On the average number of colors in the non-equivalent colorings of a graph

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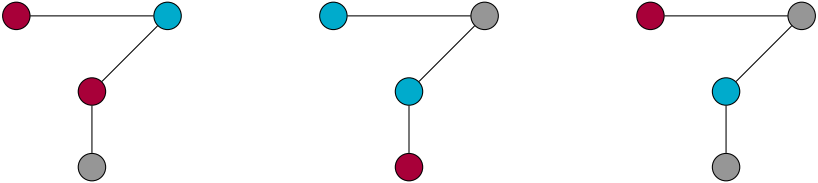
Plan

Colorings

Definition 1

A proper vertex coloring of a graph G is an assignment of colors to the vertices of G such that each vertex has only one color and every pair of adjacent vertices have different colors.

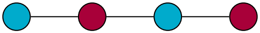
- A coloring induces a partition of the vertices of the graph.
- Two colorings are equivalent if they induce the same partition.



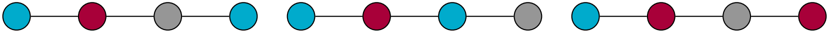
- Colorings 1 and 2 are equivalent but 3 is not.

AvCol, the big picture

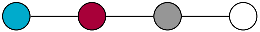
$$S(\mathsf{P}_4, 2) = 1$$



$$S(\mathsf{P}_4, 3) = 3$$



$$S(\mathsf{P}_4, 4) = 1$$



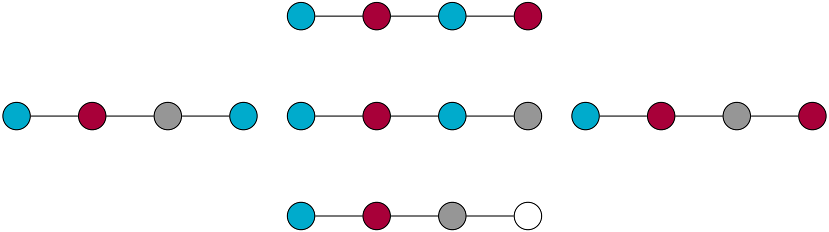
$$\mathcal{B}(\mathsf{P}_4) = 1 + 3 + 1 = 5$$

$$\mathcal{T}(\mathsf{P}_4) = 1 \times 2 + 3 \times 3 + 1 \times 4 = 15$$

$$\mathcal{A}(\mathsf{P}_4) = \frac{15}{5} = 3$$

NumCol

- We note $\mathcal{B}(G)$ (NumCol), the number of non equivalent colorings of a graph.
- If G is the empty graph, $\mathcal{B}(G)$ is actually the number of non-equivalent partitions of a set of order n, that is, the nth Bell number.
- Given a fixed number of colors k, $S(G, k)$ is the number of non-equivalent colorings of a graph G using exactly k colors.



AvCol

- We note $\mathcal{T}(G)$ (TotCol), the total number of colors in the non-equivalent colorings of G.

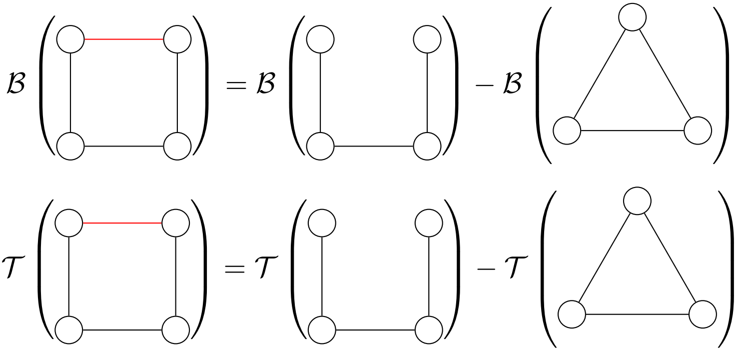
$$\mathcal{T}(G) = \sum_{k=1}^{n} kS(G,k)$$

- $\mathcal{A}(G)$ (AvCol) is the average number of colors in the non-equivalent colorings of G.

$$\mathcal{A}(G) = \frac{\mathcal{T}(G)}{\mathcal{B}(G)}$$

Computing AvCol

- Given a graph G and an edge uv (not) in G, $G - \{uv\}$ $(G + \{uv\})$ is the graph obtained from G by removing (adding) uv. The graph $G_{|uv}$ is obtained by contracting the (non-)edge uv.



Known values

- $\mathcal{A}(\mathrm{K}_n) = n$
- $\mathcal{A}(\overline{\mathrm{K}}_n) = \frac{\mathrm{B}_{n+1} - \mathrm{B}_n}{\mathrm{B}_n}$
- Let T be a tree on n vertices, $\mathcal{A}(T) = \frac{\mathcal{T}(T)}{\mathcal{B}(T)} = \frac{\mathrm{B}_n}{\mathrm{B}_{n-1}}$

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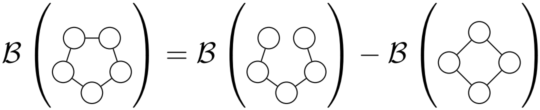

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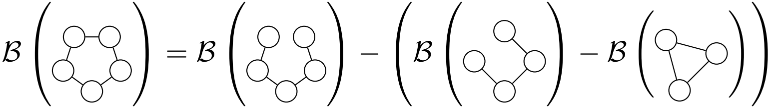

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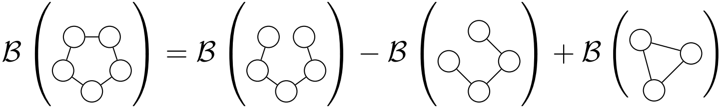

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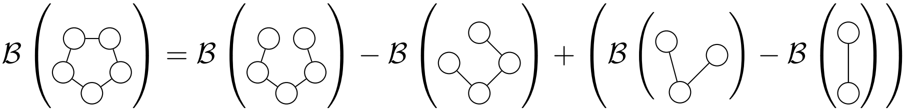

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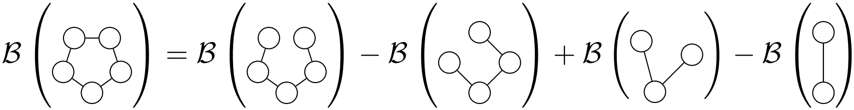

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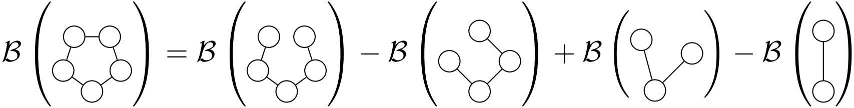

- If $n \geq 3$, $\mathcal{A}(\mathrm{C}_n) = \frac{\sum_{i=2}^{n} (-1)^{n-i} \mathcal{T}(\mathrm{P}_i)}{\sum_{i=2}^{n} (-1)^{n-i} \mathcal{B}(\mathrm{P}_i)}$

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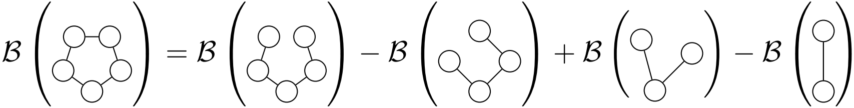

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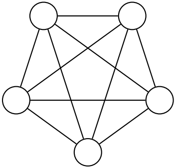

Upper bound with fixed maximum degree

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Conjecture

Let G be a graph with n vertices and maximum degree $\Delta(G) \neq 2$.

$$\mathcal{A}(G) \leq \mathcal{A}\left(\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor \mathsf{K}_{\Delta(G)+1} \cup \mathsf{K}_{n \bmod (\Delta(G)+1)}\right)$$

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- We tested all graphs of order $n \leq 12$ (166 122 463 891 graphs) and found only one counter-example :

$$\mathcal{A}(2\mathrm{K}_4 \cup \mathrm{K}_2) < \mathcal{A}(\overline{\mathrm{C}}_6 \cup \mathrm{K}_4)$$

Upper bound when $\Delta(G) = 2$

Let U_n ($n \geq 3$) be the following graph.

$$\mathrm{U}_n = \begin{cases} \frac{n}{3}\mathrm{K}_3 & \text{if } n \bmod 3 = 0, \text{ and } n \geq 3, \\ \frac{n-1}{3}\mathrm{K}_3 \cup \mathrm{K}_1 & \text{if } n = 4 \text{ or } n = 7, \\ \frac{n-4}{3}\mathrm{K}_3 \cup \mathrm{C}_4 & \text{if } n \bmod 3 = 1, \text{ and } n \geq 10, \\ \frac{n-5}{3}\mathrm{K}_3 \cup \mathrm{C}_5 & \text{if } n \bmod 3 = 2, \text{ and } n \geq 5. \end{cases}$$

Theorem

Let G be a graph of order $n \geq 3$ and maximum degree $\Delta(G) = 2$, then,

$$\mathcal{A}(G) \leq \mathcal{A}(\mathrm{U}_n)$$

with equality only if $G \simeq \mathrm{U}_n$.

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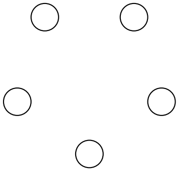
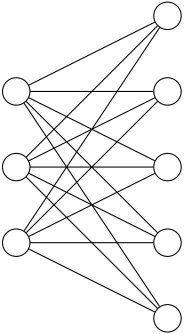
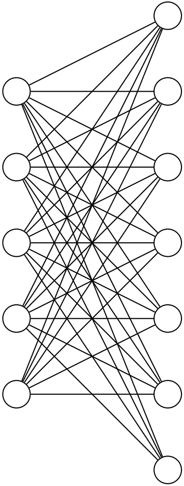


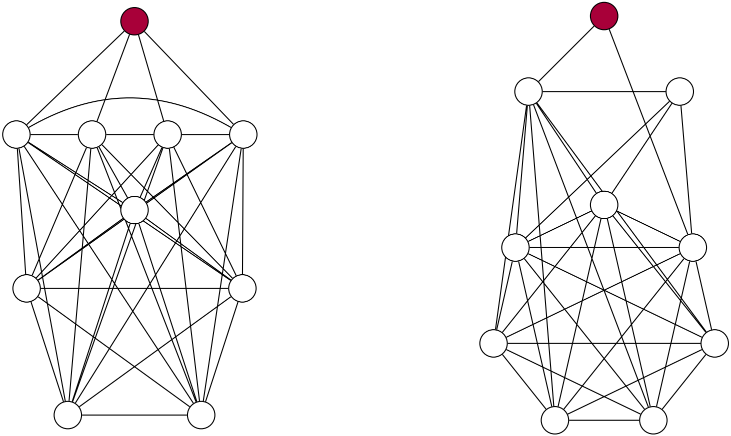
Removing an edge

- Any edge removal in these graphs strictly increases $\mathcal{A}(G)$.

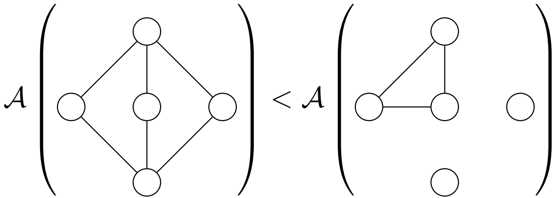



Isolating a vertex

- Removing all incident edges to the red vertices will not decrease $\mathcal{A}(G)$.



Adding isolated vertices



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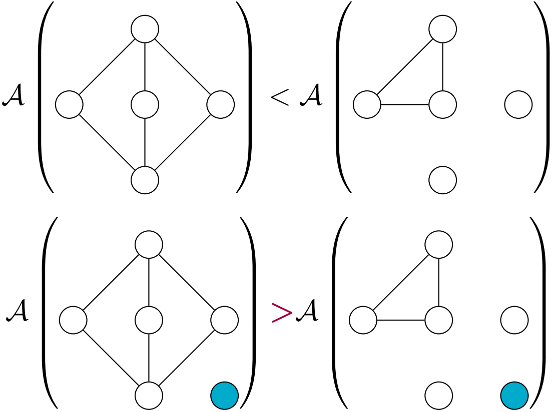

Some properties

- If v is a dominant vertex of G, $\mathcal{A}(G) = \mathcal{A}(G - v) + 1$.
- If v is a vertex of degree at most 4 or a simplicial vertex, $\mathcal{A}(G) > \mathcal{A}(G - v)$.
- If v is a simplicial vertex and u is a neighbor of v, $\mathcal{A}(G) > \mathcal{A}(G - uv)$.

Sketch of proof for chordal graphs - Fixed $\chi(G)$

- If G is a chordal graph, $\mathcal{A}(G) \geq \mathcal{A}(\mathrm{K}_{\chi(G)} \cup (n - \chi(G))\mathrm{K}_1)$.

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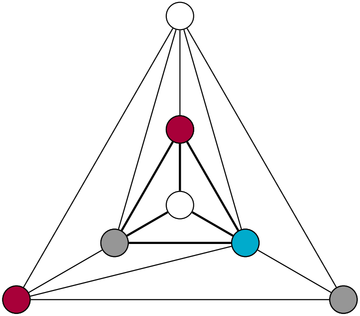

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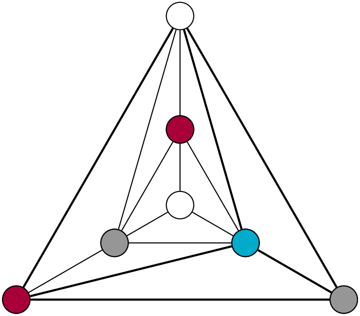

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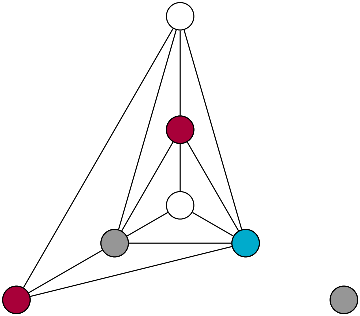
- We can reduce the graph to $Ki_{n,\chi(G)}$ by removing repeatedly removing an edge adjacent to a simplicial vertex.



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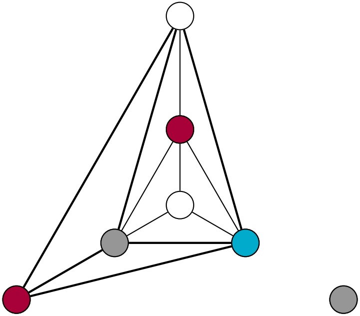

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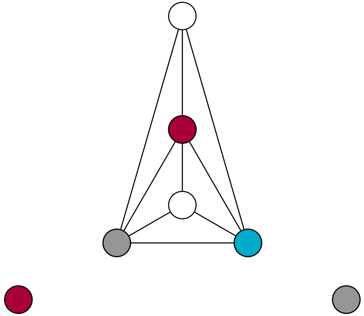

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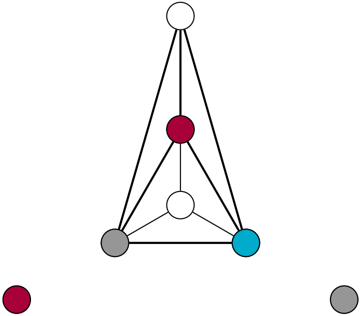

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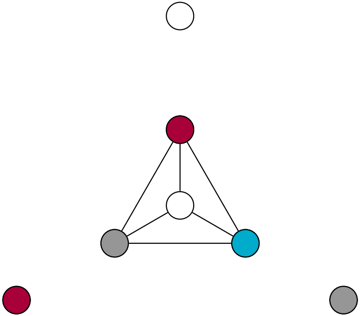

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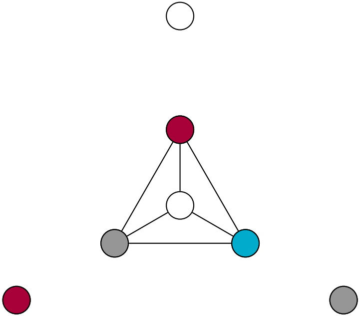

$Ki_{7,4}$

Lower bound with fixed chromatic number

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Conjecture

Let G be a graph with n vertices.

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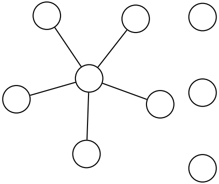
Lower bound with fixed maximum degree



$Si_{9,6}$

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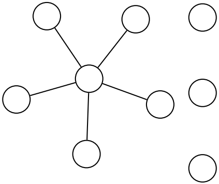

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- If $\Delta(G) \in \{1, 2, n-1\}$ or G is a chordal graph, $\mathcal{A}(G) \geq \mathcal{A}(Si_{n,\Delta(G)})$.

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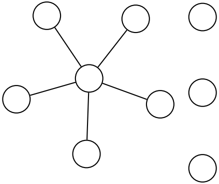

$Si_{9,6}$

- $\mathcal{A}(\overline{\mathsf{K}}_n) \leq \mathcal{A}(Si_{n,m})$.
- If $\Delta(G) \in \{1, 2, n-1\}$ or G is a chordal graph, $\mathcal{A}(G) \geq \mathcal{A}(Si_{n,\Delta(G)})$.

Conjecture

Let G be a graph with n vertices.

$$\mathcal{A}(G) \geq \mathcal{A}(Si_{n,\Delta(G)})$$

Summary of open problems

- Is it true that $\mathcal{A}(G) \leq \mathcal{A}\left(\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor \mathrm{K}_{\Delta(G)+1} \cup \mathrm{K}_{n \bmod (\Delta(G)+1)}\right)$?
- Is it true that $\mathcal{A}(\overline{\mathrm{K}}_n) \leq \mathcal{A}(G)$?
- Is it true that $\mathcal{A}(G) \geq \mathcal{A}(Ki_{n,\chi(G)})$?
- Is it true that $\mathcal{A}(G) \geq \mathcal{A}(Si_{n,\Delta(G)+1})$?
- Let v be a vertex of a graph G, is it true that $\mathcal{A}(G) > \mathcal{A}(G - v)$?

Bibliography I

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Colorings

- We consider simple undirected graphs.

Definition 2

A vertex-coloring of a graph is an assignment of colors to its vertices. A proper coloring is a coloring such that adjacent vertices have different colors.

- A coloring induces a partition of the vertices of a graph.
- We say that two colorings are equivalent if they induce the same partition of the vertices.

Bell numbers

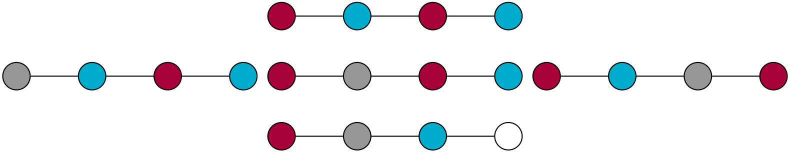
Definition 3

The *n^{th} Bell number* (B_n) is the number of non-equivalent partitions of a set with n elements.

- If the set is the set of vertices of a graph, we could extend this definition by forbidding adjacent vertices from being in the same partition.
- This is the number of non-equivalent proper colorings of a graph or NumCol ($\mathcal{B}(G)$).
- We can also define the number of non-equivalent proper colorings of a graph with k colors as $S(G,k)$. Thus, $\mathcal{B}(G) = \sum_{k=1} nS(G,k)$.

Avcol

- We write $\mathcal{T}(G) = \sum_{k=1}^{n} S(G,k)$ the total number of colors in the non-equivalent colorings of a graph G.
- We define $\mathcal{A}(G) = \frac{\mathcal{T}(G)}{\mathcal{B}(G)}$ as the average number of colors in the non-equivalent colorings of G.



- $\mathcal{B}(\mathrm{P}_4) = 5$, $\mathcal{T}(\mathrm{P}_4) = 15$, $\mathcal{A}(\mathrm{P}_4) = \frac{15}{5} = 3$