Maximizing the eccentric connectivity index for graphs with given order and diameter

Pierre Hauweele

Joint work with Alain Hertz, Hadrien Mélot,
Bernard Ries and Gauvain Devillez

Algorithms Lab, Computer Science Departement
Faculty of Sciences, University of Mons

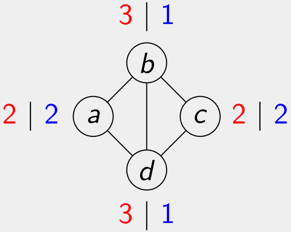
JGA — 2019

Definition

The Eccentric Connectivity Index of a graph $G = (V, E)$, denoted by $\xi^c(G)$, is

$$\xi^c(G) = \sum_{v \in V} \deg(v)\epsilon(v). \quad \text{Alternatively, } \xi^c(G) = \sum_{uv \in E} (\epsilon(u) + \epsilon(v)).$$

Example



$$\xi^c(G) = 2 \cdot 2 + 3 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 = 14$$

Problem

Among connected graphs of order n and diameter D, what is the maximum possible value for ξ^c?

Problem

Among connected graphs of order n and diameter D, what is the maximum possible value for ξ^c?

Theorem (Morgan, Mukwembi, and Swart 2011)

Let G be a connected graph of order n and diameter D. Then,

$$\xi^c(G) \leq D(n-D)^2 + \mathcal{O}(n^2).$$

The lollipops $L_{n,D}$ attain this bound.

Problem

Among connected graphs of order n and diameter D, what is the maximum possible value for ξ^c?

Theorem (Morgan, Mukwembi, and Swart 2011)

Let G be a connected graph of order n and diameter D. Then,

$$\xi^c(G) \leq D(n-D)^2 + \mathcal{O}(n^2).$$

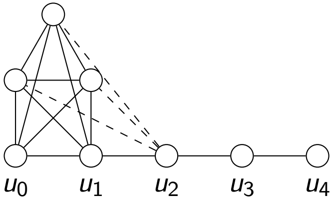
The lollipops $L_{n,D}$ attain this bound.

What about an exact bound ?

Definition

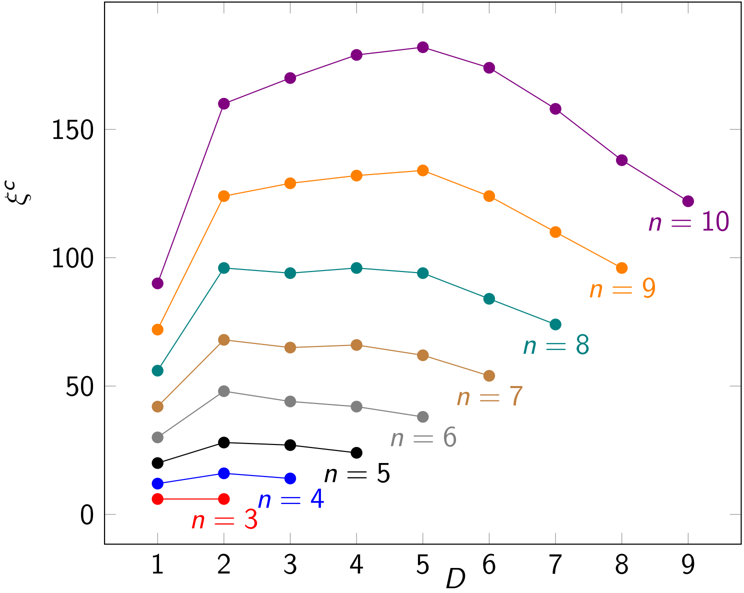
Let n, D and k be integers such that $n \geq 4$, $3 \leq D \leq n-1$ and $0 \leq k \leq n-D-1$, and let $\mathrm{E}_{n,D,k}$ be the graph (of order n and diameter D) constructed from a path $u_0 - u_1 - \ldots - u_D$ by joining each vertex of a clique K_{n-D-1} to u_0 and u_1, and k vertices of the clique to u_2.

- $\mathrm{E}_{n,D,0} \simeq L_{n,D}$, the lollipop;
- $\mathrm{E}_{n,D,n-D-1}$ is a lollipop $L_{n,D-1}$ missing an edge;
- if $D = n-1$, then $k = 0$ and $\mathrm{E}_{n,n-1,0} \simeq P_n$.

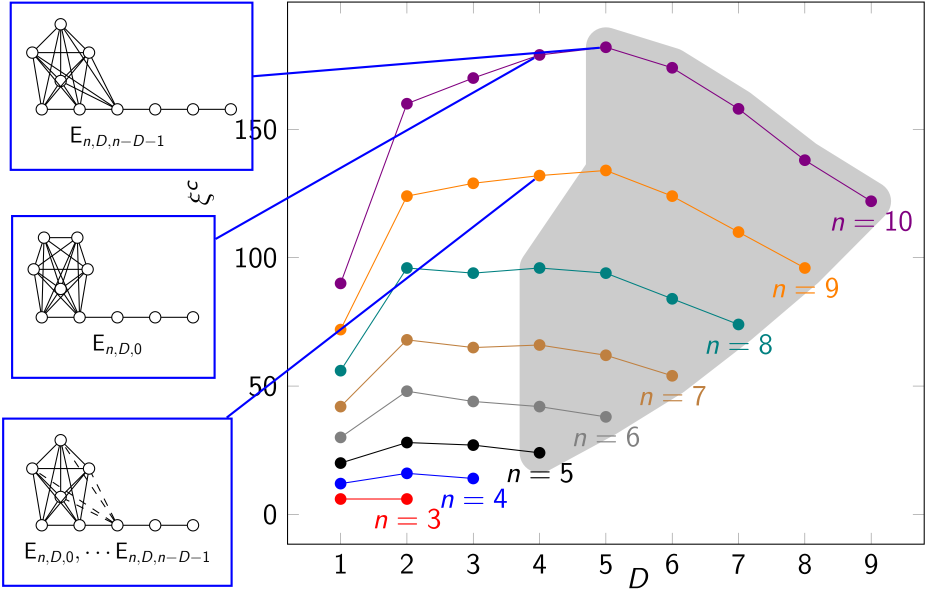


$\mathrm{E}_{8,4,k}$, dashed edges depend on k.

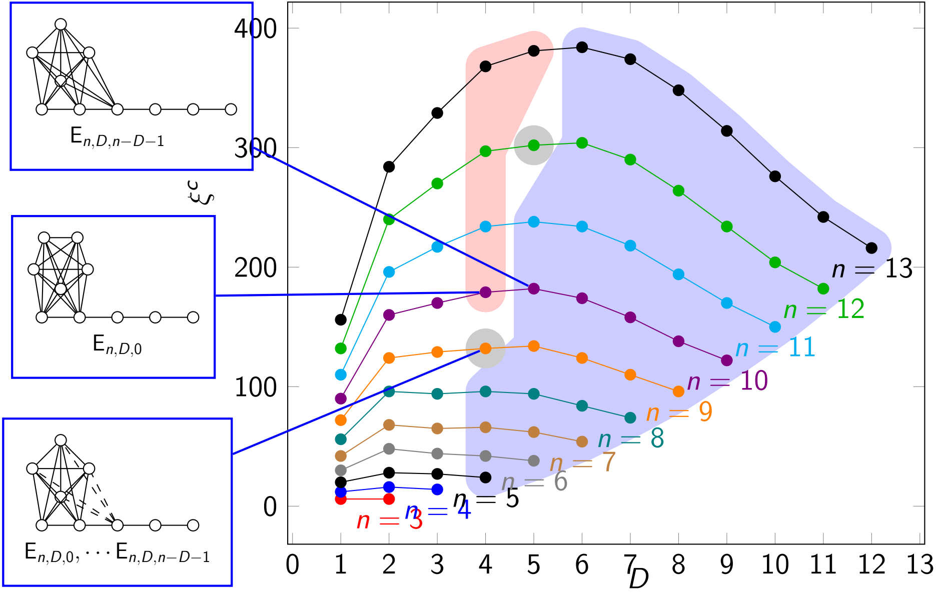
max ξ^c for given order n and diameter D



max ξ^c for given order n and diameter D



max ξ^c for given order n and diameter D



$\max \xi^c$ with order and diameter when $D \geq 3$

Theorem (H et al. 2019)

Let G be a connected graph of order $n \geq 4$ and diameter $3 \leq D \leq n-1$. Let $f(n,D) = \max\{\xi^c(\mathrm{E}_{n,D,k}) \mid k = 0, \ldots, n-D-1\}$. Then $\xi^c(G) \leq f(n,D)$ with equality if and only if G belongs to $\mathcal{C}_n^D$.

$$\mathcal{C}_n^D = \begin{cases} \{\mathrm{E}_{n,3,n-4}\} & \text{if } n = 4,5 \text{ and } D = 3; \\ \{\mathrm{E}_{n,3,2}, H_2\} & \text{if } n = 6 \text{ and } D = 3; \\ \{\mathrm{E}_{n,3,0}, \ldots, \mathrm{E}_{n,3,3}, H_3\} & \text{if } n = 7 \text{ and } D = 3; \\ \{\mathrm{E}_{n,3,0}\} & \text{if } n > 7 \text{ and } D = 3; \\ \{\mathrm{E}_{n,D,0}\} & \text{if } n > 3(D-1) \text{ and } D \geq 4; \\ \{\mathrm{E}_{n,D,0}, \ldots, \mathrm{E}_{n,D,n-D-1}\} & \text{if } n = 3(D-1) \text{ and } D \geq 4; \\ \{\mathrm{E}_{n,D,n-D-1}\} & \text{if } n < 3(D-1) \text{ and } D \geq 4. \end{cases}$$

Proof plan

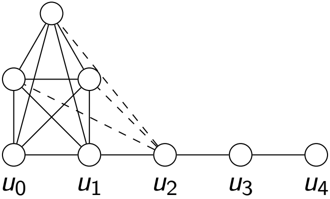
1. Compute $\xi^c(\mathrm{E}_{n,D,k})$.
2. Work out $f(n, D) = \max_k \xi^c(\mathrm{E}_{n,D,k})$ (and convince ourselves that the graphs in $\mathcal{C}_n^D$ have $\xi^c = f(n, D)$).
3. Show that, for a graph G of order n and diameter D, $\xi^c(G) \leq f(n, D)$, and if it attains the bound, then it is isomorphic to a graph in $\mathcal{C}_n^D$.

1. Compute $\xi^c(\mathrm{E}_{n,D,k})$

Lemma

Let n, D and k be integers such that $n \geq 4$, $3 \leq D \leq n-1$ and $0 \leq k \leq n-D-1$, then

$$\xi^c(\mathrm{E}_{n,D,k}) = 2\sum_{i=0}^{D-1} \max\{i, D-i\} + \Big(n-D-1\Big)\Big(2D-1+D(n-D)\Big) + k\Big(3D-n-3\Big).$$



2. Work out $f(n, D) = \max_k \xi^c(\mathsf{E}_{n,D,k})$

$$\xi^c(\mathsf{E}_{n,D,k}) = 2\sum_{i=0}^{D-1} \max\{i, D-i\} + \Big(n - D - 1\Big)\Big(2D - 1 + D(n-D)\Big) + k\Big(3D - n - 3\Big).$$

$$\mathcal{C}_n^D = \begin{cases} [\ldots] & \\ \{\mathsf{E}_{n,D,0}\} & \text{if } n > 3(D-1) \text{ and } D \geq 4; \\ \{\mathsf{E}_{n,D,0}, \ldots, \mathsf{E}_{n,D,n-D-1}\} & \text{if } n = 3(D-1) \text{ and } D \geq 4; \\ \{\mathsf{E}_{n,D,n-D-1}\} & \text{if } n < 3(D-1) \text{ and } D \geq 4. \end{cases}$$

2. Work out $f(n, D) = \max_k \xi^c(\mathrm{E}_{n,D,k})$

$$\xi^c(\mathrm{E}_{n,D,k}) = 2\sum_{i=0}^{D-1} \max\{i, D-i\} + \Big(n - D - 1\Big)\Big(2D - 1 + D(n-D)\Big) + k\Big(3D - n - 3\Big).$$

$$f(n, D) = 2\sum_{i=0}^{D-1} \max\{i, D-i\} + \Big(n - D - 1\Big)\Big(2D - 1 + D(n-D) + \max\{0, 3D - n - 3\}\Big)$$

3. Last step of the proof — subplan

Theorem

Let G be a connected graph of order $n \geq 4$ and diameter $3 \leq D \leq n-1$. Then $\xi^c(G) \leq f(n, D)$ with equality if and only if G belongs to $\mathcal{C}_n^D$.

1. Give an upper bound on the total weight of the vertices outside a diametral path P.
2. Extend to an upper bound on $\xi^c(G)$.
3. Prove that this bound is attained only if G is isomorphic to one of $\mathcal{C}_n^D$.

Tool lemma

Let G be a connected graph of diameter $D \geq 3$. Let P be a diametral path, and u a vertex on P such that $\epsilon(u) > L$, with L the longest distance from u to an extremity of P. Finally, let v be a vertex such that $d(u, v) = \epsilon(u)$ and let $v = w_1 - w_2 - \cdots - w_{\epsilon(u)+1} = u$ be a shortest path linking v to u. Then

- *vertices $w_1, \ldots, w_{\epsilon(u)-L}$ do not belong to P;*
- *vertex $w_{\epsilon(u)-L}$ has either no neighbor on P, or its unique neighbor on P is an extremity at distance L from u;*
- *if $\epsilon(u) - L > 1$ then $w_1, \ldots, w_{\epsilon(u)-L-1}$ have no neighbor on P.*

$\|P\| = D$

Tool lemma

Let G be a connected graph of diameter $D \geq 3$. Let P be a diametral path, and u a vertex on P such that $\epsilon(u) > L$, with L the longest distance from u to an extremity of P. Finally, let v be a vertex such that $d(u,v) = \epsilon(u)$ and let $v = w_1 - w_2 - \cdots - w_{\epsilon(u)+1} = u$ be a shortest path linking v to u. Then

- *vertices $w_1, \ldots, w_{\epsilon(u)-L}$ do not belong to P;*
- *vertex $w_{\epsilon(u)-L}$ has either no neighbor on P, or its unique neighbor on P is an extremity at distance L from u;*
- *if $\epsilon(u) - L > 1$ then $w_1, \ldots, w_{\epsilon(u)-L-1}$ have no neighbor on P.*

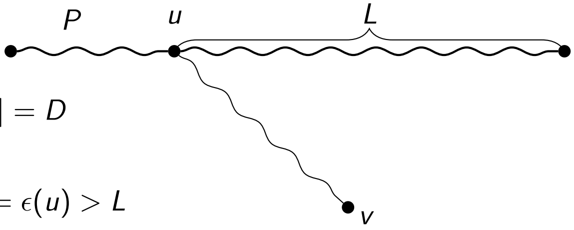
$\|P\| = D$

$\epsilon(u) > L$

Tool lemma

Let G be a connected graph of diameter $D \geq 3$. Let P be a diametral path, and u a vertex on P such that $\epsilon(u) > L$, with L the longest distance from u to an extremity of P. Finally, let v be a vertex such that $d(u, v) = \epsilon(u)$ and let $v = w_1 - w_2 - \cdots - w_{\epsilon(u)+1} = u$ be a shortest path linking v to u. Then

- *vertices $w_1, \ldots, w_{\epsilon(u)-L}$ do not belong to P;*
- *vertex $w_{\epsilon(u)-L}$ has either no neighbor on P, or its unique neighbor on P is an extremity at distance L from u;*
- *if $\epsilon(u) - L > 1$ then $w_1, \ldots, w_{\epsilon(u)-L-1}$ have no neighbor on P.*



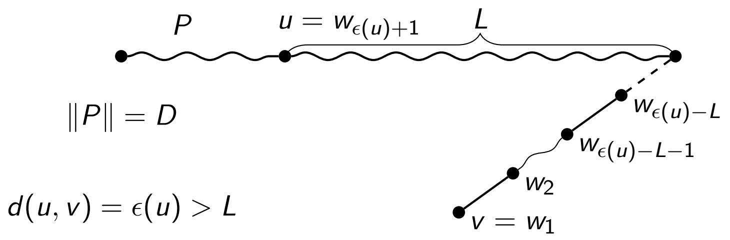
$\|P\| = D$

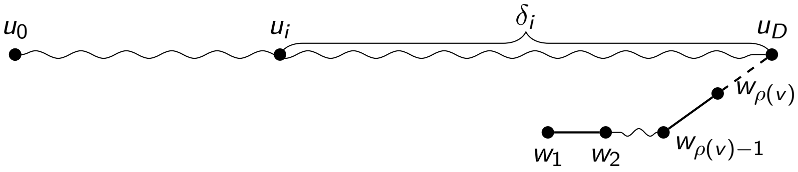
$d(u, v) = \epsilon(u) > L$

Tool lemma

Let G be a connected graph of diameter $D \geq 3$. Let P be a diametral path, and u a vertex on P such that $\epsilon(u) > L$, with L the longest distance from u to an extremity of P. Finally, let v be a vertex such that $d(u,v) = \epsilon(u)$ and let $v = w_1 - w_2 - \cdots - w_{\epsilon(u)+1} = u$ be a shortest path linking v to u. Then

- *vertices $w_1, \ldots, w_{\epsilon(u)-L}$ do not belong to P;*
- *vertex $w_{\epsilon(u)-L}$ has either no neighbor on P, or its unique neighbor on P is an extremity at distance L from u;*
- *if $\epsilon(u) - L > 1$ then $w_1, \ldots, w_{\epsilon(u)-L-1}$ have no neighbor on P.*





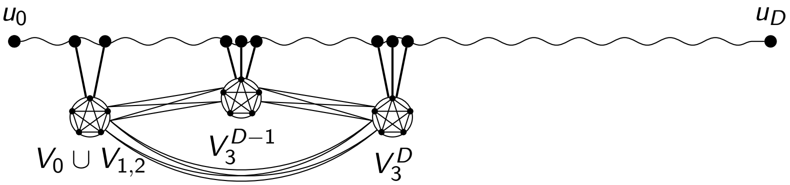
$$r_i = \epsilon(u_i) - \delta_i.$$

$$r^* = \max_{i=0}^{D} r_i).$$

Claim (weight outside P)

$$\sum_{v \notin P} \mathcal{W}(v) \leq (n-D-1)D(n-D) + n_3^{D-1}(2D-n-1) - Dn_3^D - 2Dr^* + D\min\{1, \rho^*\} - \sum_{v \in V_{1,2} \cup V_3^D \cup V_3^{D-1}} (2D-1)\rho(v).$$

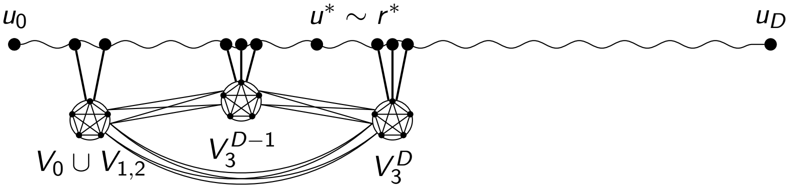
3.1. Bound on the weight outside P



$$\mathcal{W}(V_0 \cup V_{1,2}) \leq D(n-D)(n-D-1-n_3^{D-1}-n_3^D)$$
$$\mathcal{W}(V_3^{D-1} \cup V_3^D) \leq (n-D+1)\Big((D-1)n_3^{D-1}+Dn_3^D\Big)$$

$$V_{1,2} = \{v \notin P \mid |N(v) \cap P| \in \{1,2\}\},$$
$$V_3^{D-1} = \{v \notin P \mid |N(v) \cap P| = 3, \epsilon(v) \leq D-1\},$$
$$V_3^D = \{v \notin P \mid |N(v) \cap P| = 3, \epsilon(v) = D\},$$

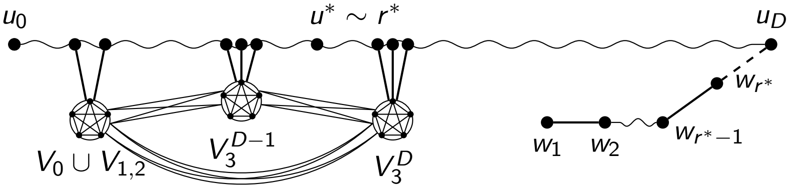
3.1. Bound on the weight outside P



$$\mathcal{W}(V_0 \cup V_{1,2}) \leq D(n-D)(n-D-1-n_3^{D-1}-n_3^D)$$
$$\mathcal{W}(V_3^{D-1} \cup V_3^D) \leq (n-D+1)\Big((D-1)n_3^{D-1} + Dn_3^D\Big)$$

$$V_{1,2} = \{v \notin P \mid |N(v) \cap P| \in \{1,2\}\},$$
$$V_3^{D-1} = \{v \notin P \mid |N(v) \cap P| = 3, \epsilon(v) \leq D-1\},$$
$$V_3^D = \{v \notin P \mid |N(v) \cap P| = 3, \epsilon(v) = D\},$$

3.1. Bound on the weight outside P



$$\mathcal{W}(V_0 \cup V_{1,2}) \leq D(n-D)(n-D-1-n_3^{D-1}-n_3^D) - 2Dr^* + D\min\{1, \rho^*\}$$

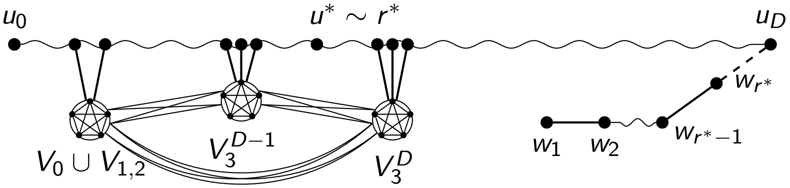
$$\mathcal{W}(V_3^{D-1} \cup V_3^D) \leq (n-D+1)\Big((D-1)n_3^{D-1} + Dn_3^D\Big)$$

$$V_{1,2} = \{v \notin P \mid |N(v) \cap P| \in \{1,2\}\},$$

$$V_3^{D-1} = \{v \notin P \mid |N(v) \cap P| = 3, \epsilon(v) \leq D-1\},$$

$$V_3^D = \{v \notin P \mid |N(v) \cap P| = 3, \epsilon(v) = D\},$$

3.1. Bound on the weight outside P



$$\mathcal{W}(V_0 \cup V_{1,2}) \leq D(n-D)(n-D-1-n_3^{D-1}-n_3^D) - 2Dr^* + D\min\{1,\rho^*\}$$

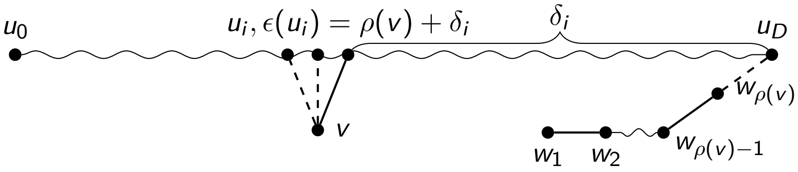
$$\mathcal{W}(V_3^{D-1} \cup V_3^D) \leq (n-D+1)\Big((D-1)n_3^{D-1} + Dn_3^D\Big)$$

We get a bound on the total weight of the vertices outside P

$$\begin{aligned} B &= \mathcal{W}(V_0 \cup V_{1,2} \cup V_3^{D-1} \cup V_3^D) \\ &= (n-D-1)D(n-D) + n_3^{D-1}(2D-n-1) + Dn_3^D - 2Dr^* \\ &\quad + D\min\{1,\rho^*\}. \end{aligned}$$

Can only be reached if all vertices outside P are pairwise adjacent.

3.2. Improving the bound on the weight outside of P



$\rho(v) = \max\{r_i \mid u_i \text{ is adjacent to } v\}, r_i = \epsilon(u_i) - \delta_i.$

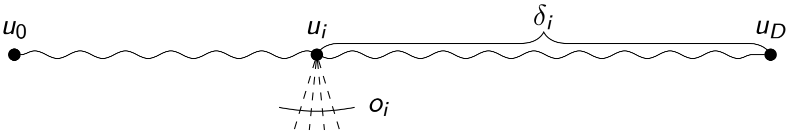
$$
\begin{aligned}
&B - \sum_{v \in V_{1,2} \cup V_3^D} 2D\rho(v) - \sum_{v \in V_3^{D-1}} (2D-1)\rho(v) - 2Dn_3^D \\
&\leq (n-D-1)D(n-D) + n_3^{D-1}(2D-n-1) - Dn_3^D - 2Dr^* \\
&\quad + D\min\{1, \rho^*\} - \sum_{v \in V_{1,2} \cup V_3^D \cup V_3^{D-1}} (2D-1)\rho(v).
\end{aligned}
$$

Which is the claim.

Claim (total weight on G)

$$\xi^c(G) \leq (n-D-1)D(n-D) + n_3^{D-1}(2D-n-1) - Dn_3^D + 2\sum_{i=0}^{D-1}\delta_i + \sum_{i=0}^{D}\delta_i o_i.$$

Bounding the weight on P



Now we compute a bound on the total weight of P.

$$\begin{aligned}\mathcal{W}(P) &= 2D + D(o_0 + o_D) + \sum_{i=1}^{D-1} (\delta_i + r_i)(2 + o_i) \\ &= 2\sum_{i=0}^{D-1} \delta_i + 2\sum_{i=1}^{D-1} r_i + \sum_{i=1}^{D-1} r_i o_i + \sum_{i=0}^{D} \delta_i o_i.\end{aligned}$$

We bound this, so as to remove the r_i's.

$$\mathcal{W}(P) \leq 2\sum_{i=0}^{D-1} \delta_i + \sum_{i=0}^{D} \delta_i o_i + 2r^*(D-1) + \sum_{v \in V_{1,2} \cup V_3^{D-1} \cup V_3^D} 3\rho(v).$$

3.3. Upper bound on $\xi^c(G)$

Summing the bounds from the two claims and rewriting, we have

$$\xi^c(G) \leq A_1 + A_2,$$

$$\text{with } A_1 = (n-D-1)D(n-D) + n_3^{D-1}(2D-n-1) - Dn_3^D + 2\sum_{i=0}^{D-1}\delta_i + \sum_{i=0}^{D}\delta_i o_i$$

$$A_2 = -\sum_{v \in V_{1,2} \cup V_3^D \cup V_3^{D-1}} (2D-4)\rho(v) - 2r^* + D\min\{1, \rho^*\}.$$

- If $r^* = 0$, then $A_2 = 0$, which implies $A_1 + A_2 = A_1$.
- If $\rho^* > 0$, then $A_2 \leq 4 - 2D - 2r^* + D = 4 - D - 2r^* < 0$, which implies $A_1 + A_2 < A_1$.
- If $r^* > 0$ and $\rho^* = 0$, then $A_2 = -2r^* < 0$, which implies $A_1 + A_2 < A_1$.

3.4. The bound is attained only if G is one of $\mathcal{C}_n^D$

In summary, the best possible bound is A_1 and this bound is attained only if the upper bound of Claim (weight outside P) is reached with $r^* = 0$. As shown in the proof of the claim, this implies $n_0 = 0$, $\epsilon(v) = D$ for all vertices in $V_{1,2}$, and all vertices in $V_{1,2} \cup V_3^{D-1}$ are pairwise adjacent.

We only need to prove that $A_1 = f(n, D)$ and that the graphs G with $\xi^c(G) = A_1 = f(n, D)$ are exactly those in C_n^D. $\longrightarrow$ bound and minimize $f(n, D) - A_1$.

Maximizing ξ^c for a fixed order

Morgan, Mukwembi, and Swart 2011 also gave an asymptotic bound for maximizing ξ^c given the order only.

Theorem (Morgan, Mukwembi, and Swart 2011)

Let G be a connected graph of order n. Then,

$$\xi^c(G) \leq \frac{4}{27}n^3 + \mathcal{O}(n^2).$$

Theorem (H et al. 2019)

Let ξ_n^{c*} be the largest eccentric connectivity index among all graphs of order n. The only graphs that attain ξ_n^{c*} are the following:

n	ξ_n^{c*}	*optimal graphs*
3	6	K_3 and P_3
4	16	$\overline{\mathrm{M}}_4$
5	30	$\overline{\mathrm{M}}_5$ and H_1
6	48	$\overline{\mathrm{M}}_6$
7	68	$\overline{\mathrm{M}}_7$
8	96	$\overline{\mathrm{M}}_8$ and $\mathrm{E}_{8,4,3}$
≥ 9	$g(n)$	$\mathrm{E}_{n,\lceil\frac{n+1}{3}\rceil+1,n-\lceil\frac{n+1}{3}\rceil-2}$.

This is obtained as a corollary of our previous results by a simple analysis of

$$\max_D f(n, D).$$

Maximizing ξ^c with given order and size

Conjecture (H et al. 2019)

Let n and m be two integers such that $n \geq 4$ and $m \leq \binom{n-1}{2}$. Also, let

$$D = \left\lfloor \frac{2n+1-\sqrt{17+8(m-n)}}{2} \right\rfloor \text{ and } k = m - \binom{n-D+1}{2} - D + 1.$$

Then, the largest eccentric connectivity index among all graphs of order n and size m is attained with $\mathsf{E}_{n,D,k}$. Moreover,

- *if $D > 3$, then $\xi^c(G) < \xi^c(\mathsf{E}_{n,D,k})$ for all other graphs G of order n and size m.*
- *if $D = 3$ and $k = n-4$, then the only other graphs G with $\xi^c(G) = \xi^c(\mathsf{E}_{n,D,k})$ are those obtained by considering a path $u_0 - u_1 - u_2 - u_3$, and by joining $1 \leq i \leq n-3$ vertices of a clique K_{n-4} to u_0, u_1, u_2 and the $n-4-i$ other vertices of K_{n-4} to u_1, u_2, u_3.*

Appendix

H, P. et al. (2019). “Maximum eccentric connectivity index for graphs with given diameter”. In: *Discrete Applied Mathematics*. DOI: https://doi.org/10.1016/j.dam.2019.04.031.

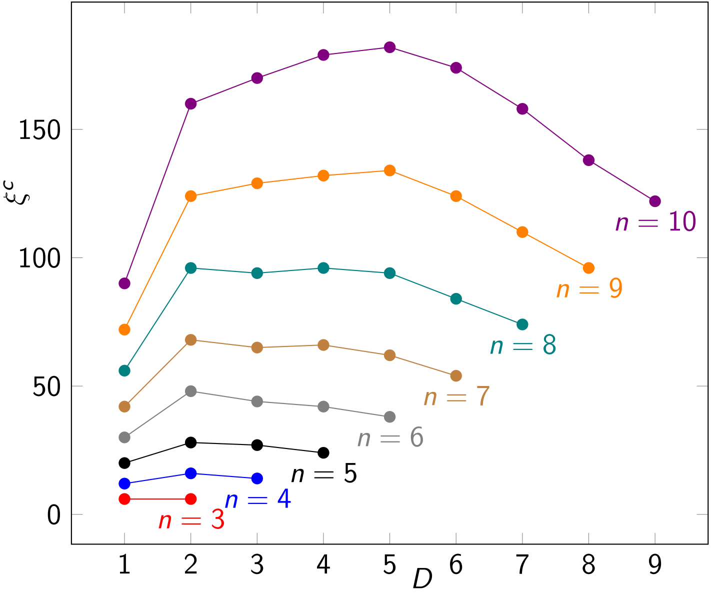
Morgan, M.J, S. Mukwembi, and H.C Swart (2011). “On the eccentric connectivity index of a graph”. In: *Discrete Mathematics* 311, pp. 1229 –1234.

– (2012). “A lower bound on the eccentric connectivity index of a graph”. In: *Discrete Applied Mathematics* 160, pp. 248 –258.

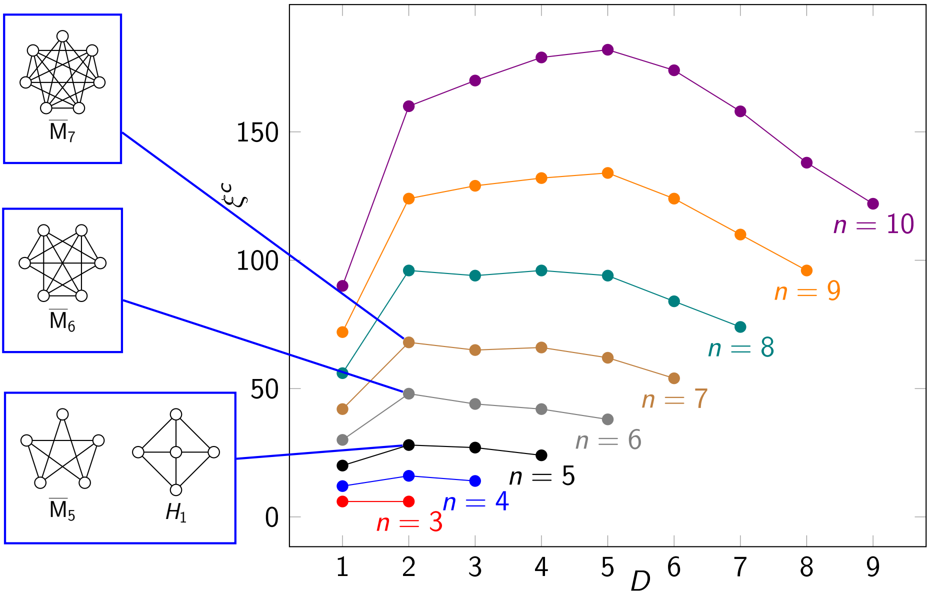
Zhang, J., Z. Liu, and B. Zhou (2014). “On the maximal eccentric connectivity indices of graphs”. In: *Appl. Math. J. Chinese Univ.* 29, pp. 374 –378.

Zhou, B. and Z. Du (2010). “On Eccentric Connectivity Index”. In: *MATCH Commun. Math. Comput. Chem.* 63, pp. 181 –198.

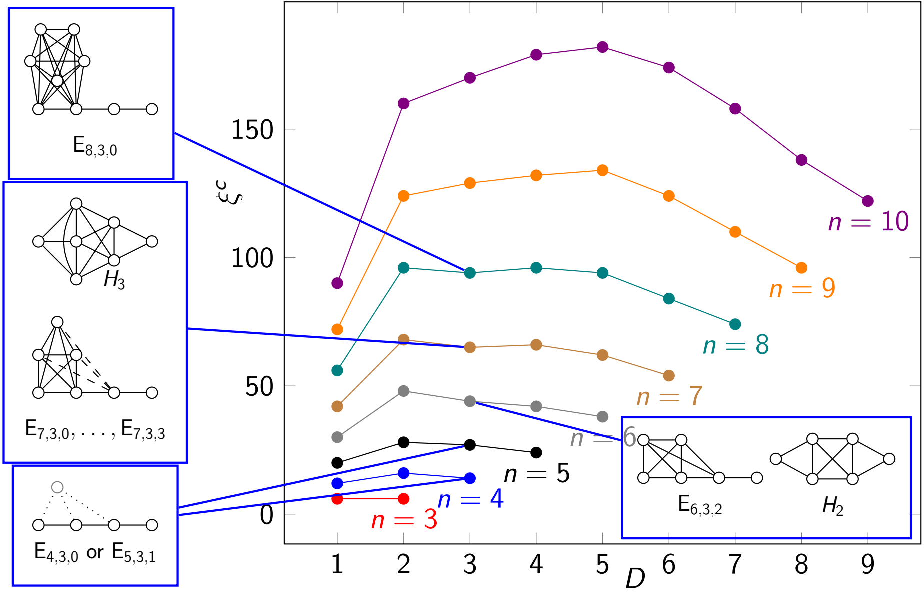
Maximum values of ξ^c for given order n and diameter D



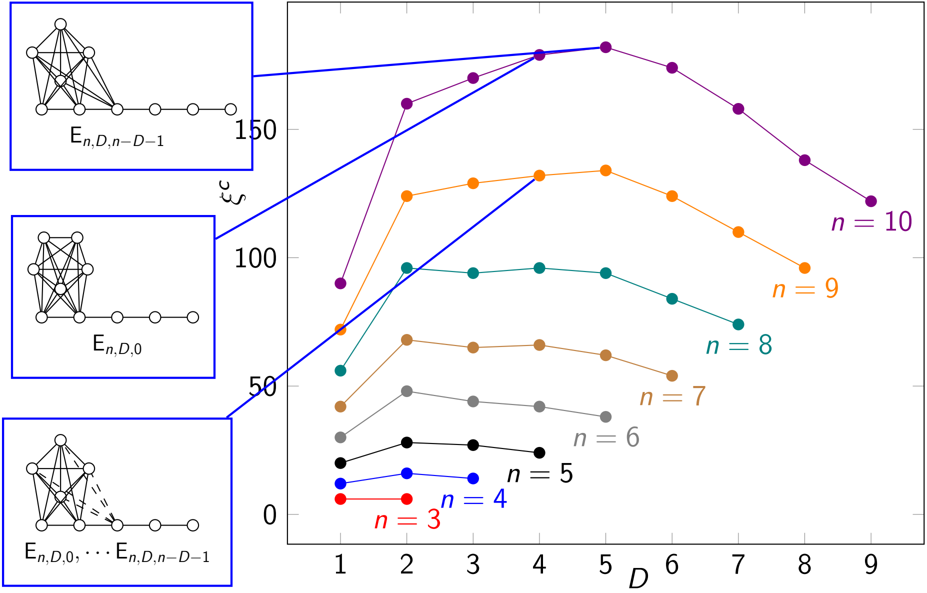
Maximum values of ξ^c for given order n and diameter D



Maximum values of ξ^c for given order n and diameter D



Maximum values of ξ^c for given order n and diameter D

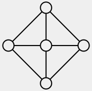


$\max \xi^c$ with given order and diameter when $D = 2$

Theorem (H et al. 2019)

Let G be a connected graph of order $n \geq 4$ and diameter 2. Then,

$$\xi^c(G) \leq 2n^2 - 4n - 2(n \bmod 2)$$

with equality if and only if $G \simeq \overline{\mathrm{M}}_n$, or $n = 5$ and $G \simeq$ .

Zhou and Du 2010

- Complete graphs: $\xi^c(\mathrm{K}_n) = n(n-1)$
- Complete bipartite graphs: $\xi^c(\mathrm{K}_{a,b}) = 4ab$ for $a, b \geq 2$
- Stars: $\xi^c(\mathrm{S}_n) = 3(n-1)$
- Cycles: $\xi^c(\mathrm{C}_n) = 2n\lfloor \frac{n}{2} \rfloor$
- Paths: $\xi^c(\mathrm{P}_n) = \lfloor \frac{3(n-1)^2+1}{2} \rfloor$

Theorem (Zhou and Du 2010)

Let G be a connected graph of order $n \geq 4$, then

$$\xi^c(G) \geq 3(n-1),$$

with equality if and only if $G \simeq S_n$.

Theorem (Zhou and Du 2010)

Let G be an n-vertex connected graph with m edges, where $n-1 \leq m \leq \binom{n}{2}$. Let $a = \left\lfloor \frac{2n-1-\sqrt{(2n-1)^2-8m}}{2} \right\rfloor$. Then

$$\xi^c(G) \geq 4m - a(n-1)$$

with equality if and only if $G \in \mathbf{G}_{(n,m)}$.
$\mathbf{G}_{(n,m)}$ is the set of graphs $K_a \vee H$, where H is a graph with $n-a$ vertices and $m - \binom{a}{2} - a(n-a)$ edges.

Theorem (Morgan, Mukwembi, and Swart 2012)

Let $G = (V, E)$ be a connected graph of order n, and diameter $D \geq 3$. Then

$$\xi^c(G) \geq \xi^c(V_{n,D}),$$

where $V_{n,D}$ is the volcano graph, obtained from a path P_{D+1} and a set S of $n - D - 1$ vertices, by joining each vertex in S to a central vertex of P_{d+1}.

Degree distance

The degree distance D' of a graph G is

$$\sum_{uv \in E} (\deg(u) + \deg(v)) d(u, v).$$

Theorem (Zhou and Du 2010)

Let $G = (V, E)$ be a connected graph with $n \geq 2$ vertices. Then

$$\xi^c(G) \geq \frac{1}{n-1} D'(G),$$

with equality if and only if $G = K_n$.

Eccentric Connectivity Index

- Sharma, Goswani and Madan introduced ξ^c in 1997 in Chemistry;
- Useful as a discriminating topological descriptor for Structure Properties and Structure Activity studies;
- Since 1997, more than 200 chemical papers about ξ^c: applications in drug design, prediction of anti-HIV activities, etc.

Eccentric Connectivity Index

- Sharma, Goswani and Madan introduced ξ^c in 1997 in Chemistry;
- Useful as a discriminating topological descriptor for Structure Properties and Structure Activity studies;
- Since 1997, more than 200 chemical papers about ξ^c: applications in drug design, prediction of anti-HIV activities, etc.
- However, the first mathematical paper with extremal properties on ξ^c was published only in 2010;
- Since 2010, about a dozen papers containing bounds on ξ^c.

Problem

Among connected graphs of order n and size m, what is the maximum possible value for ξ^c?

Maximizing ξ^c given order and size

Conjecture (Zhang, Liu, and Zhou 2014)

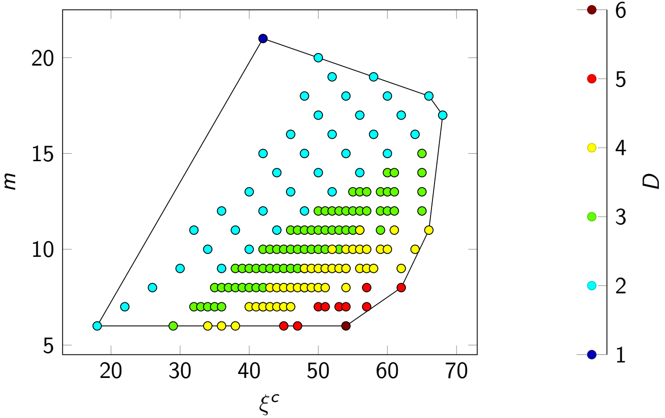
Let G be a graph of order n and size m such that $d_{n,m} \geq 3$. Then,

$$\xi^c(G) \leq \xi^c(E_{n,m}),$$

with equality if and only if $G \simeq E_{n,m}$.

- The authors prove that the conjecture is true when $m = n-1, n, \ldots, n+4$ (if n is large enough).
- It misses some corner cases (we'll come to it later).

Polytope for $n = 7$ with points colored by the diameter D



Upper bound on ξ^c for connected graphs with fixed size

We define $E_{n,m}$ as follows :

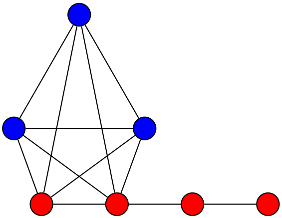
$$n = 7, m = 14$$

Upper bound on ξ^c for connected graphs with fixed size

We define $E_{n,m}$ as follows :

- The biggest possible clique without disconnecting the graph, leaving a path with the remaining vertices.

$n = 7, m = 14$

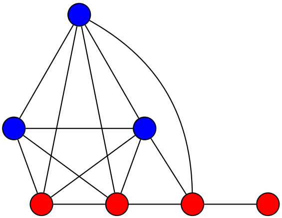


Upper bound on ξ^c for connected graphs with fixed size

We define $E_{n,m}$ as follows :

- The biggest possible clique without disconnecting the graph, leaving a path with the remaining vertices.
- Add remaining edges between vertices of the clique and the first vertex of the path.

$n = 7, m = 14$

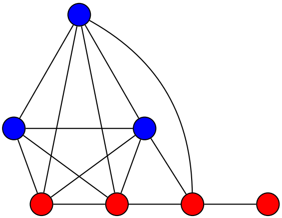


Upper bound on ξ^c for connected graphs with fixed size

We define $E_{n,m}$ as follows :

- The biggest possible clique without disconnecting the graph, leaving a path with the remaining vertices.
- Add remaining edges between vertices of the clique and the first vertex of the path.

$n = 7, m = 14$



This graph is unique for given n and m. We define $d_{n,m}$ as the diameter of $E_{n,m}$.

Bounds for connected graphs

Invariant(s)	Lower bound	Upper bound
—	✓	✗
m (size)	✓	✓
D (diameter)	✓	✓
p (number of pending vertices)	✓	✗
δ (minimum degree)	✗	✓
D (diameter) and m (size)	✓(with conditions on m and D)	
D' (degree distance)	✓	✗
M_1 (Zagreb index) and m	✗	✓
W (Wiener index)	✗	✓

Bounds for trees

Invariant(s)	Lower bound	Upper bound
—	✓	✓
D (diameter)	✓	✓
r (radius)	✗	✓
p (number of pending vertices)	✓	✓
Δ (maximum degree)	✓	✓
β (matching number)	✓	✓
α (stability number)	✓	✓
W (Wiener index)	✗	✓

Bounds for other graph classes

Unicyclic graphs

Invariant(s)	Lower bound	Upper bound
—	✓	✓
g (girth)	✓	✓

Bicyclic graphs

Invariant(s)	Lower bound	Upper bound
—	✓	✗

k-regular graphs

Invariant(s)	Lower bound	Upper bound
$k \geq 3$ fixed	✓	✓