On the Termination of Dynamics in Sequential Games

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Abstract

We consider n-player non-zero sum games played on finite trees (i.e., sequential games), in which the players have the right to repeatedly update their respective strategies (for instance, to improve the outcome wrt to the current strategy profile). This generates a dynamics in the game which may eventually stabilise to a Nash Equilibrium (as with Kukushkin's lazy improvement), and we argue that it is interesting to study the conditions that guarantee such a dynamics to terminate.

We build on the works of Le Roux and Pauly who have studied one such dynamics, namely the Lazy Improvement Dynamics. We extend these works by first defining a turn-based dynamics, proving that it terminates on subgame perfect equilibria, and showing that several variants do not terminate. Second, we define a variant of Kukushkin's lazy improvement where the players may now form coalitions to change strategies. We show how properties of the players' preferences on the outcomes affect the termination of this dynamics, and we thereby characterise classes of games where it always terminates (in particular two-player games).

1. Introduction

Since the seminal works of Morgenstern and von Neuman in the forties [vNM44], game theory has emerged as a prominent paradigm to model the behaviour of rational and selfish agents acting in a competitive setting. The first and main application of game theory is to be found in the domain of economics where the agents can model companies or investors who are competing

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for profits, to gain access to market, etc. Since then, game theory has evolved into a fully developed mathematical theory and has recently found many applications in computer science. In this setting, the agents usually model different components of a computer system (and of its environment), that have their own objective to fulfil. For example, game theory has been applied to analyse peer-to-peer file transfer protocols [NRTV07] where the agents want to maximise the amount of downloaded data in order to obtain, say, a given file; while minimising the amount of uploaded data to save bandwidth. Another application is that of controller synthesis where the two, fully antagonistic, agents are the system and its environment, and where the game objective models the control objective.

The most basic model to describe the interaction between the players is that of games in *strategic form* (aka matrix games), where all the players choose simultaneously an action (or strategy) from a finite set, and where they all obtain a payoff (usually modelled by a real value) which depends on their joint choice of actions (or strategy profile). Strategic games are one-shot games, in the sense that the players play only one action, simultaneously. An alternative model where players play in turn is that of *sequential games*, which we consider in this work. Such a game is played on a finite tree whose inner nodes are labelled by players, whose edges correspond to the possible actions of the players, and whose leaves are associated with *outcomes*. The game starts at the root, and, at each step of the game, the player who owns the current node picks an edge (i.e. an action) from this node, and the game moves to the destination node of the edge. The game ends when a leaf is reached, and the players obtain the outcome associated with that leaf. Moreover, each player has a *preference* over the possible outcomes, and each player's aim is thus to obtain the best outcome according to this preference relation.

Arguably the most natural question about these games is to determine how rational and selfish players would act (assuming that they have full knowledge of the other players's possible actions and of the outcomes). Several answers to this question have been provided in the literature, such as the famous notion of Nash Equilibrium [Nas50], which is a situation (strategy profile) in which no player has an incentive to change his choice of action alone (because such a choice would not be profitable to him). Apart from Nash equilibria, other *solution concepts* have been proposed like Subgame Perfect Equilibria. All these classical notions are defined in [OR94].

This traditional view on game theory can be qualified as *static*, in the sense that all players chose their strategies (thereby forming a strategy profile that can qualify as one of the equilibria listed above), and the game is then played *once* according to these strategies. It is, however, also natural to consider a more *dynamic* setting, in which the players can play the game repeatedly, updating their respective strategies after each play, in order to try and improve the outcome at the next play.

Contribution. In this paper, we continue a series of works that aim at connecting these static and dynamic views. That is, we want to study (in the case of extensive form games) the long term behaviour of the dynamics in which players update repeatedly their strategies and characterise when such dynamics converge, and to what kind of strategy profiles (i.e., do these stable profiles correspond to some notions of equilibria?). Obviously, this will depend on how the players update their strategies between two plays of the game. Our results consist in identifying minimal conditions on the updates (modelling potential rational behaviours of the players) for which we can guarantee convergence to some form of equilibria after a bounded number of updates. Our work is an extension of a previous paper by Le Roux and Pauly [LRP16], where they study extensively the so-called *Lazy Improvement Dynamics*. Intuitively, in this dynamics, a single player can update his strategy at a time, in order to improve his outcome, and *only* in nodes that have an influence on the final outcome (lazy update). Their main result is that this dynamics terminate on Nash Equilibria when the preferences of the players are acyclic. Our contribution consists in considering a broader family of dynamics and characterising their termination. More precisely:

- We start (Section 3) by considering all dynamics that respect the *subgame improvement* property, where players update their strategies only if this yields a better outcome in all subgames that are rooted in a node where a change has been performed. We argue that this can be considered as a rational behaviour of the players: improving in the subgames can be regarded as an incentive, even when this does not improve the global outcome of the game. Indeed, such an improvement can turn out to be profitable if one of the other players later deviates from its current choice (this is the same intuition as the one behind the notion of Subgame Perfect Equilibrium). Note that such dynamics have not been considered at all in [LRP16]. We show that, in all games where the preferences of the players are acyclic, these dynamics terminate and for some of these dynamics, the terminal profiles are exactly the Subgame Perfect Equilibria of the game.
- Then, we consider (Section 4) all the dynamics that respect the *improvement* property, where all players that change their respective strategy improve the outcome (from the point of view of their respective preferences). Among these dynamics are several ones that have already been studied by Le Roux and Pauly [LRP16] such as the Lazy Improvement Dynamics. We complete the picture (see Table 1, page 11), in particular we consider the dynamics that satisfies the *improvement* and the *laziness* properties but does not restrict the update to be performed by a single player, contrary to the *Lazy Improvement Dynamics* of Le Roux and Pauly. Thus in our dynamics, players play lazily but are allowed to form coalitions to obtain an outcome which is better for all the players of the coalition. We give necessary and sufficient conditions on the preferences of the players, to ensure termination (on Strong Nash Equilibria), in several classes of games (among which 2 player games).

Related works. The most related work is in the paper by Le Roux and Pauly [LRP16] that we extend here, as already explained. This work is inspired by the notions of *potential* and *semi-potential* introduced respectively by Monderer and Shapley [MS96] (see also [NRTV07]); and Kukushkin [Kuk02]. Note also that the idea of repeatedly playing a game and updating the players strategies between two plays is also behind evolutionary game theory, but in this case, the rules governing the updates are inspired from Darwinian evolution [MSP73, Wei95]. The best reply dynamics is one of the classical dynamics in evolutionary game theory. At each step, it consists in letting the players choose their best reply in the current strategy profile. Potential games [MS96] are games in which there exists a global function (called potential) which expresses the incentive of the players in changing their strategies. An important result in potential games is that the best reply dynamics terminates in every finite potential game [MS96]. The termination is a consequence of the fact that every change improves the global potential. The notion of semi-potential was also introduced by Kukushkin [Kuk02], in order to consider a larger class of games, while preserving some termination properties, considering restricted dynamics. The main objective of the present paper is to establish similar termination results, but for different dynamics and without assuming the existence of a (semi-)potential.

While our dynamics involve some notion of repetition, we do not fit in the classical setting of repeated games. Indeed, the payoff function of repeated games takes into account the whole sequence of strategy profiles, while in our setting players update their strategies based only on the current profile. Hence, in particular, we do not consider the concept of punishment which is central in the proof of the Folk Theorem for repeated game [LR89] (see also [OR94]).

In Section 4, we allow players to form coalitions, in order to further improve their individual payoffs. As far as we understand, this is different from the model of cooperative games (or coalitional games) [OR94] where the payoffs are associated with all the possible coalitions. In cooperative games, the solution concepts aim at optimising the partition of the players into coalitions, while in our games, the players retain their individual and selfish incentive.

2. Preliminaries

We start by defining all the notions upon which we build our results, mainly the notions of games, strategy profiles, equilibria and dynamics.

2.1. Sequential games

We consider *sequential games* (also called extensive form games), which are n-player non-zero sum games played on finite trees. Before giving the formal definition, we discuss an example to help the reader with intuition.

Example 1. *Figure 1 shows an example of sequential game, with two players denoted* 1 *and* 2*. Intuitively, each node of the tree is controlled by either of the players, and the game is played by moving a token along the branches of the tree, from the root node, down to the leaves, which are labelled by an outcome (in this case, x, y or z). The tree edges are labelled by the actions that the player controlling the origin node can play. For example, in the root node, Player* 1 *can choose to play r, in which case the game reaches the second node, controlled by Player 2, who can chose to play l. The payoff for both players is then y. We also associate a preference relation with each player that indicates how he ranks the payoffs. In the example of Figure 1, Player 1 prefers z to x and x to y (noted $y \prec_1 x \prec_1 z$), and Player 2 prefers y to z and z to x.*

Let us now give a formal definition of sequential games. The definitions and notations of this section are inspired by [Osb09]:

Definition 1. *A* sequential *or* extensive form *game G is a tuple*

$$\langle N, A, H, O, d, p, (\prec_i)_{i \in N} \rangle,$$

where:

- *N is a non-empty finite set of **players**;*
- *A is a non-empty finite set of **actions**;*
- *H is a finite set of finite sequences of A which is prefix-closed. That is, the empty sequence ε is a member of H; and $h = a^1 \dots a^k \in H$ implies that $h' = a^1 \dots a^\ell \in H$ for all $\ell < k$. Each member of H is called a node. A node $h = a^1 \dots a^k \in H$ is* terminal *if $\forall a \in A$, $a^1 \dots a^k a \notin H$. The set of terminal nodes is denoted by Z.*
- *O is the non-empty set of **outcomes**,*
- *$d : H \setminus Z \to N$ associates a player with each nonterminal node;*
- *$p : Z \to O$ associates an outcome with each terminal node;*
- *For all $i \in N$: $\prec_i$ is a binary relation over O, modelling the preferences of Player i. We write $x \prec_i y$ and $x \not\prec_i y$ when $(x, y) \in \prec_i$ and $(x, y) \notin \prec_i$ respectively*

Observe that, in this definition, we make no further hypothesis on the preference relations (in particular, we do not assume transitivity, so the preferences are not necessarily orders). We will rather show how different assumptions influence the termination of dynamics. In the rest of the paper, we say that a preference $\prec$ is *cyclic* iff there are outcomes x_1, $x_2, \dots$, x_n s.t. $x_1 \prec x_2 \prec \cdots \prec x_n \prec x_1$. Otherwise, the preference is said to be *acyclic*. One might wonder how *cyclic* preferences would correspond to *rational* behaviours of players in a game. However, some authors argue that such preferences can make sense in practice. The following example is due to Laraki and Zamir [LZ, p. 12]:

Example 2. *Let us assume we need to choose between three means of transportation to get to work: car, bus or underground. The travel times for car, bus and underground are respectively 18, 14 and 10 minutes. However, the car is more comfortable than the bus and the underground. Also, the bus is more comfortable than the ever-crowded underground. A rational player could decide*

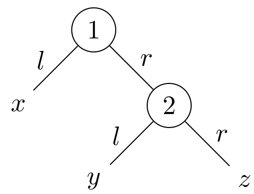

Figure 1: A sequential game with two players.

that: (i) when the difference in travel time between two transportation means is less than 5 minutes, the most comfortable one will be chosen; (ii) otherwise, the fastest one is chosen. Thus, the player prefers the car to the bus, because the difference in time is only four minutes and the car is more comfortable than the bus. He also prefers the bus to the underground, for the same reasons. However, when facing the choice between car and underground, the car is chosen because the difference in time is more than 5 minutes so speed is more important. Hence:

$$\text{car} \prec \text{underground} \prec \text{bus} \prec \text{car},$$

which is a clear case of a cyclic preference that models an arguably rational choice.

From now on, we fix a sequential game $G = \langle N, A, H, O, d, p, (\prec_i)_{i \in N} \rangle$. Then, we let $H_i = \{h \in H \setminus Z \mid d(h) = i\}$ be the set of **nodes** belonging to player i. A **strategy** $s_i : H_i \to A$ of player i is a function associating an action with all nodes belonging to player i, s.t. for all $h \in H_i$: $hs_i(h) \in H$, i.e. $s_i(h)$ is a legal action from h. Then, a tuple $s = (s_i)_{i \in N}$ associating one strategy with each player is called a **strategy profile** and we let $Strat_G$ be the set of all strategy profiles in G. For all strategy profiles s, we denote by $\langle s \rangle$ the **outcome** of s, which is the outcome of the terminal node obtained when all players play according to s. Formally, $\langle s \rangle = p(h)$ where $h = a^1 \dots a^k \in Z$ is s.t. for all $0 \leq \ell < k$: $d(a^1 \dots a^\ell) = i$ implies $a^{\ell+1} = s_i(a^1 \dots a^\ell)$. Let $s = (s_i)_{i \in N}$ be a strategy profile, and let s_j^* be a Player j strategy. Then, we denote by (s_{-j}, s_j^*) the strategy profile $(s_1, ..., s_{j-1}, s_j^*, s_{j+1}, ..., s_{|N|})$ where all players play s_i, except Player j who plays s_j^*. Since a strategy profile $s = (s_i)_{i \in N}$ fixes a strategy for all players, we abuse notations and write, for all nodes $h \in H$, $s(h)$ to denote the action $s_{d(h)}(h)$, i.e. the action that the owner of h plays in h according to s. Let s be a strategy profile, and let $h \in H \setminus Z$ be a nonterminal node. Then, we let: (1) $H|_h = \{h' \mid hh' \in H\}$ be the **subtree** of H from h; (2) $s|_h$ be the **substrategy profile** of s from h which is s.t. $\forall hh' \in H \setminus Z$: $s(hh') = s|_h(h')$; and (3) $G|_h = \langle N, A, H|_h, O, d|_h, p|_h, (\prec_i)_{i \in N} \rangle$ be the **subgame** of G from h, with, for $h' \in H|_h$, $d|_h(h') = d(hh')$ and $p|_h(h') = p(hh')$. Then, we say that a node $h = a^1 ... a^k$ **lies along** the play induced by s if $s(\varepsilon) = a^1$ and $\forall 1 \leq \ell < k$, $s(a^1 \dots a^\ell) = a^{\ell+1}$.

Example 3. *As an example, let us consider again the game in Figure 1. In this game, both players can chose either l or r in the nodes they control. So,* *a possible strategy for Player* 1 *is* s_1 *s.t.* $s_1(\varepsilon) = r$*; and a possible strategy for Player* 2 *is* s_2 *s.t.* $s_2(r) = l$*. Then,* $\langle(s_1, s_2)\rangle = y$*. Observe that, in our examples, we denote a strategy profile by the actions taken by the players. For example, we denote the profile* $s = (s_1, s_2)$ *by* rl*. With this notation, this game has four strategy profiles :* rr*,* rl*,* lr *and* ll*.*

Equilibria. Now that we have fixed the notions of games and strategies, we turn our attention to three classical notions of *equilibria*. First, a strategy profile s^* is a **Nash Equilibrium** (NE for short) if for all players $i \in N$, for all strategies s_i of player i:

$$\left\langle(s^*_{-i}, s^*_i)\right\rangle \not\prec_i \left\langle(s^*_{-i}, s_i)\right\rangle .$$

It means that, in an NE s^*, no player has interest in deviating alone from his current choice of strategy (because no such possible deviation is profitable to him).

On the other hand, a strategy profile s^* is a **Subgame Perfect Equilibrium** (SPE for short) if, for all players $i \in N$, for all strategies s_i of player i, for all nonterminal nodes $h \in H_i$ of player i:

$$\left\langle(s^*_{-i}|_h, s^*_i|_h)\right\rangle \not\prec_i \left\langle(s^*_{-i}|_h, s_i|_h)\right\rangle$$

in the subgame $G|_h$. In other words, s^* is an NE in every subgame of G.

Finally, in [Aum59], Aumann defines a **Strong Nash Equilibrium** (SNE for short) as a strategy profile in which there is no coaltion of players that can deviate while ensuring that each member of the coaltion improves their outcome. Formally, a strategy profile s^* is an SNE if, for all coalitions of players $I \subseteq N$, for all strategy profiles s_I of the coalition I, there is $i \in I$ s.t.:

$$\left\langle(s^*_{-I}, s^*_I)\right\rangle \not\prec_i \left\langle(s^*_{-I}, s_I)\right\rangle .$$

Thus, the notion of SNE is stronger than the notion of NE. Actually, the notion of SNE has sometimes been described as 'too strong' in the literature, because there are few categories of games in which SNEs are guaranteed to exist (contrary to NEs and SPEs). This has prompted other authors to introduce alternative solution concepts such as Coalition Proof Equilibria [BPW87].

Example 4. *Coming back to the game in Figure 1, the only NE is* ll*. Moreover, it is also an SPE because* l *is an NE in the only subgame of* G*. However,* ll *is not an SNE, because if both players decide to form a coalition and play the* r *action, they obtain* z *as outcome which is better than* x *for both of them. There is actually no SNE in this game. If we consider the same game, but with following preferences for Player* 2*:* $x \prec_2 y \prec_2 z$*, then* ll *is still an NE, but not an SPE. The only SPE of this game is* rr *(which is also an NE* and *an SNE).*

2.2. Dynamics in sequential games

Let us now introduce the central notion of the work, i.e. dynamics in sequential games. As explained in the introduction, we want to study the behaviour of the players when they are allowed to repeatedly update their current strategy in a strategy profile, and characterise the cases where such repeated updates (i.e. dynamics) converge to one of the equilibria we have highlighted above. More specifically, we want to characterise when a dynamics terminates for a game G, i.e. when players cannot infinitely often update their strategy.

Definition 2. *For a sequential game G, a **dynamics** $\underset{G}{\rightharpoonup}$ is a binary relation over $Strat_G \times Strat_G$. We write $s \underset{G}{\rightharpoonup} s'$ whenever $(s, s') \in \underset{G}{\rightharpoonup}$.*

Intuitively, $s \underset{G}{\rightharpoonup} s'$ means that the dynamics $\underset{G}{\rightharpoonup}$ under consideration allows the strategy profile s to be updated into s', by the change of strategy of 0 or more players. When the context is clear, we drop the name of the game G and write $\rightharpoonup$ instead of $\underset{G}{\rightharpoonup}$. Given this definition, it is clear that a dynamics $\rightharpoonup$ corresponds to a directed graph $(Strat_G, \rightharpoonup)$, where $Strat_G$ is the set of graph nodes, and $\rightharpoonup$ is its set of edges. For example, the graphs in Figure 2 (page 9) are some possible graphs representing dynamics associated to the game in Figure 1. Then, we say that the dynamics $\rightharpoonup$ **terminates** if there is no infinite sequence of strategy profiles $\left(s^k\right)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$: $s^k \rightharpoonup s^{k+1}$. Equivalently, $\rightharpoonup$ terminates iff its corresponding graph is acyclic. Intuitively, a dynamics terminates if players can not update their strategy infinitely often, which means that the game will eventually reach stability. We say that a strategy profile s is **terminal** iff there is no s' s.t. $s \rightharpoonup s'$ (i.e. s is a deadlock in the associated graph). Finally, given a pair of strategies s and s', we let $H(s, s') = \{h \in H \mid s(h) \neq s'(h)\}$ (resp. $H_i(s, s') = \{h \in H_i \mid s(h) \neq s'(h)\}$) be the set of nodes (resp. the set of nodes belonging to player i) where the player who owns the node plays differently in s compared to s'. We extend d to set of nodes $H' \subseteq H$ in the natural way: $d(H') = \{d(h) \mid h \in H'\}$.

Families of dynamics. Let us now identify interesting families of dynamics (of whom we want to characterise the terminal profiles), by characterising how players update their strategies from one profile to another.

Definition 3 (Properties of strategy updates)**.** *Let s, s' be two strategy profiles of a game G. Then:*

1. *(s, s') verifies the **Improvement Property**, written $(s, s') \models I$ if $\forall i \in N$: $s_i \neq s'_i$ implies $\langle s\rangle \prec_i \langle s'\rangle$. That is, every player that changes his strategy improves his payoff.*
2. *(s, s') verifies the **Subgame Improvement Property**, written $(s, s') \models SI$, if $\forall i \in N$, $\forall h \in H_i(s, s')$: $\langle s|_h\rangle \prec_i \langle s'|_h\rangle$. That is, every player that changes his strategy improves his (induced) payoff in all the subgames rooted at one of the changes.*
3. *(s, s') verifies the **Laziness Property**, written $(s, s') \models L$, if $\forall h \in H(s, s')$, h lies along the play induced by s'. Intuitively, we consider such updates as* lazy *because we require that the players do not change their strategy in nodes which do not influence the payoff in the resulting profile.*

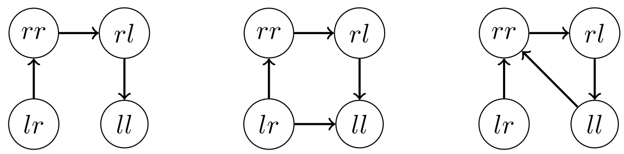

Figure 2: The graphs corresponding to the $\{I, L, 1P\}$-Dynamics (left), $\{SI, A\}$-Dynamics (middle) and $\{I, L\}$-Dynamics (right), respectively, for the game in Figure 1

4. *(s, s') verifies the **One Player Property**, written $(s, s') \models 1P$, if $\exists i \in N$ such that $\forall j \neq i$, $s_j = s'_j$. That is, at most one player updates his strategy (but he can perform changes in as many nodes as he wants).*
5. *(s, s') verifies the **Atomicity Property**, written $(s, s') \models A$, if $\exists h^* \in H$ such that $\forall h \neq h^*$, $s(h) = s'(h)$. That is, the change affects at most one node of the tree.*

Note that, except for the first two properties, those requirements do not depend on the outcome. Arguably the first three properties (Improvement, Subgame Improvement and Laziness) correspond to some kind of rational behaviours of the players, who seek to improve the outcome of the game, while performing a minimal amount of changes. On the other hand, the One Player Property is relevant because such updates do not allow players to form coalitions to improve their outcomes. Finally, the Atomicity Property is interesting *per se* because it corresponds to some kind of *minimal update*, where a single choice of a single player can be changed at a time. Because of that, dynamics respecting Atomicity will be useful in the rest of the paper to establish general results on dynamics.

Based on these properties, we can now define the dynamics that we will consider in this paper. For all $x \in \{I, SI, L, 1P, A\}$, we define the x-Dynamics as the set of all pairs (s, s') s.t. $(s, s') \models x$. Intuitively, the x-Dynamics is the most permissive dynamics where the players update their strategies respecting x. For a set $X \subseteq \{I, SI, L, 1P, A\}$, we define the X-dynamics as the intersection of all the x-dynamics with $x \in X$. Throughout the paper, we denote by $\xrightarrow{X}$ the X-Dynamics.

Example 5. *Examples of graphs associated with dynamics of particular interest (for the game in Figure 1) are displayed in Figure 2: the $\{I, L, 1P\}$-Dynamics (or Lazy Improvement Dynamics in [LRP16]) on the left; the $\{SI, A\}$-Dynamics in the middle (which will be particularly relevant to the discussion in Section 3); and the $\{I, L\}$-Dynamics on the right (which will be particularly relevant in Section 4).*

Relation between dynamics. One can observe that, following Definition 3, any update satisfying the Atomicity Property also satisfy the One Player Property. However, no such implication exist in general between the other properties:

Lemma 4. *Let s and s' be two strategies of a game G. Then:*

(1) $(s, s') \models A$ implies that $(s, s') \models 1P$; and

(2) for all $x, y \in \{I, SI, L, 1P, A\}$ s.t. $(x, y) \neq (A, 1P)$ and $x \neq y$, there exists a pair of strategies s and s' and a game G s.t.: $(s, s') \models x$ and $(s, s') \not\models y$.

Proof. Point (1) follows from Definition 3. Indeed, since only one move can be made between s and s' (because $(s, s') \models A$), then only one player can have updated his strategy.

Most of the cases in Point (2) follow trivially from Definition 3. We only provide a counter example showing that I does not imply SI and vice-versa. Let us consider the game in Figure 1. Then:

- (ll, rr) respects the improvement property (I) but not the subgame improvement (SI). Indeed, for both players, $\langle ll \rangle = x \prec \langle rr \rangle = z$, so we do have $(ll, rr) \models I$, but $\langle ll|_r \rangle = y \not\prec_2 \langle rr|_r \rangle = z$, and then player 2 does not improves his payoff in the subgame $G|_r$, so $(ll, rr) \not\models SI$.

- (lr, ll) respects the subgame improvement property (SI) but not the improvement property (I). Indeed, $(lr, ll) \models SI$, because Player 2 improves his payoff in the subgame $G|_r$ where he changes his strategy. But $(lr, ll) \not\models I$, because $\langle lr \rangle = x = \langle ll \rangle$, and thus, Player 2 does not improve (strictly) his payoff in the whole game, as required for I. □

As a consequence, we obtain direct inclusions between some dynamics. For instance $\xrightarrow{A} \subseteq \xrightarrow{1P}$ in all games.

In [LRP16], Le Roux and Pauly consider the so-called Lazy Improvement Dynamics which corresponds to our $\{I, L, 1P\}$-Dynamics. The underlying idea is to disallow players from making changes in nodes that are irrelevant (because they will not appear along the play generated by the profile), while ensuring that the payoff improves. In [LRP16], Le Roux and Pauly prove that this dynamics terminates for all games that do not have cyclic preferences and that the terminal profiles are exactly the Nash Equilibria. The results we are about to present extend theirs by considering other dynamics as announced. The rest of the paper will be structured as follows: in Section 3, we will consider dynamics which are subsets of the SI-Dynamics (like the $\{SI, A\}$-Dynamics). See line 2 in Table 1 (page 11). In Section 4, we will consider dynamics which are subsets of the I-Dynamics (like the $\{I, L\}$-Dynamics), in order to complete the results obtained by Leroux and Pauly in [LRP16]. See lines 3 to 6 in Table 1.

3. Subgame Improvement Dynamics

In this section we will focus on dynamics that respect the *Subgame Improvement Property* (see Definition 3), i.e. dynamics which are subsets of the SI-Dynamics (note that these dynamics have not been considered at all in [LRP16]). In other words, we will consider all the X-Dynamics s.t. $\{SI\} \subseteq X$.

Table 1: Characterisation of the Dynamics. A ✓in the 'Games' column means that the hypothesis is used to prove the result, the × that it is not and the • that the result does not depend on that. The 'Termination' column gives necessary (n.) and/or sufficient (s.) conditions to ensure that the dynamics terminates on *all* games in the class.

I	SI	A	L	$1P$	Games	Termination	Final Profiles	Reference
×	×	•	•	•		×	not appl.	
•	✓	•	•	•	acyclic prefs	✓	⊇ SPEs	Theorem 5
✓	•	✓	•	•	acyclic prefs	✓	⊇ NEs	Corollary 12
✓	×	×	×	•		×	not appl.	[LRP16]
✓	×	×	✓	×	swo prefs	prefs can be deeply layered (s.)	= SNEs	Theorem 19
					swo prefs	prefs out of pattern (n.)		Theorem 19
					slo prefs	prefs out of pattern (n. & s.)		Corollary 22
					swo prefs, 2 player	prefs out of pattern (n. & s.)		Corollary 22
✓	×	×	✓	✓	acyclic prefs	✓	= NEs	[LRP16]

3.1. Termination with acyclic preferences

The central result of this section is that all those dynamics terminate when the preferences of the players are acyclic[4] and some of these dynamics converge to subgame perfect equilibria, as stated in the following theorem:

Theorem 5. *Let $N, O, (\prec_i)_{i\in N}$ be respectively a set of players, a set outcomes and preferences, and let X be s.t. $SI \in X$. Then, the two following statements are equivalent:*

(1) in all *games built over $N, O, (\prec_i)_{i\in N}$ the X-Dynamics terminates; and*

(2) the preferences $(\prec_i)_{i\in N}$ are acyclic.

Moreover, for all $\{SI\} \subseteq X \subseteq \{SI, 1P, A\}$, the set of terminal profiles of the X-Dynamics is the set of SPEs of the game.

An important consequence of this theorem is that acyclic preferences form a *sufficient condition* to ensure termination of all X-dynamics with $SI \in X$. Observe that this condition is very weak because it constrains only the preferences, and not the structure (tree) of the game. We argue that this is also very reasonable, as acyclic preferences allow to capture many if not most rational preferences[5]. This condition is, however, not necessary: Theorem 5 tells us that when the preferences are cyclic, the dynamics does not terminate in *all* games embedding those preferences. Actually, one can find examples of games with cyclic preferences where the dynamics still terminate, and examples where they do not:

Example 6. *If we consider the game in Figure 1, letting now the preferences of Player 1 be $x \prec_1 y \prec_1 z \prec_1 y$ (and Player 2's unchanged). Observe that these preferences are cyclic because Player 1 prefers y to z, and z to y. Then, all the X-Dynamics with $SI \in X$ still terminate.*

However, let us further modify the example to let Player 2 have cyclic preferences too, with: $x \prec_2 y \prec_2 z \prec_2 y$. Then, the following infinite sequence of strategy profiles respects the SI-Dynamics:

$$rr, rl, rr, rl, rr \dots$$

Those examples are quite simple and could seem quite abstraits and irrationals, but they give the idea that this kind of game can, or not, terminate.

This section will be mainly devoted to proving Theorem 5. Our proof strategy works as follows. First of all, we establish termination of the $\{SI\}$-Dynamics when the preferences are acyclic (Proposition 6). This guarantees that all the X-Dynamics terminate when $SI \in X$, since all these dynamics are more restrictive.

[4] Actually, as we show at the end of the section, in the presence of players who have cyclic preferences and play lazily, the players who have acyclic preferences are still guaranteed to perform a finite number of updates only.

[5] Recall however the discussion on cyclic preferences in Example 2.

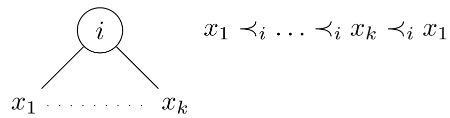

Figure 3: A game in which the $\{SI\}$-Dynamics does not terminate, with the cyclic preference $x_1 \prec_i x_2 \prec_i \cdots \prec_i x_k \prec_i x_1$.

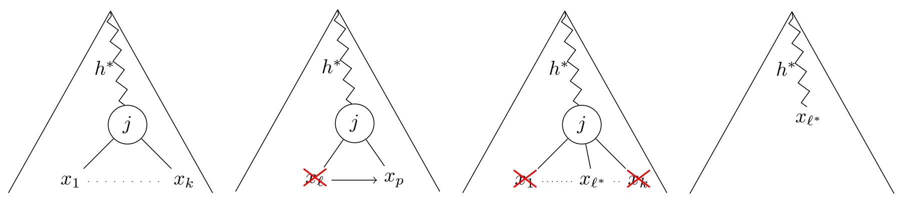

Figure 4: Steps for the reduction of the game G (proof of Proposition 6).

Second, we show that all SPEs appear as terminal profiles of the $\{SI\}$-Dynamics (Proposition 7), without guaranteeing, at that point that all terminal profiles are SPEs. Finally, we show that all the terminal profiles of the $\{SI, A\}$-Dynamics are SPEs (Proposition 8). We then argue, using specific properties of this dynamics, and relying on Definition 3, that this implies that the set of terminal profiles of all the X-dynamics such that $\{SI\} \subseteq X \subseteq \{SI, A, 1P\}$ coincide with the set of SPEs.

The $\{SI\}$-Dynamics. As mentioned, we start with two properties of the $\{SI\}$-Dynamics. Proposition 6 states that, for a fixed set of preferences, the $\{SI\}$-Dynamics terminates in all games built on these preferences iff the preferences are acyclic. Proposition 7 shows that, in these cases, the SPEs are contained in the terminal profiles of the $\{SI\}$-Dynamics.

Proposition 6. *Let $N, O, (\prec_i)_{i\in N}$ be respectively a set of players, a set of outcomes and preferences. Then, the two following statements are equivalent:*

(1) in all games G built over $N, O, (\prec_i)_{i\in N}$, the $\{SI\}$-Dynamics terminates;

(2) the preferences $(\prec_i)_{i\in N}$ are acyclic.

Proof. Throughout the proof, we fix a set N of players, a set O of outcomes and preferences $(\prec_i)_{i\in N}$. We prove the two directions of the implication separately:

(1) ⇒ (2) We proceed by contraposition. We assume that the preferences $(\prec_i)_{i\in N}$ are cyclic, i.e. there are a player $i \in N$, and outcomes x_1, x_2,..., x_k s.t.:

$$x_1 \prec_i x_2 \prec_i \cdots \prec_i x_k \prec_i x_1.$$

Then, we exhibit a game built on N, O and $(\prec_i)_{i\in N}$ in which the $\{SI\}$-Dynamics does not terminate. In this game, Player i owns the root and is the only one to play: he has to choose between one of the x_1, x_2,… x_k outcomes, as shown in Figure 3. If we let s^j be the strategy profile where i chooses outcome x_j, then the sequence:

$$s^1, s^2, \ldots, s^k, s^1, s^2, \ldots, s^k, s^1, \ldots$$

is indeed an infinite sequence of strategies that respects the $\{SI\}$-Dynamics.

$\boxed{(2) \Rightarrow (1)}$ We assume now that the preferences are acyclic, and show that the $\{SI\}$-Dynamics terminates in all games $G = \langle N, A, H, O, d, p, (\prec_i)_{i\in N}\rangle$ built over N, O, and $(\prec_i)_{i\in N}$. We establish this by induction over the number of nodes of G:

1. **Base case,** $|H| = 1$. In this case, no move is possible in the game, hence the $\{SI\}$-Dynamics terminates trivially.

2. **Inductive case** $|H| = n + 1$. For the induction hypothesis, we assume that, in all games with at most n nodes; built over N, O, and $(\prec_i)_{i\in N}$; and where $(\prec_i)_{i\in N}$ are acyclic, the $\{SI\}$-Dynamics terminates.

   Let G be a sequential game with $n + 1$ nodes built over $N, O, (\prec_i)_{i\in N}$ and let us denote by $\xrightarrow[G]{SI}$ the $\{SI\}$-Dynamics over G. We proceed by contradiction, and assume that there exists an infinite sequence of strategy profiles $\left(s^i\right)_{i\geq 1}$ such that:

$$s^1 \xrightarrow[G]{SI} s^2 \xrightarrow[G]{SI} s^3 \xrightarrow[G]{SI} \ldots$$

   Let us denote by h^* a node in G which is as in Figure 4 (left). That is h^* is s.t. $h^* \notin Z$ (h^* is not a leaf), and all successors h' of h^* are leaves (in Z). Such a node exists because G is finite. Let us denote by j the player $d(h^*)$ that controls h^*, and let $x_1, \ldots, x_p$ be all the outcomes that can be reached from h^*.

   Observe that, every time player j changes his choice in h^* along the $\left(s^i\right)_{i\geq 1}$ sequence, it means that he chooses a better outcome from the set $\{x_1, \ldots, x_k\}$, and this can happen only finitely many times along the $\left(s^i\right)_{i\geq 1}$ sequence, since there are finitely many outcomes and the preferences are acyclic. Formally, if $s^k(h^*) \neq s^\ell(h^*)$ for $k < \ell$, then $\left\langle s^k|_{h^*}\right\rangle \prec_j \left\langle s^\ell|_{h^*}\right\rangle$ by the SI property of the Dynamics, so $s^m(h^*) \neq s^k(h^*)$ for all $m \geq \ell$. Hence, $s^i(h^*)$ eventually stabilises (because there are finitely many outcomes and the preference are acyclic) and we let $x_{\ell^*} \in \{x_1, ..., x_k\}$ and $m^* \in \mathbb{N}$ be such that for all $m' \geq m^*$: $\langle s^m|_{h^*}\rangle = x_{\ell^*}$ (see Figure 4, middle right).

   Let us now consider the new game G' (with set of nodes H' and set of leaves Z') which is obtained from G by turning h^* into a leaf that yields outcome x_{ℓ^*}. Observe that G' has at most n nodes since G has $n+1$ nodes and we have removed at least one node. Thus, the induction hypothesis

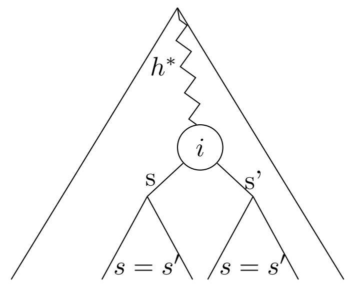

Figure 5: Illustration for the proof of Proposition 7.

applies to G' and we will obtain a contradiction by extracting from $\left(s^i\right)_{i\geq 1}$ an infinite sequence of strategy profiles from G' that respects the $\{SI\}$-Dynamics. This sequence is $\left(\tilde{s}^i\right)_{i\geq 1}$ such that for all $j \geq 1$, for all nodes $h \in H' \setminus Z'$: $\tilde{s}^j(h) = s^j(h)$. By construction of G': $\left\langle s^j \right\rangle = \left\langle \tilde{s}^j \right\rangle$ for all $j \geq m^*$. That is, all the $\tilde{s}^j$ yield the same outcome as their corresponding s^j when $j \geq m^*$, since the only difference between G and G' is that h^* has been replaced, in G' by the outcome x_{ℓ^*} that is always obtained when playing s^j in G (for $j \geq m^*$). However, since $s^j \xrightarrow[G]{SI} s^{j+1}$ for all j, then $\left(\tilde{s}^j, \tilde{s}^{j+1}\right) \models SI$ too for all $j \geq m^*$. The sequence $\left(\tilde{s}^j\right)_{j\geq m^*}$ is thus an infinite sequence of the $\{SI\}$-Dynamics in G', which is a game with at most n nodes (by construction). This contradicts the induction hypothesis, hence the infinite sequence $(s^j)_{j\geq 1}$ does not exist in G. □

Now that we have established conditions for termination of the $\{SI\}$-Dynamics, let us characterise its terminal profiles. We start by showing that all the subgame perfect equilibria of the game are terminal profiles of the dynamics:

Proposition 7. *Let G be a sequential game. Then, all SPEs of G are terminal profiles of the $\{SI\}$-Dynamics.*

Proof. We prove the contrapositive. Let s and s' be two strategy profiles of G s.t. $s \xrightarrow{SI} s'$. By Definition 3, this means that there is a node h^* belonging to some player i s.t. $s(h^*) \neq s'(h^*)$ and $\langle s|_{h^*}\rangle \prec_i \langle s'|_{h^*}\rangle$. Hence, changing the move from $s(h^*)$ to $s'(h^*)$ in such an h^* yields a better outcome for Player i in the subtree rooted in h^*, and s is thus not an SPE. □

The $\{SI, A\}$-Dynamics. We now turn our attention to the $\{SI, A\}$-Dynamics, and show in Proposition 8 that all its terminal profiles are necessarily SPEs. As announced, this result will be sufficient to prove the second part of Theorem 5, in other words to establish that all X-Dynamics such that $\{SI\} \subseteq X \subseteq \{SI, 1P, A\}$ terminate on SPEs too.

Proposition 8. *Let G be a sequential game. Then, all terminal profiles of the $\{SI, A\}$-Dynamics are SPEs of G.*

Proof. We prove the contrapositive, i.e. that every s which is not an SPE is not a terminal profile of the $\{SI, A\}$-Dynamics.

Let s be a strategy profile of G which is not an SPE. Then, there is a node h^* where some player i does not make the best choice for his payoff in the subtree rooted in h^*. Formally, there are $i \in N$, $h^* \in H_i$ and s' such that: (i) $s(h) = s'(h)$ for all nodes $h \neq h^*$; (ii) $s(h^*) \neq s'(h^*)$; and (iii) $\langle s|_{h^*} \rangle \prec_i \langle s'|_{h^*} \rangle$. Hence, $s \xrightarrow{SI,A} s'$ (by Definition 3), and s is not a terminal profile of the $\{SI, A\}$-Dynamics. □

Equipped with these three propositions, we can now prove Theorem 5:

Proof of Theorem 5. Let us consider a set of players N, a set of outcomes O, preferences $(\prec_i)_{i \in N}$ and X s.t. $SI \in X$. The X-Dynamics terminates for every game G built over $N, O, (\prec_i)_{i \in N}$ if and only if the preferences are acyclic by Proposition 6, because $\xrightarrow{X} \subseteq \xrightarrow{SI}$ by definition.

Next, let us consider X s.t. $\{SI\} \subseteq X \subseteq \{SI, A, 1P\}$. By definition of SI, A and $1P$, and using the fact that Property A implies Property $1P$ (see Lemma 4), we have: $\xrightarrow{SI,A} \subseteq \xrightarrow{X} \subseteq \xrightarrow{SI}$. Let s be a terminal profile of $\xrightarrow{X}$. Then, it is also terminal in $\xrightarrow{SI,A}$ since $\xrightarrow{SI,A} \subseteq \xrightarrow{X}$. By Proposition 8, s is thus an SPE of G. On the other hand, let s' be an SPE of G. Then by Proposition 7, s' is a terminal node of $\xrightarrow{SI}$. Since $\xrightarrow{X} \subseteq \xrightarrow{SI}$, s' is also a terminal node of $\xrightarrow{X}$. We have shown that all SPEs of G are terminal nodes of $\xrightarrow{X}$ (and vice-versa) when $\{SI\} \subseteq X \subseteq \{SI, A, 1P\}$. □

Finally, let us explain why we restrict ourselves to X-Dynamics with $X \subseteq \{SI, A, 1P\}$ in Theorem 5, when establishing a relationship between terminal profiles and SPEs. We notice that if $\{L, I\} \cap X \neq \emptyset$, (i.e. $X \not\subseteq \{SI, A, 1P\}$), then there are examples of terminal profiles of the X-Dynamics which are not SPEs. For example, consider the game in Figure 1 with the following preferences: $z \prec_1 y \prec_1 x$ and $y \prec_2 z \prec_2 x$. Then, ll is a terminal profile for the $\{I\}$-Dynamics, because any modification to the payoff occurs only if Player 1 changes his choice at the root, which will necessarily decrease his own payoff. Also, ll is a terminal profile for the $\{L\}$-Dynamics, because Player 2 will never update his choice (since the right son of the root does not occur in the play induced by ll), and, again, Player 1 will not update his choice. However, ll is not an SPE, because Player 2 does not play optimally in the subtree he controls.

3.2. Weak simulation by the $\{SI, A\}$-Dynamics

Let us now highlight an interesting property of the $\{SI, A\}$-Dynamics, namely that it weakly simulates the $\{SI\}$-Dynamics, in the following sense. In all games, every update of the $\{SI\}$-Dynamics can be split into an equivalent sequence of updates of $\{SI, A\}$-Dynamics. Formally, whenever $(s, s') \models SI$, then, we can find $s^1, \ldots, s^k$ s.t. $s \xrightarrow{SI,A} s^1 \xrightarrow{SI,A} \ldots \xrightarrow{SI,A} s^k \xrightarrow{SI,A} s'$.

We argue that this sounds like a natural property, since, if there are k nodes where the strategy profiles s and s' differ, then we should be able to perform

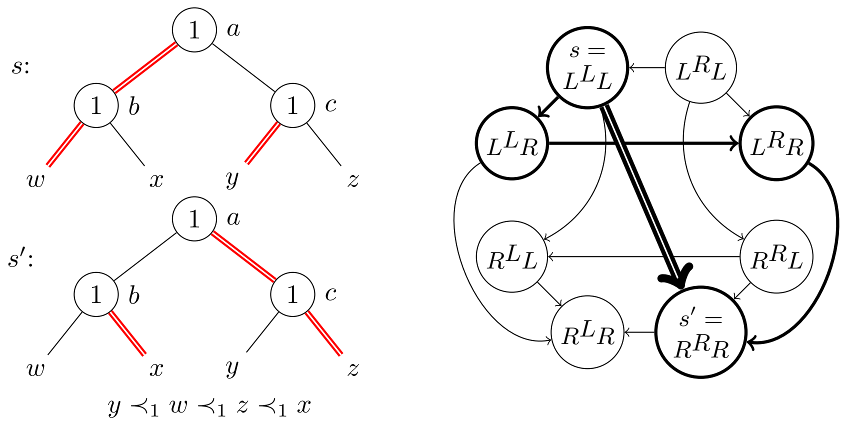

Figure 6: Left: A 1-player game, with two different strategies s and s' (a, b and c are the names of the nodes). Right: the associated graph for the $\{SI\}$-Dynamics.

theses changes one by one, according to the $\{SI\}$-Dynamics. The main difficulty (which is solve in the proof hereinafter) is to devise the right order to perform these atomic changes.

Example 7. *As an example, consider Figure 6. The left-hand part of the figure presents twice the same game with two different strategy profiles s (on top) and s'. Observe that $(s, s') \models SI$. The right-hand part of the figure presents the graph associated to the $\{SI\}$-Dynamics of the game. Strategy profiles are denoted $X^Y Z$, where Y stands for the action chosen at the root, X for the action in the left son of the root, and Z for the action in its right son. As we can see, $L^L L \xrightarrow{SI} R^R R$ can be simulated by $L^L L \xrightarrow{SI,A} L^L R \xrightarrow{SI,A} L^R R \xrightarrow{SI,A} R^R R$. However, observe that not all paths starting from $L^L L$ allow to reach $R^R R$, so the order in which we perform the atomic changes clearly matters.*

Let us now prove the following lemma that formalises these ideas:

Lemma 9. *For all sequential games G, for all s, s' s.t. $(s, s') \models SI$: there are $s^1, \dots, s^k$ s.t. $s \xrightarrow{SI,A} s^1 \xrightarrow{SI,A} \dots \xrightarrow{SI,A} s^k \xrightarrow{SI,A} s'$, where $\xrightarrow{SI,A}$ is the $\{SI, A\}$-Dynamics of G.*

Proof. Let s and s' be two strategy profiles s.t. $(s, s') \models SI$. The proof consists in showing how to build s^1 such that: $s \xrightarrow{SI,A} s^1$, $(s^1, s') \models SI$ and $|H(s^1, s')| = |H(s, s')| - 1$. Then, if $(s^1, s') \not\models A$, the same process can be applied again to yield s^2 s.t. $s \xrightarrow{SI,A} s^1 \xrightarrow{SI,A} s^2$ and $(s^2, s') \models SI$, and so forth. This process is guaranteed to terminate because, by construction: $|H(s, s')| > |H(s^1, s')| > |H(s^2, s')| > \cdots$ and $|H(s, s')|$ is finite since G is finite. Intuitively, each (s^j, s^{j+1}) step expresses one of the changes between s and s', so the strategies s^j become 'closer' to s' with j increasing.

Building such a strategy s^1 boils down to identifying a node h^*, where s and s' disagree, and which will be the (only) node where the (atomic) change will be performed when moving from s to s^1. Formally, we want a node h^* belonging to some player that we denote i and s.t.: (i) $s|_{s'(h^*)} = s'|_{s'(h^*)}$, i.e. s and s' agree in the subtree rooted in $s'(h^*)$; and (ii) for all h in $H_i(s, s')$: h^* does not lie along the play induced by $s|_h$. Let us argue that such a node always exists. First, observe that since $(s, s') \models SI$, then $H(s, s')$ is not empty. Then, we consider the node h which is the highest where s and s' disagree, i.e. $s(h) \neq s'(h)$ and, for all prefixes h' of h, $s(h') = s'(h')$. Then, either $s|_{s'(h)} = s'|_{s'(h)}$, and we let $h^* = h$; or, we consider the game $G|_{s'(h)}$ and make the same reasoning, until we manage to select a node h^*. This will eventually happen as the game G is finite.

Then, from h^*, we build the strategy profile s^1 (of G) as follows:

$$\text{for all nodes } h \in H : s^1(h) = \begin{cases} s'(h) & \text{if } h = h^* \\ s(h) & \text{otherwise.} \end{cases}$$

Clearly, $(s, s^1) \models A$. Let us show that s^1 has the desired properties:

1. First, we show that $s \xrightarrow{SI,A} s^1$. We already know that $(s, s^1) \models A$, so we must only prove that $(s, s^1) \models SI$, i.e. that $\langle s|_{h^*} \rangle \prec_i \langle s^1|_{h^*} \rangle$. We know that $s^1|_{s'(h^*)} = s|_{s'(h^*)}$ and that $s^1(h^*) = s'(h^*)$, by construction of s^1. Moreover, by construction of h^*, we know that s and s' agree in the subtree rooted in $s'(h^*)$, i.e. $s|_{s'(h^*)} = s'|_{s'(h^*)}$. Hence, s^1 and s' behave identically from h^* (included), and thus the outcome is the same from this node:

$$\langle s'|_{h^*} \rangle = \langle s^1|_{h^*} \rangle . \tag{1}$$

   On the other hand, since $s \xrightarrow{SI} s'$ and $h^* \in H(s, s')$, we know that changing from s to s' improves the outcome in h^* for Player i, i.e.

$$\langle s|_{h^*} \rangle \prec_i \langle s'|_{h^*} \rangle . \tag{2}$$

   Hence, by (1) and (2), we conclude that $\langle s|_{h^*} \rangle \prec_i \langle s^1|_{h^*} \rangle$, which implies that $s \xrightarrow{SI,A} s^1$ as explained above.

2. Second, we show that $(s^1, s') \models SI$. To establish this, we need to show that for all player $j \in N$, for all node $h \in H_j(s^1, s')$: $\langle s^1|_h \rangle \prec_j \langle s'|_h \rangle$, i.e. the outcome from h improves for j, when changing from s^1 to s'.

   Let $j \in N$ be a player and let h be a node from $H_j(s^1, s')$. Then $h \in H_j(s, s')$ by definition of s^1, hence:

$$\langle s|_h \rangle \prec_j \langle s'|_h \rangle , \tag{3}$$

   since $(s, s') \models SI$. Moreover, since h^* does not lie along the play induced by $s|_h$ (by construction of h^*):

$$\langle s^1|_h \rangle = \langle s|_h \rangle , \tag{4}$$

because s and s^1 differ only on h^*. By (3) and (4), we conclude that $\langle s^1|_h \rangle \prec_j \langle s'|_h \rangle$. Since this holds for any j and h, we conclude that $(s^1, s') \models SI$, as argued above.

3. Finally, we remark that $|H(s^1, s')| = |H(s, s')| - 1$ stems directly from the definition of s^1, as all the changes performed between s and s' are performed between s^1 and s', except for the change in h^*, that occurs now between s and s^1. □

Example 8. *Let us continue the discussion of Example 7, and let us observe that it clearly shows why the two conditions on the definition of h^* in the above proof are necessary. The first condition states that h^* must be s.t. $s|_{s'(h^*)} = s'|_{s'(h^*)}$. This shows why the first change in the decomposition cannot occur in node a. Indeed, letting $h^* = a$ yields $s^* = L^{R}L$, but s and s' dot not agree in node c. Indeed, the move $(s, L^{R}L)$ is not allowed by the $\{SI\}$-Dynamics. The second condition states that for all h in $H_i(s, s')$: h^* does not lie along the play induced by $s|_h$. A choice of h^* that would violate this condition is $h^* = b$. This yields the strategy profile $s^* = R^{L}L$, from which s' is unreachable.*

3.3. Termination in the presence of 'cyclic' players

We close this section by answering the following question: 'what happens when some players have cyclic preferences and some have not?' We call cyclic the players who have cyclic preferences and show that, although their presence is sufficient to prevent termination of the whole dynamics, players with acyclic preferences can still be guaranteed a bounded number of updates in their choices, provided that the cyclic players play lazily. Thus, in this case, any infinite sequence of updates will eventually be made up of updates from the cyclic players only. This provides some robustness to our termination result.

Let us first notice that, if we allow players with cyclic preferences to play following the $\{SI\}$-Dynamics, the result does not hold, as shown by the following example:

Example 9. *Consider the game in Figure 1 again, where we replace the preference of Player 2 by $y \prec_2 z \prec_2 y$. So, now, Player 2 has cyclic preferences, while Player 1 keeps his acyclic preferences $y \prec_1 x \prec_1 z$.*

Then, the graph associated to the $\{SI\}$-Dynamics is represented in Figure 7, left, where dotted lines represent updates of Player 2. Clearly, this graph contains a cycle in which Player 1 updates infinitely often. However, if we constrain Player 2 to play lazily, the associated graph is represented in Figure 7, right. This graph does not contains cycle in which Player 1 updates infinitely often.

Let us now establish our termination result. To this end, we fix a set of players N partitioned into the sets N_c and N_a of cyclic and acyclic players respectively; a set O of outcomes; and preferences $(\prec_i)_{i \in N}$. By 'cyclic' and 'acyclic' players, we mean that for all $i \in N_c$: $\prec_i$ is cyclic and; for all $j \in N_a$: $\prec_j$ is acyclic. We consider the $\{SI\}/\{I, L, 1P\}$-Dynamics, denoted $\leadsto$, such that, for two strategies s and s', $s \leadsto s'$ iff:

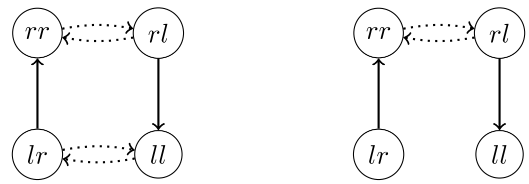

Figure 7: Graph of the $\{SI\}$-Dynamics for the game of Figure 1, with a cyclic player.

- either $H(s, s') \cap N_c = \emptyset$ and $s \xrightarrow{SI} s'$;
- or $H(s, s') = \{i\}$ for some $i \in N_c$ and $s \xrightarrow{\{I,L,1P\}} s'$

Thus, moves in the $\{SI\}/\{I, L, 1P\}$-Dynamics are of two possible kinds: either a coalition of *acyclic* players change their respective choices, according to the $\{SI\}$-Dynamics; or a single *cyclic* player plays according to the $\{I, L, 1P\}$-Dynamics. We say that $\rightsquigarrow$ *terminates for acyclic players* if there is no infinite sequence of strategy profiles $(s^k)_{k\in\mathbb{N}}$ such that: (1) for all $k \in \mathbb{N}$: $s^k \rightsquigarrow s^{k+1}$; and (2) for all $j \in \mathbb{N}$ there is $k > j$ s.t. $d(H(s^k, s^{k+1})) \cap N_a \neq \emptyset$ (i.e. the acyclic players update their strategy infinitely often). Let us show that this dynamics indeed terminates for the acyclic players:

Proposition 10. *Let $N = N_a \cup N_c$, O, $(\prec_i)_{i\in N}$ be a set of players, a set of outcomes and a set of preferences such that $\forall i \in N_c, \prec_i$ is cyclic and $\forall i \in N_a, \prec_i$ is acyclic. Then, the $\{SI\}/\{I, L, 1P\}$-Dynamics terminates for acyclic players in all games built over $N, O, (\prec_i)_{i\in N}$.*

Proof. The idea of the proof is an induction over the number of nodes of the game. If the acyclic players update their strategies only a finite number of times in all subtrees of the root (by induction hypothesis), then the only way that the acyclic players could update infinitely often in the whole game would be to have an infinite number of updates at the root of the game, which we will see is impossible.

Formally, let us consider a game G built over N, O, and $(\prec_i)_{i\in N}$, with set of nodes H, and let us prove the proposition by induction over the number of nodes of G:

1. **Base case,** $|H| = 1$. In this case, no move is possible in the game, hence the dynamics terminates trivially.

2. **Inductive case,** $|H| = n + 1$. For the induction hypothesis, we assume that, in all games with at most n nodes; built over N, O, and $(\prec_i)_{i\in N}$; the $\{SI\}/\{I, L, 1P\}$-Dynamics terminates.

   We first make the following useful observation. Let h^* be a son of the root (i.e., $h^* = a \in A$), and let $h \in H|_{h^*}$ be a node from the subtree rooted in h^*. Let us further consider an update of the $\{SI\}/\{I, L, 1P\}$-Dynamics that updates the choice in h, i.e. a pair of strategy profiles s and s' s.t. $s \rightsquigarrow s'$ and $s(h) \neq s'(h)$. Then, we claim that $s|_{h^*} \rightsquigarrow s'|_{h^*}$ holds too, i.e. the move, projected to the subtree rooted in h^* also respects the $\{SI\}/\{I, L, 1P\}$-Dynamics in this subtree. This stems directly from the definition of the $\{SI\}/\{I, L, 1P\}$-Dynamics.

Thanks to this observation, we can establish the inductive case. We proceed by contradiction and assume that there is an infinite sequence of strategy profiles $(s^k)_{k\in\mathbb{N}}$ of G s.t. $s^k \rightsquigarrow s^{k+1}$ for all $k \in \mathbb{N}$ and in which the acyclic players update their strategy infinitely often. We consider two cases:

(a) Either there is a subtree rooted in some son $h^* \in A$ of the root, and in which infinitely many updates are performed by acyclic players along the $(s^k)_{k\in\mathbb{N}}$ sequence, i.e. for all $k_0 \in \mathbb{N}$ there are $k \geq k_0$ and $h \in H|_{h^*}$ s.t. $s^k(h) \neq s^{k+1}(h)$ and $d(h) \in N_a$. But in this case we can extract an infinite sequence of strategy profiles $(\tilde{s}^k)_{k\in\mathbb{N}}$ of the subgame $G|_{h^*}$ s.t. $\tilde{s}^k \rightsquigarrow \tilde{s}^{k+1}$ for all $k \in \mathbb{N}$ (by the above observation) and in which acyclic players update their strategy infinitely often (by construction), which contradicts the induction hypothesis.

(b) Otherwise, the number of updates by acyclic players is *finite* in each subtree of the root. Since the tree is finite, it means that, after a finite number of steps, the updates by acyclic players consist in updates *at the root only*. That is, there exists $k_0 \in \mathbb{N}$ s.t. for all $k \geq k_0$: either $d(H(s^k, s^{k+1})) \in N_c$ (only one cyclic player performs an update), or $H(s^k, s^{k+1}) = \{\varepsilon\}$ (the update is performed in the root only). This means that all updates by cyclic players necessarily respect the $1P$ property (only the player owning the root performs an update); the I property (because acyclic players respect SI by hypothesis, which entails the I property when the update occurs at the root); and the L property (again, because the update occurs at the root). Hence, all players (cyclic and acyclic) play according to the $\{I, L, 1P\}$-Dynamics along the infinite suffix $(s_j)_{j\geq k_0}$ of the original sequence. However, such infinite sequences of updates in the $\{I, L, 1P\}$-Dynamics cannot exist by a result of Le Roux and Pauly [LRP16, Section 5], and we obtain again a contradiction. □

4. Improvement dynamics and coalitions

While Section 3 was devoted to characterising the X-Dynamics with $SI \in X$, we turn now attention to those where $I \in X$. Recall that Le Roux and Pauly have studied the Lazy Improvement Dynamics [LRP16] (which corresponds to our $\{I, L, 1P\}$-Dynamics) and shown that it terminates when the preferences of the players are acyclic, and reaches Nash Equilibria. Their study of the $\{I, L, 1P\}$-Dynamics was motivated by the fact that less restrictive dynamics (that are still contained in $\xrightarrow{I}$) do not always terminate, namely the $\{I\}$-

Dynamics and the $\{I, 1P\}$-Dynamics. These results appear in Table 1 (page 11) in the fourth and last lines.

Our contribution in the present section is to fill in Table 1 by the following results. First, for all the X-Dynamics with $\{I, A\} \subseteq X$, we show that acyclic preferences guarantee termination, and that the final profiles contain the Nash Equilibria (third line of the table).

Second, we consider the $\{I, L\}$-Dynamics, in the second to last line of Table 1, which can be regarded as a *coalition dynamics*, where several players can change their strategies at the same time to obtain a better outcome for *all players* taking part to the coalition. For example, in Figure 1 (page 6), the two players can make a coalition to change from ll to rr, as they both prefer z to x. We characterise families of games and conditions on the preferences where termination of the $\{I, L\}$-Dynamics is guaranteed (with the terminal profiles being exactly the *Strong Nash Equilibria* in the sense of Aumann [Aum59]).

4.1. The $\{I, A\}$-Dynamics

To complete Table 1, we consider now the third line which represents the X-Dynamics with $\{I, A\} \subseteq X$. All these Dynamics can be considered at once thanks to the next proposition:

Proposition 11. *All the X-Dynamics with $\{I, A\} \subseteq X \subseteq \{I, SI, A, L, 1P\}$ are equal.*

Proof. To understand these equalities, we focus on the $\{I, A\}$-Dynamics. This dynamics allows only one update between two profiles (because of the A) property, and the outcome must be better for the player that has changed his strategy (I property). However, for the outcome of the game to change, the atomic move must occur along the play induced by the strategy profile. Thus, the $\{I, A\}$-Dynamics verifies the Lazy Property (L). Moreover, by Lemma 4 (page 10), it also verifies the One Player Property ($1P$). Finally, as the outcome is improved for the player who performs the update, and as only one change has been done, in particular the payoff is improved in the subgame rooted at the change. Thus, the $\{I, A\}$-Dynamics also verifies the Subgame Improvement Property (SI). □

Corollary 12. *Let N, O and $(\prec_i)_{i\in N}$ be respectively a set of players, a set of outcomes and preferences. Then, the two following statements are equivalent:*

(1) in all games G built over $N, O, (\prec_i)_{i\in N}$, the $\{I, A\}$-Dynamics terminates;

(2) the preferences $(\prec_i)_{i\in N}$ are acyclic.

Proof. By Theorem 5 (page 12) as the $\{I, A\}$-Dynamics verifies the Subgame Improvement Property, this dynamics terminates for every game over some $N, O, (\prec_i)_{i\in N}$ if the preferences are acyclic.

Moreover, if the preferences are cyclic, i.e. there are a player $i \in N$, and outcomes x_1, x_2,..., x_k s.t.:

$$x_1 \prec_i x_2 \prec_i \cdots \prec_i x_k \prec_i x_1.$$

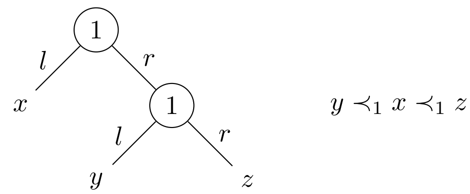

Figure 8: Example showing that the terminal profiles of $\{I, A\}$-Dynamics are not always NEs. In this case, the strategy profile ll is terminal but not an NE.

Then we can build the game in Figure 3 (page 13), in which we can build a infinite sequence of strategies respecting the $\{I, A\}$-Dynamics as in the proof of Proposition 6. □

Now that we have established termination, let us turn our attention to the terminal profiles. It turns out that they contain all Nash Equilibria of the game, but that some terminal profiles are not Nash Equilibria:

Proposition 13. *Let G be a sequential game. Then, the set of NEs of G is* strictly *contained in the set of terminal profiles of* $\xrightarrow{\{I,A\}}$.

Proof. We first show the inclusion, i.e. that all NEs are terminal profiles. This is immediate, since, if s is an NE, then no player can improve the outcome by updating his strategy alone from s. Hence, s is terminal in $\xrightarrow{\{I,A\}}$.

Next, the game in Figure 8 allows to establish the strict inclusion. Observe that the profile ll is terminal for $\xrightarrow{\{I,A\}}$, since no update can improve the outcome for the unique player, Player 1: changing his choice at the root will yield outcome y which is worse than x, and changing in the right son of the root, does not change the outcome. But ll is not a Nash equilibrium, because Player 1 could (without the restriction on the A property) decide to update the profile to rr which gives a better profile z. □

4.2. The $\{I, L\}$-Dynamics

Let us now turn our attention to the $\{I, L\}$-Dynamics. First, we observe that the terminal profiles of this dynamics are exactly the Strong Nash Equilibria (see Section 2 for the definition). We believe the observation is interesting, since, as already stated, the concept of SNE has sometimes been deemed 'too strong' in the literature:

Proposition 14. *Let G be a sequential game with set of players N, set of outcomes O and preference orders $(\prec_i)_{i \in N}$. Then, the set of terminal profiles of the $\{I, L\}$-Dynamics is the set of Strong Nash Equilibria of G.*

Proof. This property stems from the definition of Strong Nash Equilibria, which states that, in an SNE, no coalition of players has an incentive to deviate.

Let s be a strategy profile in G which is not terminal, and let s' be s.t. $s \xrightarrow{I,L} s'$. Hence, $\langle s \rangle \prec_i \langle s' \rangle$ for all players i s.t. $s_i \neq s'_i$ (by the I property of the move). Thus, the set of players $\{i \mid s_i \neq s'_i\}$ forms a coalition in which all players can improve their payoff, and s is thus not an SNE. Thus, all SNEs are terminal profiles.

On the other hand, let s be a strategy profile which is not an SNE. Hence, there is a coalition $C \subseteq N$, and a strategy profile s' s.t. $s_i = s'_i$ if $i \notin C$; and $\langle s \rangle \prec_i \langle s' \rangle$ for all $i \in C$. Observe that, in the definition of SNEs, not all players of the coalition have to update their strategy (but they have to get a better outcome). However, all players who have updated their strategy between s and s' are in C, so they have all improved the outcome (for their respective preference order), hence the move from s to s' respects I. But this move might not respect L, since some of the players of the coalition that perform an update might be doing so in nodes that do not affect the outcome in s' but still get a better outcome. Hence, we consider $C' \subseteq C$ containing all the players $i \in C$ whose update is performed in a node that lies on the path induced by s'. Moreover, players $i \in C'$ could update their strategy in several nodes, some along the play induced by s', some not. Then, let s'' be the strategy profile s.t. (i) for all $i \in C'$: $s''_i(h) = s'_i(h)$ if h is along the play induced by s', and $s''_i(h) = s_i(h)$ otherwise; and (ii) for all $i \notin C'$: $s''_i = s_i$. Then, now, $s \xrightarrow{I,L} s''$. Hence, s is not terminal. This shows that all terminal profiles are SNEs. □

As can be seen from Table 1, the conditions to ensure termination are more involved and require a finer characterisation of the preferences. We start by introducing several notions of orders that will be useful to define those conditions

Definition 15 (Strict Orders)**.**

1. *A* ***strict linear order*** *(slo for short) over a set O is a total, irreflexive and transitive binary relation over O. This is a natural way to see orders: for example, the usual orders $<$ over $\mathbb{N}$ or $\mathbb{R}$ are strict linear orders.*

2. *A* ***strict weak order*** *(swo for short) $<$ over a set O is an irreflexive and transitive binary relation over O that provides the transitivity of incomparability. Formally, for $x \neq y \in O$, if $\neg(x < y)$ and $\neg(y < x)$, we say that x and y are incomparable*[6]*, and we write $x \sim y$. Then, the transitivity of incomparability means that for all x, y, z: $(x \sim y \wedge y \sim z) \Rightarrow x \sim z$.*

3. *An slo $<'$ is a* ***strict linear extension of an swo*** *$<$ iff $x < y$ implies $x <' y$ for all x, y in O.*

We write $x \lesssim y$ if either $x < y$ or $x \sim y$. That is, $x \lesssim y$ iff $\neg(y < x)$. We argue that swos are quite natural to consider in our context. Indeed, the

[6]This can happen because a strict weak order is not a total relation

incomparability of two outcomes for a player reflects the indifference of the player regarding these outcomes. We can easily imagine two outcomes x and y such that Player 1 prefers x to y but Player 2 has no preference.

Let us notice that the transitivity of incomparability gives us that if $x \sim y$ and $y < z$ then $x < z$. Indeed, if $z < x$, then $y < z < x$ would imply that $y < x$, (which contradicts $x \sim y$), and if $x \sim z$, then $y \sim x \sim z$ would imply that $y \sim z$ (which contradicts $y < z$). Thus, we will, from now on, write $x \sim y < z$ to mean $x \sim y$, $x < z$ and $y < z$.

Layerability of orders. While the notions of order we have defined above make sense in our context, they are unfortunately not sufficient to ensure termination of $\{I, L\}$-Dynamics, as shown by the next example.

Example 10. *We consider again the game in Figure 1 (page 6) and his associated graph with $\{I, L\}$-Dynamics in Figure 2 (page 9, right). We can observe that the players have strict linear preferences, but the the dynamics does not terminate.*

We thus need to introduce more restrictions on outcomes and preferences to ensure termination. In [LR15], Le Roux considers a pattern over outcomes and preferences, and proves that the absence of this pattern induces some structure on the outcomes, in the case of strict linear order, that we call *layerability*. Layerability, in turn, can be used to prove termination of dynamics as we are about to see. Our first task is thus to generalise the pattern and the definition of layerability of outcomes in the case of *strict weak orders*.

Let O be a finite set of outcomes, N be a finite set of players and $(<_i)_{i \in N}$ be swos over O. We say that $(<_i)_{i \in N}$ is:

(i) **out of main pattern** for O if it satisfies:

$$\forall x, y, z \in O : \forall i, j \in N : \neg(x <_i y <_i z \text{ and } y <_j z <_j x); \qquad (5)$$

(ii) **out of secondary pattern** for O if it satisfies:

$$\forall w, x, y, z \in O : \forall i, j \in N : \neg(w <_i x <_i y <_i z \text{ and } x \sim_j z <_j w \sim_j y); \qquad (6)$$

and

(iii) **out of pattern** for O if it is out of main and secondary pattern.

Notice that, when $(<_i)_{i \in N}$ is a strict linear order, then $(<_i)_{i \in N}$ is out of pattern for O if and only if $(<_i)_{i \in N}$ is out of main pattern for O. We can now introduce a first notion of *layearbility* of a set of preferences that are swos. This notion will be used to characterise the termination of the $\{I, L\}$-Dynamics in restricted cases (see Theorem 22):

Table 2: Four preference swos $<_1$, $<_2$, $<_3$ and $<_4$ on the set of outcomes $\{u, v, w, x, y, z\}$ that can be layered. Each order corresponds to a column, and the order of the rows indicate decreasing preference order (for example, $v <_3 w$ since, in column $<_3$, v occurs in the second row, while w occurs in the first row)

	$<_1$	$<_2$	$<_3$	$<_4$
Layer ν	u	u	w	u
	v	v	v	v
	w	w	u	w
Layer μ	x	x	x	x
Layer λ	y	z	z	y
	z	y	y	z

Definition 16 (Layerability)**.** *Let O be a set of outcomes, N be a set of players and let $(<_i)_{i\in N}$ be their respective preferences that are swos. Then, $(<_i)_{i\in N}$* ***can be layered*** *for O if there is a partition $\{O_\lambda\}_{\lambda\in I}$ of O (whose elements are called* layers*) and a strict total order $<$ on I (i.e., the layers are totally ordered) s.t.:*

1. *for all pairs of layer O_λ and O_μ with $\lambda < \mu$, for all player $i \in N$, for all $x \in O_\lambda$ for all $y \in O_\mu$: $x \lesssim_i y$; and*
2. *for all layers O_λ, for all players $i, j \in N$, for all outcomes $w, x, y, z \in O_\lambda$: $\neg\big((x <_i y) \wedge (x <_j y) \wedge (w <_i z) \wedge (z <_j w)\big)$.*

The intuition between the notion of layer is as follows: point 1 tells us that the ordering of the layers is compatible with the preference relation of *all* players. That is, if we pick x in some layer O_λ and y in some layer O_μ, with $\lambda < \mu$ (i.e. O_μ is 'better' than O_λ), then *all* players will prefer y to x (or has no preference between x and y). However, with this point alone in the definition, one could put all the outcomes in the same layer (i.e., the partition would be trivially $\{O\}$). Point 2 is more involved: it ensures that the disagreement of the players on some outcomes is also reflected in the layers. That is, in all layers O_λ, we cannot find two players i and j that agree on a pair of elements x and y from O_λ (because they both prefer y to x) but disagree on pair of elements w and z from O_λ (because player i prefers z to w but player j prefers w to z). This is illustrated by the following example.

Example 11. *Consider the set of outcomes $\{u, v, w, x, y, z\}$ and four preference orders $<_1$, $<_2$, $<_3$ and $<_4$ depicted in Table 2. With these orders, the outcomes can be distributed in three layers. In other words the partition is $\{\{y, z\}, \{x\}, \{u, v, w\}\}$, and the order on the layers is $\{y, z\} < \{x\} < \{u, v, w\}$. On this example, the intuition given above can be verified: while the four players do not agree on the u, v and w outcomes, they agree that these outcomes are all better than x, which is always better than y and z. Moreover, in layer ν, we have the following situation: for players 1, 2 and 4: u is the best outcome,*

Table 3: Three preferences where the three players disagree two by two (as before, the order of the rows indicate decreasing preference order). These orders can be layered (by putting all outcomes in the same layer) but not deeply layered.

$<_1$	$<_2$	$<_3$
z	y	x
$x \sim y$	$x \sim z$	$y \sim z$

and w is the worst; and for player 3: w is the best outcome, and u is the worst. That is, either the players agree completely on the preferences on the outcomes, or they totally disagree.

However, this notion of layerability will not be sufficient in general, and we introduce a stronger notion that we call *deep layerability*:

Definition 17 (Deep layerability)**.** *Let O be a set of outcomes, N be a set of players and let $(<_i)_{i\in N}$ be their respective preferences that are swos. Then, $(<_i)_{i\in N}$* ***can be deeply layered*** *for O if there is a partition $\{O_\lambda\}_{\lambda\in I}$ of O (whose elements are called* layers*) and a strict total order $<$ on I s.t.:*

1. *$\{O_\lambda\}_{\lambda\in I}$ and $<$ respect the conditions of Definition 16; and, additionally*
2. *for all layer O_λ, there is a linear extension $(<_i^\lambda)_{i\in N}$ of $(<_i)_{i\in N}$ over O_λ such that for all players $i, j \in N$, for all outcomes $w, x, y, z \in O_\lambda$:*
   $\neg\big((x <_i^\lambda y) \wedge (x <_j^\lambda y) \wedge (w <_i^\lambda z) \wedge (z <_j^\lambda w)\big)$.

As can be seen immediately, Definition 17 is stronger than Definition 16. The point of Definition 17 is to avoid situations where three players disagree two by two, as shown by the following example:

Example 12. *Let us consider the following three-player example. Let $O = \{x, y, z\}$ be the set of outcomes, and assume the preferences are $<_1$, $<_2$, and $<_3$ with: $x \sim_1 y <_1 z$, $x \sim_2 z <_2 y$ and $y \sim_3 z <_3 x$. This situation is represented in Table 3. These orders can be layered by putting all outcomes in the same layer, but cannot be deeply layered.*

Termination of the $\{I, L\}$-Dynamics in the general case. Equipped with these definitions, we can now characterise the termination of the $\{I, L\}$-Dynamics, as shown in Table 1.

We start by discussing an example that shows why some of the techniques previously developed by Le Roux in [LR15] do not extend immediately here. Recall that in [LR15], Le Roux relied on the existence of a pattern in the preference orders of the players to characterise layerability, which, in turn, can be used to characterise termination. As a matter of fact, in the case of *strict linear orders*, the following equivalence holds:

Proposition 18 ([LR15])**.** *Let O be a finite set of outcomes, N a finite set of players and $(<_i)_{i\in N}$ slos. Then $(<_i)_{i\in N}$ is out of main pattern for O if and only if $(<_i)_{i\in N}$ can be layered for O.*

Table 4: Out of pattern strict weak preference orders that can not be layered (the order of the rows indicate decreasing preference order).

$<_1$	$<_2$	$<_3$
z	$z \sim x$	x
y	y	$y \sim z$
x		

Remember, however, that, in the case of slos, being out of main pattern implies being out of secondary pattern. Thus, Le Roux's result still holds when replacing 'out of main pattern' by 'out of pattern'. We thus seek to extend this result to swos. Unfortunately, it does not hold in this case, as shown by the following example:

Example 13. *Consider a set $N = \{1, 2, 3\}$ of three players, a set $O = \{x, y, z\}$ of outcomes, and assume the preferences are $<_1, <_2$ and $<_3$ with: $x <_1 y <_1 z$, $y <_2 z \sim_2 x$ and $y \sim_3 z <_3 x$, as shown in Table 4. These preferences are indeed swos* out of pattern *(thus they are out of main and secondary patterns as they satisfy both* (5) *and* (6)*), but they cannot be layered.*

It is thus not surprising that the characterisation of termination of the $\{I, L\}$-Dynamics is more intricate in the case of swos: in this case, 'deep layerability' is a necessary condition to termination[7], and 'out of pattern' is a sufficient condition. However, none of these characterisation are sufficient and necessary condition. The following Theorem is the main result of this section and sums up these conditions:

Theorem 19. *Let O be a finite set of outcomes, N be a finite set of players and $(<_i)_{i\in N}$ be swos. Then, the following implications hold:* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), *and* (4) $\not\Rightarrow$ (3) *with:*

(1) : $(<_i)_{i\in N}$ can be deeply layered for O;

(2) : the $\{I, L\}$-Dynamics terminates in all games built over O, N and $(<_i)_{i\in N}$;

(3) : all games built over O, N and $(<_i)_{i\in N}$ admit an SNE;

(4) : $(<_i)_{i\in N}$ is out of pattern for O;

Proof. We prove the different implications (or lack thereof) separately:

(1) $\Rightarrow$ (2) For this point, the idea consists in a reduction to a two player game where both players play according to $\{I, L, 1P\}$-Dynamics, i.e., they play lazily and never form coalitions. This is the point of the proof where we exploit heavily the properties of deep layers: as $(<_i)_{i\in N}$ can be deeply layered for O,

[7] Observe that in our original conference paper [BGHR17], this condition was missing.

we know that no player or coalition of players will make a change in order to reach an outcome which is in a lower layer than the current outcome. Indeed, by definition of layers, every player prefers every outcome of an upper layer to any outcome of a lower layer (see Definition 16). Then, we can consider that, at some point along the dynamics, the outcome that will be reached will always belong to the same (deep) layer O_λ because the game is finite.

Moreover, in that deep layer, we can linearise the preferences (see Definition 17, point 2) and make two teams of players: the first team will comprise all players who have the same preference order as player 1; and the second team, all other players. We can then regard each of those teams as a unique player, since all the players in the same team agree on preferences. The resulting game is a game G' with two players that never make coalitions, and play according to the $\{I, L, 1P\}$-Dynamics. By [LRP16], we know that this dynamics terminates for the game G' and we can lift this result to the original game to conclude that the $\{I, L\}$-Dynamics terminates for the game G.

Formally, let N, O be respectively set of players and set of outcomes, and $(<_i)_{i\in N}$ swos that can be deeply layered for O. Let $\{O_\lambda\}_{\lambda\in I}$ be the deep layer partition respecting the conditions of Definition 17.

In order to obtain a contradiction, let us suppose that there is an infinite sequence of strategy profiles $(s^n)_{n\in\mathbb{N}}$ such that for all $n \in N$: $s^n \xrightarrow{\{I,L\}} s^{n+1}$. Let O_μ be a layer. Let y be an outcome in O_μ, and assume that there is $n_0 \in \mathbb{N}$ s.t. $\langle s^{n_0}\rangle = y$, i.e. y is the outcome of the sequence at step n_0. Then, we claim that, for all $m \geq n_0$: $\langle s^m\rangle \notin O_\lambda$ for all $\lambda < \mu$. That is, all the outcomes that will be reached in the sequence from position n_0 will necessarily belong to some layer O_ν that is 'better than or equal' to O_μ (with $\nu \geq \mu$). This stems from the fact that each step in $(s^n)_{n\in\mathbb{N}}$ sequence respects the I property; and from the fact that all outcomes in some layer O_μ are better that all outcomes in some layer O_λ when $\lambda < \mu$, and this for all players (see point 1 of Definition 16). In other words, when some layer is left along the sequence of outcomes generated by $(s^n)_{n\in\mathbb{N}}$, it will never be visited again in the rest of the sequence. Hence, since the number of layers is finite, there is some layer that contains all the outcomes generated by the sequence from some point on. That is, we let $\lambda^* \in I$ and $n^* \in \mathbb{N}$ be s.t. $\langle s^m\rangle \in O_{\lambda^*}$ for all $m \geq n^*$.

Next, we let $(<^*_i)_{i\in N}$ be the linear extensions of $(<_i)_{i\in N}$ for O_{λ^*} (hence $<^*_i$ is defined only over O_{λ^*}). Such linear extensions are guaranteed to exist because we have assumed deep layerability (hence, $O_{\lambda*}$ is a deep layer), and by point 2 of Definition 17. Then, let us consider the set of players N_a and N_b s.t.:

$$\begin{aligned} N_a &= \{i \in N \mid <^*_i = <^*_1\} \\ N_b &= \{i \in N \mid <^*_i = (<^*_1)^{-1}\} \\ &= N \setminus N_a, \end{aligned}$$

and let us build the sequential game $G' = \langle N', A, H, O, d', p', (\precsim_i)_{i\in N}\rangle$ such that:

- $N' = \{a, b\}$, where a represents the coalition of players in N_a and b represents the coalition of players in N_b;

- for all $h \in H'$: $d'(h) = i$ iff $d(h) \in N_i$ for $i \in \{a, b\}$;
- for all $h \in H'$: $p'(h) = p(h)$.
- $<'_a |_{O_{\lambda^*}} =<^*_1$ and $<'_b |_{O_{\lambda^*}} = (<^*_1)^{-1}$. Observe that we constrain $<'_a$ and $<'_b$ on O_{λ^*}, but those preferences are indeed defined over O.

Now, let us consider the suffix $(s^m)_{m>n^*}$. Observe that all these strategy profiles are also strategy profiles in G'. Moreover, we claim that $s^m \xrightarrow[G']{\{I,L\}} s^{m+1}$ for all $m > n^*$. Indeed, $s^m \xrightarrow[G]{\{I,L\}} s^{m+1}$, by hypothesis, hence, for all players i that update their choice between s^m and s^{m+1} (i.e., $i \in d(H(s^m, s^{m+1}))$): $\langle s^m \rangle <_i \langle s^{m+1} \rangle$ with $\langle s^m \rangle$ and $\langle s^{m+1} \rangle$ in O_λ. In particular, $\langle s^m \rangle <^*_i \langle s^{m+1} \rangle$. If $i \in N_a$, it means that $\langle s^m \rangle <'_a \langle s^{m+1} \rangle$, and then $\langle s^{m+1} \rangle <'_b \langle s^m \rangle$ (by definition of $<'_a$ and $<'_b$). If $i \notin N_a$ (i.e. $i \in N_b$), then $\langle s^m \rangle <'_b \langle s^{m+1} \rangle$, and then $\langle s^{m+1} \rangle <'_a \langle s^m \rangle$. So, $s^m \xrightarrow[G']{\{I,L\}} s^{m+1}$. Hence, in G', the corresponding updates are performed by a single player, and thus, all updates also respect the $1P$ property in G'. We conclude that the suffix $(s^m)_{m>n^*}$ is an infinite sequence of strategy profiles in G' respecting the $\{I, L, 1P\}$-Dynamics, which cannot exist by [LRP16]. Contradiction.

$\boxed{(2) \Rightarrow (3)}$ By Proposition 14, all terminal profiles of the $\{I, L\}$-Dynamics are SNEs. Hence, if the dynamics terminates, there exists an SNE in the game.

$\boxed{(3) \Rightarrow (4)}$ We prove the contrapositive, i.e. $\neg(4) \Rightarrow \neg(3)$. In this case $\neg(4)$ means that O either contains the main pattern, or the secondary pattern. We consider these two cases separately.

If $(<_i)_{i \in N}$ contains the main pattern for O, then, let a, b and c be three outcomes and let i, j be two players s.t. $a <_i b <_i c$ and $b <_j c <_j a$. Such a, b, c, i and j exist since $(<_i)_{i \in N}$ is out of main pattern, see (5). Then, let us build a game on top of O, N and $(<_i)_{i \in N}$ that has no SNE: we consider the game in Figure 1, where i is identified with player 1; j with player 2; and the a, b and c outcomes are respectively y, x and z in Figure 1. Then, the graph associated to the $\{I, L\}$-Dynamics is given in Figure 2 (right). We can observe that is has no terminal node, hence no SNE, by Proposition 14.

We proceed similarly in the case where $(<_i)_{i \in N}$ contains the secondary pattern for O. The game in Figure 9 (top, left) is indeed a game that can be built on top of any order that contains the secondary pattern (6). One can check that the graph associated to the $\{I, L\}$-Dynamics (Figure 9, top, right) contains no terminal node, so the game has no SNE.

$\boxed{(4) \not\Rightarrow (3)}$ To establish this point, it suffices to exhibit a game where the preferences $(<_i)_{i \in N}$ are out of pattern, and that admits no SNE. Such a game in given in Figure 9 (bottom). One can check that the graph associated to the $\{I, L\}$-Dynamics has no terminal node, hence the game has no SNE, by Proposition 14. $\square$

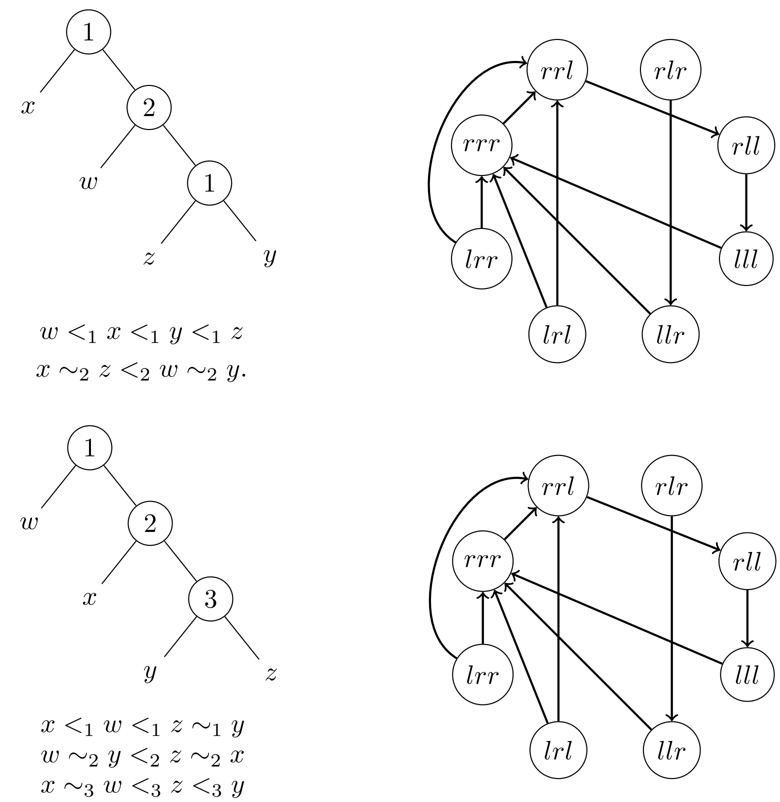

Figure 9: Two counter-examples with their associated graphs.

Termination of the $\{I, L\}$-Dynamics for two players or strict weak preference orders. While the conditions for termination we have established in Theorem 19 are rather intricate, we can show that they are much simpler in two special cases, namely: (i) when the preference orders $(<_i)_{i\in N}$ are *strict linear orders*; or (ii) when there are only two players in the game, i.e. $N = \{1, 2\}$. In those cases, the condition stating that '$(<_i)_{i\in N}$ is out of pattern' becomes a necessary and sufficient condition to the termination of the $\{I, L\}$-Dynamics. This last part of the section is dedicated to proving these results.

We start by considering the case of two-player games. For the sake of readability, we write $<_{1,2}$ instead of $(<_i)_{\{1,2\}}$. Then, we show that having the preference swos $<_{1,2}$ out of pattern is equivalent to the layerability of these swos (which is not the case in general, see Theorem 19):

Lemma 20. *Let O be a finite set of outcomes, $N = \{1, 2\}$, and $<_{1,2}$ swos over O. The following statements are equivalent:*

(1) $<_{1,2}$ are out of pattern for O;

(2) there exist strict linear extension $<'_{1,2}$ of $<_{1,2}$ that can be layered for O;

(3) $<_{1,2}$ can be layered for O.

The proof of this proposition is tedious and technical, and we have thus chosen to present it in Appendix Appendix A.

Then, based on this proposition, and on the fact that, in the case of slos, the notions and layerability and deep layerability collapse, we obtain the following:

Lemma 21. *Let N, O and $(<_i)_{i\in N}$ be respectively a set of players, a set of outcomes and preferences of the players O s.t.: either $N = \{1, 2\}$ and $<_{1,2}$ are swos ; or $(<_i)_{i\in N}$ are slos. Then: $(<_i)_{i\in N}$ can be layered for O iff $(<_i)_{i\in N}$ can be deeply layered for O.*

Proof. First, remember that if $(<_i)_{i\in N}$ can be deeply layered for O, then $(<_i)_{i\in N}$ can be layered for O, by Definition 16 and Definition 17. Let us now prove the reverse implication.

In the case where $(<_i)_{i\in N}$ are slos, the result stems immediately from the definitions of layerability and deep layerability (more precisely, from the fact that point 2 in Definition 17 is then equivalent to point 2 in Definition 16, as already observed). In the case where there are only two players, the result follows from Lemma 20: since $<_{1,2}$ can be layered for O, there are strict linear extensions $<'_{1,2}$ of $<_{1,2}$ that can be layered for O. Let then $(O_\lambda)_{\lambda\in I}$ be the layers corresponding to $<'_{1,2}$. Hence, these layers respect points 1 and 2 of Definition 16, with the orders $<'_{1,2}$. But since $<'_{1,2}$ is a linear extension of $<_{1,2}$, these layers $(O_\lambda)_{\lambda\in I}$ also respect the same points for the preference order $<_{1,2}$. Finally, one can check that they also form deep layers for $<_{1,2}$, by letting $<^{\lambda}_{1,2} = <_{1,2}$ in point 2 of Definition 17. □

As a consequence, we can now specialise the results of Theorem 19 to these two special cases:

Theorem 22. *Let N, O and $(<_i)_{i \in N}$ be respectively a set of players, a set of outcomes and preferences of the players O s.t.: either $N = \{1, 2\}$ and $<_{1,2}$ are swos ; or $(<_i)_{i \in N}$ are slos. Then, the following are equivalent:*

(1) The $\{I, L\}$-Dynamics terminates in all games built over O, N and $(<_i)_{i \in N}$;

(2) All game built over O, N and $(<_i)_{i \in N}$ admit an SNE;

(3) $(<_i)_{i \in N}$ is out of pattern for O.

Proof. One can readily see that the conditions of this Theorem are special cases of the conditions of Theorem 19. Hence, by Theorem 19, we know that: (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Hence, it remains to show that (3) $\Rightarrow$ (1).

In the case of two-player games, Lemma 20 tells us that (3) implies that $(<_i)_{i \in N}$ can be layered for O. This also holds in the case where $(<_i)_{i \in N}$ are slos by [LR15], see Proposition 18 above. By Lemma 21, the layerability property of $(<_i)_{i \in N}$ implies deep layerability in both cases of two-player games and slos. By Theorem 19 again, deep layerability implies point (1) of the present Theorem. Hence, (3) $\Rightarrow$ (1), and the three points of the Theorem are equivalent. □

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Appendix A. Proof of Lemma 20

Throughout this section, we assume a set of players $N = \{1,2\}$, a set of outcomes O and preferences $(<_i)_{i\in\{1,2\}}$ for the players, which we denote $<_{1,2}$ for the sake of readability. In order to establish Lemma 20, we start by several auxiliary results.

The first Lemma shows that if a set of outcomes can be (deeply) layered, then so can all its subsets:

Lemma 23. *If $\exists O' \subseteq O$ such that $(<_i |_{O'})_{i\in N}$ can not be layered (resp. deeply layered) for O', then $(<_i)_{i\in N}$ can not be layered (resp. deeply layered) for O.*

Proof. We prove the contrapositive, i.e.: if O can be layered (resp. deeply layered), then every subset O' of O can be layered (resp. deeply layered). Let $O' \subseteq O$ and $(O_\lambda)_{\lambda\in I}$ be according to 1, 2 of Definition 16 (respectively to points 1, 2 of Definition 16 and point 2 of Definition 17). Let $(O'_\lambda)_{\lambda\in I}$ be a partition of O' such that $\forall\lambda \in I$: $O'_\lambda \subseteq O_\lambda$. Then, one can check that all conditions of Definition 16 (respectively Definition 17) are still satisfied. □

Next, we seek to establish sufficient conditions to obtain strict linear extensions of the preference orders that are out of pattern, as this will be important to establish Lemma 20. We start with a Lemma stating that, when carefully partitioning O into O_1 and O_2, the 'out of pattern' property on O_1 and O_2 of (linear extensions) of $<_{1,2}$ carries on to O.

Lemma 24. *Let O_1, O_2 be a partition of O s.t. $\forall x \in O_1$, $\forall y \in O_2$: $x \lesssim_{1,2} y$. Then, the following statements are equivalent.*

(1) there are strict linear extensions $<''_{1,2}$ of $<_{1,2}$ that are out of pattern for O;

(2) there are strict linear extensions $<'_{1,2}$ of $<_{1,2}$ that are out of pattern for both O_1 and O_2.

Proof. Assume [8] $O = O_1 \uplus O_2$ such that $\forall x \in O_1$, $\forall y \in O_2$: $x \lesssim_{1,2} y$.

$\boxed{(1) \Rightarrow (2)}$ This implication is immediate by Lemma 23.

$\boxed{(2) \Rightarrow (1)}$ Let $<'_{1,2}$ be the linear extension of $<_{1,2}$ that can be layered for O_1 and for O_2. We let $<''_{1,2} |_{O_1} =<'_{1,2}$, $<''_{1,2} |_{O_2} =<'_{1,2}$ and $\forall x \in O_1$, $\forall y \in O_2$, $x <_{1,2} y$. Then $<''_{1,2}$ is a linear extension of $<_{1,2}$ that is out of pattern for O. □

As a consequence, we obtain:

Corollary 25. *If $\exists x \in O$ s.t. $\forall y \in O$: $y \lesssim_1 x$ and $y \lesssim_2 x$ (resp. $x \lesssim_1 y$ and $x \lesssim_2 y$), the following statements are equivalent.*

(1) there are strict linear extensions $<''_{1,2}$ of $<_{1,2}$ that are out of pattern for O

[8] Where $\uplus$ denotes the disjoint union, i.e. $O = O_1 \cup O_2$ assuming $O_1 \cap O_2 = \emptyset$.

(2) there are strict linear extensions $<'_{1,2}$ of $<_{1,2}$ that are out of pattern for $O \setminus \{x\}$.

The next Lemma is another sufficient condition to obtain strict linear extensions that are out of pattern:

Lemma 26. *If there are $x, y \in O$ s.t.: $x \sim_1 y$ and for all $z \in O$: $\neg(x <_2 z <_2 y$ or $y <_2 z <_2 x)$, then the following statements are equivalent.*

(1) there is a strict linear extension $<''_{1,2}$ of $<_{1,2}$ that is out of pattern for O

(2) there is a strict linear extension $<'_{1,2}$ of $<_{1,2}$ that is out of pattern for $O \setminus \{x\}$.

Proof. Let $x, y \in O$ be such that $x \sim_1 y$ and $\forall z \in O\ \neg(x <_2 z <_2 y \bigvee y <_2 z <_2 x)$.

$\boxed{(1) \Rightarrow (2)}$ This implication is obvious by Lemma 23.

$\boxed{(2) \Rightarrow (1)}$ Let $<'_{1,2}$ be a linear extension of $<_{1,2}$ that can be layered for $O \setminus \{x\}$. If $x \lesssim_2 y$ (resp. $y <_2 x$), then we let $y <''_1 x$, $x <''_2 y$ (resp. $x <''_1 y, y <''_2 x$), and $\forall w, z \in O \setminus \{x, y\}$: $x <''_i w$ iff $y <'_i w$; and $z <''_i w$ iff $z <'_i w$. Then $<''_{1,2}$ is a linear extension of $<_{1,2}$ out of pattern for O. □

As a consequence, we obtain:

Corollary 27. *If there are $x, y \in O$ s.t. $x \sim_{1,2} y$, the following statements are equivalent:*

(1) there is a strict linear extension $\exists <''_{1,2}$ of $<_{1,2}$ that is out of pattern for O;

(2) there is a strict linear extension $\exists <'_{1,2}$ of $<_{1,2}$ that is out of pattern for $O \setminus \{x\}$.

We are now ready to prove Lemma 20:

Lemma 20 *Let O be a finite set of outcomes, $N = \{1, 2\}$, and $<_{1,2}$ swos over O. The following statements are equivalent:*

(1) $<_{1,2}$ are out of pattern for O;

(2) there exist $<'_{1,2}$, strict linear extensions of $<_{1,2}$ that can be layered for O;

(3) $<_{1,2}$ can be layered for O.

Proof. $\boxed{(2) \Rightarrow (3)}$ Let $<'_{1,2}$ be the linear extension of $<_{1,2}$ and $\{O_\lambda\}_{\lambda \in I'}$ be the layers. For the layers of O with $<_{1,2}$, we chose $\{O_\lambda\}_{\lambda \in I} = \{O_\lambda\}_{\lambda \in I'}$, the same partition over O. Then we have:

1. $\lambda < \mu$ implies that $\forall i \in N$, $\forall x \in O_\lambda$, $\forall y \in O_\mu$: $x \lesssim_i y$. Indeed, as $\lambda < \mu$ and $\{O_\lambda\}_{\lambda \in I'}$ is a partition over O such that $<'_{1,2}$ can be layered, we have that $x <'_i y$. Moreover, as $<'_{1,2}$ is a linear extension of $<_{1,2}$, it means that $x \lesssim_{1,2} y$.

2. $\forall \lambda \in I, \forall i, j \in N, \forall w, x, y, z \in O_\lambda$: $\neg\big((x <_i y) \wedge (x <_j y) \wedge (w <_i z) \wedge (z <_j w)\big)$.

   As $\{O_\lambda\}_{\lambda \in I'}$ is a partition over O such that $<'_{1,2}$ can be layered, $\neg\big((x <'_i y) \wedge (x <'_j y) \wedge (w <'_i z) \wedge (z <'_j w)\big)$. Then we have: $\neg\big((x <_i y) \wedge (x <_j y) \wedge (w <_i z) \wedge (z <_j w)\big)$.

$\boxed{(3) \Rightarrow (1)}$ Let us prove that, if $<_{1,2}$ is not out of pattern for O, then $<_{1,2}$ can not be layered for O.

By Lemma 23, it is sufficient to show that $O = \{x, y, z\}$ and $<_{1,2,}$ such that $x <_1 y <_1 z$ and $y <_2 z <_2 x$ can not be layered, and the same for $O = \{w, x, y, z\}$ with $<_{1,2}$ such that $w <_1 x <_1 y <_1 z$ and $x \sim_2 z <_2 w \sim_2 y$.

First, if $x <_1 y <_1 z$ and $y <_2 z <_2 x$, if we want to split O into a partition $(O_\lambda)_{\lambda \in I}$, then for λ the minimum of I, $x \in O_\lambda$, because $x <_1 y$ and $x <_1 z$. Otherwise, first condition of Definition 16 can not be satisfied. But if $x \in O_\lambda$, then $y, z \in O_\lambda$, because $y <_2 x$ and $z <_2 x$, in order to satisfy the first condition of Definition 16. But in this case, the second condition of Definition 16 is not satisfied because $y <_1 z$, $y <_2 z$, $x <_1 y$ and $y <_2 x$.

Second, if $O = \{w, x, y, z\}$ and $w <_1 x <_1 y <_1 z$ and $x \sim_2 z <_2 w \sim_2 y$, if we want to divide O into a partition $(O_\lambda)_{\lambda \in I}$, then for λ the minimum of I, $w \in O_\lambda$, because $w <_1 x$, $w <_1 y$ and $w <_1 z$. Otherwise, the first condition of Definition 16 can not be satisfied. But if $w \in O_\lambda$, then $x, z \in O_\lambda$, because $x <_2 w$ and $z <_2 w$, in order to satisfy the first condition of Definition 16, and in this case $y \in O_\lambda$ because $y <_1 z$. But in this case, second condition of Definition 16 is not satisfied because $x <_1 y$, $x <_2 y$, $w <_z y$ and $z <_2 w$.

$\boxed{(1) \Rightarrow (2)}$ We will prove that if $<_{1,2}$ is out of pattern for O, then $\exists <'_{1,2}$, a linear extension of $<_{1,2}$ that is out of pattern. We will prove it by induction over $|O|$. More specifically, we want to prove that, for some O, either $<_{1,2}$ is not out of pattern, or $\exists <'_{1,2}$, a linear extension of $<_{1,2}$, which is out of pattern.

Base case: If $|O| = 1$ then $<_{1,2}$ is out of pattern and can be layered.

Inductive case: Let us suppose now that if $<_{1,2}$ is out of pattern for O, then $\exists <'_{1,2}$, a linear extension of $<_{1,2}$ that is out of pattern for all O s.t. $|O| = n \geq 1$. Let O be such that $|O| = n + 1$. We will take generic preferences for Player 1 and consider every possible preferences for Player 2. In every possible case, we will show that either $<_{1,2}$ is not out of pattern for O or $\exists <'_{1,2}$ a linear extension out of pattern for O.

O can be partitioned according to $<_1$ into $Lev^1, \ldots, Lev^m$ (equivalence classes), which will be called *levels*, such that:

1. $\forall i \in \{1, \ldots m\}$, $\forall x, y \in Lev^i$: $x \sim_1 y$; and
2. $\forall i, j \in \{1, \ldots m\}$ such that $i < j$, $\forall x \in Lev^i, y \in Lev^j$: $x <_1 y$.

Note that levels and layers are different concepts. In particular, the levels are defined by considering only one player. The idea is simply to say that all outcomes of the same level are equivalent for Player 1, and an higher level is

Table A.5: Generic preferences of Player 1.

	$<_1$
Lev^m	$z_1 \sim ... \sim z_{k''}$
	$\vee$
	$\vdots$
	$\vee$
Lev^2	$y_1 \sim ... \sim y_{k'}$
	$\vee$
Lev^1	$x_1 \sim ... \sim x_k$

better according to his preference. We can write the preferences of Player 1 as in Table A.5. Notice that, for every strict weak order $<_{1,2}$ there is a unique division into levels. For example, if Player 1 has the following preferences : $w \sim x < y < z$, we define three levels such that $Lev^1 = \{w, x\}$, $Lev^2 = \{y\}$ and $Lev^3 = \{z\}$.

We now want to consider all possible preferences for Player 2. To do so, we we consider all possible cases depending on properties of the best outcomes for Player 2. Let $Best_2 = \{x \in O \mid \forall y \in O,\ y \lesssim_2 x\}$ be the best outcomes of Player 2. Then (i) either one of them is also one of the best outcomes for player 1; if not (ii) one of them is one of the worst outcome for Player 1, and in this case we will distinguish three subcases; or (iii) none of these two cases. Formally, the five cases are as follow :

1. $\exists x \in Lev^m \cap Best_2$.
2. $Lev^m \cap Best_2 = \emptyset$ and $\exists x \in Lev^1 \cap Best_2$. We split this case in three subcases :
   (a) $|Best_2| = 1$.
   (b) $\exists y \in Lev^1$ such that $y \neq x$ and $y \in Best_2$.
   (c) If we are not in one of these two subcases, it means that $\exists y \in Lev^i$, with $1 < i < m$ such that $y \in Best_2$.
3. $(Lev^m \cup Lev^1) \cap Best_2 = \emptyset$.

Let us now consider these five cases:

1. $\exists x \in Lev^m \cap Best_2$.

   It means that x is one of the best outcomes for Player 1, and also one of the best for Player 2. Intuitively, if we want to split the outcomes into layers, we can make a layer containing only outcome x (even if there are y and z such that $x \sim_1 y$ and $x \sim_2 z$), and consider only the layers of $O \setminus \{x\}$. Formally, we know by Corollary 25 that the following statements are equivalent :

(a) $\exists <''_{1,2}$, a linear extensions of $<_{1,2}$ that is out of pattern for O; and

(b) $\exists <'_{1,2}$, a linear extensions of $<_{1,2}$ that is out of pattern for $O \setminus \{x\}$.

Then, if (1b) is true, then (1a) is true, and our proposition holds too. If (1b) if false, then, as $|O \setminus \{x\}| = n$, we can use the induction hypothesis, and then $<_{1,2}$ is not out of pattern for $O \setminus \{x\}$. In particular, $<_{1,2}$ is not out of pattern for O, and our proposition is then true.

So, if $\exists x \in Lev^m \cap Best_2$, we have that $\Big[<_{1,2}$ is out of pattern for $O \Rightarrow \exists <'_{1,2}$ a strict linear extensions of $<_{1,2}$ that can be layered for $O \Big]$.

2. $Lev^m \cap Best_2 = \emptyset$ and $\exists x \in Lev^1 \cap Best_2$. As announced, we consider three further cases:

   (a) $|Best_2| = 1$.

   Two cases are possible :

   - $\forall y, z \in O,\ y <_1 z \Rightarrow z \lesssim_2 y$
   - $\exists y, z \in O,\ y <_1 z$ and $y <_2 z$

   The first case means that the orders never agree on two outcomes, and the linear extension will retain this disagreement. Let $<'_{1,2}$ be such that :

   – For x, y such that $x <_1 y$ (resp. $x <_2 y$), $x <'_1 y$ (resp. $x <'_2 y$).
   – For x, y such that $x <_1 y$ and $x \sim_2 y$ (resp. $x <_2 y$ and $x \sim_1 y$), $y <'_2 x$ (resp. $y <'_1 x$)
   – For x, y such that $x \sim_1 y$ and $x \sim_2 y$, we decide randomly $x <'_1 y$ and $y <'_2 x$.

   Notice that these three cases catch every possible case for two outcomes x and y, and that the definition of $<'_{1,2}$ implies that $\forall x, y \in O,\ x <_1 y \Leftrightarrow y <_2 x$, with $<'_{1,2}$ linear extension of $<_{1,2}$. Then, we can easily see that $<'_{1,2}$ can be layered by making a unique layer.

   For the second case, if $\exists y, z \in O$ such that $y <_1 z$ and $y <_2 z$, two more cases are possible : either $x <_1 y <_1 z$ or $x \sim_1 y <_1 z$.

   – If $x <_1 y <_1 z$, as $y <_2 z <_2 x$ (because $Best_2 = \{x\}$), then O is not out of pattern for $<_{1,2}$, and the implication is then true.
   – If $x \sim_1 y <_1 z$, let us consider y such that $\forall y' \sim_1 y,\ y \lesssim_2 y'$. In this case, either $\forall w \in O,\ y \leq_2 w$, and then, as y is one of the worst for both players, by Corollary 25 and induction hypothesis, we can find a linear extension that can be layered, and our proposition holds. The last case to deal with is when $\exists w \in O$ such that $w <_2 y <_2 z <_2 x$. But w can be such that $w <_1 z$, $w \sim_1 z$ or $z <_1 w$. These three situations are represented in Table A.6.
   In every case, we can see that $<_{1,2}$ is not out of pattern for O : In the first case, we have the main pattern because $x <_1 w <_1 z$

Table A.6: If $<'_{1,2}$ has the main pattern

$<_1=<'_1$	$<_2$	$<_1=<'_1$	$<_2$	$<_1=<'_1$	$<_2$
	$\boldsymbol{x}$		x	$\boldsymbol{w}$	x
$\boldsymbol{z}$	$\boldsymbol{z}$	$z \quad w$	z	$\boldsymbol{z}$	$\boldsymbol{z}$
$\boldsymbol{w}$	y		y		$\boldsymbol{y}$
$\boldsymbol{x} \quad y$	$\boldsymbol{w}$	$x \quad y$	w	$x \quad \boldsymbol{y}$	$\boldsymbol{w}$

and $w <_2 z <_2 x$, in the second case, we have the secondary pattern because $x \sim_1 y <_1 z \sim_1 w$ and $w <_2 y <_2 z <_2 x$, and in the last case, we have the main pattern because $y <_1 z <_1 w$ and $w <_2 y <_2 z$. Then, in every case, the proposition is true.

(b) $\exists y \in Lev^1$ such that $y \neq x$ and $y \in Best_2$.

It means that $x \sim_2 y$ (as, by definition of $Best_2$, we have that $x \lesssim_2 y$ and $y \lesssim_2 x$). Then by Corollary 27, the implication is true in this case.

(c) The two previous subcases do not hold.

Then let $y \in Best_2 \cap Lev^i$, with $1 < i < m$ and such that $\forall j > i,\ Lev^j \cap Best_2 = \emptyset$. First, observe that, if $\exists y' \in Best_2 \cap Lev^i$, then by Corollary 27, the implication is true in this case. So let us consider that this is not the case.

If we consider a new preferences $<'_{1,2}$ such that :

i. $<_{1,2}$ is out of pattern for $O \ \Rightarrow <'_{1,2}$ is out of pattern for O.

ii. If $<'_{1,2}$ is out of pattern, then $\exists \ <''_{1,2}$ a strict linear extension of $<'_{1,2}$ that is out of pattern for O, which is a linear extension of $<_{1,2}$.

we will have proved that $\Big[<_{1,2}$ is out of pattern for $O \Rightarrow \exists <'_{1,2}$ a strict linear extensions of $<_{1,2}$ that can be layered for $O \Big]$.

Let $<'_{1,2}$ be such that :

- $<'_1 = <_1$
- $\forall z \in Best_2 \setminus \{y\},\ y <'_2 z$ (in particular, $y <'_2 x$)
- $\forall z, z' \neq y,\ z <'_2 z' \Leftrightarrow z <_2 z'$ (except for y, the order remains the same).

We will now prove that this order verifies the three conditions above.

i. $<_{1,2}$ is out of pattern for $O \ \Rightarrow <'_{1,2}$ is out of pattern for O

Let us suppose that $<'_{1,2}$ is not out of pattern for O, and proove that $<_{1,2}$ is also not out of pattern.

The only things that we know is that $x <_1 y,\ x, y \in Best_2$, and if $\exists z \in Best_2,\ x <_1 z <_1 y$. We can illustrate it as in Table A.7. What we want to do now is to distinguish every possible way for the pattern to appears in $<'_{1,2}$ and show that, in every case, $<_{1,2}$ is not out of pattern.

Table A.7: General situation for 2c

$<_1=<'_1$	$<_2$	$<'_2$
	x y z	x z
y		y
z		
x		

Table A.8: If $<'_{1,2}$ has the secondary pattern

$<_1=<'_1$	$<_2$	$<'_2$
	x y z	x $\boldsymbol{z}$
$\boldsymbol{y}$ $\boldsymbol{y'}$		$\boldsymbol{y}$
$\boldsymbol{z}$ $\boldsymbol{z'}$	z'	$\boldsymbol{z'}$
x	y'	$\boldsymbol{y'}$

- First case, let us suppose that $<'_{1,2}$ has the the secondary pattern. If this happens without y and some $z \in Best_2$, then it is clear that the same pattern appears in $<_{1,2}$. Moreover, by definition of $<'_2$, $\forall z \in O$, $\neg(z \sim'_2 y)$. Then the secondary pattern has to appear this way : $z \sim'_1 z' <'_1 y \sim'_1 y'$ and $y' <'_2 z' <'_2 y <'_2 z$. This preferences are illustrated in Table A.8.
  We want to prove that in this case, $<_{1,2}$ is not out of pattern. Remember that $y \notin Lev^m$. Then, let $w \in Lev^m$, and we know that $w \notin Best_2$, so $w <_2 y$. Two (not disjoint) cases are possible : either $y' <_2 w$ or $w <_2 z'$. In the first case, we have $z <_1 y' <_1 w$ and $y' <_2 w <_2 z$, then the pattern appears in $<_{1,2}$, and in the second case we have $z' <_1 y <_1 w$ and $w <_2 z' <_2 y$, and the pattern appears also in $<_{1,2}$. These two cases are described in Table A.9.
  Notice that this remains true if the secondary pattern appears with x instead of z, as we just need that $z <_1 y$.
- Second case, let us suppose that $<'_{1,2}$ has the main pattern. Once again, if it happens without y and some $z \in Best_2$, this pattern is also in $<_{1,2}$ as the preferences are the same. So we will have something like $z' <'_2 y <'_2 z$. To obtain the

Table A.9: Main pattern in $<_{1,2}$ if $<'_{1,2}$ has the secondary pattern

$<_1=<'_1$	$<_2$	$<'_2$	$<_1=<'_1$	$<_2$	$<'_2$
$\boldsymbol{w}$	x y $\boldsymbol{z}$	x z	$\boldsymbol{w}$	x $\boldsymbol{y}$ z	x z
y $\boldsymbol{y'}$	$\boldsymbol{w}$	y	$\boldsymbol{y}$ y'		y
$\boldsymbol{z}$ z'		z'	z $\boldsymbol{z'}$	$\boldsymbol{z'}$	z'
x	$\boldsymbol{y'}$	y'	x	$\boldsymbol{w}$	y'

Table A.10: If $<'_{1,2}$ has the main pattern

$<_1=<'_1$	$<_2$	$<'_2$
	$x \quad y$	$\boldsymbol{x}$
$\boldsymbol{y}$		$\boldsymbol{y}$
$\boldsymbol{z'}$	z'	$\boldsymbol{z'}$
$\boldsymbol{x}$		

Table A.11: Pattern in $<_{1,2}$ if $<'_{1,2}$ has the main pattern

$<_1=<'_1$	$<_2$
$\boldsymbol{w}$	$x \quad \boldsymbol{y}$
$\boldsymbol{y}$	
$\boldsymbol{z'}$	$\boldsymbol{z'}$
x	$\boldsymbol{w}$

$<_1=<'_1$	$<_2$
$\boldsymbol{w}$	$\boldsymbol{x} \quad \boldsymbol{y}$
$\boldsymbol{y}$	
$\boldsymbol{z'}$	$\boldsymbol{z'} \quad \boldsymbol{w}$
$\boldsymbol{x}$	

$<_1=<'_1$	$<_2$
$\boldsymbol{w}$	$\boldsymbol{x} \quad y$
y	$\boldsymbol{w}$
$\boldsymbol{z'}$	$\boldsymbol{z'}$
$\boldsymbol{x}$	

pattern, and as $z <'_1 y$, we then have $z <'_1 z' <'_1 y$. Notice that, in this case, we have also the pattern with x instead of z, so we will use x in order to simplify. These preferences are illustrated in Table A.10

Once again, as $y \notin Lev^m$, then $\exists w \in Lev^1$, and we know that $w \notin Best_2$. Then three cases are possible : $w <_2 z'$, $w \sim_2 z'$ or $z' <_2 w$. In all cases, we will show that $<_{1,2}$ is not out of pattern. In the first case, we have $z' <_1 y <_1 w$ and $w <_2 z' <_2 y$, which is the main pattern in $<_{1,2}$. In the second case, we have $x <_1 z' <_1 y <_1 w$ and $z' \sim_2 w <_2 x \sim_2 y$, which is the secondary pattern in $<_{1,2}$, and in the third case, we have $x <_1 z' <_1 w$ and $z' <_2 w <_2 x$, which is the main pattern in $<_{1,2}$. These three cases are illustrated in Table A.11.

□

So we have proved that, if $<_{1,2}$ is out of pattern for O, then $<'_{1,2}$ is out of pattern for O. However, $<'_{1,2}$ is not (necessary) a linear order. So we still have to prove 2(c)ii to obtain what we need.

ii. If $<'_{1,2}$ is out of pattern, then $\exists <''_{1,2}$ a strict linear extension of $<'_{1,2}$ that is out of pattern for O, which is a linear extension of $<_{1,2}$.

By definition of $<'_{1,2}$, it is clear that every linear extension of $<'_{1,2}$ will be a linear extension of $<_{1,2}$.

Let $Best'_2 = \{y \in O \mid \forall z \in O, z \leq'_2 y\}$.

Two cases are possible. Either $|Best'_2| = 1$, and then, by 2a, we already know that the implication is true. Indeed, $\exists <''_{1,2}$, linear extension of $<'_{1,2}$ that is out of pattern for O if $<'_{1,2}$ is out of pattern for O.

If not, we consider $y' \in Lev^i \cap Best'_2$, $y \neq x$, such that $\forall j >$

Table A.12: Example of division between O_1 and O_2

	$<'_1$			$<'_2$			
O_1		y			x		
O_1					y		
O_1		y'		y	$\sim$	x'	
O_1	$x \sim$	x'	$\sim x''$		w		O_2
		w			x''		O_2

$i,\ Lev^j \cap Best'_2 = \emptyset$. Then, we will do the same as previously: define a new order $<^2_{1,2}$ such that $y' <^2_2 z$ for $z \in Best'_2$. We will obtain the same result, which is that $<^2_{1,2}$ is out of pattern if $<'_{1,2}$ is out of pattern, and iterate this process until $|Best^n_2| = 1$. In this case, by 2a, our implication will be true, and so we will have what we need, in other words that if $<_{1,2}$ is out of pattern, there exists a linear extension that is out of pattern for O.

3. $(Lev^m \cup Lev^1) \cap Best_2 = \emptyset$.

As $Best_2 \neq \emptyset$, $\exists x \in Lev^i$ with $1 < i < m$ such that $x \in Best_2$. If there are several $1 < i < m$, such that $Lev^i \cap Best_2 \neq \emptyset$, we consider $x \in Lev^i$ such that i is the smallest. In other words, $\forall j < i,\ Lev^j \cap Best_2 = \emptyset$.

Two cases are possible:

- Either $\forall j < i < k, w \in Lev^j$ and $y \in Lev^k$, $w \lesssim_2 y$.
  Then, let:

  $$O_1 = \{y \in Lev^k \mid k > i\} \cup \{x' \in Lev^i \mid \exists k > i, y \in Lev^k \text{ s.t. } y \lesssim_2 x'\}$$

  and

  $$O_2 = \{w \in Lev^j \mid j < i\} \cup \{x' \in Lev^i \mid \forall k > i, y \in Lev^k, x' <_2 y\} = O \backslash O_1.$$

  An example is given in Table A.12. Then, $\forall y \in O_1, z \in O_2, z \lesssim_{1,2} y$. By Lemma 24, the proposition is true in this case.

- Or $\exists j < i < k, w \in Lev^j$ and $y \in Lev^k$ such that $y <_2 w$.

  We can suppose that $w <_2 x$ because, either $w \in Lev^1$, and we have already dealt with this case previously, or $1 < j < i$, and as we have chosen $x \in Lev^i$ such that i is the smallest, we can not have $w \sim_2 x$.

  Then $w <_1 x <_1 y$ and $y <_2 w <_2 x$, we do have the pattern so the proposition is true in this last case. □