Euclidean solutions in Einstein-Yang-Mills-dilaton theory

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Abstract

We present arguments for the existence of a new type of solutions of the Euclidean Einstein-Yang-Mills-dilaton theory in $d = 4$ dimensions. Possesing nonvanishing nonabelian charges, these nonselfdual configurations have no counterparts on the Lorentzian section. They provide, however, new saddle points in the Euclidean path integral.

1 Introduction

Following the discovery by Bartnik and McKinnon (BK) of particle-like solutions of the four-dimensional Einstein-Yang-Mills (EYM) equations [1], there has been much interest in classical solutions of Einstein gravity with nonabelian matter sources (for a review see [2]). These include hairy black holes solutions, which led to a revision of some of the basic concepts of black hole physics based on the uniqueness and no-hair theorems [3].

However, most of the investigations in the literature have been carried out for the physically most relevant case of a spacetime with Lorentzian signature. The selfdual YM instantons [4] although presenting interesting features, have a vanishing energy momentum tensor and do not disturb the spacetime geometry. The case of nonabelian gravitating instantons *i.e.* nonselfdual solutions of the field equations with Euclidean metric signature, received only scarce attention. Such configurations play an important role in the Euclidean approach to quantum gravity, where quantum amplitudes are defined by sum over positive definite metrics [5]. In the saddle point approximation, these sums are dominated by gravitational instantons with appropriate boundary conditions.

In the Einstein-Maxwell theory, the Euclidean solutions are found analytically continuing the Lorentzian signature configurations [5, 6]. The situation is more complicated for a nonabelian matter content. First, due to the absence of closed form solutions, the field equations should be solved numerically. However, for nonabelian configurations with magnetic fields only, the analytical continuation in time coordinate has no effects at the level of the field equations which have the same form on both Lorentzian and Euclidean sections. In particular, the BK particle-like solutions (and the corresponding black hole generalizations) extremize the EYM action also on the Euclidean section. The Euclidean action of these solutions can straightforwardly be evaluated [7]; in particular the black hole solutions satisfy the fist law of thermodynamics and the standard $S = A_H/4$ entropy-area relation.

The situation is more complicated for nonabelian solutions possessing both electric and magnetic nonabelian fields. The crucial point there is that the asymptotic value of the electric nonabelian potential A_t has a direct physical relevance and cannot take arbitrary values. Restricting to a nonanabelian SU(2) field, one finds that the asymptotically flat Lorentzian solutions necessarily have a vanishing electric field $A_t = 0$ and no nonabelian magnetic charge [8, 9]. However, here we present both numerical and analytical arguments for the existence of nontrivial solutions with both electric and magnetic charges for an Euclidean spacetime signature, within the same metric ansatz. Possessing a finite mass and action, these nonselfdual configurations have no Lorentzian counterparts and provide new saddle points in the Euclidean path integral. Both particle-like solutions with a R^4 topology and Euclidean black holes with a topology $S^2 \times R^2$ are found. In all cases, the nonabelian fields are nonselfdual and curve the Euclidean geometry.

The paper is structured as follows: in the next Section we present the general framework and analyse the field equations and boundary conditions. In Section 3 we present our numerical results, while in Section 4 we analyze the Euclidean action and present a brief discussion of the thermodynamics of solutions. We conclude in Section 5 with a discussion of our results.

2 General framework and equations of motion

2.1 Action principle

Our study of the Euclidean Einstein-Yang-Mills-dilaton (EYMd) system is based upon the following generalization of the EYM action

$$I = -\int_{\mathcal{M}} d^4x\sqrt{g}\left(\frac{R}{16\pi G} - \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}e^{2\gamma\phi}\mathrm{Tr}(F_{ab}F^{ab})\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K. \tag{1}$$

This type of action arises from various unified theories including superstring models and, apart from gravity and YM fields, contains also a dilaton field ϕ with coupling constant γ. (For example, $\gamma = 1/2$ and $\gamma = 1/\sqrt{3}$ are the cases corresponding to string theory and Kaluza-Klein reduction of the $d = 5$ EYM theory, respectively. The usual EYM system is found for $\phi = 0$, $\gamma = 0$.) Since changing the sign of γ is equivalent to changing the sign of ϕ, it is sufficient to consider only $\gamma > 0$.

The field strength tensor is given by $F_{ab} = \partial_a A_b - \partial_b A_a - ig[A_a, A_b]$, with g the Yang-Mills coupling constant. The last term in (1) is the Hawking-Gibbons surface term [5], where K is the trace of the extrinsic curvature for the boundary $\partial\mathcal{M}$ and h is the induced metric of the boundary.

Asking for stationary points of this action, one finds the EYMd field equations

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab}\ , \quad \nabla^2\phi - \gamma e^{2\gamma\phi}Tr(F_{ab}F^{ab}) = 0\ , \quad \nabla_a(e^{2\gamma\phi}F^{ab}) - ige^{2\gamma\phi}[A_a, F^{ab}] = 0\ , \tag{2}$$

where the energy-momentum tensor is defined by

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial_c\phi\partial^c\phi + 2e^{2\gamma\phi}\mathrm{Tr}(F_{ac}F_{bd}g^{cd} - \frac{1}{4}g_{ab}F_{cd}F^{cd}). \tag{3}$$

2.2 The ansatz

We consider the following spherically symmetric metric with Euclidean signature

$$ds^2 = \sigma^2(r)N(r)d\tau^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4}$$

where r is the radial coordinate, θ and φ are the angular coordinates with the usual range, while τ corresponds to the Euclidean time.

In this paper we'll discuss two types of solutions. The first type of configurations, which usually corresponds to analytical continuations of Lorentzian globally regular, particle-like solutions have $0 \leq r < \infty$ and a trivial topology, the Killing vector $\partial/\partial\tau$ presenting no fixed points sets (*i.e.* $g_{\tau\tau} > 0$ for any r). For the second type of solutions, the fixed point set of the Euclidean time symmetry is of two dimensions (a "bolt") and the range of the radial coordinate is restricted to $0 < r_h \leq r < \infty$, with $N(r_h) = 0$ and a finite nonzero $\sigma(r_h)$.

For the first type of solutions, the periodicity β of the Euclidean time coordinate is arbitrary, while for bolt solutions β is fixed by regularity requirements to

$$\beta = \frac{4\pi}{\sigma(r_h)N'(r_h)}\ , \tag{5}$$

where a prime denotes the derivative with respect to r.

Without any loss of generality, the SU(2) ansatz can be written in the form

$$A_a dx^a = \frac{1}{2g}\{u(r)\tau_3 d\tau + w(r)\tau_1 d\theta + (\cot\theta\tau_3 + w(r)\tau_2)\sin\theta d\varphi\}, \tag{6}$$

(where the τ_i are the standard Pauli matrices), being described by two functions $w(r)$ and $u(r)$ which we shall refer to as magnetic and electric potential, respectively.

Therefore the spherically symmetric EYMd system is described by the following one-dimensional effective action

$$\mathcal{L} = \frac{1}{4\pi G}\sigma m' - \left[\frac{e^{2\gamma\phi}}{g^2}(\sigma N w'^2 + \frac{1}{2r^2}\sigma(1-\omega^2)^2 \quad + \frac{1}{2\sigma}r^2u'^2 + \frac{1}{\sigma N}\omega^2u^2) - \frac{1}{2}\sigma N r^2(\phi')^2,\right] \tag{7}$$

the field equations reducing to a set of four non-linear differential equations

$$\begin{gathered} m' = 4\pi G\Big[\frac{e^{2\gamma\phi}}{g^2}\Big(Nw'^2 + \frac{(1-w^2)^2}{2r^2} - \frac{1}{2\sigma^2}r^2u'^2 - \frac{1}{\sigma^2 N}w^2u^2\Big) + \frac{1}{2}Nr^2\phi'^2\Big], \\ \sigma' = \frac{8\pi G\sigma}{r}(\frac{1}{2}Nr^2\phi'^2 + 2\frac{e^{2\gamma\phi}}{g^2}(w'^2 - \frac{1}{\sigma^2N^2}w^2u^2)), \quad (\sigma N e^{2\gamma\phi}w')' = e^{2\gamma\phi}(\frac{1}{r^2}\sigma w(w^2-1) + \frac{wu^2}{\sigma N}), \\ (e^{2\gamma\phi}\frac{r^2u'}{\sigma})' = \frac{2w^2u}{\sigma N}e^{2\gamma\phi}, \quad (\sigma N r^2\phi')' = \frac{2\gamma e^{2\gamma\phi}}{g^2}\Big(\sigma N w'^2 + \frac{\sigma(1-w^2)^2}{2r^2} + \frac{r^2u'^2}{2\sigma} + \frac{w^2u^2}{\sigma N}\Big). \end{gathered} \tag{8}$$

We are interested in asymptotically flat regular solutions whose mass function $m(r)$ approaches a constant finite value as $r \to \infty$, which will have also a finite Euclidean action.

Solutions of the above system are already known in a few particular cases. The embedded U(1) configurations correspond to [10, 11] (in units $4\pi G = g = 1$)

$$\begin{gathered} ds^2 = \frac{dr^2}{\lambda^2} + R^2(d\theta^2 + \sin^2\theta d\varphi^2) + \lambda^2 d\tau^2, \quad \lambda^2 = (1-\frac{r_+}{r})(1-\frac{r_-}{r})^{(1-\gamma^2)/(1+\gamma^2)}, \\ R = r(1-\frac{r_-}{r})^{\gamma^2/(1+\gamma^2)}, \quad e^{2\phi} = (1-\frac{r_-}{r})^{2\gamma/(1+\gamma^2)}, \quad w = 0, \end{gathered} \tag{9}$$

with an arbitrary value of the electric potential $u(r) = \Phi$, r_+ being a positive constant and $r_- = (1+\gamma^2)/r_+$.

In the case $\gamma = 0$ one can set $\phi = 0$ and we find the selfdual YM dyonic solution $w = r/\sinh r$, $u = \coth r - 1/r$ in a flat spacetime background. For bolt configurations with $N = 1 - 2M/r$, $\sigma \equiv 1$, the analogous of this solution solves the equations

$$w' = wu/N, \quad r^2u' = w^2 - 1, \tag{10}$$

and corresponds to a selfdual dyonic instanton in a Schwarzschild background. Except for the region near $r = r_h$, the numerical solution of these equations looks very similar to the flat space counterpart.

For a truncation $A_\tau = 0$ of the YM ansatz, the purely magnetic Euclidean configurations are found by taking $t \to i\tau$ in the Lorentzian solutions describing EYMd particle-like and black hole solutions discussed in [12, 13]. Within a stationary ansatz, this analytical continuation has no effect at the level of equations of motion. Features of the corresponding EYM Euclidean solutions have been discussed in [7]. For $u(r) = 0$ and $\gamma = 1/2$, the lowest energy solution corresponds to a BPS configuration in a version of $\mathcal{N} = 4$, $D = 4$ gauged supergravity [14]. Excited configurations indexed by the node number of w exist as well. However, all these solutions have zero magnetic charge.

In fact, for any γ, the only spherically symmetric Lorentzian configurations with reasonable asymptotics are the purely magnetic EYMd particle-like and black hole solutions [12, 13]. For finite mass solutions, the electric potential should vanish identically, the arguments presented in [8, 9] being not affected by the presence of a dilaton. First, one can see from eqs. (8) that for $A_\tau \neq 0$, the magnitude at infinity of the electric potential should be nonzero. However, for Lorentzian signature, this would imply from the $w-$equation an oscilatory asymptotic behavior of the magnetic gauge function w and thus an infinite mass. Therefore $u(\infty) = 0$, which means a purely magnetic solution $u(r) \equiv 0$ and $w(\infty) = \pm 1$, *i.e.* no magnetic charge.

However, the situation is different for an Euclidean spacetime signature. In this case, the electric potential qualitatively behaves as a Higgs field and nontrivial electrically charged solutions may exist [1]. Thus we expect these solutions to present a magnetic charge and to share a number of common features with the usual monopoles.

[1] Note that the electric potential-Higgs field analogy is not complete, since the A_τ couples also to the $g_{\tau\tau}$ metric component.

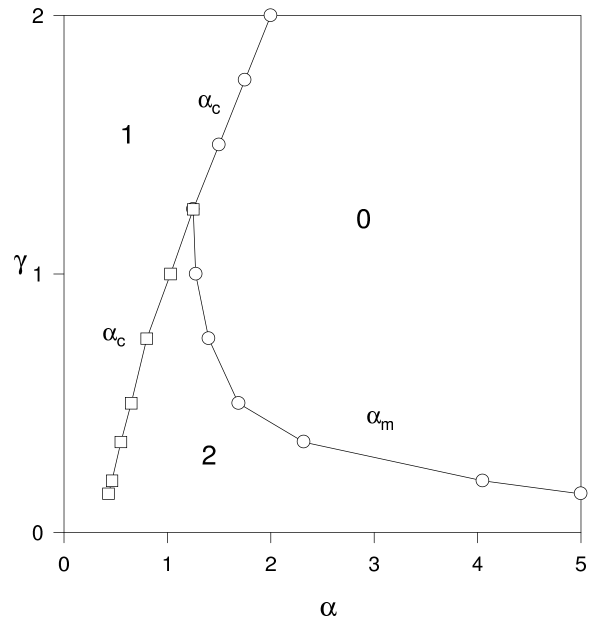


Figure 1. The domain of existence in the (α, γ)-plane of the non-abelian solutions with R^4 topology. The labels 0, 1, 2 refer to the number of solutions we found.

2.3 Boundary conditions

The field equations imply the following behaviour for $r \to 0$ in terms of four parameters $(b, \phi_0, \sigma_0, u_1)$:

$$m(r) = 8\pi G \frac{e^{2\gamma\phi_0}}{g^2}(b^2 - \frac{u_1}{4\sigma_0^2})r^3 + O(r^4), \quad \sigma(r) = \sigma_0(1 + 4\pi G \frac{e^{2\gamma\phi_0}}{g^2}(b^2 - \frac{u_1}{4\sigma_0^2})r^2) + O(r^4), \tag{11}$$
$$w(r) = 1 - br^2 + O(r^4), \quad u(r) = u_1 r + O(r^3), \quad \phi(r) = \phi_0 + 2\gamma \frac{e^{2\gamma\phi_0}}{g^2}(b^2 + \frac{u_1}{4\sigma_0^2})r^2 + O(r^4).$$

The corresponding expansion as $r \to r_h$ for bolt solutions is

$$m(r) = r_h/2 + m_1(r - r_h) + O(r - r_h)^2, \quad \sigma(r) = \sigma_h + \sigma_1(r - r_h) + O(r - r_h)^2,$$
$$w(r) = w_h + \omega_1(r - r_h) + O(r - r_h)^2, \quad u(r) = u_1(r - r_h) + u_2(r - r_h)^2 + O(r - r_h)^3, \tag{12}$$
$$\phi(r) = \phi_h + \phi_1(r - r_h) + O(r - r_h)^2,$$

where

$$m_1 = 4\pi G \frac{e^{2\gamma\phi_h}}{g^2}(\frac{(1 - w_h^2)^2}{2r_h^2} - \frac{r_h^2 u_1^2}{2\sigma_h^2}), \quad \sigma_1 = \frac{4\pi G \sigma_h}{r_h}\left(\frac{e^{2\gamma\phi_h}}{g^2}(w_1^2 - \frac{w_h^2 u_1^2}{N_1^2 \sigma_h^2}) + r_h^2 \phi_1^2\right), \tag{13}$$

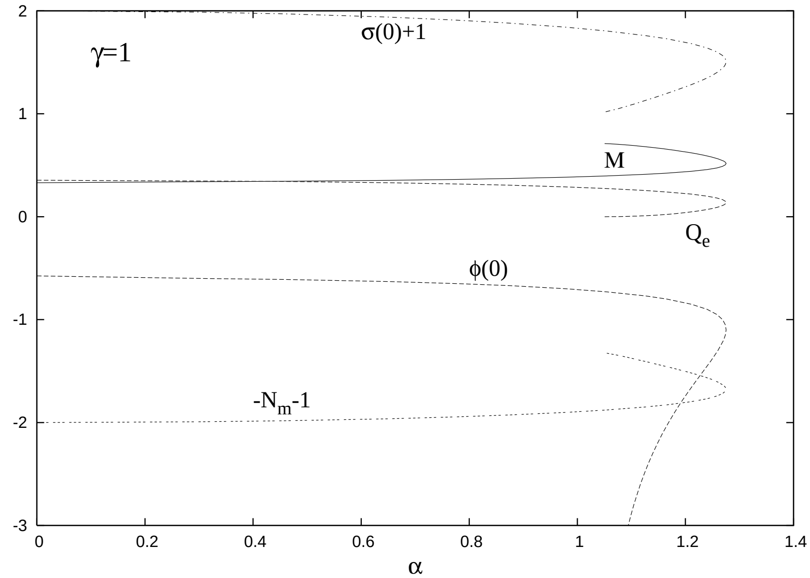
$$w_1 = \frac{w_h(w_h^2 - 1)}{N_1 r_h^2}, \quad u_2 = \frac{w_h^2 u_1}{r_h^2 N_1}, \quad \phi_1 = \frac{2\gamma e^{2\gamma\phi_h}}{\sigma_h r_h^2 N_1 g^2}(\frac{\sigma_h}{2r_h^2}(1 - w_h^2)^2 + \frac{r_h^2 u_1^2}{2\sigma_h}), \quad N_1 = (1 - 2m_1)/r_h. \tag{14}$$

$w_h, \sigma_h, u_1, \phi_h$ being free parameters.

The analysis of the field equations as $r \to \infty$ gives the following expression in terms of the constants $M,\ \Phi,\ \sigma_2,\ Q_d,\ Q_e$

$$N(r) = 1 - \frac{2M}{r} + .., \quad \sigma(r) = 1 - \frac{2\pi G \sigma_2}{r^2} + .., \quad u(r) = \Phi - \frac{Q_e}{r} + .., \quad \phi(r) = -\frac{Q_d}{r} + .., \quad w(r) = e^{-\Phi r} + .., \tag{15}$$

which is shared by both classes of solutions. The physical significance of the quantity Φ is that gives the electrostatic potential difference between the origin and infinity, while M corresponds to the mass of



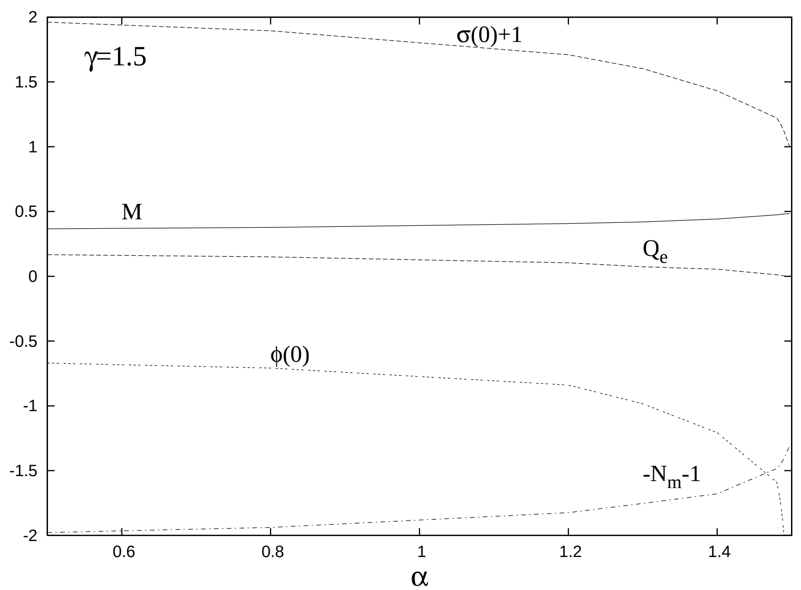


Figure 2. The mass-parameter M, the nonabelian electric charge Q_e, the values of the metric function $\sigma(r)$ and dilaton function $\phi(r)$ at the origin as well as the minimal value N_m of the metric function $N(r)$ are represented as function of the parameter α for two values of γ. Here we consider solutions with R^4 topology.

configurations. These solutions possess also nonabelian electric and magnetic charges computed *e.g.* according to

$$\begin{pmatrix} \mathbf{Q_E} \\ \mathbf{Q_M} \end{pmatrix} = \frac{1}{4\pi} \int dS_\mu^{(k)} \sqrt{g}\, \mathrm{Tr}\Big\{ \begin{pmatrix} F^{\mu\tau} \\ \tilde{F}^{\mu\tau} \end{pmatrix} \tau_3 \Big\}, \tag{16}$$

which are conserved from the Gauss flux theorem. Thus the solutions with the asymptotic form (15) have a unit magnetic charge and a electric charge Q_e. As discussed in Section 4, the dilaton charge Q_d which enters the asymptotics (15) is fixed by other relevant quantities.

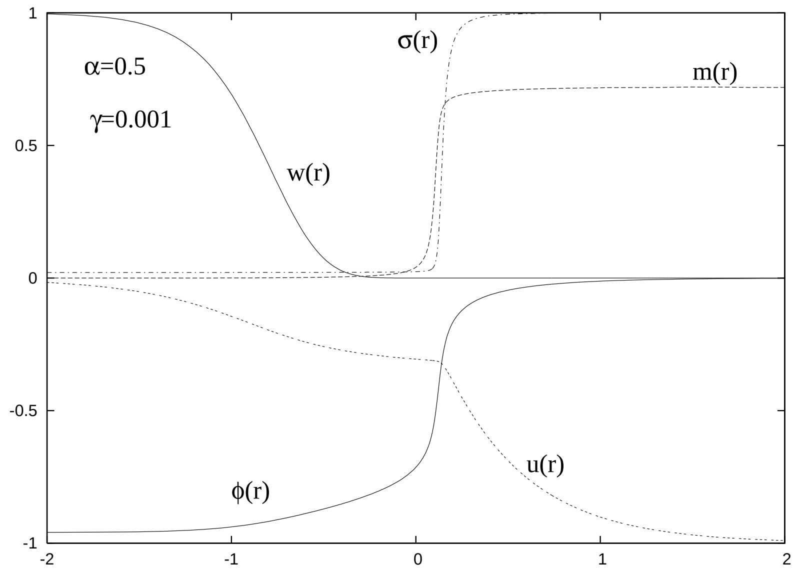


Figure 3. The profiles of a topologically R^4 EYMd solution are plotted for $\alpha = 0.5$, $\gamma = 0.001$.

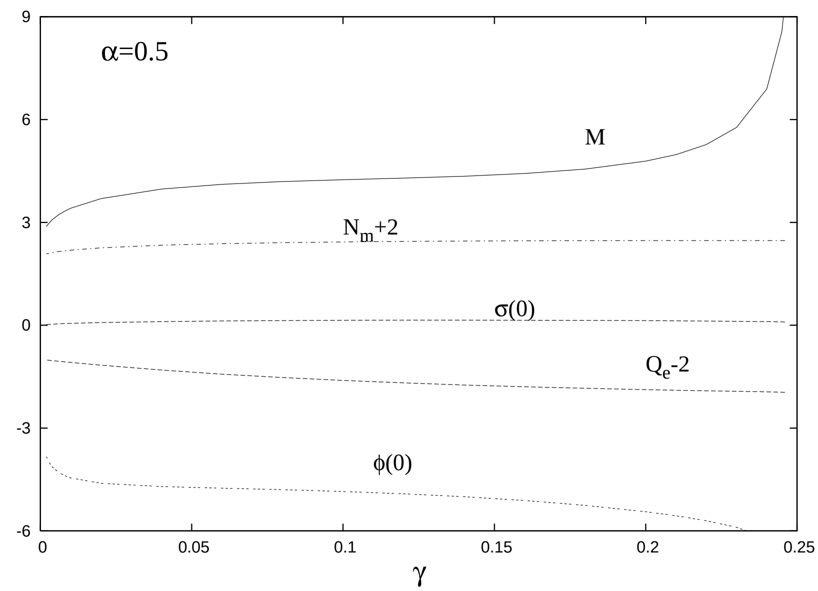


Figure 4. The same as Figure 2 for a fixed value of α and a varying dilaton coupling constant γ.

3 Numerical solutions

Although an analytic or approximate solution appears to be intractable, here we present arguments for the existence of Euclidean solutions satisfying the boundary conditions displayed above.

To perform numerical computations and order-of-magnitude estimations, it is useful to have a new set of dimensionless variables. This is obtained by using the following rescaling $r \to r/(\Phi g)$, $m \to \Phi g m$,

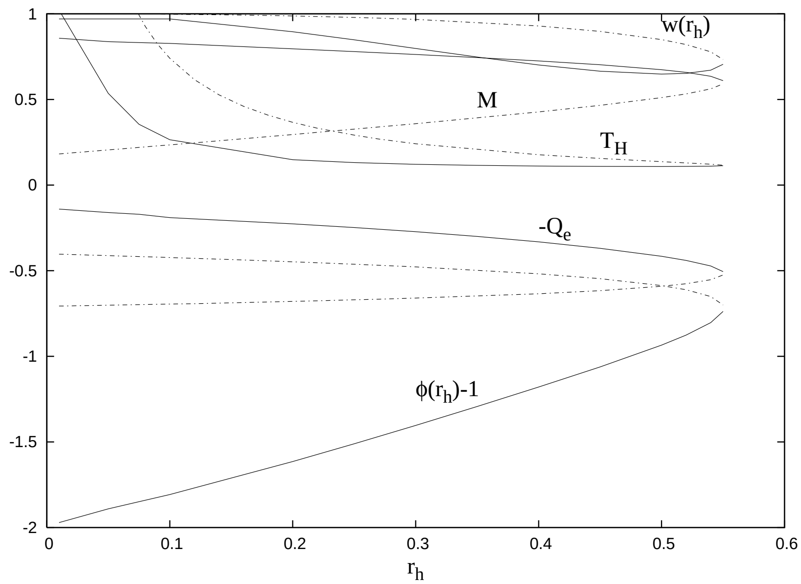


Figure 5. The mass-parameter M, the nonablelian electric charge Q_e, the values of the metric function $\sigma(r)$, the magnetic potential $w(r)$ and dilaton function $\phi(r)$ for $r = r_h$ are represented as function of r_h for $\gamma = 0.5$, $\alpha = 1.0$ bolt solutions.

$u \to u/(\Phi g)$, where Φ is the asymptotic magnitude of the electric potential. We consider also a rescaling of the dilaton and dilaton coupling constant $\phi \to \phi/\Phi$, $\gamma \to \Phi\gamma$. As a result, the field equations depend only on the coupling constants $\alpha = \sqrt{4\pi G}\Phi$ and the rescaled dilaton constant γ.

3.1 Topologically R^4 solutions

The equations of motion (8) have been solved for a large set of the parameters $(\alpha,\ \gamma)$, looking for solutions interpolating between the asymptotics (11) and (15). For all solutions we studied, the gauge functions $w(r)$ and $u(r)$, the metric functions $m(r)$, $\sigma(r)$ and the dilaton $\phi(r)$ interpolate monotonically between the corresponding values at $r = 0$ and the asymptotic values at infinity, without presenting any local extrema. Note also that for both topologically R^4 and $R^2 \times S^2$ configurations, the asymptotic value of $m(r)$ was found to take positive values only.

The domain of existence of the solutions in the (α, γ)-plane is rather involved. According to the case $\gamma < \gamma_c$ or $\gamma > \gamma_c$ (with $\gamma_c \approx 1.25$) the pattern of the solutions is quite different, as indicated on Figure 1, where the numbers in the various domains of the graphic refers to the number of non-abelian solutions in the corresponding domain we found.

First we discuss the case $\gamma < \gamma_c$. If we fix γ in this range and increase α we construct a branch of solutions which exists up to a maximal value $\alpha_m(\gamma)$. Different quantities characterizing the solutions are reported on Figure 2 for $\gamma = 1$. Then, our numerical analysis indicates that a second branch of solutions exists for $\alpha \in [\alpha_c, \alpha_m]$ [2]. The solutions of the first and of the second branch coincide in the limit $\alpha \to \alpha_m$. In the limit $\alpha \to \alpha_c$, the solutions of the second branch tends to an Einstein-Maxwell-dilaton configuration (9). This is characterized in the limit $\alpha \to \alpha_c$ (~ 1.034 in the case $\gamma = 1.0$) by the fact that $\phi(0) \to \infty$, $Q_e \to 0$, $\sigma(r) \to 1$, $w(r) \to 0$. A few of these properties are visible on Figure 2 but the claim is further demonstrated by comparing the profiles.

It is natural to try to understand how these two branches of solutions behave for α fixed and for γ decreasing to zero. Clearly, the solutions of the first branch tends to the flat space selfdual YM solution mentioned in the previous section, *i.e.* with $\phi(r) = 0$, $N(r) = \sigma(r) = 1$, $M = 0$, $Q_e = 1$ and the functions

[2] Although we cannot exclude the existence of other secondary branches, these would have a very small extension in α. These solutions are likely to exist at least for values of γ around $1/\sqrt{3}$ (see the remarks in Section 5). Finding them is a difficult numerical problem, which is not aimed in this report.

$w(r), u(r)$ given by the PS dyon. The numerical study of the limiting solution (i.e. for $\gamma \to 0$) of the second branch strongly suggests that it ends up into a discontinuous configuration indicated on Figure 3 for $\gamma = 0.001$. We see in particular that the dilaton function ϕ and the metric function σ presents a pronounced step at an intermediate value, say $r = r_i$ of the radial variable. At the same value of r, the metric function N develops a deep minimum. This critical phenomenon becomes more severe while γ deceases and the numerical integration strongly suggests that another solution of Eq. (10) is approached on the interval $[r_i, \infty]$: namely $\sigma(r) = 1, w(r) = 0, \phi(r) = 0, u(r) = -1 + 1/r$ and m is constant. The counterpart of this result for Lorentzian case [15, 16] is that gravitating monopoles and dyons form two branches of solutions, the second of which bifurcates into a Reissner-Nordström solution.

From Figure 1 we see that the solutions of the second branch exist on a small interval of γ. The evolution of several parameters characterizing the solutions of the second branch are reported on Figure 4 as functions of γ.

In the case $\gamma > \gamma_c$, the pattern is simpler, only one solution is available and the branch of non-abelian solutions bifurcate directly into a EMD solution for $\alpha \to \alpha_c$. This is illustrated by Figure 2 for $\gamma = 1.5$.

3.2 Solutions with $R^2 \times S^2$ topology

According to the standard arguments, one can expect black hole generalisations of the configurations discussed above to exist at least for small values r_h. This is confirmed by our numerical analysis and black hole solutions seem to exist for all values of α for which solutions with R^4 topology were constructed.

The first step is to determine the domain of existence of solutions in the parameter space determined by (α, γ, r_h). Since this constitutes a considerable task, we limit here our numerical study to the case $\gamma = 0.5$, although solutions with other values of the dilaton coupling constant have been studied as well. It is useful to notice that, corresponding to this value, we have $\alpha_m \simeq 1.6$ and $\alpha_c \simeq 0.8$.

It turns out that black holes solutions exist in a finite domain of the (α, r_h)-plane. In other words, if we fix α, we are able to construct a first family of solutions up to a maximal value $r_{h,max}$ which depends on α. As an example for $\alpha = 0.1$ and $\alpha = 1.0$ we find respectively $r_{h,max} \approx 0.75$ and $r_{h,max} \approx 0.55$. In the limit $r_h \to 0$ the solutions on this branch, characterized by the dashed lines in Figure 5, approach the corresponding regular solution on $r \in]0, \infty[$.

However, the pattern of these black holes solutions for large values of r_h is more sophisticated and seems to be intimately related to the number of topologically trivial solutions available for the corresponding value of α. Two cases can be distinguished, according to $\alpha < \alpha_c$ and $\alpha_c < \alpha < \alpha_m$ for which respectively one and two R^4 solutions exist (see previous section).

For $\alpha < \alpha_c$ the branch of black hole solutions exists up to a maximal value of $x_{h,max}$. The corresponding value $w(r_h)$ decreases and the solutions ends up into a configuration with $w(r) \equiv 0$ when the value of $r_{h,max}$ is reached.

For $\alpha_c < \alpha < \alpha_m$, a second branch of black hole solutions exist. In the limit $r_h \to r_{h,max}$ the two solutions converge to a common solution. In the limit $r_h \to 0$ the two branches approach the two R^4 solutions available for this value of α. This is demonstrated on Figure 5 where the mass, the electric charge, the temperature $T = 1/\beta$ and the values $w(r_h)$, $\phi(r_h)$ are plotted as functions of r_h for $\gamma = 0.5$ and $\alpha = 1.0$; there dashed (resp. dotted) lines are used for the solutions of the first (resp. second) branch.

We also studied the evolution of the solutions in the $\gamma \to 0$ limit. For the solutions of the main branch we observe that the self-dual solution (10) is approached. Namely, the field of the dilaton uniformly approaches $\phi(r) = 0$. The action of the selfdual dyon (10) is $4\pi\beta/g^2$ which, different from Charap-Duff solution [4] $u = M/r^2$, $w = \sqrt{N}$, depends on M (since $\beta = 8\pi M$ for Schwarzschild background). This limiting configuration has also unit magnetic and electric charges. A detailed study of the solutions of (10) will be presented elsewhere.

The situation is definitely different when the solutions of the second branch are examined in the same limit. Here the function $N(r)$ develops a local maximum and a local minimum respectively at $r = r_{max}$ and $r = r_{min}$, with $r_h < r_{max} < r_{min} < \infty$. The value $N(r_{min})$ got deeper and deeper for decreasing γ. Our numerics strongly suggests that the minimum of N reaches the value zero in the $\gamma = 0$-limit and that the solution approaches the self-dual solution (10) corresponding to $M = r_{min}/2$ on the interval $r \in [r_{min}, \infty]$. At the same time the profiles stay non trivial on $r \in [r_h, r_{min}]$.

4 Euclidean action and thermodynamics

Accordingly to Gibbons and Hawking [5], thermodynamic functions including the entropy can be computed directly from the saddle point approximation to the gravitational partition function (namely the generating functional analytically continued to the Euclidean spacetime). In the semiclassical approximation, the dominant contribution to the path integral will come from the neighborhood of saddle points of the action, that is, of classical solution; the zeroth order contribution to $\log Z$ will be $-I$.

The Euclidean action of these solutions can be computed by using the standard techniques in the literature. Taking the trace of the metric equations of motion yields $R = 8\pi G(\partial_a\phi\partial^a\phi)$, so the first two terms in the action cancel. The dilaton equation of motion shows that the third term is a total derivative. Thus the bulk action is proportional to the dilaton charge $I_B = -\beta Q_d/2\gamma$, and the total action of any solution can be recast as a boundary term [3] (where n^μ is a unit outward pointing normal to the boundary)

$$I = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}e^{\phi/\gamma}\nabla_\mu(e^{-\phi/\gamma}n^\mu), \tag{17}$$

which diverges as $r \to \infty$. In the traditional Euclidean path integral approach to black hole thermodynamics [5], one has to choose a suitable reference background and substract it in order to get a finite Euclidean action. The reference background here is the flat four dimensional metric, although there are some ambiguities related to the presence of gauge field charges [6]. In a more recent approach, no reference background is required, the action being regularized by adding suitable coordinate invariant boundary surface counterterms to the gravitational action. The suitable expression in our case is [17] $I_{ct} = \frac{1}{8\pi G}\int d^3x\sqrt{h}\sqrt{2\mathcal{R}}$, where $\mathcal{R}$ is the Ricci scalar of the boundary metric. Varying the total action with respect to the boundary metric $h_{\mu\nu}$, one computes also a divergence-free boundary stress-tensor. The solutions' mass is the conserved charge associated with the Killing vector $\partial/\partial\tau$ of the boundary metric (see *e.g.* [18] and the more general approach in recent work [19]). By employing this approach or by subtracting the Minkowski background contribution, one finds $I = \beta(M - Q_d/\gamma)/2$.

However, the dilaton charge is not an independent quantity, as proven by an alternative computation of total action (see also [20] for the case of a spacetime with Lorentzian signature). By integrating the Killing identity $\nabla^a\nabla_b K_a = R_{bc}K^c$, for the Killing field $K^a = \delta^a_\tau$, together with the Einstein equation

$$\frac{1}{8\pi G}R^\tau_\tau - 2e^{2\gamma\phi}\mathrm{Tr}(F_{\mu\tau}F^{\mu\tau}) = \frac{R}{16\pi G} - \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}e^{2\gamma\phi}\mathrm{Tr}(F_{ab}F^{ab}) , \tag{18}$$

it is possible to isolate the bulk action contribution at infinity and at $r = 0$ or $r = r_h$. By using the YM equations and the asymptotic expansion (15) we find

$$I = \beta(M - T\frac{A_H}{4G} - \Phi Q_e), \tag{19}$$

which implies

$$Q_d = \gamma(M - 2T\frac{A_H}{4G} - 2\Phi Q_e), \tag{20}$$

$A_H = 4\pi r_h^2$ being the event horizon area (with $A_H = 0$ for topologically trivial solutions). Here we should remark that the variation of the action (1) will give the correct equations of motion only if the gauge potential A_a is held fixed on the boundary ∂M. This imposes the boundary condition $\delta A_a = 0$ on ∂M. Thus the action (1) is appropriate to study the ensemble with fixed electric potential and fixed magnetic charge [4].

[3] In this Section we do not consider rescaled variables.

[4] To study the canonical enesemble with fixed magnetic *a*nd electric charges, we have to add a boundary term to impose fixed Q_e as boundary condition at infinity [6]. The appropriate action in this case is $\tilde{I} = I - \frac{1}{4\pi G}\int_{\partial M} d^3x\sqrt{h}n_a\mathrm{Tr}(F^{ab}A_b)$.

This is the grand canonical ensemble, at fixed temperature and fixed potential. The grand canonical (Gibbs) potential is $W = I/\beta = E - TS - \Phi Q_E$.

By using the approach in [21] it can be proven that these solutions satisfy the first law of thermodynamics $dM = TdS - \Phi dQ_e$. As a result, one finds the thermodynamic quantities

$$E = \left(\frac{\partial I}{\partial \beta}\right)_{\Phi} - \frac{\Phi}{\beta}\left(\frac{\partial I}{\partial \Phi}\right)_{\beta} = M, \quad S = \beta\left(\frac{\partial I}{\partial \beta}\right)_{\Phi} - I = \frac{A_H}{4G}\ , \tag{21}$$

while the electric charge defined as $-\frac{1}{\beta}\left(\partial I/\partial \Phi\right)_{\beta}$ is just Q_e.

As concerning the thermodynamic stability of these solutions, for the parameters range we studied so far, we found that all bolt solutions have a negative specific heat $C = T(\partial S/\partial T)_{\Phi} < 0$.

5 Further remarks

In this paper we have investigated the basic properties of a new type of dyonic solutions in Euclidean EYMd theory. For an Euclidean spacetime signature, this results in solution with a nonvanishing electric potential. The existence of these configurations originates in the fact that, different from the abelian case, the asymptotic magnitude of the nonabelian electric potential has a physical significance and it cannot be gauged away without making the configurations time-dependent. For the first type of configurations, the resulting manifolds have (if one identifies imaginary time) topology $R^3 \times S^1$ and Euler number $\chi = 0$ in contrast to the Euclidean black hole (-nonextremal) case with $\chi = 2$, thus possesing thermodynamic properties.

It is obvious that these solutions have no reasonable Lorentzian counterparts. However, they possess a finite mass and action and give new saddle points of the EYMd Euclidean path integral. Their basic properties are governed by the asymptotic magnitude of the electric potential and the value of the dilaton coupling constant.

It is worthwhile to remark that for a dilaton coupling constant $\gamma = 1/\sqrt{3}$, these solutions can be uplifted to $d = 5$, according to (with $z-$ the extra dimension)

$$ds_5^2 = e^{-\gamma\phi}\left(\sigma^2(r)N(r)d\tau^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) + e^{2\gamma\phi}dz^2, \tag{22}$$

to become solutions in a five dimensional Euclidean EYM theory. The five dimensional YM potential is still given by (6). Now one can perform a Kaluza-Klein reduction to $d = 4$ with respect the Killing vector $\partial/\partial\tau$. One finds in this way solutions in a four dimensional EYMd-Higgs theory, the action principle (1) being supplemented by the Higgs term $e^{-2\gamma\bar{\phi}}Tr(D_a\Phi D^a\Phi)$ [22, 23], $\bar{\phi}$ being the new $d = 4$ dilaton field

$$\bar{\phi} = -\frac{\phi}{2} + \frac{1}{2\gamma}\log N + \frac{1}{\gamma}\log\sigma. \tag{23}$$

The SU(2) field has a purely magnetic potential

$$A_a dx^a = \frac{1}{2g}\left\{w(r)\tau_1 d\theta + (\cot\theta\tau_3 + w(r)\tau_2)\sin\theta d\varphi\right\}, \tag{24}$$

while the Higgs field is given by $\Phi = u(r)\tau_3/2$. Since the new nonabelian field does not present an electric potential and $\partial/\partial z$ is still a Killing vector, these are solutions on both Lorentzian and Euclidean sections, with

$$ds_4^2 = e^{-3\gamma\phi/2}\sqrt{N}\sigma(r)\left(\frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \pm e^{3\gamma\phi}dz^2\right). \tag{25}$$

The solutions "dual" to topologically R^4 Euclidean configurations would describe globally regular monopole configurations, originally studied in [22], and thus presenting a complicated branch structure, with a critical value of α.

Returning to the case of a general γ, a discussion of possible generalizations of this work should start with the radially excited solutions, for which the gauge function $\omega(r)$ possesses nodes. We expect also the existence axially symmetric solutions, with a nontrivial θ-coordinate dependence. The case of rotating generalizations of the BK solutions is particularly interesting, since no spinning EYM solitons appear to exist on the Lorentzian section [24, 25]. The nonexistence of a Lorentzian rotating generalization of the BK solution can be viewed as a consequence of the impossibility to obtain regular, electrically charged nonabelian solutions without a Higgs field. This obstacle is avoided for an Euclidean signature and our preliminary numerical results indicate the existence of axially symmetric EYMd solitons with a nonzero extradiagonal metric component $g_{\varphi\tau}$, usually associated with rotation for a Lorentzian signature spacetime. Since $\partial/\partial\varphi$ is still a Killing vector for these solutions with R^4 topology, there is also a new conserved quantity, J (however, the magnetic charge is zero in this case). Similar "bolt" configurations generalizing for an Euclidean signature the rotating EYMd black holes found in [26] are very likely to exist.

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