraints as the master one-form, *i.e.* $\tau(\epsilon) = -\epsilon$ and $(\epsilon)^\dagger = -\epsilon$.

↪ The closure of the gauge transformations reads

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \ = \ \delta_{\epsilon_{12}} \ , \qquad \epsilon_{12} \ = \ [\epsilon_1, \epsilon_2]_\star \ ,$$

defining the algebra $\mathfrak{hs}(4)$.

Classical action principle (1)

Starting from $\{X^\alpha\}$ defined locally on $\mathscr{B}_\xi$ (base manifold $\mathscr{B} = \cup_\xi \mathscr{B}_\xi$ of dim. $\hat{p}+1$) satisfying some unfolded constraints with given Q-structure,

$\hookrightarrow$ off-shell extensions based on sigma models with maps

$$\phi_\xi \ : \ T[1]\mathscr{B}_\xi \ \rightarrow \ M \ ,$$

between two $\mathbb{N}$-graded manifolds, from the parity-shifted tangent bundle $T[1]\mathscr{B}$ to a *target space* M : differential $\mathbb{N}$-graded symplectic manifold with two-form $\mathscr{O}$, Q-structure $\mathscr{Q}$ and Hamiltonian $\mathscr{H}$ with :

$$\deg(\mathscr{O}) \ = \ \hat{p}+2 \ , \quad \deg(\mathscr{Q}) \ = \ 1 \ , \quad \deg(\mathscr{H}) \ = \ \hat{p}+1 \ .$$

Hamiltonian bulk action

$$S^{\text{cl}}_{\text{bulk}}[\phi|\mathscr{B}] \;=\; \sum_{\xi}\int_{\mathscr{B}_\xi}\mathscr{L}^{\text{cl}}_{\xi} \;=\; \sum_{\xi}\int_{\mathscr{B}_\xi}\pi\,\phi^*_\xi(\vartheta-\mathscr{H})\;,$$

where $\phi_\xi := \phi|_{\mathscr{B}_\xi}$ and $\pi : \Omega(T[1]\mathscr{B}) \to \Omega(\mathscr{B})$ degree-preserving homomorphism that takes k-forms on $T[1]\mathscr{B}$ of degree p to p-forms on $\mathscr{B}$, *viz.*

$$\pi \;:\; \Omega^{[k|p]}(T[1]\mathscr{B}) \to \Omega^{[p]}(\mathscr{B})\;.$$

ϕ intertwines the actions of the exterior derivative d in $\Omega(\mathscr{B})$ and the Lie derivative $\mathscr{L}_q = i_q \circ \mathrm{d} - \mathrm{d}\circ i_q$ in $\Omega(T[1]\mathscr{B})$ along the canonical Q-structure on $T[1]\mathscr{B}$ as follows :

$$\mathrm{d}\circ\pi \;=\; \pi\circ\mathrm{d} \;=\; \pi\circ\mathscr{L}_q\;,\quad q \;:=\; \theta^\mu\partial_\mu\;.$$

Equipping $T[1]\mathscr{B}$ with coordinates

$$(x^\mu, \theta^\mu) \ , \qquad \deg(x^\mu, \theta^\mu) \ = \ (0, 1) \ ,$$

one has

$$\pi(f(x^\mu, \theta^\mu; dx^\mu, d\theta^\mu)) \ = \ f(x^\mu, dx^\mu; dx^\mu, 0) \ .$$

Thus the exterior differential d , which has form-degree 1, has degree 1, *i.e.*

$$\deg(\mathrm{d}) \ = \ \deg(q) \ = \ 1 \ .$$

The assumption that the sigma-model maps ϕ have vanishing intrinsic degree implies

$$\Omega^{[k|p]}(M) \stackrel{\phi^*}{\to} \Omega^{[k|p]}(T[1]\mathscr{B}) \stackrel{\pi}{\to} \Omega^{[p]}(\mathscr{B}) ,$$

that is, the pull-back ϕ^* of a k-form of $\mathbb{N}$-degree p on M is a ditto on $T[1]\mathscr{B}$, in its turn sent by π to a p-form on $\mathscr{B}$; the condition that M is $\mathbb{N}$-graded (instead of $\mathbb{Z}$-graded) and $\deg(\mathrm{d}) = 1$ implies that $p \geqslant k$. Thus, since

$$\mathscr{O} = \mathrm{d}\vartheta \in \Omega^{[2|\hat{p}+2]}(M) , \quad \vartheta \in \Omega^{[1|\hat{p}+1]}(M) , \quad \mathscr{H} \in \Omega^{[0|\hat{p}+1]}(M) ,$$

it follows that

$$\pi\phi^*(\vartheta - \mathscr{H}) \in \Omega^{[\hat{p}+1]}(\mathscr{B}) ,$$

which can then be integrated by decomposing $\mathscr{B}$ into charts $\mathscr{B}_\xi$.

Classical action principle (2)

$\hookrightarrow$ Classical action principle of Hamiltonian type :

$$S^{\rm cl}_{\rm bulk}[\phi|\mathscr{B}] \;=\; \sum_{\xi}\int_{\mathscr{B}_\xi}\mathscr{L}^{\rm cl}_{\xi} \;=\; \sum_{\xi}\int_{\mathscr{B}_\xi}\pi\,\phi^*_{\xi}(\vartheta-\mathscr{H})\;,$$

where ϑ is a pre-symplectic form.

$\hookrightarrow$ Writing $\vartheta = {\rm d}Z^i\vartheta_i\,,\ \mathscr{O}=\frac{1}{2}\,{\rm d}Z^i{\rm d}Z^j\,\widetilde{\mathscr{O}}_{ij}=\frac{1}{2}\,{\rm d}Z^i\mathscr{O}_{ij}\,{\rm d}Z^j$ and defining the associated graded Poisson bracket

$$\{A,B\}^{[-\hat p]} = (-1)^{\hat p+(\hat p+i+1)A}\;\partial_i A\,\mathscr{P}^{ik}\,\partial_j B$$

where $\mathscr{P}^{ik}\mathscr{O}_{kj}=(-1)^{\hat p}\delta^i_j$, then ...

Classical action principle (3)

- ... the variation of the Lagrangian :

$$\delta \mathscr{L}^{\rm cl}_{\rm bulk} \;=\; \delta Z^i \mathscr{R}^j \widetilde{\mathscr{O}}_{ij} + {\rm d}\left(\delta Z^i \vartheta_i\right) \;,$$

where generalized curvatures and Hamiltonian vector field

$$\mathscr{R}^i \;=\; {\rm d}Z^i + \mathscr{Q}^i \;, \qquad \mathscr{Q}^i \;=\; (-1)^{\hat{p}+1} \mathscr{P}^{ij} \partial_j \mathscr{H} \;,$$
$$\overrightarrow{\mathscr{Q}} \;=\; \mathscr{Q}^i \vec{\partial}_i \;, \qquad \deg(\overrightarrow{\mathscr{Q}}) \;=\; 1 \;.$$

- Variational principle $\Longrightarrow$ $\mathscr{R}^i \approx 0$, whose Cartan integrability on shell requires $\overrightarrow{\mathscr{Q}}$ to be a Hamiltonian Q-structure

$$\mathscr{L}_{\overrightarrow{\mathscr{Q}}} \overrightarrow{\mathscr{Q}} \;\equiv\; 0 \quad \Leftrightarrow \quad \mathscr{Q}^j \partial_j \mathscr{Q}^i \;\equiv\; 0 \quad \Leftrightarrow \quad \partial_i \{\mathscr{H}, \mathscr{H}\}^{[-\hat{p}]} \;\equiv\; 0 \;.$$

Classical action principle (4)

Nilpotency of $\overrightarrow{\mathscr{Q}}$ with suitable boundary conditions on the fields and gauge parameters ensure invariance of the action under

$$\begin{aligned} \delta_\varepsilon Z^i &= \mathrm{d}\varepsilon^i - \varepsilon^j \partial_j \mathscr{Q}^i + \tfrac{1}{2}\,\varepsilon^k \mathscr{R}^l\,\partial_l \widetilde{\mathscr{O}}_{kj}\,\mathscr{P}^{ji}\;, \\ \delta_\varepsilon \mathscr{L}^{\mathrm{cl}}_{\mathrm{bulk}} &= \mathrm{d}K_\varepsilon\;, \qquad K_\varepsilon \;=\; \varepsilon^i \mathscr{R}^j \widetilde{\mathscr{O}}_{ij} + \delta_\varepsilon Z^i \vartheta_i\;. \end{aligned}$$

Closure of gauge transformations :

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] Z^i \;=\; \delta_{\varepsilon_{12}} Z^i - \overrightarrow{\mathscr{R}} \varepsilon^i_{12}\;,$$

where $\overrightarrow{\mathscr{R}} = \mathscr{R}^i \partial_i$ and

$$\varepsilon^i_{12} \;=\; -\tfrac{1}{2}\,[\overrightarrow{\varepsilon}_1, \overrightarrow{\varepsilon}_2]\,\mathscr{Q}^i\;.$$

Classical action principle (5)

- *Under certain extra assumptions* on ϑ and $\mathscr{H}$, the action can be defined globally by gluing together the locally defined fields and gauge parameters along chart boundaries using gauge transitions $\delta_t Z^i$ and $\delta_t \varepsilon^i$ with parameters $\{t^i\} = t^{\xi}_{\xi'}$ defined on overlaps.

  Assumptions :

$$(i) \quad \delta_t K_\varepsilon \;=\; 0\ , \quad (ii) \quad \partial_j \partial_k \overrightarrow{t}\, \mathscr{Q}^i \;=\; 0\ , \quad (iii) \quad K_t \;\equiv\; 0\ .$$

- Assumption $(i) \Longrightarrow$ cancellation of contributions to $\delta_\varepsilon S^{\rm cl}_{\rm bulk}$ from chart boundaries in the interior of $\mathscr{B}$, s.t. the variational principle implies the BC on fields and gauge parameters

$$K_\varepsilon|_{\partial\mathscr{B}} \;\equiv\; 0\ .$$

Classical action principle (6)

- Assumptions (*ii*) and (*iii*) ensure compatibility between gauge transformations and gauge transitions in the sense that performing a transition transformation on fields and gauge parameters between two adjacent charts and moving along the gauge orbit are two operations that commute. $\hookrightarrow$ Gives access to $\delta_{\varepsilon_\xi}\, t^{\xi}_{\xi'}$ and $\delta_{t^{\xi}_{\xi'}}\, \varepsilon_\xi$.
- The $\{t^{\xi}_{\xi'}\}'s \rightsquigarrow$ subalgebra of Cartan transformations that preserve the Lagrangian density, *i.e.* selects the transitions.
- Assuming there are no constants of total degree $\hat{p}+2$ on M , the condition $\partial_i\{\mathscr{H},\mathscr{H}\}^{[-\hat{p}]} \equiv 0$ is equivalent to the *structure equation*

$$\{\mathscr{H},\mathscr{H}\}^{[-\hat{p}]} \;\equiv\; 0 \quad\Leftrightarrow\quad (-1)^{i(\hat{p}+1)}\,\partial_i\mathscr{H}\,\mathscr{P}^{ij}\partial_j\mathscr{H} \;\equiv\; 0\,.$$

Correct amplitudes for unbroken HS

- In the case of Vasiliev's model : PSM action with bulk + boundary deformations.

$$S^{Tot.} \quad = \quad S^{bulk}[X, P] + S^{bound.}[X] \, .$$

  Reproduces the full nonlinear equations, same content perturbatively.

- With the addition of suitable boundary deformations built from the zero-forms of X , $Z = \int DX DP \exp[\frac{i}{\hbar} S^T]$ reproduces, to lowest order in $\hbar$, the correct N-point functions of the free $O(N)$ model on boundary [Colombo,Sundell], $(N = 2, 3)$ then [Didenko,Skvortsov] $N \geqslant 4$.

AKSZ quantization

Classical coordinates $Z^i \equiv Z^{i\,\langle 0\rangle}_{[p_i]}$ on M is extended into coordinates on $\boldsymbol{M}$:

$$\left\{ Z^{i\,\langle g\rangle}_{[p_i-g]}\ , \qquad Z^{\langle -1-g\rangle}_{i[\hat{p}+1-p_i+g]} \ := \ \left(Z^{i\,\langle g\rangle}_{[p_i-g]} \right)^+ \right\}\ , \qquad g \ = \ 0,\ldots,p_i\ ,$$

$O^{\langle g\rangle}_{[p]}$: ghost number g and form degree p.

Total degree and Graβmann parity (for classical theories consisting of only bosonic fields) :

$$|\cdot| \ := \ \mathrm{deg}(\cdot) + \mathrm{gh}(\cdot)\ , \quad \mathrm{Gr}(\cdot) \ = \ |\cdot| \mod 2\ .$$

So,

$$|Z^{i\,\langle g\rangle}_{[p_i-g]}| \ = \ p_i\ , \qquad |Z^{\langle -1-g\rangle}_{i[\hat{p}+1-p_i+g]}| \ = \ \hat{p} - p_i\ .$$

Integrate a total form on a p-chain

Given a differential form $L \in \Omega(\boldsymbol{M})$ of fixed total degree $|L|$, described locally on $\boldsymbol{M}$ by a function $L(Z, Z^+, dZ, dZ^+)$, with pull-back

$$\pi \boldsymbol{\phi}^*(L) \equiv \sum_{p=0}^{\hat{p}+1} [\pi \boldsymbol{\phi}^*(L)]^{\langle |L|-p \rangle}_{[p]} \qquad \in \ \Omega(\mathscr{B})$$

and a q-cycle $\mathscr{C} \subseteq \mathscr{B}$, the integral

$$I(L|\mathscr{C}) \ := \ \sum_\xi \int_{B_\xi \cap \mathscr{C}} \pi \, \boldsymbol{\phi}^*{}_\xi(L) \ := \ \sum_\xi \int_{B_\xi \cap \mathscr{C}} [\pi \boldsymbol{\phi}^* L]^{\langle |L|-q \rangle}_{[q]}$$

$$i.e. \quad \mathrm{gh}(I(L|\mathscr{C})) \ = \ |L| - q \ .$$

The *canonical coordinates* $Z^i = (X^\alpha, P_\alpha)$ of M induce supercoordinates $\boldsymbol{Z}^i = (\boldsymbol{X}^\alpha, \boldsymbol{P}_\alpha)$ of $\boldsymbol{M}$ of fixed total degree :

$$\boldsymbol{X}^\alpha = \underbrace{X^{\alpha\,\langle p_\alpha\rangle}_{[0]} + X^{\alpha\,\langle p_\alpha-1\rangle}_{[1]} + \ldots + X^{\alpha\,\langle 0\rangle}_{[p_\alpha]}}_{\text{fields}} + \underbrace{P^{\alpha\,\langle -1\rangle}_{[p_\alpha+1]} + P^{\alpha\,\langle -2\rangle}_{[p_\alpha+2]} + \ldots + P^{\alpha\,\langle p_\alpha-\hat{p}-1\rangle}_{[\hat{p}+1]}}_{\text{anti-fields}} \quad ,$$

$$\boldsymbol{P}_\alpha = \underbrace{P^{\langle \hat{p}-p_\alpha\rangle}_{\alpha\ [0]} + P^{\langle \hat{p}-p_\alpha-1\rangle}_{\alpha\ [1]} + \ldots + P^{\langle 0\rangle}_{\alpha\ [\hat{p}-p_\alpha]}}_{\text{fields}} + \underbrace{X^{\langle -1\rangle}_{\alpha\ [\hat{p}-p_\alpha+1]} + X^{\langle -2\rangle}_{\alpha\ [\hat{p}-p_\alpha+2]} + \ldots + X^{\langle -p_\alpha-1\rangle}_{\alpha\ [\hat{p}+1]}}_{\text{anti-fields}} \quad .$$

Symplectic and pre-symplectic forms $\boldsymbol{O}$ and $\boldsymbol{\vartheta}$ on $\boldsymbol{M}$:

$$\boldsymbol{O} \;=\; \left[(-1)^{\alpha+1}\mathrm{d}\boldsymbol{X}^{\alpha}\mathrm{d}\boldsymbol{P}_{\alpha}\right]^{\langle 0\rangle}_{[\hat{p}+2]} \;=\; \mathrm{d}\boldsymbol{\vartheta}\;, \qquad \boldsymbol{\vartheta} \;=\; \left[\mathrm{d}\boldsymbol{X}^{\alpha}\boldsymbol{P}_{\alpha}\right]^{\langle 0\rangle}_{[\hat{p}+1]}\;,$$

and we denote the corresponding *graded Poisson bracket* on $\boldsymbol{M}$ by

$$\{\cdot,\cdot\} \;\equiv\; \{\cdot,\cdot\}^{\langle 0\rangle}_{[-\hat{p}]}\;,$$

and graded Poisson bracket on $\mathrm{Maps}\,[T[1]\mathscr{B},\boldsymbol{M}]$, referred to as the BV bracket, is denoted by

$$(\cdot,\cdot) \;\equiv\; (\cdot,\cdot)^{\langle 1\rangle}_{[0]}\;,$$

with quantum numbers $\mathrm{gh}\,((\cdot,\cdot)) = 1$ and $\deg\,((\cdot,\cdot)) = 0\,.$

BV bracket induced from Poisson bracket.

As observed by AKSZ, the BV bracket $(\cdot,\cdot)$ on $\mathrm{Maps}\,[T[1]\mathscr{B},\boldsymbol{M}]$ is induced from the graded Poisson bracket $\{\cdot,\cdot\}$ on $\Omega^{[0]}(\boldsymbol{M})$ via the formula

$$(\,I(F|\mathscr{B})\,,\,\boldsymbol{\phi}^*(F')\,)\;\equiv\;\boldsymbol{\phi}^*(\{F,F'\})\;.$$

It follows that the BV-adjoint action of the pre-symplectic form is related to the exterior derivative as follows :

$$(\,I(\mathrm{d}\boldsymbol{X}^\alpha\boldsymbol{P}_\alpha|\mathscr{B})\,,\,\boldsymbol{\phi}^*(L)\,)\;\equiv\;\mathrm{d}\,\boldsymbol{\phi}^*(L)\;\equiv\;\boldsymbol{\phi}^*(\mathrm{d}\,L)\;,$$

for $L\in\Omega(\boldsymbol{M})$.

Superfunctionals

Functionals built from ultra-local superfunctionals $\boldsymbol{\phi}^*(\boldsymbol{G})$ where $\boldsymbol{G} \in \Omega(\boldsymbol{M})$ have local representatives of the form $\boldsymbol{G} = G(\boldsymbol{Z}^i, d\boldsymbol{Z}^i)$ where $G \in \Omega(M)$. In particular, if $\boldsymbol{F}, \boldsymbol{F}'$ are superfunctions it follows that

$$\{\boldsymbol{F}, \boldsymbol{F}'\} \;=\; \big(\{F, F'\}_{[-\hat{p}]}(Z^i)\big)\big|_{Z^i \to \boldsymbol{Z}^i} \;,$$

where $\{F, F'\}_{[-\hat{p}]}$ denotes the Poisson bracket evaluated in the classical target space M.

$$\boldsymbol{S}_{\text{bulk}}[\boldsymbol{\phi}|\mathscr{B}] := I(\boldsymbol{L}|\mathscr{B}) = \sum_{\xi}\int_{\mathscr{B}_\xi}\pi\boldsymbol{\phi}^*_\xi(\boldsymbol{L}) , \quad \boldsymbol{L} := \mathrm{d}\boldsymbol{X}^\alpha\boldsymbol{P}_\alpha - \mathscr{H}(\boldsymbol{X},\boldsymbol{P}) ,$$

with $\mathscr{H}$ being a solution to the classical structure equation obeying $\mathscr{H}|_{P_\alpha=0} = 0$. Defining

$$\mathrm{s}(\cdot) := (\boldsymbol{S}_{\text{bulk}}, (\cdot)) ,$$

one has

$$\mathrm{s}\boldsymbol{Z}^i = \boldsymbol{R}^i ,$$

where the generalized supercurvatures

$$\boldsymbol{R}^i := \mathrm{d}\boldsymbol{Z}^i + \boldsymbol{Q}^i , \qquad \boldsymbol{Q}^i := \mathscr{Q}^i(\boldsymbol{Z}^j) = (-1)^{\hat{p}+1}\mathscr{P}^{ij}\partial_j\mathscr{H}(\boldsymbol{Z}^i) .$$

The locally-defined field configurations form equivalence classes modulo gauge transformations (in canonical coordinates)

$$\delta_{\boldsymbol{\varepsilon}} \boldsymbol{Z}^i \quad := \quad \mathrm{d}\boldsymbol{\varepsilon}^i - \boldsymbol{\varepsilon}^j \partial_j \boldsymbol{Q}^i \;\; ,$$

where the parameters have total degree $|\boldsymbol{\varepsilon}^i| = |\boldsymbol{Z}^i| - 1$ and expansions into components with fixed ghost numbers and form degrees given by the suspension of $\boldsymbol{X}^\alpha$ and $\boldsymbol{P}_\alpha$ with one unit of form degree, and zero units of ghost number.

As in the classical case, the AKSZ action can be defined globally using fiber bundle-type geometries :

(I) the local representatives $\boldsymbol{Z}^i_\xi$ are glued together using transition functions with parameters $\boldsymbol{t}^{i,\xi}_{\xi'} = (\boldsymbol{t}^\alpha, 0)^\xi_{\xi'}$ obeying

$$(\overrightarrow{\boldsymbol{P}} - 1)\overrightarrow{\boldsymbol{t}}\mathscr{H} \ \equiv \ 0 \,,$$

(II) the following Dirichlet conditions are imposed :

$$\boldsymbol{P}_\alpha|_{\partial B} \ = \ 0 \ .$$

The AKSZ relation between the BV bracket and the Poisson bracket $\Rightarrow$

$$(\boldsymbol{S}_{\text{bulk}}, \boldsymbol{S}_{\text{bulk}}) \;=\; (-1)^{\hat{p}} \sum_{\xi} \oint_{\partial B_\xi} \pi \boldsymbol{\phi}^*_\xi \,(\boldsymbol{R}^\alpha \boldsymbol{P}_\alpha - 2\boldsymbol{L}) \;=\; 0 \;,$$

where the latter equality follows from the boundary conditions and the facts that $\delta_{\boldsymbol{t}} \boldsymbol{L} \equiv \boldsymbol{K}_{\boldsymbol{t}} \equiv 0$ and that

$$\delta_{\boldsymbol{t}} \boldsymbol{P}_\alpha \;=\; -(-1)^\alpha \overrightarrow{\boldsymbol{t}} \partial_\alpha \mathscr{H} \;, \qquad \delta_{\boldsymbol{t}} \boldsymbol{R}^\alpha \;=\; (-1)^{\hat{p}(\alpha+1)} \overrightarrow{\boldsymbol{R}}_X \overrightarrow{\boldsymbol{t}} \partial^\alpha \mathscr{H} \;,$$

where $\overrightarrow{\boldsymbol{R}}_X := \boldsymbol{R}^\alpha \partial_\alpha$, implying

$$\delta_{\boldsymbol{t}} (\boldsymbol{R}^\alpha \boldsymbol{P}_\alpha) \;\equiv\; \overrightarrow{\boldsymbol{R}}_X \overrightarrow{\boldsymbol{t}} (\overrightarrow{\boldsymbol{P}} - 1) \mathscr{H} \;\equiv\; 0 \;.$$

↱ The AKSZ action $\boldsymbol{S}_{\text{bulk}}$ solves the classical BV master equation

$$(\boldsymbol{S}_{\text{bulk}}, \boldsymbol{S}_{\text{bulk}}) \;=\; 0 \quad \Leftrightarrow \quad \mathrm{s}^2 \;=\; 0 \;,$$

subject to the functional boundary condition

$$\boldsymbol{S}_{\text{bulk}}[\boldsymbol{\phi}|B]|_{\boldsymbol{\phi}=\phi} \;=\; S^{\text{cl}}_{\text{bulk}}[\phi|B] \;.$$