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Review

A Concise Review of State Estimation Techniques for Partial Differential Equation Systems

Ivan Francisco Yupanqui Tello [1,2,†], Alain Vande Wouwer [1,*,†] and Daniel Coutinho [2,†]

[1] Systems, Estimation, Control, and Optimization (SECO), University of Mons, 7000 Mons, Belgium; ivanfrancisco.yupanquitello@umons.ac.be
[2] Postgraduate Program in Engineering of Automation and Systems, Federal University of Santa Catarina, Florianopolis 88040-900, SC, Brazil; daniel.coutinho@ufsc.br
* Correspondence: alain.vandewouwer@umons.ac.be
† These authors contributed equally to this work.

Abstract: While state estimation techniques are routinely applied to systems represented by ordinary differential equation (ODE) models, it remains a challenging task to design an observer for a distributed parameter system described by partial differential equations (PDEs). Indeed, PDE systems present a number of unique challenges related to the space-time dependence of the states, and well-established methods for ODE systems do not translate directly. However, the steady progresses in computational power allows executing increasingly sophisticated algorithms, and the field of state estimation for PDE systems has received revived interest in the last decades, also from a theoretical point of view. This paper provides a concise overview of some of the available methods for the design of state observers, or software sensors, for linear and semilinear PDE systems based on both early and late lumping approaches.

Keywords: distributed parameter systems; partial differential equations; observers; state estimation; monitoring; early lumping; late lumping

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1. Introduction

For many physical systems, the states, inputs, and outputs may depend on spatial coordinates, which define a position in a multidimensional space. These systems are modeled by partial differential equations (PDEs) and, along with systems modeled by integral equations and delay differential equations, are called distributed parameter systems (DPSs). They differ from lumped parameter systems (LPSs), modeled by ordinary differential equations (ODEs), in the spatial dependence of their states, inputs and outputs. The theory of systems described by PDEs has undergone an extensive development and is further supported by the availability of sophisticated numerical tools and almost unlimited computational power.

The PDE description is an essential feature for modeling and analysis of processes where the spatial distribution of the process variables has to be taken into account. Over the years, it has evolved as one of the fundamental mathematical descriptions of many technical processes and scientific observations arising from chemical and physical principles (e.g., conservation laws as well as mass, energy and momentum balances).

DPS examples range from heating and cooling systems [1,2], fluid heat exchangers [3,4], biochemical reactors and similar processes [5–8], mechanical systems [9–11], resource recovery applications (such as under ground oil and water and coal reservoirs) [12,13] to the prediction and control of atmospheric pollution [14], epidemiological modeling [15,16] for the study of infectious diseases spread and the control of forest and meadow fires [17]. The time and space dependence makes the analysis of systems modeled by PDEs more complex than in the lumped-parameter case also depending on the type of boundary conditions.

The on-line state estimation of DPSs modeled by PDEs is particularly a delicate problem in view of the system dimensionality and the fact that providing a comprehensive set of sensors is either physically impossible or too costly [18]. In such a case, the internal states have to be estimated on the basis of the mathematical model and (available) online measurements provided by sensors located at strategic positions in the spatial domain. In general, measurements for PDE systems are either in domain (provided by sensors located inside the domain of the PDE system in a pointwise or piecewise manner) or boundary (provided by sensors located at the boundary of the PDE system).

The problem of designing observers for DPSs modeled by PDEs (in the following, we will use DPSs and PDE systems interchangeably, by abuse of language, but for the sake of simplicity) has been an active field of research over the last years. Thus, the literature related to the subject matter is plentiful but scattered. These works address a diverse class of systems with widely different approaches, symbolism, measurement processes and boundary conditions associated with them. As classical surveys systematizing the various techniques in this field, one can cite [19–21] and the references therein, or the monograph [22], where a broad class of estimation techniques for DPSs is presented.

This paper aims at giving an overview of well recognized methodologies for state estimation of (semilinear) PDE systems. In addition, we identify challenges for future research in this field. The remainder of this paper is organized as follows. A brief introduction to PDE systems is presented in Section 2, while Sections 3 and 4 survey the early and late lumping approaches, respectively, Section 5 describes the applied perspectives of the subject, and Section 6 concludes this paper.

2. PDE Systems

The evolution of DPS states governed by PDEs is described in an infinite-dimensional space (and therefore PDE systems are often referred as infinite dimensional systems). To simplify the presentation and discussion, the PDE system of interest is represented by a general class of semilinear systems. In particular, we consider the following (abstract) differential equations on a Hilbert space $\mathcal{H}$:

$$\begin{aligned} &\partial_t x(z,t) = \mathcal{A}x(z,t) + \mathcal{B}u_d(t) + F(x(z,t)), \;\; x(z,0) = x_0(z) \in \mathcal{H}, \\ &\mathfrak{B}x(z,t) = u_b(t), \end{aligned} \tag{1}$$

for $(z,t) \in \Omega \times (0,\infty)$, where $\Omega = (a,b) \subset \mathbb{R}$ in the unidimensional case or an open subset in $\mathbb{R}^n$ in higher dimensions, $x(\cdot,t) \in \mathcal{H}$ denotes the state, $u_d(t) \in \mathbb{R}^{n_d}$ is the known distributed exogenous input and $u_b(t) \in \mathbb{R}^{n_b}$ is the boundary known exogenous input, $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is a differential linear operator, $\mathcal{B} : \mathcal{D}(\mathcal{B}) \to \mathbb{R}^{n_d}$ is the distributed input operator, $\mathfrak{B} : \mathcal{D}(\mathfrak{B}) \to \mathbb{R}^{n_b}$ is the boundary input operator, and $F(\cdot)$ is a nonlinear function from $\mathcal{H}$ into itself. We also consider with respect to system (1) that:

- The nonlinear function $F(\cdot) : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz continuous, i.e., there exists a positive constant $l^F = l^F(\rho)$, where $\rho > 0$, such that

$$\|F(x) - F(\hat{x})\| \leq l^F \|x - \hat{x}\| \tag{2}$$

  holds for all $x, \hat{x} \in \mathcal{H}$ with $\|x\|, \|\hat{x}\| \leq \rho$.
- The output measurement vector is defined by means of

$$y(t) = \begin{bmatrix} y_d(t) & y_b(t) \end{bmatrix}^T \tag{3}$$

  with

$$\begin{aligned} y_d(t) &= \mathcal{C}_d x(z,t), \\ y_b(t) &= \mathcal{C}_b x(z,t), \end{aligned} \tag{4}$$

defining the distributed and boundary measurement vectors, where $\mathcal{C}_d : \mathcal{D}(\mathcal{C}_d) \to \mathbb{R}^{\eta_d}$ and $\mathcal{C}_b : \mathcal{D}(\mathcal{C}_b) \to \mathbb{R}^{\eta_b}$ are the distributed and boundary output operators, respectively. For convenience, the total measurement may be alternatively represented as $y(t) = \mathcal{C}x(z,t)$ with $y \in \mathbb{R}^{n_y}$, $n_y = \eta_d + \eta_b$, and $\mathcal{C}$ denoting the total output operator.

For the purpose of implementing observer design techniques for PDE systems, the dimension of the system must be reduced. This process is called lumping, and there are two kinds of lumping [20,23]: early lumping and late lumping. The easiest, most straightforward approach, termed early lumping, simply discretizes the PDE system at the earliest opportunity into an approximate model consisting of a set of ODEs.

Subsequently, the appropriate design methods for LPSs are applied directly to design state observers without recourse to DPS theory at all. This approach, however, has inherent drawbacks. First, conditions for detectability and observability, which should depend only on the placement of sensors, can also depend on the method of lumping and the location of discretization points. Second, one quickly loses the physical essence of the problem, and the observer design may fail to take advantage of natural properties of the system.

The alternative approach, late lumping, takes full advantage of the available PDE theory and uses the original PDE model for observer design. It is only at the last stages, after the observer design has been made, that the resulting observer equations are lumped for reasons of numerical integration and implementation. Late lumping allows the designer to take advantage of all the natural features of the problem and to understand the system structure more thoroughly. However, this approach requires a deeper knowledge of DPS theory. The following sections review both approaches.

3. Early Lumping Approach

In the early lumping approach, the PDE system is first approximated by a finite-dimensional system, obtained by means of model reduction methods, for which the observer is designed. Therefore, model reduction may be considered as one of the fundamental issues in early lumping DPS state estimation. An important concept with this respect is the time–space separation framework [24], where some forms of discretization in space and in time are operated separately. Whereas the classical finite difference methods, which are commonly used to solve PDE systems numerically, do not yield low-order approximations, the method of weighted residuals (MWR) [25] is the most popular method for model reduction.

3.1. Model Order Reduction Using MWR

Consider the abstract formulation of the PDE system in (1) with $\mathcal{H} = \mathbf{L}_2(\Omega)$. The unknown solution $x(z,t)$ is approximated by a function $x^*(z,t)$

$$x(z,t) \approx x^*(z,t) = \sum_{n=1}^{N} a_n(t)\phi_n(z), \tag{5}$$

where the set $\{\phi_n(z) \in \mathcal{H} : n = 1,\ldots,N\}$ constitutes a basis of a finite dimensional subspace on $\mathcal{H}$, and the a_n coefficients are functions of time. The spatial functions $\phi_n(z)$ are ordered, and those with a small index n represent low spatial frequency solution features, whereas the higher-index functions correspond to high frequency information. The approximate solution to the problem is expressed in terms of N basis functions, thus, neglecting faster solution modes (with indexes over N). The coefficients $a_n(t)$ must be defined in such a way that $x^*(z,t)$ is a good approximation to the solution of (1), i.e., $x^*(z,t) \to \mathcal{D}(\mathcal{A})$, $\forall t \geq 0$, and the residual

$$R(z,t) = \partial_t x^*(z,t) - \mathcal{A}x^*(z,t) - F(x^*(z,t)) \tag{6}$$

approaches zero in some average sense over the spatial domain Ω by requiring that

$$\langle w_i(z), R(z,t)\rangle = 0, \quad i = 1,\ldots,N \tag{7}$$

where $\{w_i(z) : i = 1, \ldots, N\}$ is a set of weighting functions to be appropriately defined.

The choice of weighting functions can lead to several different methods. In the following, only the two most popular techniques in systems theory are considered:

1. In the Galerkin method, the weighting functions $w_1(z), \ldots, w_N(z)$ are chosen equal to the basis functions, i.e.,

$$w_i(z) = \phi_i(z), \quad i = 1, 2, \ldots, N. \tag{8}$$

This technique makes the residual orthogonal to each basis function and, therefore, provides the best possible solution in the space defined by the N functions $\phi_1, \ldots, \phi_N(z)$. Indeed, as $N \to \infty$, the residual $R(z,t)$ tends to zero, as it is orthogonal to every function in a complete set of functions.

2. In collocation methods, the weighting functions $w_i(z)$ are chosen to be Dirac functions $\delta(z - z_i)$, so that

$$R(z = z_i, t) = 0, \quad i = 1, \ldots, N, \tag{9}$$

yielding a set of differential equations to be solved at N points in the spatial domain. The collocation method has been further refined in [25,26] and has been shown to be extremely powerful. In particular, orthogonal collocation makes use of orthogonal polynomials as basis functions and the optimal location of the collocation points are chosen at the roots of orthogonal polynomials (e.g., [27]).

One of the key issues in developing MWR solutions is the selection of the order N of the approximations, i.e., the decision on the fast modes $\{\phi_n : n = N+1, \ldots, \infty\}$, which are neglected in the course of model reduction. To avoid resorting to high-order approximations, inertial manifold (IM) methods [28,29], are appealing. An IM is a finite-dimensional Lipschitz manifold, which attracts every trajectory exponentially and on which the PDE system can be approximated.

However, IMs exist only for certain classes of PDE systems, e.g., parabolic PDEs, and their analytical derivation can be difficult or even impossible. To overcome these problems, the concept of an approximate inertial manifold (AIM) has been introduced in [30,31]. In [29], an efficient approximation method was developed based on singular perturbations.

The selection of the spatial basis functions (BFs) $\phi_1(z), \ldots, \phi_N(z)$ is of paramount importance for the accuracy of the approximation. In particular, the BFs should be selected according to the characteristics of the spatial domain and boundary conditions. Some typical choices are:

1. **Eigenfunctions** of linear spatial operators are suitable for parabolic PDE systems, as their eigenspectrum displays a separation between slow and fast modes [32].
2. **Orthogonal polynomials** are well suited for the model reduction of nonlinear PDE systems. In general, Legendre, Hermite, Laguerre and Chebyshev polynomials are good candidates for the spectral approximation of such systems, depending on the definition of the spatial domain and boundary conditions [33–35]. When the spatial domain has a complex shape, or the spatial activity of the solution requires it, it might be necessary to resort to finite element decompositions. In this latter case, the basis functions are no longer global but defined on local elements;
3. **Empirical basis functions** are used in Proper Orthogonal Decomposition (POD). These functions, which capture the system dynamics, can be computed either from experimental data collected on the real process or from numerical results obtained from the simulation of a detailed model of the process [36–38].

An overview of these approximation techniques is given in the book [39], alongside a MATLAB library.

3.2. State Observer Design

Whatever the method used to obtain a reduced model described by ODEs, it will have the following state space representation (or otherwise its time-discretized version)

$$\begin{aligned} \dot{a}(t) &= Aa(t) + B_d u_d(t) + F_a(a(t)) \\ B_b a(t) &= u_b(t) \\ y_a(t) &= Ca(t). \end{aligned} \tag{10}$$

Then, the following Luenberger observer may be proposed to estimate the state vector $a(t)$:

$$\begin{aligned} \dot{\hat{a}}(t) &= A\hat{a}(t) + B_d u_d(t) + F_a(\hat{a}(t)) + L(y_a(t) - \hat{y}_a(t)) \\ B_b \hat{a}(t) &= u_b(t) \\ \hat{y}_a &= C\hat{a}(t) \end{aligned} \tag{11}$$

where $\hat{a}(t) \in \mathbb{R}^N$ is the observer state and $L \in \mathbb{R}^{N \times n_y}$ is the output injection gain. Hence, the estimation $\hat{x}(z,t)$ of $x(z,t)$ can be obtained from

$$\hat{x}(z,t) = \sum_{n=1}^{N} \hat{a}_n(t)\phi_n(z). \tag{12}$$

Thus, the early lumping approach leads to observer design methods for LPSs, which however, often include additional elements such as the influence of sensor locations. In the following, we highlight some works addressing the early lumping approach for state estimation.

In [40], the orthogonal collocation method is applied to convert the PDE model that describes the dynamics of a bioreactor into an ODE system. The observability of the system is studied based on the linearized ODE model, and a criterion of relative observability is discussed and analyzed as a function of the number and position of the substrate sensors located along the reactor.

A systematic approach to efficiently reconstruct the state in distributed process systems from a limited, and usually reduced, number of sensors is presented in [41]. The model reduction is achieved using POD and the selection of the most appropriate type (and number) of measurements by the solution of a max/min optimization problem.

Torres et al. [42] presented a model of water-hammer dynamics based on the collocation method. This model is shown to fairly represent possible leak effects in a pipeline and, thus, to be useful for the purpose of observer-based leak detection. In [43], an extended Kalman filter (EKF) is presented for estimating the fluid flow through a plate valve of a reciprocating positive displacement pump. An ODE model was constructed by POD and Galerkin projection. A technique for ensuring observability was also presented.

Recently, Ghattassi et al. [44] designed a reduced order observer for a nonlinear PDE model consisting of a radiative transfer equation coupled with a nonlinear heat equation in a two-dimensional domain. The original system is approximated by an N-dimensional ODE system using both discontinuous and continuous Galerkin methods. Thanks to the differential mean value theorem, the error dynamic system is cast in terms of a linear parameter varying system description, and the linear matrix inequality (LMI) framework is used to derive sufficient convex conditions ensuring global convergence of the error dynamics.

4. Late Lumping Approach

In the late-lumping approach, the distributed nature of the system model is preserved for state estimation resulting in an infinite-dimensional state observer, which is later lumped for implementation purposes. This approach theoretically leads to state estimators with a better performance, since no approximation of the model is made. Nevertheless, it requires the manipulation of less conventional mathematical tools and methods. Considering the system representation as defined in (1), the general form of a Luenberger-like infinite dimensional observer is given by

$$\begin{aligned}
\partial_t \hat{x}(z,t) &= \mathcal{A}\hat{x}(z,t) + \mathcal{B}u_d(t) + F(\hat{x}(z,t)) + \mathcal{L}(y(t) - \hat{y}(t)), \;\; \hat{x}(z,0) = \hat{x}_0(z) \in \mathcal{H} \\
\mathfrak{B}\hat{x}(z,t) &= u_b(t) + l_b(y_b(t) - \hat{y}_b(t)) \\
\hat{y}_d(t) &= \mathcal{C}_d\hat{x}(z,t) \\
\hat{y}_b(t) &= \mathcal{C}_b\hat{x}(z,t) \\
\hat{y}(t) &= \begin{bmatrix} \hat{y}_d(t) & \hat{y}_b(t) \end{bmatrix}^T.
\end{aligned} \tag{13}$$

where $\hat{x}(z,t)$ is the estimate of $x(z,t)$, and $\mathcal{L} : \mathbb{R}^{n_y} \to \mathcal{H}$ and $l_b : \mathbb{R}^{n_y} \to \mathbb{R}^{n_b}$ are the output injection gains to be designed.

Now, let

$$e(z,t) = x(z,t) - \hat{x}(z,t) \tag{14}$$

be the estimation error vector. Then, the estimation error dynamics can be cast as follows:

$$\begin{aligned}
\partial_t e(z,t) &= (\mathcal{A} - \mathcal{L}\mathcal{C})e(z,t) + F(x(z,t)) - F(\hat{x}(z,t)) \\
(\mathfrak{B} - l_b\mathcal{C}_b)e(z,t) &= 0, \qquad e(z,0) = e_0(z) \in \mathcal{H}.
\end{aligned} \tag{15}$$

Thus, the output injection gain $\mathcal{L}$ is designed in order to guarantee the stability of the estimation error dynamics as described in (15).

Infinite dimensional observer synthesis of DPSs is usually based on semigroup theory, Lyapunov approach or using the backstepping technique, which are summarized in the sequel.

4.1. Synthesis Based on the Semigroup Theory

The study of operator semigroups is a mature area of functional analysis, which is still very active. In this framework, the DPS is rewritten as a sum of linear and/or nonlinear operator semigroups defined in Hilbert or Banach spaces. The study of observation operators for such semigroups is relatively more recent. Crucial to the developments of these studies are the concepts of admissibility, observability and detectability [45–47].

The linear state observer design (i.e., $F(x) \equiv 0$) is an extension of the well-known lumped parameter Luenberger observer design to the infinite dimensional system described in (1). Thus, we have to determine an output injection operator $\mathcal{L}$ in such a way that the operator $\mathcal{A}_L = \mathcal{A} - \mathcal{L}\mathcal{C}$, with domain $\mathcal{D}(\mathcal{A}_L) = \mathcal{D}(\mathcal{A}) \cap \ker(\mathfrak{B} + l_b\mathcal{C})$, generates a stable strongly continuous semigroup $\mathcal{S}_L(t)$, i.e., there exist positive real scalars M and α such that

$$\|\mathcal{S}_L(t)\| \leq M\mathrm{e}^{-\alpha t}, \;\; \forall t \geq 0. \tag{16}$$

This approach is described extensively in [46,48,49] for general DPSs. It is worth mentioning that when $\mathcal{A}$ is a Riesz-spectral operator, its spectral properties allow using the modal output injection gain [49,50]. Concerning the design of observers for semilinear PDEs using the abstract infinite dimensional description, the dissipativity-based observer design [51], provides an effective mean for dealing with complex nonlinearities yielding local and global convergent results through the use of modal measurement injection approach as addressed in [49].

PDE systems may be subject to random disturbances, and the problem of estimating the state from noisy observations is important in practical applications. By far the largest part of the literature on infinite-dimensional filtering consists in a formal analysis of the Kalman filtering problem for linear distributed parameter systems. The most important generalization of the Kalman–Bucy filter to linear infinite dimensional systems has been provided in the seminal work of Curtain and Pritchard [49,52,53], where they investigated the infinite dimensional Riccati equation in the context of evolution operators. This generalizes semigroups for abstract (stochastic) evolution equations and allows establishing the Kalman filter in Hilbert spaces.

Although filtering for linear PDE systems is well explored, not as much theory has been developed for nonlinear PDEs. The extended Kalman filter (EKF) in infinite dimensions has been applied computationally to particular nonlinear PDE systems [54,55]. However, no proof exists for the existence of the continuous-time EKF for nonlinear PDE systems. In [56], a first result in extending the EKF from finite-dimensional systems to infinite dimensions is provided. Therein, as in (linear) Kalman filter design [49], the output injection gain in (13) is taken to be time variant and computed as

$$\mathcal{L}(t) = \hat{\Pi}(t)\mathcal{C}^*\mathcal{V}^{-1}, \quad t \in [0, t_f] \tag{17}$$

where $\hat{\Pi}(t) : [0, t_f] \to L(\mathcal{H})$ is a self adjoint non-negative operator and the unique solution of the infinite-dimensional differential Riccati equation

$$\begin{aligned} \left\langle \dot{\hat{\Pi}}(t)\xi_1, \xi_2 \right\rangle &= \langle \hat{\Pi}(t)\xi_1, (\mathcal{A} + DF(\hat{x}(\cdot,t)))^*\xi_2 \rangle + \langle (\mathcal{A} + DF(\hat{x}(\cdot,t)))^*\xi_1, \hat{\Pi}(t)\xi_2 \rangle \\ &\quad + \langle \mathcal{W}\xi_1, \xi_2 \rangle - \left\langle \mathcal{C}\hat{\Pi}(t)\xi_1, \mathcal{V}^{-1}\mathcal{C}\hat{\Pi}(t)\xi_2 \right\rangle, \\ \hat{\Pi}(0) &= \hat{\Pi}_0, \end{aligned} \tag{18}$$

for all $\xi_1,\ \xi_2 \in \mathcal{D}(\mathcal{A}^*)$, $t \in [0, t_f]$, with $\mathcal{V},\ \mathcal{W}$ being self adjoint non-negative operators, DF the Fréchet-derivative of $F(\cdot)$ on $\mathcal{H}$ and $\mathcal{A}^*$ the adjoint operator of $\mathcal{A}$. We emphasize that the Riccati Equation (18) and the observer Equation (13) in principle have to be solved simultaneously because (18) depend on $\hat{x}(z, t)$.

4.2. Synthesis Based on the Lyapunov Theory

In the Lyapunov-based observer synthesis, the convergence of the observer is established by analyzing the stability of the estimation error dynamics in (15) considering an appropriate Lyapunov functional that satisfies $V(0) = 0$ and for which there exist positive real scalars a and b such that

$$a\|e\| \le V(e) \le b\|e\| \tag{19}$$

for all $e \in \mathcal{H}$, with e denoting the estimation error. Furthermore, if there also exists a positive real scalar c such that for all $t \ge 0$ the time derivative of $V(e)$ along the trajectories of the estimation error dynamics defined in (15) satisfies

$$\dot{V}(e(z,t)) \le -c\|e(z,t)\|, \tag{20}$$

then, it can be easily shown that

$$\|e(z,t)\| \le \sqrt{\frac{b}{a}}\|e_0(z)\|\mathrm{e}^{-\frac{c}{b}t}, \quad t \ge 0. \tag{21}$$

Thus, considering an appropriate parametrization of the Lyapunov functional candidate satisfying (19), the stability condition in (20) can be expressed in terms of linear matrix inequality (LMI) constraints. The resulting LMIs are numerically solved using standard interior-point algorithms, and the solution also provides the observer output injection operator $\mathcal{L}$.

This approach has been applied in [57,58] to the design of boundary observers for linear and quasi-linear first order hyperbolic systems. The latter references introduce sufficient conditions ensuring the exponential stability of the error dynamics considering only information from PDE boundaries and the following Lyapunov functional candidate

$$V(e(z,t)) = \int_\Omega e^T(z,t)Pe(z,t)\mathrm{e}^{-\mu z}dz. \tag{22}$$

Fridman and Orlov [59] address the exponential stability and L_2-gain analysis for scalar PDEs, governed by semilinear partial differential equations of parabolic and hyperbolic types. LMI-based sufficient conditions are proposed to ensure the system exponential stability while providing a guaranteed decay rate. The proposed conditions are then utilized to synthesize an H_∞ static observer-based boundary control.

The design of an observer for a class of scalar semilinear PDEs with linear diffusion-convection transport, nonlinear reaction, and boundary measurements is proposed in [60]. The problem is addressed within a weighted Lyapunov functional framework, where the weight function is regarded as a design degree of freedom and the associated algebraic inequality convergence conditions are handled considering an LMI procedure.

More recently, Sum-of-Squares (SOS) optimization methods have been applied to the parametrization of positive Lyapunov functionals using SOS polynomials, which renders convex the problem of observer design. It should be emphasized that algorithms for solving SOS programs are fully automated in MATLAB toolboxes, such as SOSTOOLS [61] and YALMIP [62]. Then, the SOS problem is parsed into an SDP formulation that can be solved by LMI solvers such as SeDuMi [63]. In this framework, Gahlawat and Peet [64,65] designed observer-based controllers for a class of scalar linear parabolic PDEs ($\mathcal{H} = \mathbf{L}_2(0,1)$) with boundary measurements considering the following Lyapunov functional

$$V(e(z,t)) = \langle e(\cdot,t), \mathcal{P}e(\cdot,t)\rangle \tag{23}$$

with

$$(\mathcal{P}e)(z,t) = M(z)e(z,t) + \int_0^z K_1(z,\xi)e(\xi,t)dz + \int_z^1 K_2(z,\xi)e(\xi,t)dz. \tag{24}$$

4.3. Synthesis Based on Backstepping Technique

The backstepping method for the stabilization of PDE systems, as it is known today, was first introduced in the seminal work of Smyshlyaev and Krstic [66]. This approach, first developed for a general 1-D linear reaction-diffusion-advection PDE on $\mathcal{H} = \mathbf{L}_2(\Omega)$, is based on a constructive strategy with a design in the continuous space domain. In the context of state observer design for linear PDE systems (i.e., $F(x) = 0$), the backstepping technique considers an invertible integral Volterra type transformation

$$e(z,t) = w(z,t) - \int_0^z P(z,\xi)w(\xi,t)d\xi \tag{25}$$

to match the state estimation error dynamics into a target system as shown below

$$\begin{aligned} \partial_t w(z,t) &= \mathcal{A}_t w(z,t) \\ \mathfrak{B}_t w(z,t) &= 0 \end{aligned} \tag{26}$$

which has the desired stability properties, where $\mathcal{A}_t$ and $\mathcal{B}_t$ stand respectively for operators of the same structure as $\mathcal{A}$ and $\mathfrak{B}$ with parameter values defining prescribed features of stability.

By substituting (26) into (15), we obtain a set of conditions on the integral kernel $P(z,\xi)$ in the form of the hyperbolic PDE on a limited space domain. The design conditions can be solved numerically or in some cases analytically. By the principle of equivalence, the stability of the target system allows concluding on the stability of the estimation error dynamics. The strength of this approach lies in its structural simplicity and a fairly wide range of applications for various classes of systems modeled by PDEs.

In particular, for systems governed by parabolic PDEs defined on a one-dimensional (1D) spatial domain, a systematic observer design approach using boundary sensing is introduced in [67]. Recently, a backstepping-based observer design has been presented in [68] for reaction-diffusion processes with spatially varying reaction coefficients and a weighted average of the state over the spatial domain as the measured output.

In [69,70], the backstepping-based observer design is addressed for reaction-diffusion processes evolving in multidimensional spatial domains, while the sliding mode theory and a backstepping-like procedure for parabolic processes is applied in [71,72] by adopting a nonconventional target system for the error dynamics, embedding certain discontinuous output injection terms related to sliding mode design.

More recently, multivariable systems of coupled PDEs have been considered in a backstepping-based boundary control and observer design settings. The most intensive efforts of the current literature seems, however, to be oriented towards coupled processes of transport type [73–77].

4.4. Robust Synthesis

Sliding mode observers follow the ideas developed in sliding mode control and have shown satisfactory performances when robustness with respect to small dynamic uncertainties and disturbances is one of the desired design features. In these observers, the output injection operator is a discontinuous mapping from the output euclidean space $\mathbb{R}^{n_y}$ into the Hilbert state space $\mathcal{H}$. Utkin and Drakunov introduced sliding mode observer for LPSs [78] and since then various generalizations for different classes of PDE systems have been developed over the last two decades.

In [79], Orlov presented a model reference adaptive control for DPSs described by second-order PDEs of parabolic and hyperbolic types. In the design process of the controller, a sliding mode-based state derivative observer is constructed, which estimates the derivative of the spatial variable. More recently, several works [80–82] have expanded these results and suggested sliding mode observers for a specific class of DPSs considering only a limited number of online available measurements. In this connection, the assessment of the asymptotic convergence of the observation error using Lyapunov arguments plays a fundamental role.

Another approach is provided by interval estimation, which is well suited to the situation where robustness with respect to uncertainties is considered: using input–output information, an observer can be constructed to evaluate the set of admissible values (interval) for the state at each instant of time. The interval width is linked to the size of the model uncertainty. There are several approaches to design interval/set-membership estimators for PDE systems within which we can cite [83–85].

5. Applied Research Perspectives

The field of state estimation of PDE systems is approaching a state of maturity as theory is available for a wide range of practical problems. Furthermore, in view of the evolution of computational and embedded hardware the implementation and real–time evaluation have become tractable issues, which could be verified by realizations in the areas mentioned in Section 1. As it can be noted from the works cited previously in this paper, there are, at present, a large number of original reported applications of state estimation in systems modeled by PDEs.

However, upon closer examination, one should note that most of these reported studies involve mathematical modeling and simulation, and only a relatively small number of applications of on-line state estimation to laboratory processes have been reported. In addition, to the authors' knowledge, there have been essentially no application of distributed on-line state estimation and control to large-scale industrial processes. Thus, while off-line procedures, such as process identification and optimal design, appear to be in relatively wide use, industrial real-time applications are still underdeveloped.

Most applications of state estimation techniques to initial-boundary value problems are limited to systems whose spatial extension is considered in one direction only. For instance, a tubular chemical reactor will usually be considered as a 1D system, along the axial direction, assuming radial symmetry and negligible variations in the radial direction. This makes sense as monitoring of the time evolution of the spatial profiles of the state variables usually focuses on the main direction of activities/variations, so as to ensure safe and optimal operation.

Nonetheless, some developments have been reported for 2D systems, or even 3D systems, but they remain rare. Probably the first report of the application of a distributed observer to a real system in two spatial dimensions is the work of Lausterer [86] where the temperature distribution in a heated ingot is estimated using a late lumping Kalman filter. More recently, the authors of [73] proposed an observer for the velocity, pressure, electric potential and current fields in a magnetohydrodynamic (MHD) channel flow.

The observer is based on the linearized 3D MHD equations, combined with the linear injection of the output estimation error with observer gains designed using backstepping in Fourier space. Another example is provided by a data-driven robust observer for a 2D-Boussinesq equation modeling the coupled airflow and temperature in ventilation of air conditioning processes as reported in [87].

On the other hand, over the recent years, new application domains have emerged that may also support the development of novel theoretical concepts. In this connection, state estimation techniques appear promising for large scale networks of interconnected processes, where the use of PDEs enables a compact system description to develop state estimation strategies. Examples of this kind of application may be found in pedestrian dynamics [88], traffic congestion [89,90], multi–agent systems representing mobile robots or other interconnected dynamic devices [91,92].

Table 1 gives an overview of a collection of research papers classified by publication dates (from older to more recent) with indication of the methodology used and the type of applications. The following notation is used: EL: Early lumping, LL: Late lumping, OE: optimal estimation, SGT: Semigroup theory, LM: Lyapunov method, BT: Backstepping technique, RE: Robust state estimation, SIM: Simulation, RA: Real Application, and 2/3D: 2 or 3 dimensions. The group of papers OE includes Kalman-filtering-based observers not applying SGT, and receding horizon estimators.

Table 1. An overview of state estimation developments (left column lists the papers by chronological order, while the other columns indicate the methodology and the type of applications—EL: Early lumping, LL: Late lumping, OE: optimal estimation, SGT: Semigroup theory, LM: Lyapunov method, BT: Backstepping technique, RE: Robust state estimation, SIM: Simulation, RA: Real Application, 2/3D: 2 or 3 dimensions.).

Paper	EL	LL	OE	SGT	LM	BT	RE	SIM	RA	2/3D
[86]		X	X						X	X
[40]	X							X		
[93]		X	X					X		
[41]	X							X		
[67]		X				X		X		
[94]		X	X					X	X	
[57]		X			X			X	X	
[50]	X			X				X	X	
[42]	X		X					X		
[73]		X				X				X
[59]		X			X		X			
[71]		X				X	X	X		
[65]		X			X			X		
[74]		X				X			X	
[75]		X				X		X	X	
[58]		X			X			X		
[60]		X			X			X	X	
[70]		X				X	X	X		X

Table 1. *Cont.*

Paper	EL	LL	OE	SGT	LM	BT	RE	SIM	RA	2/3D
[76]		X				X		X		
[68]		X				X		X		
[77]		X				X		X		
[64]		X			X			X		
[51]		X		X				X		
[80]		X					X	X	X	
[54]		X	X					X		
[72]		X				X	X	X		
[55]		X		X					X	
[43]	X		X					X	X	X
[44]	X						X	X		X
[84]	X						X	X		
[81]	X						X	X		
[82]		X			X		X	X		
[83]		X			X		X	X		
[87]	X						X	X		X

6. Concluding Remarks

This paper presents a brief survey of the advances in state estimation of PDE systems. While the early lumping approach provides a direct path to the use of the realm of methods available for ODE systems, the late lumping approach offers the possibility to develop tailor made solutions, exploiting the problem structure. Even though the research field associated to DPS systems has enjoyed a renewed enthusiasm in the last two decades, we must observe that real-life applications still largely lag behind the theory.

There are many reasons for this observation, starting with the sophisticated mathematical framework required for the development of the late lumping observers, but also the lack of adequacy of some theoretical results, which are cast into idealized problem classes that do not correspond to the real-life models, combining different model equation types, e.g., PDEs, ODEs, AEs and instrumentation configurations. In these cases, late lumping and the use of standard state estimation techniques remain quite popular. Alternatively, late lumping associated with optimization-based methods, such as receding horizon estimation [93,94], which have not been covered in this review, can be the the best route for the practicing engineer.

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Abbreviations

The following abbreviations are used in this manuscript:

PDEs	Partial differential equations
ODEs	Ordinary differential equations
DPSs	Distributed parameter systems
MWR	Method of weighted residuals
BFs	Basis functions
FEM	Finite element method
FDM	Finite difference method
POD	Proper orthogonal decomposition
EFs	Eigenfunctions
OPs	Orthogonal polynomials
SVD	Singular value decomposition
EKF	Extended Kalman filter
RTE	Radiative transfer equation
NHE	Nonlinear heat equation
LMIs	Linear matrix inequalities
SOSs	Sum of squares
DMVT	Differential mean value theorem
LPV	Linear parameter varying
EL	Early lumping
LL	Late lumping
SGT	Semigroup theory
LM	Lyapunov method
BT	Backstepping technique
REE	Robust state estimation
SIM	Simulation
RA	Real Application
2/3D	2 or 3 dimensions

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