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Linear connections with a propagating spin-3 field in gravity

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We show that Fronsdal's Lagrangian for a free massless spin-3 gauge field in Minkowski spacetime is contained in a general Yang-Mills–like Lagrangian of metric-affine gravity (MAG), the gauge theory of the general affine group in the presence of a metric. Because of the geometric character of MAG, this can best be seen by using Vasiliev's frame formalism for higher-spin gauge fields in which the spin-3 frame is identified with the *tracefree nonmetricity* one-form associated with the shear generators of $GL(n, \mathbb{R})$. Furthermore, for specific gravitational gauge models in the framework of full nonlinear MAG, exact solutions are constructed, featuring propagating massless and massive spin-3 fields.

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I. INTRODUCTION

Metric-affine gravity (MAG, see [1] for a review) constitutes a rich and natural framework for the study of gravitational phenomena at high energy, when spacetime is expected to lose its Riemannian character. Thanks to its geometric formulation, it is also a promising candidate theory for the unification of gravity with the other fundamental forces based on Yang-Mills–like actions for internal gauge groups. Its spacetime can be seen as a generalization of the spacetime of Weyl's unified theory of gravitation and electromagnetism [2]. There, spacetime is described by a manifold in which not only the direction but also the norm of vectors are affected by parallel transport, thereby providing a ''true infinitesimal geometry.'' By adopting a metric-affine spacetime (L_n, g), instead of the usual Riemannian spacetime V_n of Einstein's general relativity, one naturally extends the latter by introducing torsion T and nonmetricity Q, besides the Levi-Civita connection, still conserving a classical, smooth spacetime. The connection one-form Γ in (L_n, g) takes a value in the Lie algebra $gl(n, \mathbb{R})$ of the general linear group $GL(n, \mathbb{R})$, the subgroup of the affine gauge group of MAG. More precisely, a metric-affine spacetime is described by a metric $g_{\alpha\beta}$, a coframe field ϑ^α and an independent connection $\Gamma_\alpha{}^\beta$ that generally carries torsion $T^\alpha := D\vartheta^\alpha$ and nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, where D denotes the $GL(n, \mathbb{R})$-covariant exterior derivative.

The idea that the metricity condition $Q_{\alpha\beta} = 0$ may become operational at low energy after spontaneous symmetry breaking is attractive and has been investigated for some time (see [1] and references therein). That the totally symmetric piece of the nonmetricity may become massive after the spontaneous symmetry breaking of $GL(n, \mathbb{R})$ down to its Lorentz subgroup $SO(1, n-1)$, leaving the metric as a massless Goldstone field, was studied recently in [3]. There, it was suggested that this totally symmetric and traceless piece of Q should behave as a massless spin-3 gauge field at the Planck energy.

It is well known that the nonmetricity Q contains a spin-3 piece, the totally symmetric and traceless piece of Q being called `trinom` in [1]. However, it is only in the recent work [4] that this idea was taken seriously: Fronsdal's action [5] for a massless spin-3 field was written such that on shell the propagating spin-3 field coincides with `trinom`. The latter field then acquired mass by a specific Brout-Englert-Higgs (BEH) mechanism based on the spontaneous breaking $GL(n, \mathbb{R})/SO(1, n-1)$ viewed as a small part of a more general BEH mechanism by which the full diffeomorphism group $G = \mathrm{Diff}(n, R)$ is broken down to its Lorentz subgroup H. Besides the metric being regarded as a Goldstone field, specific parameters characterizing the coset space G/H were interpreted [4] as higher-spin connections, in the context of which it seemed plausible indeed to assume the Lorentz group as stability group H.

In the present work, we elaborate on the idea that the totally symmetric and traceless part of the nonmetricity could represent a massless spin-3 gauge field. After a brief review of metric-affine geometry in Sec. II, we show in Sec. III that MAG houses indeed such a field by exhibiting Fronsdal's theory for a massless spin-3 field as a subsector of linearized MAG. Because of the geometric nature of MAG, it is actually more convenient to consider Vasiliev's

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Lagrangian [6] for a massless framelike spin-3 field in Minkowski spacetime. The crucial step is to identify the traceless nonmetricity—the component of the nonmetricity which lies along the shear generator of $GL(n,\mathbb{R})$—with Vasiliev's spin-3 framelike field, thereby providing another geometrical interpretation for the latter field and showing that Fronsdal's spin-3 theory is hidden in MAG.

To take care of this observation, in Sec. IV different types of field Lagrangians of MAG and the corresponding field equations will be investigated in the sector of vanishing torsion. Our main aim is to show that the field equations of MAG provide solutions for propagating spin-3 fields.

As a first ansatz we present in (63) (Sec. IV A) a nonmetricity $Q_{\alpha\beta}$ that is pretty much adapted to describe propagating modes of the totally symmetric spin-3 field ${}^{(1)}Q_{(\alpha\beta\gamma)}$. Furthermore, we investigate particular Lagrangians to exhibit the different propagation behavior of massless as well as of massive modes.

In Sec. IV B we consider a Yang-Mills–like Lagrangian for the pure spin-3 field ${}^{(1)}Q_{\alpha\beta\gamma}$ and show that, in vacuum, this field configuration is just trivial since from a field theoretical point of view kinetic terms of the nonmetricity are missing. This reminds us of the situation in the Einstein-Cartan theory where torsion is proportional to the spin of matter, just mediating some type of contact interaction.

The situation can be improved in Sec. IV C by adding curvature dependent terms to the Lagrangian (82). Adding a Hilbert-Einstein–type Lagrangian supports the existence of massless spin-3 modes. Because of the curvature $\sim DQ_{\alpha}{}^{\beta}$ and the second field equation $\sim DR_{\alpha}{}^{\beta}$, the Bianchi identities will "freeze" out the genuine dynamical degrees of freedom of the fields. Hence, a field Lagrangian such as (91) leads still to a second field equation which is algebraic in the field strengths, cf. (94). In that case we would like to call such fields *pseudopropagating*. Provided the coupling constants will be adjusted suitably, the second field equation will be fulfilled without further constraints and the first field equation reduces to an Einstein equation with a cosmological constant in a Riemannian spacetime.

If we supplement the Lagrangian (91) with further pieces of the nonmetricity, cf. Sec. IV D, additionally massive modes can be generated, at least for particular choices of the coupling constants.

After identifying the Vasiliev field $e_{\alpha\beta\gamma}$ with the trace-free nonmetricity $\overset{\circ}{\mathcal{Q}}_{\alpha\beta\gamma}$ of MAG, it is natural to consider field Lagrangians quadratic in the strain curvature $Z_{\alpha\beta}$ yielding genuine dynamical degrees of freedom, cf. Sec. IV E. For this particular consideration a slightly modified Kerr-Schild ansatz for the nonmetricity will be considered in which the propagation will be characterized by the field ℓ, cf. (70). Consequently, this type of approach will convert the nonlinear second field equation into a *linear* partial differential equation of second order. Accordingly, we derive $\Box{}^{(1)}Q_{(\alpha\beta\gamma)} = 0$ for the components of the spin-3 field for massless modes ($\ell^2 = 0$). Observe that in general relativity this method implies that the full nonlinear Einstein tensor equals its linearized part. In this sense the Kerr-Schild ansatz leads to an "exact linearization," cf. Gürses *et al.* [7]. This linearizing property of the Kerr-Schild ansatz can also be applied successfully in MAG. We generalize the Kerr-Schild form ℓ in (158). Then field configurations with *massive* spin-3 character can also be generated. An example of such a simple toy model can be found in Sec. IV E 2.

The conclusions are outlined in Sec. V and some technical results are relegated to the appendixes.

II. METRIC-AFFINE GEOMETRY

A. Notation and conventions

In this section we will summarize shortly the main properties of an n-dimensional metric-affine spacetime. At each point of spacetime, we have a coframe ϑ^{α} spanning the cotangent space; the frame (or anholonomic) indices $\alpha, \beta, \gamma \ldots$ run over $0, 1, \ldots, n-1$. We denote local coordinates by x^i; (holonomic) coordinate indices are $i, j, k, \ldots = 0, 1, \ldots, n-1$. Most of our formalism is correct for arbitrary n. However, in this article we will mainly concentrate on $n = 4$. We can decompose the coframe with respect to a coordinate coframe according to $\vartheta^{\alpha} = e_i{}^{\alpha} dx^i$. For the frame e_{α}, spanning the tangent space, we have $e_{\alpha} = e^i{}_{\alpha}\partial_i$. If $\rfloor$ denotes the interior product, then we have the duality condition $e_{\alpha}\rfloor\vartheta^{\beta} = \delta_{\alpha}^{\beta}$. Symmetrization will be denoted by parentheses $(\alpha\beta) := \frac{1}{2}\alpha\beta + \frac{1}{2}\beta\alpha$, antisymmetrization by brackets $[\alpha\beta] := \frac{1}{2}\alpha\beta - \frac{1}{2}\beta\alpha$, and analogously for p indices with the factor $\frac{1}{p!}$; see Schouten [8]. Indices excluded from (anti)symmetrization are surrounded by vertical strokes: $(\alpha|\gamma|\beta) := \frac{1}{2}\alpha\gamma\beta + \frac{1}{2}\beta\gamma\alpha$, etc.

We assume the existence of a metric

$$g = g_{\alpha\beta}\vartheta^{\alpha} \otimes \vartheta^{\beta} \quad \text{with} \quad g_{ij} = e_i{}^{\alpha} e_j{}^{\beta} g_{\alpha\beta}. \tag{1}$$

Choosing *orthonormal* (co)frames $e_i{}^{\alpha} =^{*} \overset{\circ}{e}_i{}^{\alpha}$, we have the condition

$$g_{\alpha\beta} \overset{*}{=} g_{ij}\overset{\circ}{e}{}^i{}_{\alpha}\overset{\circ}{e}{}^j{}_{\beta} = o_{\alpha\beta} := \mathrm{diag}(-1, +1, \ldots, +1), \tag{2}$$

whereas the *holonomic* gauge is defined by $C^{\alpha} := d\vartheta^{\alpha} =^{*} 0$, that is,

$$\vartheta^{\alpha} \overset{*}{=} \delta_i^{\alpha} dx^i, \qquad e_{\alpha} \overset{*}{=} \delta_{\alpha}^i \partial_i. \tag{3}$$

When a metric is present, we can introduce the Hodge star operator $\star$. If we denote exterior products of the coframe ϑ^{α} as $\vartheta^{\alpha\beta} := \vartheta^{\alpha} \wedge \vartheta^{\beta}$, $\vartheta^{\alpha\beta\gamma} := \vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \vartheta^{\gamma}$, etc., then we can introduce, as an alternative to the theta basis, the eta basis according to

$$\eta := {}^\star 1, \qquad \eta^\alpha := {}^\star\vartheta^\alpha,$$
$$\eta^{\alpha\beta} := {}^\star\vartheta^{\alpha\beta}, \qquad \eta^{\alpha\beta\gamma} := {}^\star\vartheta^{\alpha\beta\gamma}, \text{ etc.;} \tag{4}$$

see [1,9]. This basis can be very convenient if the * is involved in formulas.

Furthermore, the manifold will be assumed to carry a metric-independent linear connection $\Gamma_\alpha{}^\beta$, see Kobayashi & Nomizu [10] or Frankel [11], that generally supports the torsion $T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$ and the nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$. Here d denotes the exterior derivative and D the $GL(n,\mathbb{R})$ gauge-covariant exterior derivative.

It is advantageous to split the connection into Riemannian and non-Riemannian parts. If we introduce the distortion one-form $N_\alpha{}^\beta$, the connection reads

$$\Gamma_\alpha{}^\beta = \tilde\Gamma_\alpha{}^\beta + N_\alpha{}^\beta. \tag{5}$$

In the following, the tilde always denotes the purely Riemannian contribution. Torsion and nonmetricity can be recovered from $N_\alpha{}^\beta$ by

$$Q_{\alpha\beta} = 2N_{(\alpha\beta)} \quad \text{and} \quad T^\alpha = N_\beta{}^\alpha \wedge \vartheta^\beta. \tag{6}$$

Explicitly, the distortion one-form $N_\alpha{}^\beta$ can be expressed in terms of torsion and nonmetricity as

$$N_{\alpha\beta} = -e_{[\alpha}\rfloor T_{\beta]} + \tfrac{1}{2}(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\vartheta^\gamma + (e_{[\alpha}\rfloor Q_{\beta]\gamma})\vartheta^\gamma + \tfrac{1}{2}Q_{\alpha\beta}. \tag{7}$$

Furthermore, it will be helpful to separate this into

$$N_{\alpha\beta} = N_{[\alpha\beta]} + \tfrac{1}{2}\overset{\circ}{Q}_{\alpha\beta} + \tfrac{1}{2}Qg_{\alpha\beta}, \tag{8}$$

with $Q := Q^\alpha{}_\alpha/n$, $\overset{\circ}{Q}_{\alpha\beta} := Q_{\alpha\beta} - Qg_{\alpha\beta}$, and $g^{\alpha\beta}\overset{\circ}{Q}_{\alpha\beta} = 0$.

For $n = 4$, the traceless nonmetricity $\overset{\circ}{Q}_{\alpha\beta} = \overset{\circ}{Q}_{\gamma\alpha\beta}\vartheta^\gamma$ has 36 independent components that can be decomposed under $O(1,3)$ as $36 = 16 \oplus 16 \oplus 4$:

$$\overset{\circ}{Q}_{\alpha\beta} = {}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta}. \tag{9}$$

Then, we have the following irreducible decomposition of the components of the nonmetricity one-form $Q_{\alpha\beta} = Q_{\gamma\alpha\beta}\vartheta^\gamma$ with respect to the (pseudo)orthogonal group, cf. [1,12],

$$\underbrace{Q_{\alpha\beta}}_{\square\otimes\square\!\square} = \underbrace{{}^{(1)}Q_{\alpha\beta}}_{\text{spin 3}} \oplus \underbrace{{}^{(2)}Q_{\alpha\beta}}_{\text{spin 2}} \oplus \underbrace{{}^{(3)}Q_{\alpha\beta}}_{\text{spin 1}} \oplus \underbrace{{}^{(4)}Q_{\alpha\beta}}_{\text{spin 1}}, \tag{10}$$

where we have marked the *leading* spin content of the fields. We have also given the decomposition of the $GL(n,\mathbb{R})$-reducible components $Q_{\gamma\alpha\beta}$ into irreducible representations of the (pseudo)orthogonal group, so that the Young diagrams on the right-hand side of the above equality label $O(1, n-1)$-irreducible representations. (Note the multiplicity 2 of the irreducible vector representation.) The names of our corresponding computer macros are $Q_{\alpha\beta} =$ `trinom` + `binom` + `vecnom` + `conom`. Defining $\Lambda_\alpha := e^\beta\rfloor\overset{\circ}{Q}_{\alpha\beta}$, we have explicitly

$$\begin{aligned}{}^{(1)}Q_{\alpha\beta} &= \left[\overset{\circ}{Q}_{(\gamma\alpha\beta)} - \frac{2}{n+2}\Lambda_{(\gamma}g_{\alpha\beta)}\right]\vartheta^\gamma \\ &= \left[Q_{(\gamma\alpha\beta)} - \frac{2}{n+2}\Lambda_{(\gamma}g_{\alpha\beta)} - Q_{(\gamma}g_{\alpha\beta)}\right]\vartheta^\gamma,\end{aligned} \tag{11}$$

$${}^{(2)}Q_{\alpha\beta} = \frac{2}{3}\left[\overset{\circ}{Q}_{\gamma\alpha\beta} - \overset{\circ}{Q}_{(\alpha\beta)\gamma} + \frac{1}{n-1}(\Lambda_\gamma g_{\alpha\beta} - \Lambda_{(\alpha}g_{\beta)\gamma})\right]\vartheta^\gamma, \tag{12}$$

$${}^{(3)}Q_{\alpha\beta} = \frac{2n}{(n-1)(n+2)}\left[\Lambda_{(\alpha}g_{\beta)\gamma} - \frac{1}{n}\Lambda_\gamma g_{\alpha\beta}\right]\vartheta^\gamma, \tag{13}$$

$${}^{(4)}Q_{\alpha\beta} = g_{\alpha\beta}Q_\gamma\vartheta^\gamma. \tag{14}$$

The irreducible part ${}^{(1)}Q_{\alpha\beta} = {}^{(1)}Q_{\gamma\alpha\beta}\vartheta^\gamma$ (`trinom`) corresponds to the *totally symmetric* piece ${}^{(1)}Q_{\gamma\alpha\beta} = {}^{(1)}Q_{(\gamma\alpha\beta)}$ of the nonmetricity in which the *traces* have been *subtracted out*,

$$\begin{aligned}{}^{(1)}Q_{\gamma\alpha\beta} &\overset{n=4}{=} \overset{\circ}{Q}_{(\gamma\alpha\beta)} - \tfrac{1}{3}\Lambda_{(\gamma}g_{\alpha\beta)} \\ &= Q_{(\gamma\alpha\beta)} - \tfrac{1}{3}g_{(\gamma\alpha}(3Q_{\beta)} + \Lambda_{\beta)}).\end{aligned} \tag{15}$$

The tracelessness of ${}^{(1)}Q_{\gamma\alpha\beta}$ means $g^{\alpha\beta(1)}Q_{\gamma\alpha\beta} = 0$ and $g^{\gamma\alpha(1)}Q_{\gamma\alpha\beta} = 0$. The second term on the right-hand side of (15) takes care of $g^{\alpha\beta}Q_{\gamma\alpha\beta} = nQ_\gamma$ and the third term of $g^{\gamma\alpha}\overset{\circ}{Q}_{\gamma\alpha\beta} = g^{\gamma\alpha}\overset{\circ}{Q}_{\gamma\beta\alpha} = \Lambda_\beta$. The totally symmetric piece $Q_{(\gamma\alpha\beta)}$ plays an important role in the recent gravitational theory proposed in [4]. In the following we will focus on the properties of ${}^{(1)}Q_{\alpha\beta}$, which, as we have seen, carries leading spin 3.

The curvature two-form is defined by $R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta = \frac{1}{2}R_{\gamma\delta\alpha}{}^\beta\vartheta^\gamma \wedge \vartheta^\delta$. Associated with it is the Ricci one-form $\mathrm{Ric}_\alpha := e_\beta\rfloor R_\alpha{}^\beta = \mathrm{Ric}_{\gamma\alpha}\vartheta^\gamma$. Then the components of the Ricci tensor read $\mathrm{Ric}_{\alpha\beta} = R_{\gamma\alpha\beta}{}^\gamma$. The Einstein $(n-1)$-form is given by $G_\alpha := \frac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma}$.

With the help of (5) and (6) we can decompose the total curvature $R_\alpha{}^\beta$ into Riemannian and post-Riemannian pieces:

$$R_\alpha{}^\beta = \tilde R_\alpha{}^\beta + \tilde D N_\alpha{}^\beta - N_\alpha{}^\gamma \wedge N_\gamma{}^\beta. \tag{16}$$

In a metric-affine spacetime, the curvature two-form can be split into a symmetric (strain) piece $Z_{\alpha\beta} := R_{(\alpha\beta)}$ and an antisymmetric (rotational) piece $W_{\alpha\beta} := R_{[\alpha\beta]}$:

$$R_{\alpha\beta} = Z_{\alpha\beta} + W_{\alpha\beta}. \tag{17}$$

In turn, from $Z_{\alpha\beta}$, we can subtract out the trace $Z := Z_\alpha{}^\alpha$ and arrive thereby at the shear curvature

$$\mathcal{Z}_{\alpha\beta} := Z_{\alpha\beta} - \frac{1}{n} Z g_{\alpha\beta}, \qquad \mathcal{Z}_\alpha{}^\alpha = 0. \tag{18}$$

The Einstein $(n-1)$-form depends only on the rotational curvature:

$$\begin{aligned} G_\alpha &= \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{[\beta\gamma]} = \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge W^{\beta\gamma} \\ &= \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge ({}^{(4)}W^{\beta\gamma} + {}^{(5)}W^{\beta\gamma} + {}^{(6)}W^{\beta\gamma}). \end{aligned} \tag{19}$$

If we decompose G_α with respect to the $(n-1)$-form basis η^β, namely $G_\alpha = G_{\alpha\beta}\eta^\beta$, then the $G_{\alpha\beta}$ denote the components of the Einstein tensor and $G_{\alpha\beta} = W_{\gamma\alpha\beta}{}^\gamma - \frac{1}{2} g_{\alpha\beta} W_{\gamma\delta}{}^{\delta\gamma}$.

In analogy to the Ricci one-form, we can define a Ricci-type one-form (the "Rizzi" one-form) for $Z_\alpha{}^\beta$ and $\mathcal{Z}_\alpha{}^\beta$, respectively:

$$\mathrm{Riz}_\alpha := e_\beta \rfloor Z_\alpha{}^\beta \quad \text{and} \quad \mathcal{R}\mathrm{iz}_\alpha := e_\beta \rfloor \mathcal{Z}_\alpha{}^\beta. \tag{20}$$

In components, we have $\mathrm{Riz}_{\alpha\beta} = Z_{\gamma\alpha\beta}{}^\gamma$ and $\mathcal{R}\mathrm{iz}_{\alpha\beta} = \mathcal{Z}_{\gamma\alpha\beta}{}^\gamma$.

The zeroth Bianchi identity

$$DQ_{\alpha\beta} \equiv 2Z_{\alpha\beta} \tag{21}$$

links the nonmetricity to the strain curvature. After some reordering (see Appendix B), we can isolate a purely Riemannian covariant derivative according to

$$\tilde{D}\mathcal{Q}_{\alpha\beta} - N_{[\alpha\gamma]} \wedge \mathcal{Q}^\gamma{}_\beta - N_{[\beta\gamma]} \wedge \mathcal{Q}_\alpha{}^\gamma = 2\mathcal{Z}_{\alpha\beta}. \tag{22}$$

Note that, in the case of $N_{[\alpha\beta]} = 0$, the shear curvature is completely determined by the Riemannian exterior covariant derivative of the tracefree nonmetricity.

B. Field equations

The field equations of MAG have been derived in a first-order Lagrangian formalism where the geometrical variables $\{g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta\}$ are minimally coupled to matter fields, collectively denoted Ψ, such that the total Lagrangian, i.e., the geometrical part V plus the matter part L_{matter}, results in

$$\begin{aligned} L_{\mathrm{total}} &= V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) \\ &\quad + L_{\mathrm{matter}}(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi). \end{aligned} \tag{23}$$

Using the definitions of the excitations,

$$\begin{aligned} M^{\alpha\beta} &= -2\frac{\partial V}{\partial Q_{\alpha\beta}}, \qquad H_\alpha = -\frac{\partial V}{\partial T^\alpha}, \\ H^\alpha{}_\beta &= -\frac{\partial V}{\partial R_\alpha{}^\beta}, \end{aligned} \tag{24}$$

the field equations of metric-affine gravity can be expressed in a very concise form [1]:

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta} \qquad (\delta/\delta g_{\alpha\beta}), \tag{25}$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha \qquad (\delta/\delta\vartheta^\alpha), \tag{26}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta = \Delta^\alpha{}_\beta \qquad (\delta/\delta\Gamma_\alpha{}^\beta), \tag{27}$$

$$\frac{\delta L}{\delta \Psi} = 0 \qquad \text{(matter)}. \tag{28}$$

As a side remark, we discuss shortly the type of matter that couples directly to the nonmetricity $Q_{\alpha\beta}$; see also [13]. If we go over from the original geometrical variables $g_{\alpha\beta}$, ϑ^α, $\Gamma_\alpha{}^\beta$ to the alternative variables $g_{\alpha\beta}$, ϑ^α, T^α, $Q_{\alpha\beta}$, then, with the help of Lagrangian multipliers (see [1]), we find as a response to the variation of the torsion T^α and the nonmetricity $Q_{\alpha\beta}$

$$\delta L_{\mathrm{matter}} = \cdots + \delta T^\alpha \wedge \mu_\alpha + \tfrac{1}{2}\delta Q_{\alpha\beta} \wedge \Xi^{\alpha\beta}. \tag{29}$$

Here the dots subsume the variations with respect to $g_{\alpha\beta}$ and ϑ^α. Hence, for the hypermomentum with its definition $\delta L_{\mathrm{matter}} = \cdots + \delta\Gamma_\alpha{}^\beta \wedge \Delta^\alpha{}_\beta$, we get

$$\Delta^\alpha{}_\beta = \vartheta^\alpha \wedge \mu_\beta + \Xi^\alpha{}_\beta, \tag{30}$$

where $\tau_{\alpha\beta} := \Delta_{[\alpha\beta]}$ is the spin current and the strain-type current $\Xi^\alpha{}_\beta$ is symmetric: $\Xi_{\alpha\beta} = \Xi_{\beta\alpha}$. In a hydrodynamic representation (see Obukhov and Tresguerres [14]), a convective ansatz for the strain-type current reads $\Xi_{\alpha\beta} = \xi_{\alpha\beta}(v\rfloor\eta)$, where $v = v^\alpha e_\alpha$ is the velocity of the fluid and η the volume n-form; moreover, $\xi_{\alpha\beta} = \xi_{\beta\alpha}$. Accordingly, it is the *material strain-type current* $\Xi^{\alpha\beta}$ that couples to the *nonmetricity* $Q_{\alpha\beta}$. More specifically, the dilation current $\Delta^\gamma{}_\gamma$ couples to the Weyl covector Q and the shear-type current $\mathcal{Z}^{\alpha\beta} := \Xi^{\alpha\beta} - \frac{1}{n} g^{\alpha\beta}\Xi^\gamma{}_\gamma$ to the tracefree nonmetricity $\mathcal{Q}_{\alpha\beta}$.

On the right-hand sides of each of the three gauge field equations (25)–(27), we identify the material currents as sources; on the left-hand side there are typical Yang-Mills–like terms governing the gauge fields, their first derivatives, and the corresponding nonlinear gauge field currents. These gauge currents turn out to be the metrical (Hilbert) energy-momentum of the gauge fields

$$\begin{aligned} m^{\alpha\beta} &:= 2\frac{\partial V}{\partial g_{\alpha\beta}} \\ &= \vartheta^{(\alpha} \wedge E^{\beta)} + Q^{(\beta}{}_\gamma \wedge M^{\alpha)\gamma} - T^{(\alpha} \wedge H^{\beta)} \\ &\quad - R_\gamma{}^{(\alpha} \wedge H^{|\gamma|\beta)} + R^{(\beta|\gamma|} \wedge H^{\alpha)}{}_\gamma, \end{aligned} \tag{31}$$

the canonical (Noether) energy-momentum of the gauge fields

$$\begin{aligned} E_\alpha &:= \frac{\partial V}{\partial \vartheta^\alpha} \\ &= e_\alpha\rfloor V + (e_\alpha\rfloor T^\beta) \wedge H_\beta + (e_\alpha\rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma \\ &\quad + \frac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma}) M^{\beta\gamma}, \end{aligned} \tag{32}$$

and the hypermomentum of the gauge fields

$$E^{\alpha}{}_{\beta} := \frac{\partial V}{\partial \Gamma_{\alpha}{}^{\beta}} = -\vartheta^{\alpha} \wedge H_{\beta} - g_{\beta\gamma} M^{\alpha\gamma}, \qquad (33)$$

respectively.

The *most general* parity-conserving MAG Lagrangian, at most bilinear in $\{Q_{\alpha\beta}, T^{\alpha}, R^{\alpha}{}_{\beta}\}$, has been investigated by Esser [15] and reads

$$\begin{aligned} V_{\mathrm{MAG}} = \frac{1}{2\kappa}\Big[& -a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda_0 \eta + T^{\alpha} \wedge {}^{\star}\Big(\sum_{I=1}^{3} a_I{}^{(I)} T_{\alpha}\Big) + Q_{\alpha\beta} \wedge {}^{\star}\Big(\sum_{I=1}^{4} b_I{}^{(I)} Q^{\alpha\beta}\Big) + 2\Big(\sum_{I=2}^{4} c_I{}^{(I)} Q_{\alpha\beta}\Big) \wedge \vartheta^{\alpha} \wedge {}^{\star}T^{\beta} \\ & + b_5({}^{(3)}Q_{\alpha\gamma} \wedge \vartheta^{\alpha}) \wedge {}^{\star}({}^{(4)}Q^{\beta\gamma} \wedge \vartheta_{\beta})\Big] - \frac{1}{2\rho} R^{\alpha\beta} \wedge {}^{\star}\Big[\sum_{I=1}^{6} w_I{}^{(I)} W_{\alpha\beta} + \sum_{I=1}^{5} z_I{}^{(I)} Z_{\alpha\beta} + w_7 \vartheta_{\alpha} \wedge (e_{\gamma}\rfloor^{(5)} W^{\gamma}{}_{\beta}) \\ & + z_6 \vartheta_{\gamma} \wedge (e_{\alpha}\rfloor^{(2)} Z^{\gamma}{}_{\beta}) + \sum_{I=7}^{9} z_I \vartheta_{\alpha} \wedge (e_{\gamma}\rfloor^{(I-4)} Z^{\gamma}{}_{\beta})\Big]. \end{aligned} \qquad (34)$$

One should also consult Refs. [12,15–18] and the literature quoted there.

Here κ is the dimensionful "weak" Newton-Einstein gravitational constant, λ_0 the "bare" cosmological constant, and ρ the dimensionless "strong" gravity coupling constant. The constants $a_0, \ldots, a_3$, $b_1, \ldots, b_5$, c_2, c_3, c_4, $w_1, \ldots, w_7$, $z_1, \ldots, z_9$ are dimensionless and give a weight for the different contributions of each linearly independent term entering the Lagrangian. Actually, we will not consider the complete Lagrangian (34). Instead, we choose a simplified version with

$$w_7 = z_6 = z_7 = z_8 = z_9 = 0 \qquad (35)$$

whose effect is to decouple $Z^{\alpha}{}_{\beta}$ from $W^{\alpha}{}_{\beta}$ in the Lagrangian. Taking (36) into account, the various excitations $\{M^{\alpha\beta}, H_{\alpha}, H^{\alpha}{}_{\beta}\}$ are found to be

$$\begin{aligned} M^{\alpha\beta} = & -\frac{2}{\kappa} {}^{\star}\Big(\sum_{I=1}^{4} b_I^{(I)} Q^{\alpha\beta}\Big) - \frac{2}{\kappa}\Big[c_2 \vartheta^{(\alpha} \wedge {}^{\star(1)}T^{\beta)} \\ & + c_3 \vartheta^{(\alpha} \wedge {}^{\star(2)}T^{\beta)} + \frac{1}{4}(c_3 - c_4) {}^{\star}T g^{\alpha\beta}\Big] \\ & - \frac{b_5}{\kappa}\Big[\vartheta^{(\alpha} \wedge {}^{\star}(Q \wedge \vartheta^{\beta)}) - \frac{1}{4} g^{\alpha\beta} {}^{\star}(3Q + \Lambda)\Big], \end{aligned} \qquad (36)$$

$$H_{\alpha} = -\frac{1}{\kappa} {}^{\star}\Big(\sum_{I=1}^{3} a_I{}^{(I)} T_{\alpha} + \sum_{K=2}^{4} c_K{}^{(K)} Q_{\alpha\beta} \wedge \vartheta^{\beta}\Big), \qquad (37)$$

$$H^{\alpha}{}_{\beta} = \frac{a_0}{2\kappa} \eta^{\alpha}{}_{\beta} + \sum_{I=1}^{6} w_I {}^{\star(I)} W^{\alpha}{}_{\beta} + \sum_{K=1}^{5} z_K {}^{\star(K)} Z^{\alpha}{}_{\beta}. \qquad (38)$$

The general structure of the excitations can be found in [1]; compare also [15].

III. MASSLESS SPIN-3 THEORY IN MAG

In this section, we show that, as was expected from the decomposition of $Q_{\alpha\beta}$, the action of MAG in the free limit and in Minkowski spacetime indeed incorporates Fronsdal's action for a massless spin-3 gauge field, the latter field being dynamically represented by ${}^{(1)}Q_{\alpha\beta}$.

As was first shown by Fronsdal [5] in 1978, a massless integer-spin gauge field in Minkowski spacetime is described by a totally symmetric tensor $h_{i_1 \ldots i_s}$ subject to the double tracelessness condition (for $s \geq 4$) $o^{i_1 i_2} o^{i_3 i_4} h_{i_1 \ldots i_s} = 0$. A quadratic Lagrangian for a free spin-s field is fixed unambiguously in the form $L^s = h \vec{L} h$ (where $\vec{L}$ is some second-order differential operator) by the requirement of gauge invariance under the Abelian gauge transformations $\delta h_{i_1 \ldots i_s} = s \partial_{(i_1} \lambda_{i_2 \ldots i_s)}$. The gauge parameters $\lambda_{i_1 \ldots i_{s-1}}$ are rank-$(s-1)$ totally symmetric traceless tensors, with $\lambda_{i_1 \ldots i_{s-1}} = \lambda_{(i_1 \ldots i_{s-1})}$ and $o^{i_1 i_2} \lambda_{i_1 \ldots i_{s-1}} = 0$. This formulation is parallel [19] to the metric formulation of gravity.

In 1980, in view of extending supergravity theories by the addition of high-spin gauge fields, Vasiliev proposed a framelike reformulation of Fronsdal's theory by using generalized vielbeins and spin connections [6].

In the next subsections, we briefly review Fronsdal's and Vasiliev's approaches for the massless spin-3 gauge field. Both approaches will be needed when showing the occurrence of a massless spin-3 sector in MAG.

A. Massless spin-3 field in Fronsdal's approach

The action given in [5] for a totally symmetric massless spin-3 gauge field $h_{ijk} = h_{(ijk)}$ in Minkowski spacetime reads

$$\begin{aligned} S[h_{ijk}] = -\frac{1}{2} \int d^n x \Big[& \partial_{\ell} h_{ijk} \partial^{\ell} h^{ijk} - 3 \partial_j h^{\ell}{}_{\ell i} \partial^j h_k{}^{ki} \\ & + 6 \partial_j h^{\ell}{}_{\ell i} \partial_k h^{jki} - 3 \partial_j h^j{}_{ik} \partial_{\ell} h^{\ell ik} \\ & - \frac{3}{2} \partial^j h^{\ell}{}_{\ell j} \partial_i h^k{}_k{}^i \Big]. \end{aligned} \qquad (39)$$

It is invariant under the gauge transformations

$$\delta h_{ijk} = 3 \partial_{(i} \lambda_{jk)}, \qquad \lambda_{ij} = \lambda_{(ij)}, \qquad o^{ij} \lambda_{ij} = 0. \qquad (40)$$

The corresponding source-free field equations are equivalent to

$$\mathcal{F}_{ijk} := \Box h_{ijk} - 3\partial^{\ell}\partial_{(i}h_{jk)\ell} + 3\partial_{(i}\partial_{j}h_{k)\ell}{}^{\ell} = 0. \quad (41)$$

It is possible to reach the harmonic gauge

$$D_{jk} := \partial^{i}h_{ijk} - \partial_{(j}h_{k)i}{}^{i} = 0, \qquad \delta D_{jk} = \Box\lambda_{jk} \quad (42)$$

in which the field equations take the canonical massless Klein-Gordon form $\Box h_{ijk} = 0$. By a residual gauge transformation with parameter $\bar{\lambda}_{ij}$ obeying $\Box\bar{\lambda}_{ij} = 0$, it is possible to set the trace of the gauge field to zero, yielding

$$\Box h_{ijk} = 0, \qquad \partial^{i}h_{ijk} = 0, \qquad o^{ij}h_{ijk} = 0. \quad (43)$$

Actually, some residual gauge transformations $\delta h_{ijk} = 3\partial_{(i}\tilde{\lambda}_{jk)}$ are still allowed in (43). As shown in [19], this gauge theory leads to the correct number of physical degrees of freedom, that is, to the dimension of the irreducible representation of the little group $O(n-2)$ corresponding to the one-row Young diagram of length $s = 3$.

The counting of physical degrees of freedom can also be done by using the gauge-invariant spin-3 Weinberg tensor $\mathcal{K}$ [20] (see also [19]) which is the projection of $\partial_i\partial_k\partial_m h_{n\ell j}$ on the tensor field irreducible under $GL(n,\mathbb{R})$ with symmetries labeled by the Young tableau

i	k	m
j	ℓ	n

.

Since $\partial_i\partial_k\partial_m h_{n\ell j}$ is already symmetric in all indices of the two rows of the above Young tableau, it only remains to antisymmetrize over the three pairs $(ij, k\ell, mn)$. This corresponds to taking 3 curls of the symmetric tensor field $h_{n\ell j}$ and yields a curvaturelike tensor

$$\mathcal{K}_{ijk\ell mn} := 8\partial_{[i|}\partial_{[k|}\partial_{[m}h_{n]|\ell]|j]}. \quad (44)$$

In fact, the source-free Fronsdal equations (41) imply the Ricci-flat–like equations

$$\mathcal{F} = 0 \quad \Rightarrow \quad \mathrm{Tr}\mathcal{K} = 0 \quad \Leftrightarrow \quad o^{ik}\mathcal{K}_{ijk\ell mn} = 0. \quad (45)$$

Conversely, it was shown in [21] that the Ricci-flat–like equations $\mathrm{Tr}\mathcal{K} = 0$ imply[1] the Fronsdal equations $\mathcal{F} = 0$. This was obtained by combining various former results [23,26,27]. Using the definition of $\mathcal{K}$, the equations (45) give the following set of first-order field equations:

$$\begin{cases} \partial_{[i}\mathcal{K}_{jk]\ell mno} = 0, \\ \partial^{i}\mathcal{K}_{ijk\ell mn} = 0, \end{cases} \qquad \text{where } \mathrm{Tr}\mathcal{K} = 0. \quad (46)$$

When $n = 4$, the above equations correspond to the (spin-3) Bargmann-Wigner equations [28], originally expressed in terms of two-component tensor spinors in the representation $(3,0)\oplus(0,3)$ of $SL(2,\mathbb{C})$. See also [26] for a careful analysis of Fronsdal's spin-3 gauge theory using the Weinberg tensor $\mathcal{K}$ (denoted R_6 in [26]).

In the massless spin-1 case, the Bargmann-Wigner equations read

$$\begin{cases} \partial_{[i}F_{jk]} = 0, \\ \partial^{i}F_{ij} = 0, \end{cases} \quad (47)$$

which are nothing but the source-free Maxwell equations. They imply $\Box F = 0$ and $F = dA$, where, as usual, $F = \frac{1}{2}F_{ij}dx^i\wedge dx^j$ and $A = A_i dx^i$. They are invariant under $\delta A = d\lambda$. The tensor F transforms into the representation $(1,0)\oplus(0,1)$ of $SL(2,\mathbb{C})$. One can choose the Lorentz gauge-fixing condition $\partial^i A_i = 0$ and look for solutions of the source-free Maxwell equations with the ansatz

$$A = A_i dx^i = \Phi(x)k = \Phi(x)k_i dx^i, \quad (48)$$

where k_i are the constant components of a one-form k, which is null: $k\wedge{}^\star k = 0$. We may choose the vector dual to the one-form k in the z direction: $k^i = (E,0,0,E)$. The Lorentz condition ${}^\star d{}^\star A = 0$ implies the equation $k\wedge{}^\star d\Phi = 0$, which is satisfied with $\Phi = \phi(\xi^a)e^{ik\cdot x}$ where $k\cdot x := k_i x^i = -Et + Ez$ and where $\phi(\xi^a)$ is a function of the transverse coordinates $\xi^1 = x$, $\xi^2 = y$. (Implicitly, the real component of Φ must be taken.) Then, the d'Alembertian equation $(d^\star d^\star + {}^\star d^\star d)A = 0$ is verified if $\phi(\xi^a)$ is a harmonic function in the (x,y) plane, $\Delta_{x,y}\phi(\xi^a) = 0$. The monochromatic plane-wave solution $A_i = \phi(\xi^a)k_i e^{ik\cdot x}$ displayed here characterizes an electromagnetic pure-radiation field F (also called a *null field*) since we have the vanishing of the two invariants $F\wedge F$ and $F\wedge{}^\star F$. Note also that we have $A\wedge dA = 0$, which implies by Frobenius's theorem that the vector dual to A is hypersurface orthogonal, the surface being described by the equation $\Sigma \equiv k\cdot x + \mathrm{const} = 0$.

With the pure-radiation massless spin-1 solution $F = dA$ displayed above, it is simple to construct helicity-3 plane-wave solutions of the Bargmann-Wigner equations (46):

$$\begin{aligned} &h_{ijk} = \Phi k_i k_j k_k, \qquad && k_i = (-E,0,0,E), \\ &\Phi = \phi(\xi^a)e^{ik\cdot x}, \qquad && \Delta_\xi\phi(\xi^a) = 0 = \Box\Phi. \end{aligned} \quad (49)$$

Indeed, computing the spin-3 Weinberg tensor $\mathcal{K}_{ijk\ell mn}$, we find

$$\mathcal{K}_{ijk\ell mn} = -8(k_{[i|}k_{[k|}k_{[m}\partial_{n]}\partial_{|\ell]}\partial_{|j]}\phi)e^{ik\cdot x}. \quad (50)$$

By using the properties of k and ϕ, it can be shown that the Bargmann-Wigner equations (46) are obeyed. Hence, on shell, the field strength $\mathcal{K}$ is a propagating massless helicity-3 field. It gives a representation of $SL(2,\mathbb{C})$ labeled by $(0,3)\oplus(3,0)$ and satisfies the massless Klein-Gordon equation $\Box\mathcal{K} = 0$. In the van der Waerden 2-

[1]More details, references and general results for tensor gauge fields transforming in arbitrary irreducible representations of $GL(n,\mathbb{R})$ can be found in [22]. Note that, by introducing a pure gauge field (sometimes referred to as a "compensator"), it is possible to write a local (but higher-derivative) action for spin-3 [23] that is invariant under unconstrained gauge transformations. Recently, this action was generalized to the arbitrary spin-s case by further adding an auxiliary field [24] (see also [25] for an older non "minimal" version of it).

spinor notation, the monochromatic plane-wave solution written above corresponds to a $\mathcal{K}$ that is equivalent to a totally symmetric 6-spinor with all 6 null directions coinciding. The (6 times repeated) null spinor represents the lightlike wave covector k_i, cf. [26]. Finally, note that (i) the equations (46) hold in arbitrary dimension $n > 2$ and (ii) the gauge potentials h_{ijk} given in (49) satisfy the equations (43).

Actually, we can put the plane-wave solutions (49) in exactly the same form as that found by Obukhov [29] for metric-affine gravity; see also Pasic and Vassiliev [30]. One must identify Obukhov's one-form u with our one-form $k\Phi$ and his H with our Φ, so that the nonmetricity reads $Q_{\alpha\beta} = k_\alpha k_\beta u = \Phi k_\alpha k_\beta k$. Then, as done in [29], it is straightforward to add torsion by taking $\Gamma_{\alpha\beta} = k_{[\alpha}\varphi_{\beta]}k + k_\alpha k_\beta u$, where $\varphi_\alpha = \partial_\alpha H$. Similarly, one can choose the coframe and metric as in [29], since they only depend on the function H. The only component of the curvature $W_{\alpha\beta}$ that remains is the Weyl piece ${}^{(1)}W_{\alpha\beta}$. In conclusion, with the identifications explained here, we have made the exact correspondence between our plane-wave solutions (49) and those of Obukhov [29].

B. Vasiliev's approach to a massless spin-3 field

Fronsdal's action for a massless spin-s gauge field in Minkowski spacetime was elegantly rewritten by Vasiliev [6] in a first-order framelike formalism. In the particular spin-3 case, the set of bosonic fields consists of a generalized vielbein $e_{i\alpha\beta}$ and a generalized spin connection $\omega_{i\gamma\alpha\beta}$. They obey the following algebraic identities:

$$e_{i\alpha\beta} = e_{i\beta\alpha}, \qquad o^{\alpha\beta}e_{i\alpha\beta} = 0, \qquad \omega_{i\gamma\alpha\beta} = \omega_{i\gamma\beta\alpha},$$
$$\omega_{i(\gamma\alpha\beta)} = 0, \qquad o^{\alpha\beta}\omega_{i\gamma\alpha\beta} = 0, \qquad o^{\gamma\alpha}\omega_{i\gamma\alpha\beta} = 0. \tag{51}$$

The action was originally written in four dimensions as [6]

$$S[e, \omega] = \int d^4x \varepsilon^{ijk\ell}\varepsilon_{\alpha\beta\gamma k}\omega_\ell{}^{\alpha\beta\delta}\left(\partial_i e_{j\delta}{}^\gamma - \frac{1}{2}\omega_{ij\delta}{}^\gamma\right). \tag{52}$$

As in the Einstein-Cartan theory of gravitation (see [31,32]), the connection is a nonpropagating field. One can solve the source-free field equations for $\omega_{i\gamma\alpha\beta}$ and express it in terms of the framelike field $e_{i\alpha\beta}$. Inserting the result back in the action (52) and multiplying by $1/\rho$ for further purpose, one obtains an action in second-order formalism, in a form valid in any number of spacetime dimensions,[2]

$$S[e_{\gamma\alpha\beta}] = \frac{1}{\rho}\int d^n x\left[\left(\frac{1}{4}\mathcal{Z}_{\gamma\delta\alpha\beta} - \mathcal{Z}_{\alpha\gamma\delta\beta}\right)\mathcal{Z}^{\gamma\delta\alpha\beta} - (2\mathcal{Z}_{\gamma\alpha\beta}{}^\gamma + \mathcal{Z}_{\gamma\beta\alpha}{}^\gamma)\mathcal{Z}_\delta{}^{\alpha\beta\delta}\right], \tag{53}$$

where $\mathcal{Z}_{\alpha\beta} = \frac{1}{2}\mathcal{Z}_{\gamma\delta\alpha\beta}\vartheta^\gamma \wedge \vartheta^\delta = \frac{1}{2}\mathcal{Z}_{ij\alpha\beta}dx^i \wedge dx^j$ is the curvaturelike two-form constructed from the one-form $e_{\alpha\beta} = e_{i\alpha\beta}dx^i = e_{\gamma\alpha\beta}\vartheta^\gamma$ by exterior differentiation:

$$\mathcal{Z}_{\alpha\beta} := de_{\alpha\beta} \quad \Leftrightarrow \quad \mathcal{Z}_{\gamma\delta\alpha\beta} = 2\partial_{[\gamma}e_{\delta]\alpha\beta}. \tag{54}$$

Note that $\partial_\gamma := e^i_\gamma\partial_i$. The action (53) is invariant under the gauge transformations

$$\delta e_{\gamma\alpha\beta} = \partial_\gamma\hat{\xi}_{\alpha\beta} + \hat{a}_{\gamma\alpha\beta}, \tag{55}$$

where $\hat{a}_{\gamma\alpha\beta}$ is traceless, $o^{\alpha\beta}\hat{a}_{\gamma\alpha\beta} = 0$ and $o^{\gamma\alpha}\hat{a}_{\gamma\alpha\beta} = 0$, and it transforms in the (2,1) module of $O(1, n-1)$ denoted by the Young tableau

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma \\ \cline{1-1}\end{array}.$$

The gauge parameter $\hat{\xi}_{\alpha\beta}$ is symmetric $\hat{\xi}_{\alpha\beta} = \hat{\xi}_{\beta\alpha}$ and traceless $o^{\alpha\beta}\hat{\xi}_{\alpha\beta} = 0$, i.e., it transforms in the (2,0) module

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline\end{array}$$

of $O(1, n-1)$.

Because of the gauge symmetry $\delta_{\hat{a}}e_{\gamma\alpha\beta} = \hat{a}_{\gamma\alpha\beta}$, only the totally symmetric component of $e_{\gamma\alpha\beta}$ survives in the action, yielding Fronsdal's action (up to an inessential overall constant factor) for $h_{\gamma\alpha\beta} \equiv e_{(\gamma\alpha\beta)}$, invariant under $\delta h_{\gamma\alpha\beta} = 3\partial_{(\gamma}\lambda_{\alpha\beta)}$, with $\lambda_{\alpha\beta} = \frac{1}{3}\hat{\xi}_{\alpha\beta}$ [6].

C. Fronsdal's action in MAG

As we anticipated by using the notation $\mathcal{Z}_{\alpha\beta}$ for the curvaturelike two-form of Vasiliev's spin-3 vierbein one-form $e_{\alpha\beta}$, the Lagrangian in (53) is contained in a general MAG Lagrangian (34) taken at quadratic order and evaluated in flat spacetime. The crucial point is to identify Vasiliev's spin-3 frame field with the traceless nonmetricity:

$$e_{\alpha\beta} = \frac{1}{2}\mathcal{Q}_{\alpha\beta}$$
$$= \Gamma_{(\alpha\beta)} - \frac{1}{n}g_{\alpha\beta}\Gamma_\gamma{}^\gamma - \left(\tilde{\Gamma}_{(\alpha\beta)} - \frac{1}{n}g_{\alpha\beta}\tilde{\Gamma}_\gamma{}^\gamma\right)$$
$$\overset{*}{=} \Gamma_{(\alpha\beta)} - \frac{1}{n}o_{\alpha\beta}\Gamma_\gamma{}^\gamma, \tag{56}$$

where the tilde denotes the Riemannian connection and the star refers to orthonormal coordinates. Then, taking the traceless part of the zeroth Bianchi identity $\frac{1}{2}DQ_{\alpha\beta} \equiv Z_{\alpha\beta}$ and recalling the definition (18) of the shear curvature $\mathcal{Z}_{\alpha\beta}$, one finds (the irreducible decomposition is listed in Appendix A)

[2] See Eq. (20) of [33] with the identification $f_{mn\bar{a}\bar{b}} \to \mathcal{Z}_{mn\alpha\beta}$; see also [34] with $B_{mn|ab} \to \mathcal{Z}_{mn\alpha\beta}$.

$$\tfrac{1}{2}D\mathcal{Q}_{\alpha\beta} \equiv \mathcal{Z}_{\alpha\beta} = {}^{(1)}\mathcal{Z}_{\alpha\beta} + {}^{(2)}\mathcal{Z}_{\alpha\beta} + {}^{(3)}\mathcal{Z}_{\alpha\beta} + {}^{(5)}\mathcal{Z}_{\alpha\beta}. \tag{57}$$

This is an exact relation valid in each metric-affine space. If we now use *orthonormal* coordinates and *linearize*, we discover that

$$\tfrac{1}{2}d\mathcal{Q}_{\alpha\beta} = de_{\alpha\beta} = {}^{(1)}\mathcal{Z}_{\alpha\beta} + {}^{(2)}\mathcal{Z}_{\alpha\beta} + {}^{(3)}\mathcal{Z}_{\alpha\beta} + {}^{(5)}\mathcal{Z}_{\alpha\beta}. \tag{58}$$

Here $\mathcal{Z}_{\alpha\beta}$ is the curvaturelike two-form defined in (54). Of course, since the decomposition of $\mathcal{Z}_{\alpha\beta}$ is purely algebraic, it also holds at the linearized level, for $\mathcal{Z}_{\alpha\beta}$.

We can now equate the Lagrangian (34) with (53) (the former taken at quadratic order, in Minkowski spacetime). We obtain a system of linear equations for the parameters $a_0, \ldots, a_3,\ b_1, \ldots, b_5,\ c_2,\ c_3,\ c_4,\ w_1, \ldots, w_7,\ z_1, \ldots, z_9$. Obviously, only the terms $\int DZ \wedge^\star DZ$ of (34) will contribute to the action (53), so that only the constants z_1, z_2, z_3, z_4, z_5 will be nonzero *a priori*. Furthermore, one can already guess that z_4 will be vanishing because Vasiliev's action (53) involves only the traceless part $\mathcal{Z}_{\alpha\beta}$ of $\mathcal{Z}_{\alpha\beta}$, which is linearly independent from the pure trace part ${}^{(4)}\mathcal{Z}_{\alpha\beta}$.

Using Appendix A, the volume n-form η, and the Rizzi-like one-form associated with $\mathcal{Z}_{\alpha\beta}$ [namely $\not{i}z_\alpha$, cf. (20)], the Lagrangian $\mathcal{L} = L\eta = -\frac{1}{2\rho}\mathcal{Z}^{\alpha\beta} \wedge^\star (\sum_{I=1,2,3,5} z_I^{(I)}\mathcal{Z}_{\alpha\beta})$ can be written as

$$\begin{aligned} L = {} & \frac{z_1 + z_2}{8\rho}\mathcal{Z}_{\gamma\delta\alpha\beta}\mathcal{Z}^{\gamma\delta\alpha\beta} - \frac{z_1 - z_2}{4\rho}\mathcal{Z}_{\gamma\delta\alpha\beta}\mathcal{Z}^{\alpha\gamma\delta\beta} \\ & - \frac{1}{4\rho}\left[\frac{3n+4}{n(n+2)}z_1 + \frac{z_2}{n-2} - \frac{2n}{n^2-4}z_3 - \frac{2}{n}z_5\right] \\ & \times \not{i}z_{\alpha\beta}\not{i}z^{\alpha\beta} - \frac{1}{4\rho}\left[\frac{n+4}{n(n+2)}z_1 - \frac{z_2}{n-2}\right. \\ & \left. + \frac{2n}{n^2-4}z_3 - \frac{2}{n}z_5\right]\not{i}z_{\alpha\beta}\not{i}z^{\beta\alpha}. \end{aligned} \tag{59}$$

Hence (59) is equal to the Lagrangian in (53) if and only if the following equations hold:

$$z_1 = 3, \qquad z_2 = -1, \qquad z_3 = 1 - n, \qquad z_5 = 3(1 - n), \tag{60}$$

all the other constants, in particular z_4, being equal to zero. Accordingly, Vasiliev's action (53) reads

$$\begin{aligned} S^{\text{Vasiliev}}[e_{\gamma\alpha\beta}] &= S^{\text{Fronsdal}}[h_{\gamma\alpha\beta}] \\ &= -\frac{1}{2\rho}\int_{\Omega_n}\mathcal{Z}^{\alpha\beta} \wedge^\star \left(\sum_{I=1,2,3,5} z_I^{(I)}\mathcal{Z}_{\alpha\beta}\right) \end{aligned} \tag{61}$$

together with (60). Finally, the field equations turn out to be

$$\begin{aligned} 0 = {} & \frac{\delta S^{\text{Vasiliev}}}{\delta e^{\gamma\alpha\beta}} \\ = {} & -\frac{1}{\rho}\partial^\delta[2\mathcal{Z}_{\alpha[\gamma\delta]\beta} + 2\mathcal{Z}_{\beta[\gamma\delta]\alpha} - \mathcal{Z}_{\gamma\delta\alpha\beta} \\ & + 4o_{\beta[\gamma}\not{i}z_{\delta]\alpha} + 4o_{\alpha[\gamma}\not{i}z_{\delta]\beta} + 2o_{\gamma(\alpha}\not{i}z_{\beta)\delta} \\ & - 2o_{\delta(\alpha}\not{i}z_{\beta)\gamma}]. \end{aligned} \tag{62}$$

Because of the equality $S^{\text{Fronsdal}} = S^{\text{Vasiliev}}$, the equations (62) are equivalent to Fronsdal's equations (41).

It is possible to pick up a gauge in which the only irreducible part that remains of the shear curvature $\mathcal{Z}_{\gamma\delta\alpha\beta}$ is its first component ${}^{(1)}\mathcal{Z}_{\gamma\delta\alpha\beta} \propto \partial_{[\gamma}h_{\delta]\alpha\beta}$. The field $h_{\alpha\beta\gamma} \equiv e_{(\alpha\beta\gamma)}$ is the only component of the framelike field that survives in the action, while the trace $o^{\alpha\beta}h_{\alpha\beta\gamma}$ and the divergence $\partial^\alpha h_{\alpha\beta\gamma}$ both vanish in the appropriately chosen gauge. This gauge is the one for which the field equations take the form (43). As noted at the end of Sec. III A, the plane-wave solutions (49) satisfy the corresponding gauge conditions. Therefore, it is easy to see that the components ${}^{(I)}\mathcal{Z}_{\gamma\delta\alpha\beta}$, $I = 2,\ 3,\ 5$, are zero for the plane-wave solutions constructed in (49).

Up to an inessential factor of 2, we have thus identified the spin-3 gauge field in Vasiliev's frame formalism with the component of the nonmetricity one-form which lies along the shear generator of $GL(n, \mathbb{R})$. This enabled us to show in a direct way the appearance of Fronsdal's massless spin-3 action as a part of MAG's action (34), provided that the free parameters present in the latter action are picked according to (60), the remaining ones being zero altogether.

IV. SPIN-3–LIKE EXACT SOLUTIONS OF FULL NONLINEAR MAG

As we have shown in the previous section, in the gravitational gauge sector of MAG, the connection $\Gamma_\alpha{}^\beta$ already mediates particles of different spin content, from 1 to 3. Since the works of Fronsdal [5,35,36], it has been widely recognized that free massive and massless higher-spin fields consistently propagate in maximally symmetric spaces,[3] and consistent higher-spin cubic vertices have been obtained in such spaces (see [40] for a light-cone analysis and references on the problem of consistent higher-spin cubic vertices, including Yang-Mills and gravitational couplings; see [41,42] for non-Abelian massless spin-3 covariant cubic vertices in flat space; higher-derivative Abelian vertices are discussed in [43]). However, so far, no interacting Lagrangian—consistent at all orders in the coupling constants—has been written

[3]See the recent work of Buchbinder *et al.* [37] for more details and references. See also Illge & Schimming [38], Illge & Wünsch [39], and references therein, where more general backgrounds have been investigated.

that would nontrivially involve spin-3 gauge fields. Presumably an infinite number of higher-spin fields is required. The best hope in that direction is the theory initiated by Fradkin and Vasiliev [44], further developed notably in [45] and reviewed, e.g., in [46].

In the field theoretical approach proposed in [3,4], higher-spin connections arise in the context of symmetry breaking mechanisms starting from the group of analytical diffeomorphism $G = \mathrm{Diff}(n, R)$. Breaking this symmetry down to the Lorentz group $SO(1, n-1)$, e.g., those generalized connections can be identified with certain parameters of the coset space G/H and give rise to an infinite tower of higher-spin fields, cf. also [1,47,48].

Because of the identification (56) and the results of the previous section, it appears that full nonlinear MAG offers an interesting vantage point on the difficult problem of spin-3 interactions, with itself and with gravity. Therefore, an important step in that direction is to search for exact solutions of full nonlinear MAG that propagate the spin-3 field ${}^{(1)}Q_{\alpha\beta}$. Moreover, such exact solutions are, within MAG, interesting for their own sake, and, in particular, also for studying non-Riemannian cosmological models; see Puetzfeld [49,50].

A. Ansatz for the nonmetricity

To isolate the main spin-3 content of the connection, we will postulate the existence of a one-form $\ell(x)$ and a scalar field $\Phi(x)$, such that the nonmetricity can be parametrized according to

$$Q_{\alpha\beta} = \Phi \ell_\alpha \ell_\beta \ell, \tag{63}$$

with

$$\ell = \ell_\alpha \vartheta^\alpha \quad \text{and} \quad \ell^2 := g^{\alpha\beta}\ell_\alpha\ell_\beta = \ell_\alpha \ell^\alpha. \tag{64}$$

For this ansatz one should compare Obukhov [29,51] who introduced plain fronted waves in MAG; see also our considerations on the spin-3 solutions in (49).

Because of (63), the components of the one-form $Q_{\alpha\beta}$ become totally symmetric, i.e.,

$$Q_{\gamma\alpha\beta} = Q_{(\gamma\alpha\beta)} = \Phi \ell_\gamma \ell_\alpha \ell_\beta. \tag{65}$$

From there on, we will put $n = 4$. Because of (65), the irreducible pieces of the nonmetricity will simplify. Together with the one-forms

$$Q_\alpha{}^\alpha = 4Q = \Phi \ell^2 \ell, \tag{66}$$

$$\overset{\circ}{Q}{}_{\alpha\beta} = Q_{\alpha\beta} - Q g_{\alpha\beta} = \Phi(\ell_\alpha\ell_\beta - \tfrac{1}{4} g_{\alpha\beta}\ell^2)\ell, \tag{67}$$

$$\Lambda = (e^\beta \rfloor \overset{\circ}{Q}{}_{\alpha\beta})\vartheta^\alpha = \tfrac{3}{4}\Phi\ell^2\ell = 3Q, \tag{68}$$

and the two-form

$$P_\alpha := \overset{\circ}{Q}{}_{\alpha\beta} \wedge \vartheta^\beta - \tfrac{1}{3}\vartheta_\alpha \wedge \Lambda = 0, \tag{69}$$

we find for the irreducible parts of the nonmetricity

$$\begin{aligned}
{}^{(1)}Q_{\alpha\beta} &= \Phi(\ell_\alpha\ell_\beta - \tfrac{1}{6}\ell^2 g_{\alpha\beta})\ell - \tfrac{1}{3}\Phi\ell^2\ell_{(\alpha}\vartheta_{\beta)} \\
&= \Phi(\ell_\alpha\ell_\beta\ell_\gamma - \tfrac{1}{2}\ell^2 g_{(\alpha\beta}\ell_{\gamma)})\vartheta^\gamma, \\
{}^{(2)}Q_{\alpha\beta} &= -\tfrac{2}{3} e_{(\alpha}\rfloor P_{\beta)} = 0, \\
{}^{(3)}Q_{\alpha\beta} &= \tfrac{1}{3}\Phi\ell^2(\ell_{(\alpha}\vartheta_{\beta)} - \tfrac{1}{4} g_{\alpha\beta}\ell) \\
&= \tfrac{1}{3}\Phi\ell^2(g_{\gamma(\alpha}\ell_{\beta)} - \tfrac{1}{4} g_{\alpha\beta}\ell_\gamma)\vartheta^\gamma, \\
{}^{(4)}Q_{\alpha\beta} &= \tfrac{1}{4}\Phi\ell^2 g_{\alpha\beta}\ell.
\end{aligned} \tag{70}$$

Since ${}^{(1)}Q_{\alpha\beta} \neq 0$, the ansatz (63) may carry *genuine* spin 3. This is consistent with (49) and (50) and with the fact that the helicity-3 plane-wave solutions obey Bargmann-Wigner equations for spin 3. Observe that the main spin-2 contribution, mediated by the tensor part ${}^{(2)}Q_{\alpha\beta}$, vanishes identically. By using (67)–(69) and the vanishing of the torsion, $T^\alpha = 0$, it is possible to show that ${}^{(2)}Z_{\alpha\beta} = 0$ and ${}^{(3)}Z_{\alpha\beta} \sim d\Lambda$; see Appendix B, Eq. (B17).

Furthermore, we will need the Hodge duals of ${}^{(1)}Q_{\alpha\beta}$ and of the other irreducible pieces. Here the η basis (4) is very convenient. The ${}^{(1)}Q_{\alpha\beta}$, as expressed in terms of ϑ^γ, can be easily hodged:

$${}^{\star(1)}Q_{\alpha\beta} = \Phi(\ell_\alpha\ell_\beta\ell_\gamma - \tfrac{1}{2}\ell^2 g_{(\alpha\beta}\ell_{\gamma)})\eta^\gamma. \tag{71}$$

It works for the other pieces analogously. If we recall $\vartheta^\alpha \wedge \eta^\beta = g^{\alpha\beta}\eta$ (see [1]), then, by straightforward algebra, we find

$$\begin{aligned}
{}^{(1)}Q_{\alpha\beta} \wedge {}^{\star(1)}Q^{\alpha\beta} &= \tfrac{1}{2}\Phi^2\ell^6\eta, \\
{}^{(3)}Q_{\alpha\beta} \wedge {}^{\star(3)}Q^{\alpha\beta} &= \tfrac{1}{4}\Phi^2\ell^6\eta, \\
{}^{(4)}Q_{\alpha\beta} \wedge {}^{\star(4)}Q^{\alpha\beta} &= \tfrac{1}{4}\Phi^2\ell^6\eta.
\end{aligned} \tag{72}$$

We transvect (71) with ℓ^β and find

$${}^{\star(1)}Q_{\alpha\beta}\ell^\beta = \tfrac{2}{3}\Phi\ell^2(\ell_\alpha\ell_\beta - \tfrac{1}{4}\ell^2 g_{\alpha\beta})\eta^\beta. \tag{73}$$

Additionally, a couple of relations for the nonmetricity as multiplied by $\eta^{\alpha\beta}$ will be needed for simplifying the field equations. We use the ansatz (63) and the properties of the η bases, cf. [1],

$$\begin{aligned}
Q_{\gamma\alpha} \wedge \eta^\gamma{}_\beta &= \Phi\ell_\alpha\ell_\beta\ell_\gamma\eta^\gamma - \Phi\ell^2\ell_\alpha\eta_\beta, \\
Q \wedge \eta_{\alpha\beta} &= -\tfrac{1}{2}\Phi\ell^2\ell_{[\alpha}\eta_{\beta]}, \\
Q_{\gamma[\alpha} \wedge \eta^\gamma{}_{\beta]} &= -\Phi\ell^2\ell_{[\alpha}\eta_{\beta]} = 2Q \wedge \eta_{\alpha\beta}, \\
Q_{\gamma(\alpha} \wedge \eta^\gamma{}_{\beta)} &= \Phi\ell_\alpha\ell_\beta\ell_\gamma\eta^\gamma - \Phi\ell^2\ell_{(\alpha}\eta_{\beta)}.
\end{aligned} \tag{74}$$

Some consequences of the ansatz (63) that we will use over and over again are the following relations:

$$Q_{\alpha\beta} = \Phi\ell_\alpha\ell_\beta\ell, \qquad \overset{\circ}{Q}{}_{\alpha\beta} \sim \ell, \tag{75}$$

$$e_\gamma\rfloor Q_{\alpha\beta} = Q_{\gamma\alpha\beta} = Q_{(\gamma\alpha\beta)}, \qquad e_{[\gamma}\rfloor Q_{\alpha]\beta} = Q_{[\gamma\alpha]\beta} = 0, \tag{76}$$

$$\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} - e_{[\alpha}\rfloor T_{\beta]} + \tfrac{1}{2}(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\vartheta^\gamma + \tfrac{1}{2}Q_{\alpha\beta}, \quad (77)$$

$$Q = \tfrac{1}{3}\Lambda = \tfrac{1}{4}\Phi\ell^2\ell, \qquad Q\wedge\Lambda = 0, \quad (78)$$

$$P_\alpha = 0, \qquad {}^{(2)}Q_{\alpha\beta} = 0. \quad (79)$$

Furthermore, we assume for the rest of Sec. IV, similar to Boulanger and Kirsch [3,4], that the *torsion vanishes*:

$$T^\alpha = 0. \quad (80)$$

This implies that the connection (77) reduces to

$$\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} + \tfrac{1}{2}Q_{\alpha\beta} \quad \text{or} \quad N_{\alpha\beta} = \tfrac{1}{2}Q_{\alpha\beta}. \quad (81)$$

Connections of this type have been studied in a different context by Baekler *et al.* [52,53].

B. A pure ${}^{(1)}Q_{\alpha\beta}$ square Lagrangian

In order to understand a propagating connection, we consider first as a very special and degenerate case of (34) the simple field Lagrangian

$$V_{(1)Q^2} = \frac{b_1}{2\kappa}Q_{\alpha\beta}\wedge{}^{\star(1)}Q^{\alpha\beta}. \quad (82)$$

The corresponding excitations (36)–(38) turn out to be

$$M^{\alpha\beta} = -\frac{2}{\kappa}b_1{}^{\star(1)}Q^{\alpha\beta}, \qquad H_\alpha = 0, \qquad H^\alpha{}_\beta = 0, \quad (83)$$

and the gauge currents (31)–(33) read

$$m^{\alpha\beta} = \vartheta^{(\alpha}\wedge E^{\beta)} + Q^{(\beta}{}_\gamma\wedge M^{\alpha)\gamma}, \quad (84)$$

$$E_\alpha = e_\alpha\rfloor V_{(1)Q^2} + \tfrac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma})M^{\beta\gamma}, \quad (85)$$

$$E^\alpha{}_\beta = -g_{\beta\gamma}M^{\alpha\gamma} = \frac{2}{\kappa}b_1 g_{\beta\gamma}{}^{\star(1)}Q^{\alpha\gamma}. \quad (86)$$

Then the source-free field equations (25)–(27) reduce to

$$DM^{\alpha\beta} - m^{\alpha\beta} = 0, \quad (87)$$

$$E_\alpha = 0, \quad (88)$$

$$E^\alpha{}_\beta = 0. \quad (89)$$

This is a rather trivial case. Because of (86) and (89), we have

$${}^{(1)}Q^{\alpha\beta} = 0. \quad (90)$$

Thus, also $M^{\alpha\beta} = 0$, and the field equations are identically fulfilled. Consequently, the source-free field equations corresponding to the purely quadratic Lagrangian (82) do not allow for propagating spin-3 fields. Our ansatz (75) was not needed in order to achieve this result.

All this seems hardly surprising. However, we have to be aware that $Q_{\alpha\beta} = -Dg_{\alpha\beta}$ is itself a field strength. Hence a check of the triviality of the Lagrangian (82) was desirable.

C. Adding a Hilbert-Einstein–type term

Let us augment the Lagrangian (82) by a curvature piece, the simplest one being the curvature scalar, and a cosmological term. In this case the Lagrangian assumes the form

$$V_{\mathrm{R}+{}^{(1)}\mathrm{Q}^2} = -\frac{a_0}{2\kappa}R_{\alpha\beta}\wedge\eta^{\alpha\beta} - \frac{\lambda_0}{\kappa}\eta + \frac{b_1}{2\kappa}Q_{\alpha\beta}\wedge{}^{\star(1)}Q^{\alpha\beta}. \quad (91)$$

Besides the gravitational constant κ and the cosmological constant λ_0, we have $a_0 = +1$ or $= 0$ (for switching on and off) and $b_1 =$ arbitrary as dimensionless coupling constants. For this particular Lagrangian, the excitations turn out to be

$$M^{\alpha\beta} = -\frac{2}{\kappa}b_1{}^{\star(1)}Q^{\alpha\beta}, \qquad H_\alpha = 0, \qquad H^\alpha{}_\beta = \frac{a_0}{2\kappa}\eta^\alpha{}_\beta. \quad (92)$$

Substitution of (92) into the second source-free field equation yields the algebraic relation

$$\frac{a_0}{2\kappa}(Q^{\alpha\gamma}\wedge\eta_{\gamma\beta} - 2Q\wedge\eta^\alpha{}_\beta + T^\gamma\wedge\eta^\alpha{}_{\beta\gamma}) - \frac{2}{\kappa}b_1{}^{\star(1)}Q^\alpha{}_\beta = 0. \quad (93)$$

We now substitute the ansatz (71), (75), and (80) into (93):

$$\frac{a_0}{2}\left(\Phi\ell_\alpha\ell_\gamma\ell\wedge\eta^\gamma{}_\beta - \frac{1}{2}\Phi\ell^2\ell\wedge\eta_{\alpha\beta}\right) - 2b_1\Big[\Phi\ell_\alpha\ell_\beta\ell_\gamma\eta^\gamma - \frac{1}{6}\Phi\ell^2(g_{\alpha\beta}\ell_\gamma\eta^\gamma + \ell_\alpha\eta_\beta + \ell_\beta\eta_\alpha)\Big] = 0. \quad (94)$$

Transvection with ℓ^β yields

$$(-\tfrac{1}{4}a_0 + \tfrac{1}{3}b_1)\Phi\ell^4\eta_\alpha + (\tfrac{1}{4}a_0 - \tfrac{4}{3}b_1)\Phi\ell^2\ell_\alpha\ell_\gamma\eta^\gamma = 0. \quad (95)$$

The second field equation (93) is only fulfilled by the choice

$$\ell^2 = 0. \quad (96)$$

We substitute this into (94) and obtain

$$\left(\frac{a_0}{2} - 2b_1\right)\Phi\ell_\alpha\ell_\beta\ell_\gamma\eta^\gamma = 0. \quad (97)$$

The only choice for nontrivial field configurations is

$$b_1 = \frac{a_0}{4} \quad \text{and} \quad \ell^2 = 0. \quad (98)$$

What about the first field equation? Because of (96), the Hodge dual of ${}^{\star(1)}Q_{\alpha\beta}$ reduces to

$$ {}^{\star(1)}Q_{\alpha\beta} = \Phi \ell_\alpha \ell_\beta \ell_\gamma \eta^\gamma \quad \text{and} \quad {}^{(1)}Q_{\alpha\beta} \wedge {}^{\star(1)}Q^{\alpha\beta} = 0. \tag{99} $$

To simplify the gauge current E_α in (32), we need information about $(e_\alpha \rfloor Q_{\beta\gamma}) M^{\beta\gamma}$. Because of (96), this can be shown to be identically zero. Collecting our results, the first source-free field equation (26) reduces to

$$ E_\alpha = e_\alpha \rfloor V_{\mathrm{R}+{}^{(1)}\mathrm{Q}^2} + \frac{a_0}{2\kappa} (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \eta^\beta{}_\gamma = 0 \tag{100} $$

or, with $\lambda = \lambda_0 / a_0$, to

$$ G_\alpha + \lambda \eta_\alpha = 0, \tag{101} $$

where G_α is the Einstein three-form (19) that will determine the one-form ℓ and the scalar field Φ.

We can decompose the first field equation (101) into Riemannian and post-Riemannian pieces. For this purpose we start with the antisymmetric part of (16),

$$ W^{\alpha\beta} = R^{[\alpha\beta]} = \tilde{R}^{\alpha\beta} + \tilde{D} N^{[\alpha\beta]} - N^{[\alpha|\gamma|} \wedge N_\gamma{}^{\beta]} \tag{102} $$

in which (81) is substituted:

$$ W^{\alpha\beta} = \tilde{R}^{\alpha\beta} - \tfrac{1}{4} Q^{[\alpha|\gamma|} \wedge Q_\gamma{}^{\beta]}. \tag{103} $$

The last two terms vanish since $Q^{\alpha\gamma} \wedge Q_\gamma{}^\beta = \Phi^2 \ell^\alpha \ell^\gamma \ell_\gamma \ell^\beta \ell \wedge \ell = 0$. Thus,

$$ G_\alpha = \tfrac{1}{2} \eta_{\alpha\beta\gamma} \wedge W^{\beta\gamma} = \tfrac{1}{2} \eta_{\alpha\beta\gamma} \wedge \tilde{R}^{\beta\gamma} = \tilde{G}_\alpha. \tag{104} $$

Hence our field equation reads $\tilde{G}_\alpha + \lambda \eta_\alpha = 0$ or, in components of the (Riemannian) Einstein tensor,

$$ \tilde{G}_{\alpha\beta} + \lambda g_{\alpha\beta} = 0. \tag{105} $$

Observe that (101) to leading order yields

$$ D^{(1)} Q_{\alpha\beta} + \text{nonlinear terms} = 0. \tag{106} $$

To separate the maximal spin content $s = 3$ of the connection, we have to take the totally symmetric part of (81):

$$ \begin{aligned} \Gamma_{(\gamma\alpha\beta)} &= \tfrac{1}{2} Q_{(\gamma\alpha\beta)} + \tfrac{1}{2} \partial_{(\gamma} g_{\alpha\beta)} \\ &= \tfrac{1}{2} \Phi \ell_\gamma \ell_\alpha \ell_\beta + \tfrac{1}{2} \partial_{(\gamma} g_{\alpha\beta)} \overset{*}{=} \tfrac{1}{2} \Phi \ell_\gamma \ell_\alpha \ell_\beta. \end{aligned} \tag{107} $$

The star denotes the choice of an orthonormal frame. However, as we have seen, these terms drop out from (101) and only the Riemannian counterpart (105) is left.

Anyway, any solution of Einstein's field equation with a cosmological constant will generate (massless) fields with spin-3 content in the framework of MAG. It remains to be seen whether this fact is of physical relevance. In any case, it shows that higher-spin fields can be constructed from the field equations of MAG. Transvection of (105) with ℓ^β yields

$$ \tilde{G}_\alpha{}^\beta \ell_\beta = \lambda \ell_\alpha. \tag{108} $$

This is an eigenvalue equation for the eigenvector ℓ^α, and the cosmological constant λ is the corresponding eigenvalue of the (Riemannian) Einstein tensor.

D. Still more $Q_{\alpha\beta}$ square terms added for spin-3 fields with $\ell^2 \neq 0$

The gravitational sector also allows for spin-3 modes with $\ell^2 \neq 0$. We call them tentatively *massive* modes since we interpret ℓ as a wave covector. To support the connection $\Gamma_\alpha{}^\beta$ to carry massive modes of this type, the Lagrangian (91) has to be extended in order to include, besides ${}^{(1)}Q_{\alpha\beta}$, also the other irreducible pieces of the nonmetricity. These contributions will induce massive spin-3 parts in the connection. As a suitable Lagrangian with this property, we choose

$$ \begin{aligned} V_{\mathrm{R}+\mathrm{Q}^2} = &- \frac{a_0}{2\kappa} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{\lambda_0}{\kappa} \eta + \frac{1}{2\kappa} Q_{\alpha\beta} \\ &\wedge \sum_{I=1}^{4} b_I {}^{\star(I)} Q^{\alpha\beta}. \end{aligned} \tag{109} $$

The corresponding excitations are

$$ \begin{gathered} M^{\alpha\beta} = -\frac{2}{\kappa} \sum_{I=1}^{4} b_I {}^{\star(I)} Q^{\alpha\beta}, \qquad H_\alpha = 0, \\ H^\alpha{}_\beta = \frac{a_0}{2\kappa} \eta^\alpha{}_\beta. \end{gathered} \tag{110} $$

Accordingly, the second field equation (27) [with (33)] is again algebraic:

$$ \frac{a_0}{2\kappa} (Q_\alpha{}^\gamma \wedge \eta_{\gamma\beta} - 2 Q \wedge \eta_{\alpha\beta} + T^\gamma \wedge \eta_{\alpha\beta\gamma}) + M_{\alpha\beta} = 0. \tag{111} $$

Its trace, its symmetric, and its antisymmetric pieces read, respectively,

$$ M_\gamma{}^\gamma = 0 \quad \text{or} \quad b_4 Q = 0, \tag{112} $$

$$ \frac{a_0}{2\kappa} Q_{(\alpha}{}^\gamma \wedge \eta_{|\gamma|\beta)} + M_{\alpha\beta} = 0, \tag{113} $$

$$ \frac{a_0}{2\kappa} (Q_{[\alpha}{}^\gamma \wedge \eta_{|\gamma|\beta]} - 2 Q \wedge \eta_{\alpha\beta} + T^\gamma \wedge \eta_{\alpha\beta\gamma}) = 0. \tag{114} $$

In the case of vanishing torsion $T^\alpha = 0$ and the application of (75) in combination with (74), Eq. (114) vanishes identically and is thus fulfilled, and the symmetric part (113) becomes

$$ \frac{a_0}{2\kappa} \Phi (\ell_\alpha \ell_\beta \ell_\gamma \eta^\gamma - \ell^2 \ell_{(\alpha} \eta_{\beta)}) + M_{\alpha\beta} = 0. \tag{115} $$

With the ansatz (75), we find for $M_{\alpha\beta}$ in (110)

$$M_{\alpha\beta} = -\frac{2\Phi}{\kappa}\Big[b_1 \ell_\alpha \ell_\beta \ell_\gamma - \frac{1}{12}(2b_1 + b_3 - 3b_4)\ell^2 g_{\alpha\beta}\ell_\gamma - \frac{1}{3}(b_1 - b_3)\ell^2 g_{\gamma(\alpha}\ell_{\beta)}\Big]\eta^\gamma. \quad (116)$$

We substitute this into (115) and find a new form of the symmetric part of the second field equation:

$$\frac{\Phi}{6\kappa}[3(a_0 - 4b_1)\ell_\alpha\ell_\beta\ell_\gamma + (2b_1 + b_3 - 3b_4)\ell^2 g_{\alpha\beta}\ell_\gamma + (-3a_0 + 4b_1 - 4b_3)\ell^2 g_{\gamma(\alpha}\ell_{\beta)}]\eta^\gamma = 0. \quad (117)$$

The b_4-term in this equation is $\sim b_4 {}^\star Q$. Because of (112), it drops out. We transvect this equation first with ℓ^β,

$$[(\tfrac{3}{2}a_0 - 8b_1 - b_3)\ell^2\ell_\alpha\ell_\gamma + (-\tfrac{3}{2}a_0 + 2b_1 - 2b_3)\ell^4 g_{\alpha\gamma}]\eta^\gamma = 0, \quad (118)$$

and subsequently with ℓ^α,

$$-3(2b_1 + b_3)\ell^4\ell_\gamma\eta^\gamma = 0. \quad (119)$$

Provided $\ell^2 \neq 0$, we have from (112) and from (119) the relations $b_4 = 0$ and $b_3 = -2b_1$, respectively. If we substitute the latter into (118), we have finally

$$b_1 = \tfrac{1}{4}a_0, \qquad b_3 = -\tfrac{1}{2}a_0, \qquad b_4 = 0, \qquad \text{all for } \ell^2 \neq 0. \quad (120)$$

For a reformulation of the first field equation (26) [with (32)],

$$E_\alpha = e_\alpha \rfloor V_{\mathrm{R}+\mathrm{Q}^2} + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma + \tfrac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})M^{\beta\gamma} = 0, \quad (121)$$

we use (as part of $V_{\mathrm{R}+\mathrm{Q}^2}$)

$$Q_{\alpha\beta} \wedge \sum_{I=1}^{4} b_I {}^{\star(I)}Q^{\alpha\beta} = \Big(\frac{1}{2}b_1 + \frac{1}{4}b_3 + \frac{1}{4}b_4\Big)\Phi^2\ell^6\eta \quad (122)$$

and

$$\frac{1}{2}Q_{\alpha\beta\gamma}M^{\beta\gamma} = -\frac{1}{\kappa}\Big(\frac{1}{2}b_1 + \frac{1}{4}b_3 + \frac{1}{4}b_4\Big)\Phi^2\ell^4\ell_\alpha\ell_\beta\eta^\beta. \quad (123)$$

If we collect our results, (121) can be written as

$$E_\alpha = -\frac{a_0}{\kappa}G_{\alpha\beta}\eta^\beta - \frac{\lambda_0}{\kappa}\eta_\alpha + \frac{1}{2\kappa}\Big(\frac{1}{2}b_1 + \frac{1}{4}b_3 + \frac{1}{4}b_4\Big) \times \Phi^2\ell^6\eta_\alpha - \frac{1}{\kappa}\Big(\frac{1}{2}b_1 + \frac{1}{4}b_3 + \frac{1}{4}b_4\Big)\Phi^2\ell^4\ell_\alpha\ell_\beta\eta^\beta = 0. \quad (124)$$

Eventually, the first field equation reads

$$\frac{a_0}{\kappa}\Big(G_{\alpha\beta} + \frac{\lambda_0}{a_0}g_{\alpha\beta}\Big)\eta^\beta + \frac{1}{\kappa}\Big(\frac{1}{2}b_1 + \frac{1}{4}b_3 + \frac{1}{4}b_4\Big) \times \Phi^2\ell^4\Big(\ell_\alpha\ell_\beta - \frac{1}{2}\ell^2 g_{\alpha\beta}\Big)\eta^\beta = 0. \quad (125)$$

Using the parameter set (120), the expression containing b_1 etc. collapses to zero and we end up with an Einstein-type vacuum equation

$$G_{\alpha\beta}(\Gamma) + \lambda g_{\alpha\beta} = 0, \quad (126)$$

where we put again $\lambda = \lambda_0/a_0$. As in the last subsection, this equation, using our ansatz (75) and (80), reduces to the Einstein equation in Riemannian spacetime:

$$\tilde{G}_{\alpha\beta} + \lambda g_{\alpha\beta} = 0. \quad (127)$$

In our context, the Einstein three-form $G_\alpha(\Gamma)$ equals the Riemannian one $G_\alpha(\tilde{\Gamma}) \equiv \tilde{G}_\alpha$. There is a general underlying pattern. If a connection is deformed by means of an additive one-form $A_\alpha{}^\beta$ according to $\bar{\Gamma}_\alpha{}^\beta = \Gamma_\alpha{}^\beta + A_\alpha{}^\beta$, then the curvature tensor responds with

$$\bar{R}_\alpha{}^\beta = R_\alpha{}^\beta + DA_\alpha{}^\beta - A_\alpha{}^\gamma \wedge A_\gamma{}^\beta. \quad (128)$$

In the special case of a projective transformation with $A_\alpha{}^\beta = \delta_\alpha^\beta P$, we have (see [1,54])

$$\overset{\text{proj}}{R}{}_\alpha{}^\beta = R_\alpha{}^\beta + \delta_\alpha^\beta dP. \quad (129)$$

Thus,

$$\overset{\text{proj}}{W}_{\alpha\beta} := \overset{\text{proj}}{R}_{[\alpha\beta]} = R_{[\alpha\beta]} = W_{\alpha\beta} \quad \text{and} \quad \overset{\text{proj}}{G}_\alpha = G_\alpha. \quad (130)$$

The Einstein three-form is invariant under projective transformations. Therefore, a gravitational Lagrangian in MAG cannot consist of a Hilbert-Einstein–type term alone. It has to carry additional terms.

The connection of our ansatz (81), namely $\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} + \frac{1}{2}Q_{\alpha\beta}$, transforms the curvature according to

$$R_{\alpha\beta}(\Gamma) = \tilde{R}_{\alpha\beta} + \tfrac{1}{2}\tilde{D}Q_{\alpha\beta} - \tfrac{1}{4}Q_\alpha{}^\gamma \wedge Q_{\beta\gamma}. \quad (131)$$

Consequently,

$$W_{\alpha\beta}(\Gamma) = \tilde{W}_{\alpha\beta} - \tfrac{1}{4}Q_\alpha{}^\gamma \wedge Q_{\beta\gamma}, \quad (132)$$

since the last term is antisymmetric in α and β. In turn,

$$G_\alpha(\Gamma) = \tilde{G}_\alpha + \tfrac{1}{2}\eta_{\alpha\beta\gamma}Q^\beta{}_\delta \wedge Q^{\gamma\delta}. \quad (133)$$

However, in accordance with our ansatz (75), the Q-square term vanishes:

$$G_\alpha(\Gamma) = \tilde{G}_\alpha, \qquad W_{\alpha\beta}(\Gamma) = \tilde{W}_{\alpha\beta}. \quad (134)$$

E. A quadratic Lagrangian with pure strain curvature

In reminiscence of the Fronsdal Lagrangian, let us investigate a gravitational gauge model in the framework of MAG with a field Lagrangian quadratic in the (symmetric) strain curvature,[4] i.e., we will concentrate on the field Lagrangian

$$V_{Z^2} = -\frac{1}{2\rho} R^{\alpha\beta} \wedge \sum_{I=1}^{5} z_I{}^{\star(I)} Z_{\alpha\beta}. \tag{135}$$

Incidentally, such Lagrangians may be also interesting in cosmology; see Puetzfeld [49,50]. The excitations belonging to the Lagrangian (135) turn out to be

$$M^{\alpha\beta} = 0, \qquad H_\alpha = 0, \qquad H^\alpha{}_\beta = \frac{1}{\rho} \sum_{I=1}^{5} z_I{}^{\star(I)} Z^\alpha{}_\beta. \tag{136}$$

Note that $H^{\alpha\beta}$ is symmetric in α and β. The source-free field equations (26) and (27) reduce to

$$e_\alpha \rfloor V_{Z^2} + \frac{1}{\rho} (e_\alpha \rfloor Z_\beta{}^\gamma) \wedge \sum_{I=1}^{5} z_I{}^{\star(I)} Z^\beta{}_\gamma = 0, \tag{137}$$

$$D\left(\sum_{I=1}^{5} z_I{}^{\star(I)} Z^\alpha{}_\beta \right) = 0. \tag{138}$$

The trace of the second field equation (138) yields

$$2 z_4 d^\star dQ = 0 \tag{139}$$

and from its antisymmetric piece only

$$Q_{\gamma[\alpha|} \sum_{I=1}^{5} z_I{}^{\star(I)} Z_{|\beta]}{}^\gamma = 0 \tag{140}$$

is left over.

In order to get some insight into the possible solution classes, we will distinguish between $\ell^2 = 0$ and $\ell^2 \neq 0$.

1. Solutions with $\ell^2 = 0$

Let us first recall from (78) that the Weyl covector Q, for $\ell^2 = 0$, vanishes identically. Hence ${}^{(4)}Z_{\alpha\beta} = 0$. Again with our ansatz, according to (B17), we have ${}^{(2)}Z_{\alpha\beta} = 0$ and ${}^{(3)}Z_{\alpha\beta} \sim d\Lambda$. However, $\Lambda \sim Q$; see (78). Accordingly,

$${}^{(2)}Z_{\alpha\beta} = {}^{(3)}Z_{\alpha\beta} = {}^{(4)}Z_{\alpha\beta} = 0. \tag{141}$$

[4] A Lagrangian quadratic in the rotational curvature of the type $W^{\alpha\beta} \wedge {}^\star W_{\alpha\beta}$ would not have a propagating ${}^{(1)}Q_{\alpha\beta}$ piece. This can be seen as follows: The third term on the right-hand side of (7) selects all the pieces of $Q_{\alpha\beta}$, except ${}^{(1)}Q_{\alpha\beta}$. The fourth term will not contribute to give a kinetic term $d^{(1)}Q^{\alpha\beta} \wedge {}^\star d^{(1)}Q_{\alpha\beta}$ via $W^{\alpha\beta} \wedge {}^\star W_{\alpha\beta}$ because of its symmetries. Therefore, only the third term of (7) has a chance to contribute a kinetic term $dQ^{\alpha\beta} \wedge {}^\star dQ_{\alpha\beta}$; but in the third term ${}^{(1)}Q_{\alpha\beta}$ dropped out.

To find solutions of the field equations (137), we will make use of the *Kerr-Schild ansatz* for the metric, cf. [7], which will be expressed in terms of a null tetrad according to

$$\begin{aligned} g &= g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta \\ &= \vartheta^0 \otimes \vartheta^1 + \vartheta^1 \otimes \vartheta^0 - \vartheta^2 \otimes \vartheta^3 - \vartheta^3 \otimes \vartheta^2; \end{aligned} \tag{142}$$

that is, the anholonomic components of the (local) metric are given by

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{143}$$

We will introduce a set of coordinates $(\zeta, \bar\zeta, u, v)$ and choose the coframe

$$\vartheta^0 = d\zeta, \qquad \vartheta^1 = d\bar\zeta, \qquad \vartheta^2 = du, \qquad \vartheta^3 = dv + V\vartheta^2. \tag{144}$$

Then, the metric assumes the form

$$g = 2(d\zeta d\bar\zeta - du dv) - 2V(\zeta, \bar\zeta, u) du^2, \tag{145}$$

which will generate a class of pp waves, *inter alia*, cf. [29,51,55].

The key point now is to identify the propagation vector ℓ of the spin-3 field with that of the Kerr-Schild ansatz; i.e., we will choose for the *propagating* `trinom`

$$\ell = V(\zeta, \bar\zeta, u) du, \tag{146}$$

with the further property

$$\ell \wedge d\ell = 0. \tag{147}$$

[In classical general relativity the components of ℓ^{KS} are chosen to be $\ell^{\mathrm{KS}}_\alpha = (0, 0, 1, 0)$.] Hence, in the massless case, i.e., $\ell^2 = 0$, it would be advantageous to rescale the function Φ according to

$$\Phi \rightarrow \hat\Phi / V, \quad \text{with} \quad \hat\Phi = \hat\Phi(\zeta, \bar\zeta, u). \tag{148}$$

This rescaling introduces some redundancy. However, it is very convenient when one searches for exact solutions of MAG. Then, one can take, e.g., for V an exact solution of Einstein's theory (in Riemannian spacetime), but still have $\hat\Phi$ as a separate field for fulfilling the field equations of MAG.

We insert (75) and (144), together with (148), into the first field equation (137). It is fulfilled identically for arbitrary parameter values of z_I. The second field equation (138) yields just one equation for the determination of the functions V and $\hat\Phi$,

$$\begin{aligned} 0 = z_1 (&\hat\Phi_{\zeta\bar\zeta} V^2 + 2\hat\Phi_\zeta V_{\bar\zeta} V + 2\hat\Phi_{\bar\zeta} V_\zeta V + 2 V_{\zeta\bar\zeta} \hat\Phi V \\ &+ 2 V_\zeta V_{\bar\zeta} \hat\Phi) = z_1 (\hat\Phi V^2)_{\zeta\bar\zeta}. \end{aligned} \tag{149}$$

Incidentally, the choice $V = 1$, that is $\ell = du$, would lead

to $z_1\hat{\Phi}_{\zeta\bar{\zeta}} = 0$. Observe that in this case the corresponding metric g alone represents a flat spacetime whereas the pair $\{g, \hat{\Phi}\}$ yields a nonflat solution of MAG. This shows that the rescaling in (148) is a useful procedure.

Substitution of the nonmetricity (75) and the coframe (144) together with the condition of vanishing torsion yields for the massless case ($\ell^2 = 0$)

$$ {}^{(1)}W^{\alpha\beta} \neq 0, \qquad {}^{(4)}W^{\alpha\beta} \neq 0, \quad \text{and} \quad {}^{(1)}Z^{\alpha\beta} \neq 0. \tag{150}$$

It has been verified by using our REDUCE-EXCALC computer algebra programs that these are the only nonvanishing irreducible pieces of the curvature. We find, in particular, ${}^{(5)}Z^{\alpha\beta} = 0$. Moreover, the strain curvature can be written in a compact notation as

$$ {}^{(1)}Z^{\alpha\beta} = \frac{1}{2V} d(\hat{\Phi}V^2)\delta^\alpha_3\delta^\beta_3 \wedge \ell. \tag{151}$$

The partial differential equation (149) has simple polynomial solutions, *inter alia*, such as

$$ \begin{aligned} V &= f_1(u)\bar{\zeta}^2 + f_2(u)\bar{\zeta} + f_3(u) \quad \text{or} \\ V &= f_4(u)\zeta^2 + f_5(u)\zeta + f_6(u), \end{aligned} \tag{152}$$

with arbitrary wave profiles $f_1(u), \ldots, f_6(u)$.

Summarizing, the propagating *massless* spin-3 field can be characterized by the coframe (144) and by

$$ {}^{(1)}Q^{\alpha\beta} = \hat{\Phi}V(\zeta, \bar{\zeta}, u)\delta^\alpha_3\delta^\beta_3\ell = \hat{\Phi}V^2\delta^\alpha_3\delta^\beta_3\vartheta^2, \tag{153}$$

where $\hat{\Phi}$ and V are a solution of (149). A comparison with (151) shows that

$$ {}^{(1)}Z^{\alpha\beta} = \tfrac{1}{2}d{}^{(1)}Q^{\alpha\beta}; \tag{154}$$

that is, the nonmetricity ${}^{(1)}Q^{\alpha\beta}$ acts as a true potential for the strain curvature ${}^{(1)}Z^{\alpha\beta}$. The only nonzero component of the spin 3 carrying piece ${}^{(1)}Q_{\alpha\beta}$ turns out to be

$$ {}^{(1)}Q_{222} = \hat{\Phi}V^2(\zeta, \bar{\zeta}, u). \tag{155}$$

Hence, the second field equation (138) can be written symbolically as

$$ z_1\Box{}^{(1)}Q_{\alpha\beta\gamma} = 0. \tag{156}$$

We would like to mention that all results in this subsection will remain valid if one allows also for a nonzero torsion trace, in accordance with the general results of Heinicke *et al.* [12]. Hence, any torsion trace could be parametrized as

$$ {}^{(2)}T^\alpha = \Psi\vartheta^\alpha \wedge \ell, \qquad \Psi = \Psi(\zeta, \bar{\zeta}, u, v), \tag{157}$$

which is directly related to (63).

2. Solutions with $\ell^2 \neq 0$

In order to look for solutions of *massive propagating* ${}^{(1)}Q_{\alpha\beta}$, we have to choose a more general representation of the one-form ℓ, because (146) describes a null vector. As a simple modification of (146) leading to nonvanishing ℓ^2 we can choose

$$ \ell = V\vartheta^2 + m_0\vartheta^0 + m_1\vartheta^1, \tag{158}$$

where we assume for simplicity that m_0 and m_1 are constants. For the norm ℓ^2 we find

$$ \ell^2 = 2m_0m_1 \neq 0. \tag{159}$$

We could scale ℓ^2 to unity with the choice $m_0 = m_1 = 1/\sqrt{2}$. However, we will not do so.

The ansatz (75) for the nonmetricity will be written slightly modified as

$$ Q^{\alpha\beta} = \frac{\hat{\Phi}\ell^\alpha\ell^\beta}{V\ell^2}\ell, \tag{160}$$

with $\hat{\Phi} = \hat{\Phi}(u)$ and $\ell^\alpha = e^\alpha\rfloor\ell$. Even with these assumptions, it will be difficult to solve the field equations. For this reason, we assume furthermore that the scalar V is constant, too. This will lead us to a certain toy model showing that the solution manifold for the field equations (137) and (138) is not empty and allows for massive propagating modes. We inserted all this into the first and second field equations: The first field equation is fulfilled identically, provided the coupling constants are chosen according to

$$ 5z_1 + z_3 + 3(z_4 + z_5) = 0, \tag{161}$$

$$ 5z_1 + 2z_4 + z_5 = 0, \tag{162}$$

and the second field equation yields a second-order linear differential equation for $\hat{\Phi}$,

$$ (3z_1 + z_4)\hat{\Phi}_{uu} = 0. \tag{163}$$

This simple model implies two different subcases, either

$$ \hat{\Phi}(u) \text{ arbitrary}, \quad \text{with} \quad z_3 = z_5 = z_1 \quad \text{and} \quad z_4 = -3z_1, \tag{164}$$

or

$$ \begin{aligned} &\hat{\Phi}_{uu} = 0, \quad \text{with} \quad 5z_1 + z_3 + 3(z_4 + z_5) = 0 \\ &\quad \text{and} \quad 5z_1 + 2z_4 + z_5 = 0, \end{aligned} \tag{165}$$

leading to a 2-parameter class of solutions.

We find for these solutions that only the strain curvature $Z_{\alpha\beta}$ is nonvanishing and that the nonmetricity $Q_{\alpha\beta}$ is mainly nontrivial, namely

$$ W_{\alpha\beta} = 0, \qquad {}^{(2)}Z_{\alpha\beta} = 0, \qquad {}^{(2)}Q_{\alpha\beta} = 0. \tag{166}$$

All other irreducible pieces are nonvanishing.

To give an idea of the complexity of this simple toy model, we list the *massive* spin-3 part of the nonmetricity,

$$^{(1)}Q_{00} = \frac{m_1\hat{\Phi}}{6m_0V}[m_0\vartheta^0 + 3(m_1\vartheta^1 + V\vartheta^2)], \tag{167}$$

$$^{(1)}Q_{01} = \frac{\hat{\Phi}}{6V}(m_0\vartheta^0 + m_1\vartheta^1 + 2V\vartheta^2), \tag{168}$$

$$^{(1)}Q_{02} = -\frac{m_1\hat{\Phi}}{6V}\vartheta^2, \tag{169}$$

$$^{(1)}Q_{03} = -\frac{\hat{\Phi}}{6m_0V}[2m_0V\vartheta^0 + 3(m_1\vartheta^1 + V\vartheta^2)V + m_0m_1\vartheta^3], \tag{170}$$

$$^{(1)}Q_{11} = \frac{m_0\hat{\Phi}}{6m_1V}(3m_0\vartheta^0 + m_1\vartheta^1 + 3V\vartheta^2), \tag{171}$$

$$^{(1)}Q_{12} = -\frac{m_0\hat{\Phi}}{6V}\vartheta^2, \tag{172}$$

$$^{(1)}Q_{13} = -\frac{\hat{\Phi}}{6m_1V}[(3m_0\vartheta^0 + 2m_1\vartheta^1 + 3V\vartheta^2)V + m_0m_1\vartheta^3], \tag{173}$$

$$^{(1)}Q_{22} = 0, \tag{174}$$

$$^{(1)}Q_{23} = \frac{\hat{\Phi}}{6V}(m_0\vartheta^0 + m_1\vartheta^1 + 2V\vartheta^2), \tag{175}$$

$$^{(1)}Q_{33} = \frac{\hat{\Phi}}{6m_0m_1}[3(m_0\vartheta^0 + m_1\vartheta^1)V + 3V^2\vartheta^2 + 2m_0m_1\vartheta^3]. \tag{176}$$

Because of (166), the spin-2 and spin-1 carrying pieces are also nontrivial for those massive modes. A systematic exploitation of the ansatz (158) and its generalizations will be given elsewhere.

3. Rewriting the Lagrangian V_{Z^2}

It is also instructive to rewrite the Lagrangian (135) in terms of a set of different variables. The $^{(1)}Z_{\alpha\beta}$ square piece we leave as it is. Under our constraints, $^{(2)}Z_{\alpha\beta} = 0$; see (B17). The $^{(3)}Z_{\alpha\beta}$, as displayed in (B17), can be expressed in terms of $d\Lambda$. This implies

$$^{(3)}Z_{\alpha\beta} \wedge {}^{\star(3)}Z^{\alpha\beta} = \tfrac{1}{27}d\Lambda \wedge {}^{\star}d\Lambda. \tag{177}$$

Also simple is $^{(4)}Z_{\alpha\beta}$; see (A7). Thus,

$$^{(4)}Z_{\alpha\beta} \wedge {}^{\star(4)}Z^{\alpha\beta} = dQ \wedge {}^{\star}dQ. \tag{178}$$

With the definition (A8) of $^{(5)}Z_{\alpha\beta}$, we derive the identity

$$^{(5)}Z_{\alpha\beta} \wedge {}^{\star(5)}Z^{\alpha\beta} = \tfrac{3}{8}\Xi_\alpha \wedge {}^{\star}\Xi^\alpha. \tag{179}$$

Collecting our results (177)–(179), and recalling $\Lambda = 3Q$ [see (78)], the Lagrangian (135) can be put into the form

$$V_{Z^2} = -\frac{1}{2\rho}\left[z_1\,{}^{(1)}Z_{\alpha\beta} \wedge {}^{\star(1)}Z^{\alpha\beta} + \left(\frac{z_3}{3} + z_4\right)dQ \wedge {}^{\star}dQ + \frac{3}{8}z_5\Xi_\alpha \wedge {}^{\star}\Xi^\alpha\right]. \tag{180}$$

If one desires, one can also introduce the Rizzi one-form. Under our constraints, we have

$$V_{Z^2} = -\frac{1}{2\rho}\left[z_1\,{}^{(1)}Z_{\alpha\beta} \wedge {}^{\star(1)}Z^{\alpha\beta} + \left(\frac{z_3}{3} + z_4\right)dQ \wedge {}^{\star}dQ + \frac{3}{8}z_5\not{R}iz_\alpha \wedge {}^{\star}\not{R}iz^\alpha\right]. \tag{181}$$

Note that, for a consistent transition to this new Lagrangian, one has to add suitable Lagrange multiplier terms to the Lagrangian.

V. DISCUSSION

In this paper, we carefully investigated the sector of MAG related to a (free) massless spin-3 field and found exact solutions of full nonlinear MAG theory in vacuum with propagating nonmetricity $^{(1)}Q_{\alpha\beta}$.

Up to an inessential factor 2, we identified the spin-3 gauge field in Vasiliev's frame formalism with $\not{Q}_{\alpha\beta}$, the component of the nonmetricity one-form which lies along the shear generator of $GL(n,\mathbb{R}) \subset \mathbb{R}^n \rtimes GL(n,\mathbb{R})$. This enabled us to show in a direct way the appearance of Fronsdal's massless spin-3 action in flat space as a part of MAG's action, provided that the free parameters present in the latter action are picked according to (60), the remaining ones being zero altogether. Fronsdal's Lagrangian turns out to be purely quadratic in the shear curvature, a purely post-Riemannian piece of the general linear curvature. We also clarified the dynamical spin content of the plane-wave solution found in [29] by explicitly relating it to a simple propagating helicity-3 solution of the Bargmann-Wigner equations.

We then constructed several exact solutions of full nonlinear MAG in vacuum with propagating tracefree nonmetricity, some showing a massless spin-3 behavior, others presenting a massivelike spin-3 character. Note that, although we have proved the occurrence of Fronsdal's massless spin-3 Lagrangian inside MAG by choosing the only nonzero parameters as in (60), we have not shown that the Singh-Hagen massive spin-3 Lagrangian [56] could also be hosted inside MAG. This would require the introduction of a scalar field, not present in the general MAG Lagrangian (34) we have been considering here. This scalar field was introduced in [4] as a BEH field, in analogy to the Higgs field in $U(1)$ symmetry breaking.

In MAG, as in any gauge theory, the geometrical fields are coupled to matter currents. In addition to the symmetric (Hilbert) energy-momentum current, which is coupled to

the metric field, we have additionally the spin current and the dilation plus shear currents inducing torsion and nonmetricity fields, respectively.

This requires the homogeneous Lorentz group to be embedded in the larger general linear group. Having identified Vasiliev's spin-3 frame field with the traceless nonmetricity, we have gained another geometrical interpretation for the former field (the tracelessness of the Vasiliev gauge parameter $\hat{\xi}_{\alpha\beta}$ being the natural consequence of a shear transformation), but we have lost the Lorentz group as the local symmetry group of the tangent manifold [13]. Indeed, although the Weyl one-form leaves the conformal light-cone structure intact, the traceless nonmetricity (which couples to the shear current of matter) does *not* preserve the light-cone structure and the local Lorentz symmetry under parallel transport [1] with respect to the connection $\Gamma_{\alpha}{}^{\beta}$. This implies that, in our discussion, we are relating the massless spin-3 field with situations in which there is no conventional flat, special relativity limit, like e.g. in the early universe or in the microscopic domain where the coupling of the shear plus dilation current of matter to nonmetricity is expected to become non-negligible, not to mention the coupling of matter's intrinsic spin current to the torsion field. This picture is in accordance with Fronsdal's spin-3 Lagrangian inside MAG being purely quadratic in the shear curvature, hence belonging to the strong-gravity post-Riemannian part of MAG's Lagrangian.

Although there is presumably no consistent coupling between a spin-3 field and dynamical Hilbert-Einstein gravity (without resorting to an infinite tower of higher-spin fields), our results suggest that spin-3 dynamics in the framework of MAG could be well defined in the limit where strong-gravitational MAG effects prevail and where shear-type excitations of matter are expected to arise. Finally, it would be interesting to compare our results with those presented in [12].

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APPENDIX A: IRREDUCIBLE DECOMPOSITION OF THE STRAIN CURVATURE

1. In components

We have the following irreducible decomposition of the components of the strain-curvature two-form $Z_{\alpha\beta} = \frac{1}{2} Z_{\gamma\delta\alpha\beta} \vartheta^{\gamma} \wedge \vartheta^{\delta} = \frac{1}{2} Z_{ij\alpha\beta} dx^{i} \wedge dx^{j}$ with respect to the (pseudo)orthogonal group, cf. [1,12,57],

$$\underbrace{Z_{\alpha\beta}}_{\square\otimes\square} = \underbrace{{}^{(1)}Z_{\alpha\beta}}_{\square\!\square} \oplus \underbrace{{}^{(2)}Z_{\alpha\beta}}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus \underbrace{{}^{(3)}Z_{\alpha\beta}}_{\square} \oplus \underbrace{{}^{(4)}Z_{\alpha\beta}}_{\square} \oplus \underbrace{{}^{(5)}Z_{\alpha\beta}}_{\square\!\square} , \tag{A1}$$

We have given the decomposition of the $GL(n, \mathbb{R})$-reducible components $Z_{\gamma\delta\alpha\beta}$ into irreducible representations of the (pseudo)orthogonal group, so that the Young diagrams on the right-hand side of the above equality label $O(1, n-1)$-irreducible representations. (Note the multiplicity 2 of the antisymmetric rank-2 tensor irreducible representation. Indeed, $Z_{\alpha[\gamma\delta]}{}^{\alpha}$ and $Z_{\gamma\delta\alpha}{}^{\alpha}$ are linearly independent.) Accordingly,

$$\begin{aligned}
{}^{(1)}Z_{\gamma\delta\alpha\beta} &= \frac{1}{2}(\mathcal{Z}_{\gamma\delta\alpha\beta} - \mathcal{Z}_{\alpha[\gamma\delta]\beta} - \mathcal{Z}_{\beta[\gamma\delta]\alpha}) + \frac{1}{2(n+2)}(\mathcal{Z}_{\varepsilon[\gamma\alpha]}{}^{\varepsilon} g_{\beta\delta} - \mathcal{Z}_{\varepsilon[\delta\alpha]}{}^{\varepsilon} g_{\beta\gamma} + \mathcal{Z}_{\varepsilon[\gamma\beta]}{}^{\varepsilon} g_{\alpha\delta} - \mathcal{Z}_{\varepsilon[\delta\beta]}{}^{\varepsilon} g_{\alpha\gamma} \\
&\quad + 2 g_{\alpha\beta} \mathcal{Z}_{\varepsilon[\gamma\delta]}{}^{\varepsilon}) + \frac{1}{n}(\mathcal{Z}_{\varepsilon(\gamma\alpha)}{}^{\varepsilon} g_{\beta\delta} + \mathcal{Z}_{\varepsilon(\gamma\beta)}{}^{\varepsilon} g_{\alpha\delta} - \mathcal{Z}_{\varepsilon(\delta\alpha)}{}^{\varepsilon} g_{\beta\gamma} - \mathcal{Z}_{\varepsilon(\delta\beta)}{}^{\varepsilon} g_{\alpha\gamma}), \\
{}^{(2)}Z_{\gamma\delta\alpha\beta} &= \frac{1}{2(n-2)}(\mathcal{Z}_{\varepsilon[\gamma\alpha]}{}^{\varepsilon} g_{\beta\delta} + \mathcal{Z}_{\varepsilon[\gamma\beta]}{}^{\varepsilon} g_{\alpha\delta} - \mathcal{Z}_{\varepsilon[\delta\alpha]}{}^{\varepsilon} g_{\beta\gamma} - \mathcal{Z}_{\varepsilon[\delta\beta]}{}^{\varepsilon} g_{\alpha\gamma}) + \frac{3}{4}(\mathcal{Z}_{[\gamma\delta\alpha]\beta} + \mathcal{Z}_{[\gamma\delta\beta]\alpha}) \\
&\quad - \frac{1}{(n-2)} \mathcal{Z}_{\varepsilon[\gamma\delta]}{}^{\varepsilon} g_{\alpha\beta}, \\
{}^{(3)}Z_{\gamma\delta\alpha\beta} &= \frac{n}{(n+2)(n-2)}(\mathcal{Z}_{\varepsilon[\alpha\gamma]}{}^{\varepsilon} g_{\beta\delta} + \mathcal{Z}_{\varepsilon[\beta\gamma]}{}^{\varepsilon} g_{\alpha\delta} - \mathcal{Z}_{\varepsilon[\alpha\delta]}{}^{\varepsilon} g_{\beta\gamma} - \mathcal{Z}_{\varepsilon[\beta\delta]}{}^{\varepsilon} g_{\alpha\gamma}) + \frac{4}{(n+2)(n-2)} \mathcal{Z}_{\varepsilon[\gamma\delta]}{}^{\varepsilon} g_{\alpha\beta}, \\
{}^{(4)}Z_{\gamma\delta\alpha\beta} &= \frac{1}{n} Z_{\gamma\delta\varepsilon}{}^{\varepsilon} g_{\alpha\beta}, \qquad {}^{(5)}Z_{\gamma\delta\alpha\beta} = \frac{1}{n}(\mathcal{Z}_{\varepsilon(\alpha\delta)}{}^{\varepsilon} g_{\beta\gamma} + \mathcal{Z}_{\varepsilon(\beta\delta)}{}^{\varepsilon} g_{\alpha\gamma} - \mathcal{Z}_{\varepsilon(\alpha\gamma)}{}^{\varepsilon} g_{\beta\delta} - \mathcal{Z}_{\varepsilon(\beta\gamma)}{}^{\varepsilon} g_{\alpha\delta}),
\end{aligned}$$

and with the shear curvature

$$\mathcal{Z}_{\gamma\delta\alpha\beta} := Z_{\gamma\delta\alpha\beta} - \frac{1}{n} g_{\alpha\beta} Z_{\gamma\delta\varepsilon}{}^{\varepsilon}.$$

Equivalently, by introducing $\not{\mathrm{R}}\mathrm{iz}_{\alpha\beta}$, we can rewrite this as follows:

$$
\begin{aligned}
{}^{(1)}Z_{\gamma\delta\alpha\beta} &= \frac{1}{2}(\not{Z}_{\gamma\delta\alpha\beta} - \not{Z}_{\alpha[\gamma\delta]\beta} - \not{Z}_{\beta[\gamma\delta]\alpha}) + \frac{1}{2(n+2)}(\not{\mathrm{R}}\mathrm{iz}_{[\gamma\alpha]}g_{\beta\delta} - \not{\mathrm{R}}\mathrm{iz}_{[\delta\alpha]}g_{\beta\gamma} + \not{\mathrm{R}}\mathrm{iz}_{[\gamma\beta]}g_{\alpha\delta} - \not{\mathrm{R}}\mathrm{iz}_{[\delta\beta]}g_{\alpha\gamma}) \\
&\quad + \frac{1}{n+2}\not{\mathrm{R}}\mathrm{iz}_{[\gamma\delta]}g_{\alpha\beta} + \frac{1}{n}(\not{\mathrm{R}}\mathrm{iz}_{(\gamma\alpha)}g_{\beta\delta} + \not{\mathrm{R}}\mathrm{iz}_{(\gamma\beta)}g_{\alpha\delta} - \not{\mathrm{R}}\mathrm{iz}_{(\delta\alpha)}g_{\beta\gamma} - \not{\mathrm{R}}\mathrm{iz}_{(\delta\beta)}g_{\alpha\gamma}), \\
{}^{(2)}Z_{\gamma\delta\alpha\beta} &= \frac{1}{2(n-2)}[(\not{\mathrm{R}}\mathrm{iz}_{[\gamma|\beta} - \not{\mathrm{R}}\mathrm{iz}_{\beta[\gamma)}g_{|\alpha|\delta]} + (\not{\mathrm{R}}\mathrm{iz}_{[\gamma|\alpha} - \not{\mathrm{R}}\mathrm{iz}_{\alpha|\gamma]})g_{\beta|\delta]}] + \frac{3}{4}(\not{Z}_{[\gamma\delta\alpha]\beta} + \not{Z}_{[\gamma\delta\beta]\alpha}) - \frac{1}{n-2}\not{\mathrm{R}}\mathrm{iz}_{[\gamma\delta]}g_{\alpha\beta}, \\
{}^{(3)}Z_{\gamma\delta\alpha\beta} &= \frac{n}{n^2-4}[(\not{\mathrm{R}}\mathrm{iz}_{\alpha[\gamma} - \not{\mathrm{R}}\mathrm{iz}_{[\gamma|\alpha)}g_{|\beta|\delta]} + (\not{\mathrm{R}}\mathrm{iz}_{\beta[\gamma} - \not{\mathrm{R}}\mathrm{iz}_{[\gamma|\beta)}g_{|\alpha|\delta]}] + \frac{4}{n^2-4}\not{\mathrm{R}}\mathrm{iz}_{[\gamma\delta]}g_{\alpha\beta}, \\
{}^{(4)}Z_{\gamma\delta\alpha\beta} &= \frac{1}{n}Z_{\gamma\delta\rho}{}^{\rho}g_{\alpha\beta}, \qquad {}^{(5)}Z_{\gamma\delta\alpha\beta} = \frac{1}{n}[(\not{\mathrm{R}}\mathrm{iz}_{\alpha[\delta} + \not{\mathrm{R}}\mathrm{iz}_{[\delta|\alpha|})g_{\gamma]\beta} + (\not{\mathrm{R}}\mathrm{iz}_{\beta[\delta} + \not{\mathrm{R}}\mathrm{iz}_{[\delta|\beta|})g_{\gamma]\alpha}].
\end{aligned}
$$

With regard to the uniqueness of the decomposition, a remark is in order: If we simply apply the Young diagram procedure to the components $Z_{\gamma\delta\alpha\beta}$ of $Z_{\alpha\beta}$ and take traces, three of the five irreducible pieces obtained are ${}^{(1)}Z_{\alpha\beta}$, ${}^{(2)}Z_{\alpha\beta}$, and ${}^{(5)}Z_{\alpha\beta}$, as above, but the remaining two pieces are arbitrary combinations of the two irreducible subspaces involved in ${}^{(3)}Z_{\alpha\beta}$ and ${}^{(4)}Z_{\alpha\beta}$ above and hence are not canonical. Here, however, the initial decomposition (A2) with respect to the indices on the *two-form* $Z_{\alpha\beta}$ has led to a unique canonical set of irreducible pieces.

2. In exterior calculus—analogies with the irreducible decomposition of $Q_{\alpha\beta}$

We recall the definition of the tracefree shear curvature two-form

$$\not{Z}_{\alpha\beta} = Z_{\alpha\beta} - \frac{1}{n}g_{\alpha\beta}Z, \quad \text{with} \quad Z = Z_{\gamma}{}^{\gamma}. \tag{A2}$$

We cut this two-form into different pieces by contracting with e_β and transvecting with ϑ^α:

$$
\begin{aligned}
\not{Z}_{\alpha} &:= e^{\beta}\rfloor\not{Z}_{\alpha\beta} \equiv \not{\mathrm{R}}\mathrm{iz}_{\alpha}, \qquad \hat{\Delta} := \frac{1}{n-2}\vartheta^{\alpha}\wedge\not{Z}_{\alpha}, \\
S_{\alpha} &:= \not{Z}_{\alpha\beta}\wedge\vartheta^{\beta} + \vartheta_{\alpha}\wedge\hat{\Delta}.
\end{aligned} \tag{A3}
$$

We have $\vartheta^\alpha\wedge S_\alpha = 0$, $e^\alpha\rfloor S_\alpha = 0$; that is, the three-form S_α, in 4D, has $4\times 4 - 1 - 6 = 9$ independent components. Subsequently we can subtract out the trace of $\not{Z}_\alpha$:

$$\Xi_{\alpha} := \not{Z}_{\alpha} - \tfrac{1}{2}e_{\alpha}\rfloor(\vartheta^{\gamma}\wedge\not{Z}_{\gamma}). \tag{A4}$$

We have $\vartheta^\alpha\wedge\Xi_\alpha = 0$, $e^\alpha\rfloor\Xi_\alpha = 0$; that is, the one-form Ξ_α, in 4D, has $4\times 4 - 6 - 1 = 9$ independent components.

The irreducible pieces may then be written as (the number of independent components is specified for $n = 4$)

$$\text{(9 ind. comp.)} \qquad {}^{(2)}Z_{\alpha\beta} := \tfrac{1}{2}e_{(\alpha}\rfloor S_{\beta)}, \tag{A5}$$

$$\text{(6 ind. comp.)} \qquad {}^{(3)}Z_{\alpha\beta} := \frac{n}{n+2}\left(\vartheta_{(\alpha}\wedge e_{\beta)}\rfloor - \frac{2}{n}g_{\alpha\beta}\right)\hat{\Delta}, \tag{A6}$$

$$\text{(6 ind. comp.)} \qquad {}^{(4)}Z_{\alpha\beta} := \frac{1}{n}g_{\alpha\beta}Z, \tag{A7}$$

$$\text{(9 ind. comp.)} \qquad {}^{(5)}Z_{\alpha\beta} := \frac{2}{n}\vartheta_{(\alpha}\wedge\Xi_{\beta)}, \tag{A8}$$

$$\text{(30 ind. comp.)} \qquad {}^{(1)}Z_{\alpha\beta} := Z_{\alpha\beta} - {}^{(2)}Z_{\alpha\beta} - {}^{(3)}Z_{\alpha\beta} - {}^{(4)}Z_{\alpha\beta} - {}^{(5)}Z_{\alpha\beta}. \tag{A9}$$

Apparently, the forms $\{S_\alpha, \hat{\Delta}, Z, \Xi_\alpha\}$ are equivalent to the irreducible pieces $\{{}^{(2)}Z_{\alpha\beta}, {}^{(3)}Z_{\alpha\beta}, {}^{(4)}Z_{\alpha\beta}, {}^{(5)}Z_{\alpha\beta}\}$, respectively.

The strain curvature is of the type of a field strength. The corresponding "potential" is expected to be the nonmetricity $Q_{\alpha\beta}$. As we will show, the irreducible decomposition of the nonmetricity is reminiscent of that of the strain curvature. In order to underline this, we will present all definitions etc. strictly in parallel to the formulas above of the strain curvature.

We start with the tracefree nonmetricity one-form

$$\not{Q}_{\alpha\beta} = Q_{\alpha\beta} - g_{\alpha\beta}Q, \quad \text{with} \quad Q = \frac{1}{n}Q_{\gamma}{}^{\gamma}. \tag{A10}$$

We cut this two-form into different pieces by contracting with e_β and transvecting with ϑ^α:

$$
\begin{aligned}
\Lambda_{\alpha} &:= e^{\beta}\rfloor\not{Q}_{\alpha\beta}, \qquad \Lambda := \vartheta^{\alpha}\Lambda_{\alpha}, \\
P_{\alpha} &:= \not{Q}_{\alpha\beta}\wedge\vartheta^{\beta} - \frac{1}{n-1}\vartheta_{\alpha}\wedge\Lambda.
\end{aligned} \tag{A11}
$$

We have $\vartheta^\alpha\wedge P_\alpha = 0$, $e^\alpha\rfloor P_\alpha = 0$; that is, the two-form P_α, in 4D, has $6\times 4 - 4 - 4 = 16$ independent components.

The irreducible pieces may then be written as (the number of independent components is specified for $n = 4$)

$$\text{(16 ind. comp.)} \qquad {}^{(2)}Q_{\alpha\beta} := -\tfrac{2}{3}e_{(\alpha}\rfloor P_{\beta)}, \tag{A12}$$

(4 ind. comp.)

$$ {}^{(3)}Q_{\alpha\beta} := \frac{2n}{(n-1)(n-2)}\left(\vartheta_{(\alpha}e_{\beta)}\rfloor - \frac{1}{n}g_{\alpha\beta}\right)\Lambda, \tag{A13} $$

(4 ind. comp.) $$ {}^{(4)}Q_{\alpha\beta} := Qg_{\alpha\beta}, \tag{A14} $$

(16 ind. comp.)

$$ {}^{(1)}Q_{\alpha\beta} := Q_{\alpha\beta} - {}^{(2)}Q_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta} - {}^{(4)}Q_{\alpha\beta}. \tag{A15} $$

Apparently, the forms $\{P_\alpha, \Lambda, Q\}$ are equivalent to the irreducible pieces $\{{}^{(2)}Q_{\alpha\beta}, {}^{(3)}Q_{\alpha\beta}, {}^{(4)}Q_{\alpha\beta}\}$, respectively.

The analogies between the different irreducible decompositions of the forms T^α, $Q^{\alpha\beta}$, and $Z^{\alpha\beta}$ in n dimensions can be displayed in a pictorial description as follows:

$$ \begin{pmatrix} T \\ \mathfrak{V} \end{pmatrix} \quad \circ\!\!-\!\!\bullet \quad T^\alpha = \begin{pmatrix} {}^{(1)}T^\alpha \\ {}^{(2)}T^\alpha \\ {}^{(3)}T^\alpha \end{pmatrix} \tag{A16} $$

$$ \begin{pmatrix} P_\alpha \\ \Lambda \\ Q \end{pmatrix} \quad \circ\!\!-\!\!\bullet \quad Q_{\alpha\beta} = \begin{pmatrix} {}^{(1)}Q^{\alpha\beta} \\ {}^{(2)}Q^{\alpha\beta} \\ {}^{(3)}Q^{\alpha\beta} \\ {}^{(4)}Q^{\alpha\beta} \end{pmatrix} \tag{A17} $$

$$ \begin{pmatrix} S_\alpha \\ \hat{\Delta} \\ Z \\ \Xi_\alpha \end{pmatrix} \quad \circ\!\!-\!\!\bullet \quad Z_{\alpha\beta} = \begin{pmatrix} {}^{(1)}Z^{\alpha\beta} \\ {}^{(2)}Z^{\alpha\beta} \\ {}^{(3)}Z^{\alpha\beta} \\ {}^{(4)}Z^{\alpha\beta} \\ {}^{(5)}Z^{\alpha\beta} \end{pmatrix} \tag{A18} $$

where the symbol $\circ\!\!-\!\!\bullet$ denotes the correspondence between the set of forms on the left-hand side and the corresponding irreducible pieces of the field strengths on the right-hand side. Hence, the common procedure shows that we need k independent forms (generally of different degrees) to create $k+1$ irreducible pieces of the corresponding field strength. We recall the definition $T := e_\alpha\rfloor T^\alpha$ and of ${}^{(3)}T^\alpha := e^\alpha\rfloor\mathfrak{V}$, together with $\mathfrak{V} := \frac{1}{3}\vartheta^\beta \wedge T_\beta$.

For later convenience, we list the irreducible pieces as wedged with ϑ^β:

$$ \begin{aligned} &{}^{(1)}Z_{\alpha\beta}\wedge\vartheta^\beta = 0, \qquad {}^{(2)}Z_{\alpha\beta}\wedge\vartheta^\beta = S_\alpha, \\ &{}^{(3)}Z_{\alpha\beta}\wedge\vartheta^\beta = -\vartheta_\alpha\wedge\hat{\Delta}, \qquad {}^{(4)}Z_{\alpha\beta}\wedge\vartheta^\beta = \frac{1}{n}\vartheta_\alpha\wedge Z, \\ &{}^{(5)}Z_{\alpha\beta}\wedge\vartheta^\beta = 0, \qquad \mathcal{Z}_{\alpha\beta}\wedge\vartheta^\beta = S_\alpha - \vartheta_\alpha\wedge\hat{\Delta}. \end{aligned} \tag{A19} $$

We can do the analogous for the nonmetricity:

$$ \begin{aligned} &{}^{(1)}Q_{\alpha\beta}\wedge\vartheta^\beta = 0, \\ &{}^{(2)}Q_{\alpha\beta}\wedge\vartheta^\beta = P_\alpha, \\ &{}^{(3)}Q_{\alpha\beta}\wedge\vartheta^\beta = \frac{1}{n-1}\vartheta_\alpha\wedge\Lambda, \\ &{}^{(4)}Q_{\alpha\beta}\wedge\vartheta^\beta = -\vartheta_\alpha\wedge Q, \\ &\mathcal{Q}_{\alpha\beta}\wedge\vartheta^\beta = P_\alpha + \frac{1}{n-1}\vartheta_\alpha\wedge\Lambda. \end{aligned} \tag{A20} $$

APPENDIX B: ZEROTH BIANCHI IDENTITY

1. Zeroth Bianchi identity in different disguises

A link between the three-form $S_\alpha \sim {}^{(2)}Z_{\alpha\beta}$ and the two-form $P_\alpha \sim {}^{(2)}Q_{\alpha\beta}$ can be found via the zeroth Bianchi identity:

$$ DQ_{\alpha\beta} \equiv 2Z_{\alpha\beta}. \tag{B1} $$

We introduce the slashed quantities:

$$ D\mathcal{Q}_{\alpha\beta} + D(Qg_{\alpha\beta}) = 2\mathcal{Z}_{\alpha\beta} + \frac{2}{n}g_{\alpha\beta}Z_\gamma{}^\gamma \tag{B2} $$

or, since $dQ = \frac{2}{n}Z_\gamma{}^\gamma$,

$$ D\mathcal{Q}_{\alpha\beta} + dQg_{\alpha\beta} + Q\wedge Q_{\alpha\beta} = 2\mathcal{Z}_{\alpha\beta} + g_{\alpha\beta}dQ. \tag{B3} $$

Accordingly,

$$ D\mathcal{Q}_{\alpha\beta} + Q\wedge\mathcal{Q}_{\alpha\beta} = 2\mathcal{Z}_{\alpha\beta}. \tag{B4} $$

The difference between the connection $\Gamma_\alpha{}^\beta$ and the Riemannian connection $\tilde{\Gamma}_\alpha{}^\beta$ is the distortion one-form

$$ \begin{aligned} &N_\alpha{}^\beta = \Gamma_\alpha{}^\beta - \tilde{\Gamma}_\alpha{}^\beta, \\ &\text{with} \qquad N_{(\alpha\beta)} = \tfrac{1}{2}Q_{\alpha\beta}, N_\beta{}^\alpha\wedge\vartheta^\beta = T^\alpha. \end{aligned} \tag{B5} $$

If we execute the covariant exterior differentiation in (B4), we find

$$ \begin{aligned} &\tilde{D}\mathcal{Q}_{\alpha\beta} - N_{[\alpha\gamma]}\wedge\mathcal{Q}^\gamma{}_\beta - N_{[\beta\gamma]}\wedge\mathcal{Q}_\alpha{}^\gamma + Q\wedge\mathcal{Q}_{\alpha\beta} \\ &\qquad - \tfrac{1}{2}Q_{\alpha\gamma}\wedge\mathcal{Q}^\gamma{}_\beta - \tfrac{1}{2}Q_{\beta\gamma}\wedge\mathcal{Q}_\alpha{}^\gamma = 2\mathcal{Z}_{\alpha\beta}. \end{aligned} \tag{B6} $$

After some algebra, the explicit square pieces in the nonmetricity drop out. Thus,

$$ \tilde{D}\mathcal{Q}_{\alpha\beta} - N_{[\alpha\gamma]}\wedge\mathcal{Q}^\gamma{}_\beta - N_{[\beta\gamma]}\wedge\mathcal{Q}_\alpha{}^\gamma = 2\mathcal{Z}_{\alpha\beta}. \tag{B7} $$

Let us come back to (B4). We wedge from the right-hand side with ϑ^β:

$$ \begin{aligned} &D(\mathcal{Q}_{\alpha\beta}\wedge\vartheta^\beta) + Q\wedge(\mathcal{Q}_{\alpha\beta}\wedge\vartheta^\beta) + \mathcal{Q}_{\alpha\beta}\wedge T^\beta \\ &\quad = 2(\mathcal{Z}_{\alpha\beta}\wedge\vartheta^\beta). \end{aligned} \tag{B8} $$

Also, here we can provide a version with a Riemannian derivative. The simplest is to wedge (B7) from the right with ϑ^β and to note $\tilde{D}\vartheta^\alpha = 0$:

$$(\tilde{D}\mathcal{Q}_{\alpha\beta}\wedge\vartheta^{\beta})-N_{[\alpha\gamma]}\wedge\mathcal{Q}^{\gamma}{}_{\beta}\wedge\vartheta^{\beta}-N_{[\beta\gamma]}\wedge\mathcal{Q}_{\alpha}{}^{\gamma}\wedge\vartheta^{\beta}=2\mathcal{Z}_{\alpha\beta}\wedge\vartheta^{\beta}. \tag{B9}$$

Then we substitute (A19) and (A20) into (B8) and find

$$D\left(P_{\alpha}+\frac{1}{n-1}\vartheta_{\alpha}\wedge\Lambda\right)+Q\wedge\left(P_{\alpha}+\frac{1}{n-1}\vartheta_{\alpha}\wedge\Lambda\right)+\mathcal{Q}_{\alpha\beta}\wedge T^{\beta}=2(S_{\alpha}-\vartheta_{\alpha}\wedge\hat{\Delta}). \tag{B10}$$

We differentiate the sum and collect the torsion dependent terms

$$(DP_{\alpha}+Q\wedge P_{\alpha})-\frac{1}{n-1}\vartheta_{\alpha}\wedge(d\Lambda+Q\wedge\Lambda)+\left(\mathcal{Q}_{\alpha\beta}+\frac{1}{n-1}g_{\alpha\beta}\Lambda\right)\wedge T^{\beta}=2(S_{\alpha}-\vartheta_{\alpha}\wedge\hat{\Delta}). \tag{B11}$$

Our strategy is now to separate S_{α} from $\hat{\Delta}$. We contract (B11) from the left with $-\frac{1}{4}e^{\alpha}\rfloor$:

$$-\tfrac{1}{4}e^{\gamma}\rfloor\{\text{l.h.s. of (B11)}\}_{\gamma}=\tfrac{1}{2}e^{\alpha}\rfloor(\vartheta_{\alpha}\wedge\hat{\Delta})=2\hat{\Delta}-\tfrac{1}{2}\vartheta_{\alpha}\wedge(e^{\alpha}\rfloor\hat{\Delta})=\hat{\Delta} \tag{B12}$$

or

$${}^{(3)}Z_{\alpha\beta}\sim\hat{\Delta}=-\tfrac{1}{4}e^{\gamma}\rfloor\{\text{l.h.s. of (B11)}\}_{\gamma}. \tag{B13}$$

Now we can resolve (B11) with respect to S_{α}:

$${}^{(2)}Z_{\alpha\beta}\sim S_{\alpha}=\frac{1}{2}\left(DP_{\alpha}+Q\wedge P_{\alpha}-\frac{1}{n-1}\vartheta_{\alpha}\wedge(d\Lambda+Q\wedge\Lambda)+\left(\mathcal{Q}_{\alpha\beta}+\frac{1}{n-1}g_{\alpha\beta}\Lambda\right)\wedge T^{\beta}\right)+\vartheta_{\alpha}\wedge\hat{\Delta}. \tag{B14}$$

In this formula, ${}^{(2)}Z_{\alpha\beta}\sim S_{\alpha}$ is expressed in terms of nonmetricity and torsion. Note that our results (B13) and (B14) are generally valid. No constraints have been assumed so far. However, this will be done in the next subsection.

2. Consequences of the ansatz (75) and of vanishing torsion (80)

We substitute (78)–(80) into (B14):

$$S_{\alpha}=-\frac{1}{2(n-1)}\vartheta_{\alpha}\wedge d\Lambda+\vartheta_{\alpha}\wedge\hat{\Delta}. \tag{B15}$$

The two-form $\hat{\Delta}$ we take from (B13) after the constraints (78)–(80) have been substituted into the left-hand side of (B11). Thus,

$$\hat{\Delta}=\frac{1}{2(n-1)}d\Lambda\quad\text{and}\quad S_{\alpha}=0 \tag{B16}$$

or

$$\begin{aligned}&{}^{(2)}Z_{\alpha\beta}=0\quad\text{and}\\&{}^{(3)}Z_{\alpha\beta}=\frac{n}{2(n-1)(n+2)}\left(\vartheta_{(\alpha}\wedge e_{\beta)}\rfloor-\frac{2}{n}g_{\alpha\beta}\right)d\Lambda.\end{aligned} \tag{B17}$$

APPENDIX C: FIRST BIANCHI IDENTITY

Consider the first Bianchi identity,

$$DT^{\alpha}\equiv R_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}. \tag{C1}$$

The irreducible pieces of $W_{\alpha\beta}$ and $Z_{\alpha\beta}$ obey quite generally the algebraic constraints [1],

$${}^{(1)}W_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}={}^{(4)}W_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}={}^{(6)}W_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}={}^{(1)}Z_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}={}^{(5)}Z_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}=0. \tag{C2}$$

Thus,

$$DT^{\alpha}=({}^{(2)}W_{\beta}{}^{\alpha}+{}^{(3)}W_{\beta}{}^{\alpha}+{}^{(5)}W_{\beta}{}^{\alpha}+{}^{(2)}Z_{\beta}{}^{\alpha}+{}^{(3)}Z_{\beta}{}^{\alpha}+{}^{(4)}Z_{\beta}{}^{\alpha})\wedge\vartheta^{\beta}=0. \tag{C3}$$

APPENDIX D: SECOND BIANCHI IDENTITY

The second Bianchi identity reads

$$DR_{\beta}{}^{\alpha}=D(Z_{\beta}{}^{\alpha}+W_{\beta}{}^{\alpha})=D\left(\sum_{I=1}^{5}{}^{(I)}Z_{\beta}{}^{\alpha}+\sum_{I=1}^{6}{}^{(I)}W_{\beta}{}^{\alpha}\right)=D\left(\sum_{I=1}^{5}({}^{(I)}Z_{\beta}{}^{\alpha}+{}^{(I)}W_{\beta}{}^{\alpha})-\frac{W}{12}\vartheta_{\beta}\wedge\vartheta^{\alpha}\right)\equiv 0. \tag{D1}$$

Here $W:=e_{\gamma}\rfloor e_{\delta}\rfloor W^{\gamma\delta}$ is the curvature scalar and the corresponding term in (D1) represents the sixth piece of $W_{\beta}{}^{\alpha}$; see [1].

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