

Gravitating Yang-Mills dyon vortices in 4+1 spacetime dimensions

Yves Brihaye¹ and Eugen Radu²

¹*Physique-Mathématique, Université de Mons-Hainaut, Mons, Belgium*

²*Department of Mathematical Physics, National University of Ireland Maynooth, Maynooth, Ireland*

Abstract

We consider vortex-type solutions in $d = 5$ dimensions of the Einstein gravity coupled to a nonabelian SU(2) field possessing a nonzero electric part. After the dimensional reduction, this corresponds to a $d = 4$ Einstein-Yang-Mills-Higgs-U(1)-dilaton model. A general axially symmetric ansatz is presented, and the properties of the spherically symmetric solutions are analysed.

1 Introduction

Following the discovery by Bartnik and McKinnon of particle-like solutions of the four-dimensional Einstein-Yang-Mills (EYM) equations [1], there has been much interest in classical solutions of Einstein gravity with nonabelian matter sources.

A recent progress in this area was due to investigation of nontrivial configurations in higher dimensional spacetimes, particle-like and black hole solutions of EYM theories being studied in [2, 3]. These configurations are spherically symmetric in the d -dimensional spacetime and are sustained by the higher order terms in the Yang-Mills (YM) hierarchy. Without these terms, only vortex-type solutions, with a number of codimensions, are possible to exist.

The situation is exemplified by the best understood EYM-SU(2) model in $4 + 1$ dimensions. Without gravity, the pure YM theory in $d = 5$ Minkowski spacetime admits topologically stable, particle-like and vortex-type solutions obtained by uplifting the $d = 4$ YM instantons and $d = 3$ YM-Higgs monopoles. However, as found in [4], the particle spectrum become completely destroyed by gravity, while the vortices admit very non-trivial gravitating generalizations. Assuming the metric and matter fields to be independent on the extra coordinate x^5 , then the $(4 + 1)$ -dimensional static EYM system reduces effectively to a $(3 + 1)$ -dimensional Einstein-Yang-Mills-Higgs-dilaton (EYMHD) system. The solutions of this system exist for a finite range of values of the gravitational coupling constant, and in the strong gravity limit they become gravitationally closed. These gravitating vortices comprise an infinite family including the fundamental solution and its excitations.

These ideas were taken further to $4 + n$ dimensions [5], where n Higgs triplets and n dilatons appear. For two or more extra dimensions, a number of constraints result from the off-diagonal terms of the energy-momentum tensor, which implies a more complicated metric ansatz.

Here we remark that all these studies are restricted to a purely magnetic YM connection. In the four dimensional case, the electric YM potential necessarily vanishes for static asymptotically flat regular EYM solutions [6, 7].¹

Here we present numerical arguments that this result do not generalize to higher dimensions. We show that the known vortex-type solutions admit dyonic generalizations, existing up to some maximal value of the gravitational coupling constant. Beside the fundamental gravitating dyon solutions, we also construct excited configurations. To simplify the general picture, we consider the case of 4+1 dimensions only, although similar results exist for any $d > 4$.

¹However, gravitating dyon solutions exist in the presence of a Higgs field in the adjoint representation [8].

2 The model

2.1 Action principle

The five dimensional EYM-SU(2) coupled system is described by the action

$$I_5 = \int d^5x \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2g^2} \text{Tr}\{F_{MN}F^{MN}\} \right), \quad (1)$$

(throughout this letter, the indices $\{M, N, \dots\}$ will indicate the five dimensional coordinates and $\{\mu, \nu, \dots\}$ will indicate the coordinates in the four dimensional physical spacetime; also, the length of the extra dimension x^5 is taken to be one).

Here G is the gravitational constant, R is the Ricci scalar associated with the spacetime metric g_{MN} and $F_{MN} = \frac{1}{2}\tau^a F_{MN}^{(a)}$ is the gauge field strength tensor defined as

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N], \quad (2)$$

where the gauge field is $A_M = \frac{1}{2}\tau^a A_M^{(a)}$, τ^a being the Pauli matrices and g the gauge coupling constant.

Variation of the action (1) with respect to the metric g^{MN} leads to the Einstein equations

$$R_{MN} - \frac{1}{2}g_{MN}R = 8\pi G T_{MN}, \quad (3)$$

where the YM stress-energy tensor is

$$T_{MN} = 2\text{Tr}(F_{MP}F_{NQ}g^{PQ} - \frac{1}{4}g_{MN}F_{PQ}F^{PQ}). \quad (4)$$

Variation with respect to the gauge field A_μ leads to the YM equations

$$\nabla_M F^{MN} - i[A_M, F^{MN}] = 0. \quad (5)$$

In [4] it has been proven that there are no purely magnetic ($A_t = 0$) finite-energy, particle-like solutions of the above equations with a SO(4) spacetime symmetry group. Also, as explicitly shown in the Appendix 1 of Ref. [9], a spherically symmetric nontrivial YM-SU(2) ansatz with a nonzero electric potential is necessarily time dependent. To avoid these results, we are forced to modify the original action by consider higher order terms in the YM hierarchy [2], [3].

2.2 The ansatz

However, the action principle (1) presents less symmetric solutions with interesting properties. In what follows we will consider vortex-type configurations, assuming that both the matter functions and the metric functions are independent on the extra-coordinate x^5 . Without any loss of generality, we consider a five-dimensional metric parametrization

$$ds^2 = e^{-2\phi/\sqrt{3}}\gamma_{\mu\nu}dx^\mu dx^\nu + e^{4\phi/\sqrt{3}}(dx^5 + 2W_\mu dx^\mu)^2. \quad (6)$$

With this assumption, the considered theory admits an interesting Kaluza-Klein (KK) picture. While the KK reduction of the Einstein term in (1) is standard, to reduce the YM action term, is convinient to take an SU(2) ansatz

$$A = \mathcal{A}_\mu dx^\mu + g\Phi(dx^5 + 2W_\mu dx^\mu), \quad (7)$$

where W_μ is a U(1) potential, \mathcal{A}_μ is a purely four-dimensional gauge field potential, while Φ corresponds after the dimensional reduction to a triplet Higgs field.

This leads to the four dimensional action principle

$$I_4 = \int d^4x \sqrt{-\gamma} \left[\frac{1}{4\pi G} \left(\frac{\mathcal{R}}{4} - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - e^{2\sqrt{3}\phi} \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \right) - e^{2\phi/\sqrt{3}} \frac{1}{2g^2} \text{Tr}\{\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}\} \right. \\ \left. - e^{-4\phi/\sqrt{3}} \text{Tr}\{D_\mu \Phi D^\mu \Phi\} - 2e^{2\phi/\sqrt{3}} \frac{1}{g} G_{\mu\nu} \text{Tr}\{\Phi \mathcal{F}^{\mu\nu}\} - 2e^{2\phi/\sqrt{3}} G_{\mu\nu} G^{\mu\nu} \text{Tr}\{\Phi^2\} \right], \quad (8)$$

where \mathcal{R} is the Ricci scalar for the metric $\gamma_{\mu\nu}$, while $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and $G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$ are the SU(2) and U(1) field strength tensors defined in $d = 4$.

Here we consider five dimensional configurations possessing two more Killing vectors apart from $\partial/\partial x^5$, $\xi_1 = \partial/\partial\varphi$, corresponding to an axially symmetry of the four dimensional metric sector (where the azimuth angle φ range from 0 to 2π), and $\xi_2 = \partial/\partial t$, with t the time coordinate.

The five dimensional YM ansatz in this case is a straightforward generalization of the axially symmetric $d = 4$ ansatz obtained in the pioneering papers by Manton [10] and Rebbi and Rossi [11]. For the time and extra-direction translational symmetry, we choose a gauge such that $\partial A/\partial t = \partial A/\partial x^5 = 0$. However, a φ -rotation around the z -axis can be compensated by a gauge rotation

$$\mathcal{L}_\varphi A_N = D_N \Psi, \quad (9)$$

with Ψ being a Lie-algebra valued gauge function. This introduces an winding number n in the ansatz (which is a constant of motion and is restricted to be an integer) and implies the existence of a potential W with

$$F_{N\varphi} = D_N W, \quad (10)$$

where $W = A_\varphi - \Psi$.

The most general axially symmetric 5D Yang-Mills ansatz contains 15 functions: 12 magnetic and 3 electric potentials and can be easily obtained in cylindrical coordinates $x^N = (\rho, \varphi, z)$ (with $\rho = r \sin \theta$, $z = r \cos \theta$, and r, θ and φ being the usual spherical coordinates)

$$A_N = \frac{1}{2} A_N^{(\rho)}(\rho, z) \tau_\rho^n + \frac{1}{2} A_N^{(\varphi)}(\rho, z) \tau_\varphi^n + \frac{1}{2} A_N^{(z)}(\rho, z) \tau_z, \quad (11)$$

where the only φ -dependent terms are the SU(2) matrices (composed of the standard (τ_x, τ_y, τ_z) Pauli matrices)

$$\tau_\rho^n = \cos n\varphi \tau_x + \sin n\varphi \tau_y, \quad \tau_\varphi^n = -\sin n\varphi \tau_x + \cos n\varphi \tau_y, \quad (12)$$

(we can reduce this general ansatz by imposing some extra symmetries on the fields).

Transforming to spherical coordinates, it is convenient to introduce, without any loss of generality, a new SU(2) basis $(\tau_r^n, \tau_\theta^n, \tau_\varphi^n)$, with

$$\tau_r^n = \sin \theta \tau_\rho^n + \cos \theta \tau_z, \quad \tau_\theta^n = \cos \theta \tau_\rho^n - \sin \theta \tau_z, \quad (13)$$

which yields the general expression

$$A_N = \frac{1}{2} A_N^r(r, \theta) \tau_r^n + \frac{1}{2} A_N^\theta(r, \theta) \tau_\theta^n + \frac{1}{2} A_N^\varphi(r, \theta) \tau_\varphi^n. \quad (14)$$

For this parametrization $2\Psi = n\tau_z = n \cos \theta \tau_r^n - n \sin \theta \tau_\theta^n$. The gauge invariant quantities expressed in terms of these functions will be independent on the angle φ .

The assumed symmetries together with the YM equations implies the following relations for some extra-diagonal components of the energy-momentum tensor

$$T_t^5 = 2\text{Tr}\left\{\frac{1}{\sqrt{-g}} \partial_N (\sqrt{-g} A_t F^{N5})\right\}, \quad T_5^t = 2\text{Tr}\left\{\frac{1}{\sqrt{-g}} \partial_N (\sqrt{-g} A_5 F^{Nt})\right\}, \quad (15) \\ T_\varphi^t = 2\text{Tr}\left\{\frac{1}{\sqrt{-g}} \partial_N (\sqrt{-g} W F^{Nt})\right\}, \quad T_t^\varphi = 2\text{Tr}\left\{\frac{1}{\sqrt{-g}} \partial_N (\sqrt{-g} A_t F^{N\varphi})\right\}.$$

Also, one can prove the results

$$E_e = 2Tr\left\{\int_V d^3x \sqrt{-g} F_{Mt} F^{Mt}\right\} = 2Tr\left\{\oint_\infty dS_\mu \sqrt{-g} A_t F^{\mu t}\right\}, \quad (16)$$

$$E_h = 2Tr\left\{\int_V d^3x \sqrt{-g} F_{M5} F^{M5}\right\} = 2Tr\left\{\oint_\infty dS_\mu \sqrt{-g} A_5 F^{\mu 5}\right\}, \quad (17)$$

$$J = 2Tr\left\{\int_V d^3x \sqrt{-g} F_{M\varphi} F^{Mt}\right\} = 2Tr\left\{\oint_\infty dS_\mu \sqrt{-g} W F^{\mu t}\right\}, \quad (18)$$

(where the volume integral is taken over the three dimensional physical space) which implies that, for nontrivial dyonic particle-like solutions, the magnitude of the gauge potentials A_t , A_5 should be nonzero at infinity.

In this letter, we will restrict to a spherically symmetric ansatz for the four dimensional metric

$$\gamma_{\mu\nu} dx^\mu dx^\nu = \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - N(r)\sigma^2(r)dt^2 \quad (19)$$

with

$$N(r) = 1 - \frac{2m(r)}{r}.$$

This implies the existence of two more Killing vectors apart from $\partial/\partial x^5$, $\partial/\partial t$, $\partial/\partial\varphi$, originating in the SO(3) symmetry of the above line element.

The expression of the five-dimensional SU(2) connection is obtained by using the standard rule for calculating the gauge potentials for any spacetime group [12, 13, 14]. Taking into account the symmetries of the metric form (6), a straightforward computation leads to the YM ansatz

$$\begin{aligned} A_r &= 0, \quad A_\theta = (1 - K(r))\tau_\varphi^1, \quad A_\varphi = -(1 - K(r))\sin\theta\tau_\theta^1, \\ A_t &= \left(u(r) + 2J(r)H(r)\right)\tau_r^1, \quad A_5 = H(r)\tau_r^1, \end{aligned} \quad (20)$$

where $(\tau_r^1, \tau_\theta^1, \tau_\varphi^1)$ are found by taking $n = 1$ in the expressions (12), (13).

We remark that A_t , A_5 are oriented along the same direction in the isospace. Therefore, the energy-momentum tensor of nontrivial YM configurations presents also T_t^5 , T_5^t components, while $T_\varphi^t = T_t^\varphi = 0$. This implies the existence, in the five dimensional metric ansatz (6), of one extradiagonal component

$$W_\mu = J(r)\delta_\mu^t, \quad (21)$$

corresponding, in a four dimensional picture, to an U(1) electric potential. Thus, any five dimensional SU(2) dyon configuration necessarily has a nonzero $J(r)$, which cannot be gauged away. A vanishing four dimensional electric potential implies a purely magnetic SU(2) configuration in $d = 5$.

In the same approach, $K(r)$, $u(r)$ are $d = 4$ magnetic and electric SU(2)-gauge potentials, $\phi(r)$ is a dilaton while $H(r)$ is the Higgs field. The configurations discussed in this paper extremize also the action principle (8) and can be viewed as regular solutions of the four dimensional theory. The existence of a nonvanishing Abelian electric field in $d = 4$ without a singular source is not a surprise, given the specific coupling of the U(1) potentials to SU(2) and Higgs fields in the action principle (8). A similar property has been noticed in other models with a Maxwell field interacting with some matter fields (see e.g. [15]).

2.3 The equations of motion and boundary conditions

Dimensionless quantities are obtained by using the following rescaling

$$r \rightarrow r/(gH_0), \quad H(r) \rightarrow gH_0 H, \quad u(r) \rightarrow gH_0 u, \quad m(r) \rightarrow m/(gH_0), \quad (22)$$

where H_0 is the asymptotic value of $H(r)$ and we define also the coupling constant $\alpha = H_0\sqrt{4\pi G}$ (there also nontrivial solutions with $H_0 = 0$; however, that implies a purely magnetic ansatz $u(r) = 0$).

With these conventions, the field equations for the four metric variables (m , σ , ϕ , J) and the three matter functions (K , H , u) take the form

$$\begin{aligned}
m' &= e^{2\sqrt{3}\phi} \frac{r^2 J'^2}{2\sigma^2} + \frac{1}{2} r^2 N \phi'^2 + \alpha^2 \left(e^{2\phi/\sqrt{3}} T_1 + e^{-4\phi/\sqrt{3}} T_2 + \frac{e^{2\phi/\sqrt{3}}}{\sigma^2} T_3 \right), \\
\frac{\sigma'}{\sigma} &= r \phi'^2 + \frac{2\alpha^2}{r} \left(e^{2\phi/\sqrt{3}} K'^2 + e^{-4\phi/\sqrt{3}} \frac{r^2 H'^2}{2} + \frac{e^{2\phi/\sqrt{3}}}{\sigma^2 N^2} K^2 u^2 \right) \\
(e^{2\phi/\sqrt{3}} \sigma N K')' &= e^{2\phi/\sqrt{3}} \sigma \frac{K(K^2 - 1)}{r^2} + e^{-4\phi/\sqrt{3}} \sigma K H^2 - \frac{e^{2\phi/\sqrt{3}} K u^2}{\sigma N}, \\
(r^2 \sigma N \phi')' &= -\sqrt{3} e^{2\sqrt{3}\phi} \frac{r^2 J'^2}{\sigma} + \frac{2\sigma \alpha^2}{\sqrt{3}} \left(e^{2\phi/\sqrt{3}} T_1 - 2e^{-4\phi/\sqrt{3}} T_2 - e^{2\phi/\sqrt{3}} \frac{T_3}{\sigma^2} \right), \\
\left(e^{2\sqrt{3}\phi} \frac{r^2}{\sigma} (u' + 2J'H) \right)' &= 2e^{2\phi/\sqrt{3}} \frac{K^2 u}{\sigma N}, \\
0 &= \left(e^{2\sqrt{3}\phi} \frac{r^2 J'}{\sigma} + 2\alpha^2 e^{2\phi/\sqrt{3}} \frac{r^2 H}{\sigma} (u' + 2J'H) \right)' \\
(e^{-4\phi/\sqrt{3}} \sigma N r^2 H')' &= 2e^{-4\phi/\sqrt{3}} \sigma K^2 H + 2e^{2\phi/\sqrt{3}} \frac{r^2 J'}{\sigma} (u' + 2J'H)^2,
\end{aligned} \tag{23}$$

where

$$T_1 = N K'^2 + \frac{(K^2 - 1)^2}{2r^2}, \quad T_2 = \frac{1}{2} N r^2 H'^2 + K^2 H^2, \quad T_3 = \frac{r^2}{2} (u' + 2J'H)^2 + \frac{K^2 u^2}{N}.$$

Here we notice the following expressions for the extradiagonal components of the Einstein tensor

$$E_t^5 = \frac{1}{\sqrt{-g}} \frac{d}{dr} (\Gamma_{tM}^r g^{M5} \sqrt{-g}), \quad E_5^t = \frac{1}{\sqrt{-g}} \frac{d}{dr} (\Gamma_{5M}^r g^{Mt} \sqrt{-g}), \tag{24}$$

which, together with (15) imply the existence of two first first integrals of the field equations. For regular configurations, we find the simple relations

$$\begin{aligned}
u' &= -\frac{J'}{2\alpha^2 H} (e^{4\phi/\sqrt{3}} + 4\alpha^2 H^2), \\
e^{4\phi/\sqrt{3}} \left(N' J \sigma + N(2J\sigma' - \sigma(J + 2\sqrt{3}J\phi')) \right) &- 4e^{10\phi/\sqrt{3}} J' J^2 \\
&= 2\alpha^2 \left(N\sigma^2 H'(u + 2JH) + 2e^{2\phi/\sqrt{3}} J(u' + 2JH')(u + 2JH) \right).
\end{aligned} \tag{25}$$

An exact nonabelian dyon solution of the EYM equations can be constructed, it reads

$$\begin{aligned}
ds^2 &= ((\beta^2 + 1)e^{c_3 x} - \beta^2 e^{c_2 x}) \left(dx^5 + 2J(x)dt \right)^2 + dx^2 + c_1(d\theta^2 + \sin^2 \theta d\varphi^2) - \frac{e^{(c_2+c_3)x}}{(\beta^2 + 1)e^{c_3 x} - \beta^2 e^{c_2 x}} dt^2, \tag{26} \\
J(x) &= \frac{1}{2} \frac{(e^{c_3 x} - e^{c_2 x})\beta\sqrt{\beta^2 + 1}}{(\beta^2 + 1)e^{c_3 x} - \beta^2 e^{c_2 x}}, \quad K(x) = \sqrt{q}, \quad H(x) = \sqrt{h}\sqrt{\beta^2 + 1}e^{c_3 x/2}, \quad u(x) = \beta\sqrt{h}e^{c_3 x/2} - 2J(x)H(x),
\end{aligned}$$

where β is an arbitrary real parameter and

$$\begin{aligned}
c_1 &= \frac{1-q}{h} \simeq 1.12637\alpha^2, \quad c_2 = \frac{8hq}{c_3(q-1)} \simeq \frac{1.26557}{\alpha}, \quad c_3 = 2\sqrt{\frac{h(-1+h\alpha^2(1+q))}{(q-1)(h\alpha^2-1)}} \simeq \frac{0.482496}{\alpha}, \\
h &= \frac{3 + (3-2q)q + \sqrt{1+q(-6+q(-51+4(q-11)q))}}{4(1+q)\alpha^2} \simeq \frac{0.811482}{\alpha^2}, \\
q &= \frac{1}{12} \left(-7 - \frac{83}{(1259 + 18\sqrt{6657})^{1/3}} + (1259 + 18\sqrt{6657})^{1/3} \right) \simeq 0.08597.
\end{aligned} \tag{27}$$

The coordinate x used here is related to the radial coordinate r defined in (19) by $r^4 = ((\beta^2 + 1)e^{c_3 x} - \beta^2 e^{c_2 x})$. This generalizes the "warped" $AdS_3 \times S^2$ solution found in [4] which is recovered for $\beta = 0$.

However, in this paper we will study globally regular, asymptotically flat solutions of the field equations (23). That implies the following set of boundary conditions

$$K(0) = 1, \quad H(0) = 0, \quad \partial_r|_{r=0}\phi = 0, \quad N(0) = 1, \quad \partial_r|_{r=0}J = 0, \quad u(0) = 0, \quad \partial_r|_{r=0}\sigma = 0, \quad (28)$$

$$K(\infty) = 1, \quad H(\infty) = 1, \quad \phi(\infty) = 0, \quad N(\infty) = 1, \quad J(\infty) = 0, \quad u(\infty) = \gamma, \quad \sigma(\infty) = 1. \quad (29)$$

The asymptotic form of the solutions can be systematically constructed in both regions, near the origin and for large values of r . The field equations implies the following behavior as $r \rightarrow 0$ in terms of five parameters $(b, j_0, u_1, h_1, \phi_0)$

$$\begin{aligned} K(r) &= 1 - br^2 + O(r^4), \quad u(r) = u_1 r + O(r^3), \quad H(r) = h_1 r + O(r^3), \\ N(r) &= 1 - \frac{\alpha^2 e^{-4\phi_0/\sqrt{3}}}{\sigma_0^2} \left(h_1^2 \sigma_0^2 + e^{2\sqrt{3}\phi_0} (4b^2 \sigma_0^2 + u_1^2) \right) r^2 + O(r^3), \\ J(r) &= j_0 - e^{-4\phi_0/\sqrt{3}} \alpha^2 h_1 u_1 r^2 + O(r^4), \\ \sigma(r) &= \sigma_0 + \frac{e^{-4\phi_0/\sqrt{3}} \alpha^2}{2\sigma_0} \left(h_1^2 \sigma_0^2 + 2e^{2\sqrt{3}\phi_0} (4b^2 \sigma_0^2 + u_1^2) \right) r^2 + O(r^3), \\ \phi(r) &= \phi_0 + \frac{e^{-4\phi_0/\sqrt{3}} \alpha^2}{2\sqrt{3}\sigma_0^2} \left(-2h_1^2 \sigma_0^2 + e^{2\sqrt{3}\phi_0} (4b^2 \sigma_0^2 - u_1^2) \right) r^2 + O(r^3), \end{aligned} \quad (30)$$

while a similar analysis as $r \rightarrow \infty$ gives

$$\begin{aligned} K(r) &\sim e^{-\sqrt{1-\gamma^2}r}, \quad J(r) \sim \frac{2Q\alpha^2\gamma}{r}, \quad H(r) \sim 1 - \frac{Q}{r}, \quad u(r) \sim \gamma \left(1 - \frac{(1+4\alpha^2)Q}{r} \right), \\ \phi(r) &\sim \frac{\phi_1}{r}, \quad N(r) \sim 1 - \frac{2M}{r}, \quad \sigma(r) \sim 1 - \frac{\phi_1^2 + \alpha^2 Q^2}{2r}, \end{aligned} \quad (31)$$

where $\gamma, Q, M, \sigma_1, \phi_1$ are arbitrary parameters. The parameter M corresponds to the ADM mass of the four dimensional solutions. The nonabelian electric charge of the solutions is defined as $Q_e = (1 + 4\alpha^2)\gamma Q$, while the $U(1)$ electric charge is just $2\alpha^2 Q_e$.

For $u(\infty) > H(\infty)$ the asymptotic behaviour of $K(r)$ becomes oscillatory. Therefore, the value of the electric potential at infinity is restricted to be smaller than the asymptotic value of A_5 , *i.e.* $\gamma < 1$ ².

The relations (16), (25) imply also that for a vanishing γ , the functions $u(r)$ and $J(r)$ vanishes identically and we are lead to the purely magnetic configurations studied in [4].

3 Numerical results

The equations of motion (23) have been solved for a large set of physical parameters α, γ and Q_e . The values of α and Q_e are generically sufficient to specify the system. The various functions, including, *e.g.* the value of the parameter γ , can then be reconstructed from the numerical solutions. As expected, the globally regular dyonic solutions have many features in common with the purely magnetic vortex solutions discussed in [4]; they also present new features that we will pointed out in the discussion. Dyons solutions are found for any monopole configuration by slowly increasing the parameter $J(0)$, and extending up to the maximal value of γ .

The complete classification of the solutions in the space of physical parameters Q_e, α is a considerable task which is not aimed in this paper. Instead, we analyzed in details a few particular classes of solutions which, hopefully, reflect all the properties of the general pattern.

²A similar behavior has been noticed for dyons in a four dimensional (E)YMH theory [16], [8].

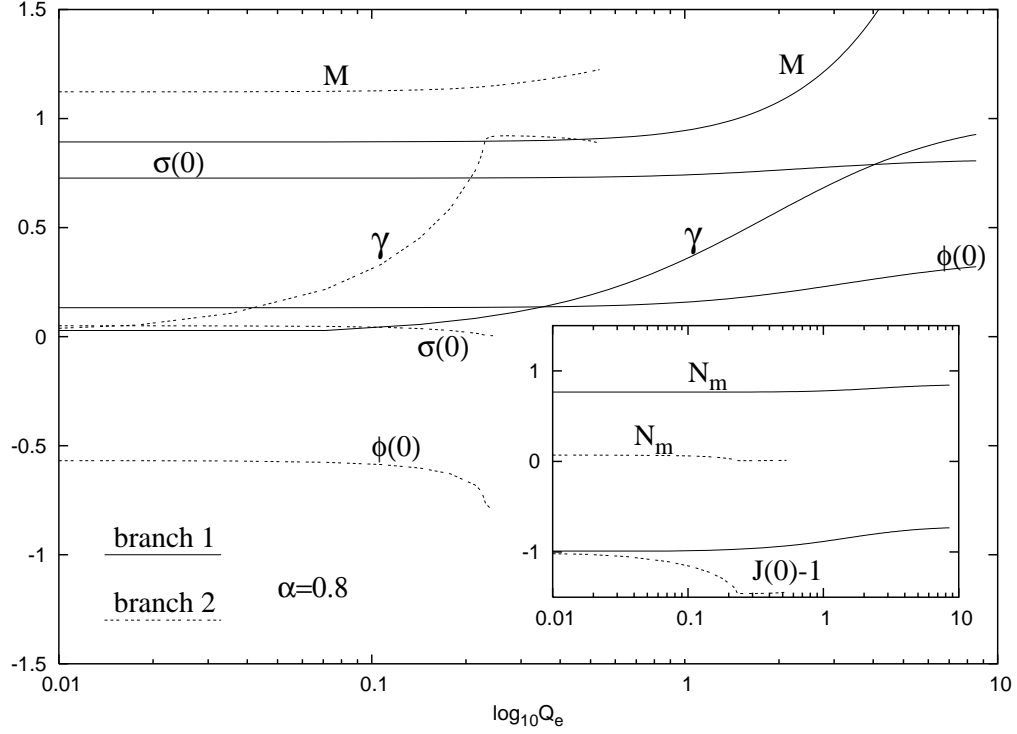


Figure 1. The values of the parameters M , $\sigma(0)$, N_m , $\phi(0)$, $J(0)$ and γ are shown as a function of Q_e for two different branches of solutions at $\alpha = 0.8$.

3.1 Known solutions

Let us first briefly summarize the pattern of solution in the known cases *i.e.* in the case $\alpha = 0$ and for a vanishing A_t -field, when $\gamma = J(r) = 0$.

For $\alpha = 0$, the gravity decouples and we find pure Einstein gravity in $d = 5$. The solutions for the metric ansatz (6), (19) are found by uplifting a family of four dimensional Einstein-Maxwell-dilaton black hole solutions discussed in [17, 18].

However, of interest here is the flat space trivial background ($\phi = J = 0$), in which case we find for the matter part the BPS dyon solutions [19] uplifted to $d = 5$. Increasing α , we expect these solutions to get deformed by gravity, at the same time the metric functions N , σ , J and the dilaton function ϕ becoming nontrivial. The case with gravity present and a vanishing electric part of the $SU(2)$ field is studied in [4] (here $J(r) = 0$). The main branch of solutions exists on the interval $\alpha \in [0, \alpha_{max}]$ with $\alpha_{max} \approx 1.268$. Then another branch of solutions (with higher ADM mass) exist for $\alpha \in [\alpha_{cr}, \alpha_{max}]$ with $\alpha_{cr} \approx 0.535$. Several branches of solutions exist in addition for very small intervals of α in the region of $\alpha = \alpha_{cr}$.

3.2 Varying Q_e

Our numerical results reveals that the branches of solutions occurring in the $A_t = 0$ limit are naturally continued in the dyon case and that for $Q_e \neq 0$ they still exist for compact intervals of α . A common feature of the dyon solutions is that the parameter γ is a monotonically increasing function of the electric charge Q_e .

In this subsection we present the results obtained for fixed α and increasing the electric charge parameter Q_e . The case Q_e fixed and α varying will be addressed in the next subsection. For definiteness we choose $\alpha = 0.8$. For this value of there are just two solutions in the purely magnetic case $A_t = 0$. The solution with the lowest ADM mass ($M_{ADM} \approx 0.893$) will be referred as belonging to the "main branch" while the solution with the highest ADM mass ($M_{ADM} \approx 1.122$) will be referred as on the "secondary branch".

In Figure 1, the quantities N_{min} , $\sigma(0)$, $J(0)$, $\phi(0)$, and $u(\infty) \equiv \gamma$ are plotted as functions of Q_e for the main and secondary branches respectively by the solid and dotted lines (where N_{min} corresponds to the minimal value of the metric function $N(r)$).

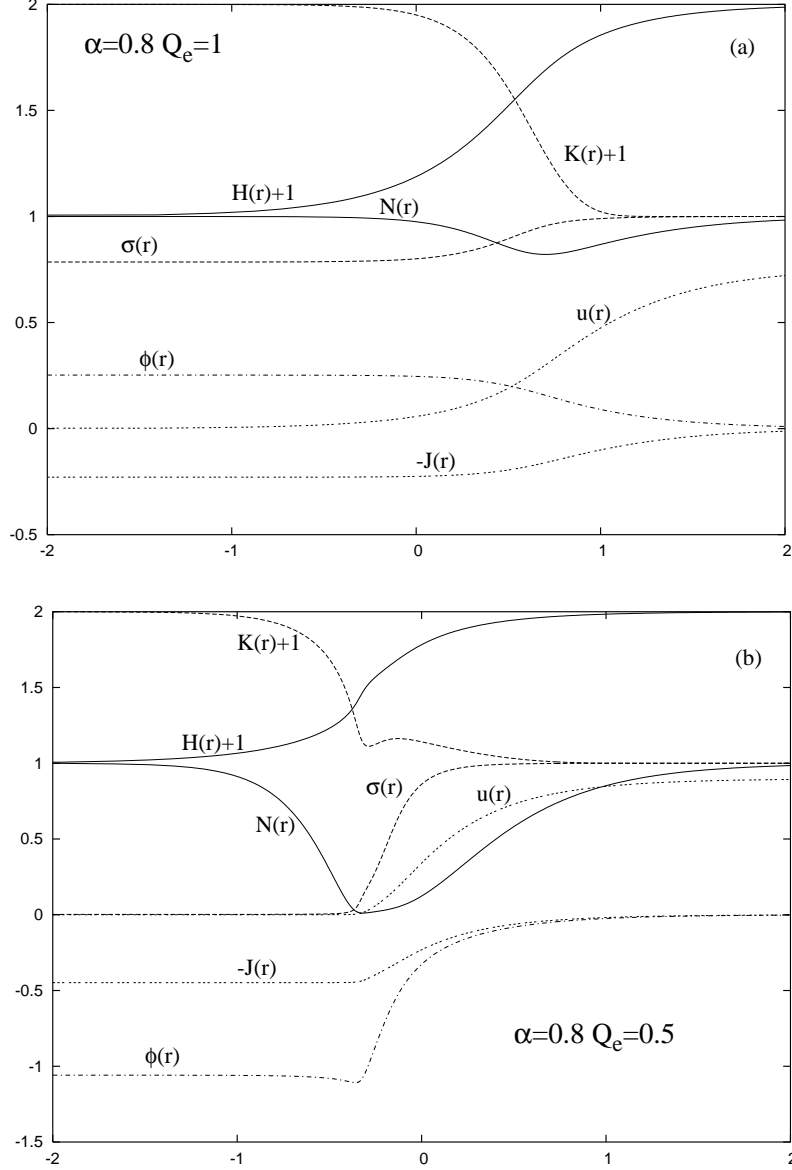


Figure 2. The metric functions $N(r)$, $\sigma(r)$, $\phi(r)$, $J(r)$ and the matter functions $K(r)$, $H(r)$ and $u(r)$ are shown for $\alpha = 0.1$ and two values of Q_e .

The ADM mass is also reported by the line labelled M . It turns out that the main solution can be constructed without problem for high values of Q_e (we constructed it up to $Q_e = 10.0$).

The quantities M , N_{min} , $\sigma(0)$, $J(0)$, $\phi(0)$ all increases with Q_e . Interestingly, the value γ also increases and approaches the critical value $\gamma = 1$ for $Q_e \gg 1$. So our numerical analysis suggests that solutions with arbitrary electric charge exist, composing the main branch.

The construction of the secondary branch of solutions is, by contrast, more involved. As demonstrated by Figure 1 (dotted lines), the solution present different behaviours according to the magnitude of the electric charge. For $\alpha = 0.8$, the limit between these two regimes occurs for $Q_e \approx 0.23$. Let us first analyze the case $Q_e < 0.23$. For such values of the electric charge, the parameter γ increases monotonically with Q_e and culminates at $\gamma \approx 0.92$ for $Q_e \approx 0.23$ (correspondingly $\sigma(0) \approx 0.004$). The naive picture would suggest that $\gamma = 1$ will be reached at some critical value of Q_e , but the scenario turns out to be different. Indeed, increasing Q_e such that $Q_e > 0.23$ leads to a new regime where, namely, both γ and $\sigma(0)$ decreases.

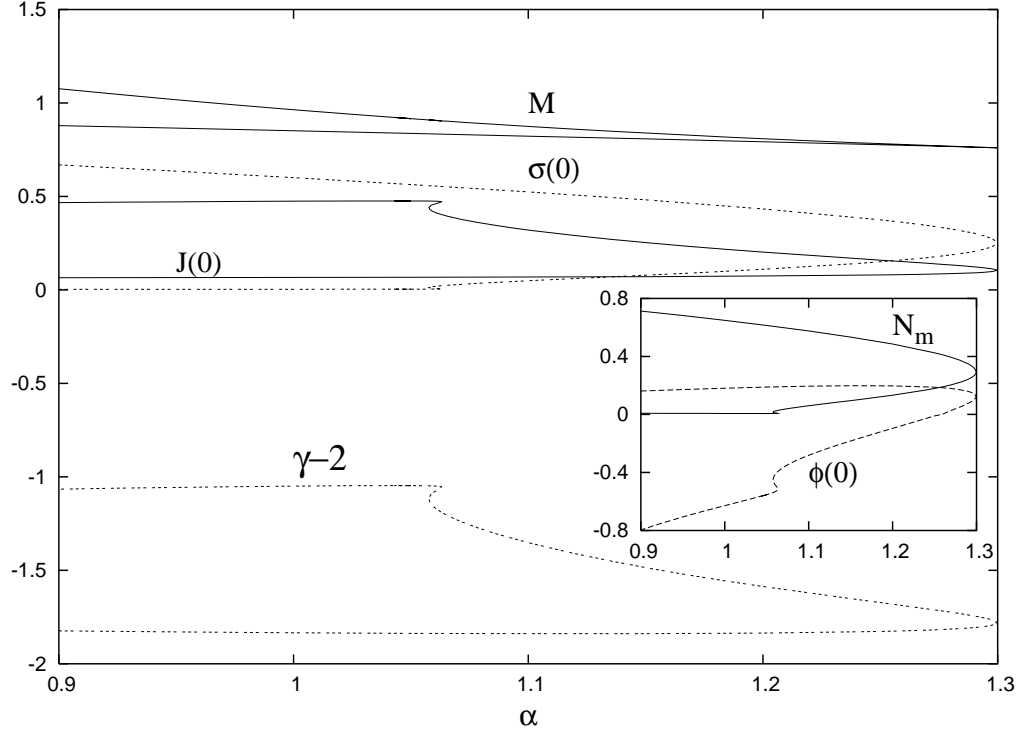


Figure 3. The values of the parameters M , $\sigma(0)$, N_m , $\phi(0)$, $J(0)$ and γ are shown as a function of α at $Q_e = 0.5$.

A further numerical study of the equations reveals that the branch of solutions stop at critical value of Q_e (in the case $\alpha = 0.8$, we find $Q_{e,c} \approx 0.65$) for which the parameter $\sigma(0)$ vanishes. Here our numerical results suggest that the secondary branch stops into a singular solution at some maximal value of Q_e .

The profiles of the different functions N , σ , K , H , ϕ , J , u are represented in Figure 2 for generic values of Q_e , namely $Q = 1$ on the main branch and for $Q_e = 0.5$ for the secondary branch. On the second graphic, we see in particular that the dilaton function $\phi(r)$ presents a minimum and that the gauge function develops an oscillation at $r \sim 0.8$.

3.3 Varying α

In this section we will pay attention to the domain of existence of the solutions in the parameter α for fixed values of the electric charge. Our numerical results suggest that, for fixed Q_e the solutions exist up to a maximal value of α , say α_{max} and that, at least for small values of the electric charge, this value increases slightly with Q_e ; for instance :

$$\alpha_{max}(Q_e = 0) \approx 1.262 \quad , \quad \alpha_{max}(Q_e = 0.5) \approx 1.2996 \quad , \quad \alpha_{max}(Q_e = 1.0) \approx 1.375 \quad , \quad (32)$$

We studied in details the case corresponding to $Q_e = 0.5$, some relevant data being presented in Figure 3.

We see in particular that on the main branch both the minimum of the metric function N and the value of the metric function σ at the origin monotonically decrease when α increases. The mass parameter M also decreases. The value of $J(0)$ increases slightly with α on this branch. For $\alpha > \alpha_{max}$ no solution exist but, along with the case $Q_e = 0$ we found a second branch of solutions for $\alpha < \alpha_{max}$.

The solutions on the second branch have a higher ADM mass. When α increases, the solutions on the second branch are characterized by increasing values of the parameters N_m and $\sigma(0)$. The parameter $\phi(0)$ become negative at some intermediate value of α . Surprisingly, we observed on this second branch the occurrence of two supplementary branches of solutions which develop on a very small interval of the parameter α in the region $\alpha \approx 1.05$. This is clearly illustrated on Figure 1. The high accuracy of our numerical routine strongly discard the possibility of a numerical artefact for these two extra branches ³.

³To integrate the equations, we used the differential equation solver COLSYS which involves a Newton-Raphson method [20].

The first integral relations (25) are also satisfied with a very good accuracy.

Again, the discussion of the parameter γ is crucial and deserves a particular attention. If we follow the values of this parameter on the *second* branch, it appears that (for increasing α) this parameter approaches very closely the value $\gamma = 1$ (which, remember, turns out to be a critical point related to the asymptotic behaviour of the function $K(r)$). However the value $\gamma = 1$ is not attained. Instead, the very tiny third branch of solution appears there for $\alpha \approx 1.06$; from this value the values of the parameter γ decreases and reaches $\gamma = 0.2217$ at $\alpha = \alpha_{max}$. To our knowledge, this constitutes a new (at least unexpected) phenomenon. It contrasts, namely, with the case of gravitating dyon studied in [8] where it was observed that the branch solutions stopped at some maximal value of the electric charge where the parameter γ reaches the limit $\gamma = 1$.

4 Further remarks

It is reasonable to expect that the EYM equations (3), (5) present also axially symmetric dyon solutions, obtained within the general metric ansatz (6)⁴. These configurations would generalize for $A_t \neq 0$ the axially symmetric monopole vortices discussed in [21].

An interesting physical question is whether these five dimensional EYM vortices can rotate or not. Working in $d = 4$, no nonperturbative rotating generalizations of the Bartnik-McKinnon solutions seem to exist [22, 23]. Here we present arguments that the $d = 5$ dyon vortices necessarily have a vanishing total angular momentum.

The total angular momentum is defined in this case as the charge associated with the axial Killing vector $\partial/\partial\varphi$, and from (18) we find that, for regular configurations it can be written as a surface integral at infinity

$$\begin{aligned} J &= \int_V T_\varphi^t \sqrt{-g} d^3x = 2Tr \left\{ \oint_\infty dS_\mu W F^{\mu t} \right\} \\ &= 2\pi \lim_{r \rightarrow \infty} \int_0^\pi d\theta \sin \theta \, r^2 [W^{(r)} F^{rt(r)} + W^{(\theta)} F^{rt(\theta)} + W^{(\varphi)} F^{rt(\varphi)}]. \end{aligned} \quad (33)$$

The evaluation of this relation is straightforward. The boundary conditions at infinity for A_5 , generalizing (29) for the axially symmetric case, are $A_5^{(r)} = H_0$, $A_5^{(\theta)} = A_5^{(\varphi)} = 0$ [21]. The requirement of a finite total mass/energy implies that the decay of the $F_{rt}^{(a)}$ as $r \rightarrow \infty$ is faster than $1/r^{1.5}$. Also, the contribution $F_{\varphi 5}^{(a)} F_{\varphi 5}^{(a)} g^{\varphi\varphi} g^{55}$ term to the total mass/energy is finite if $F_{\varphi 5}^{(a)}$ approaches zero to large r sufficiently rapid. This implies the fall-off conditions $(W_\theta, W_\varphi) \sim 1/r^{0.5+\epsilon}$ for large r (with $\epsilon > 0$). Therefore the last two terms in (33) give null contribution to the total angular momentum. The contribution of the $A_\varphi^{(r)}$ to the total angular momentum is also zero, since it vanishes asymptotically for a regular configuration.

Therefore we find

$$\lim_{r \rightarrow \infty} Tr(r^2 W F_{rt}) = -\frac{nQ_e}{2} \cos \theta \quad (34)$$

(where $Q_e = \lim_{r \rightarrow \infty} r^2 F_{rt}^{(r)}$) and the total angular momentum of the axially symmetric regular dyon vortices is clearly zero.

This is not a surprise, if we use the observation that, after the KK dimensional reduction, these five dimensional EYM configurations will correspond to $d = 4$ EYMHD-U(1) solutions, with a specific coupling between the gauge and Higgs sectors. However, as found in [22], [24], [25], the angular momentum of the $d = 4$ pure EYM dyons is zero. The inclusion of a dilaton and a Maxwell field does not change that conclusion.

The $d = 5$ solutions can be regarded as the "regularized" version of the known vacuum black string configurations, obtained by taking $F_{\mu\nu} = 0$ in (1). Black strings with an SU(2) hair were constructed recently in Ref. [26], for $A_t = 0$ however. It would be interesting to look for the dyon counterparts of these configurations, in particular for axially symmetric solutions. Different from the regular case, the axially

⁴The axially symmetry is defined in this case with respect to the Killing vector $\partial/\partial\varphi$.

symmetric dyonic black strings will have a nonvanishing angular momentum localised on the event horizon. We expect that these configurations (viewed as solutions for the action principle (8)) will share some of the properties of the pure EYM solutions discussed recently in [27].

For these solutions the geometry of the horizon is $R \times S^2$. However, in five dimensions more complex configurations are allowed, as proven by the "black ring" solution of the vacuum Einstein equations [28]. This is a rotating black hole with an event horizon of topology $S^1 \times S^2$, the rotation being required to prevent the ring from collapsing. Generalizations with a U(1) gauge field are considered in [29]. Nonabelian versions of these solutions are also likely to exist, and will necessarily present an electric part. In this case one may expect the nonabelian matter content to desingularise these solutions, leading to regular, rotating rings.

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