On o-minimal hybrid systems*

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Abstract. This paper is driven by a general motto: bisimulate a hybrid system by a finite symbolic dynamical system. In the case of o-minimal hybrid systems, the continuous and discrete components can be decoupled, and hence, the problem reduces in building a finite symbolic dynamical system for the continuous dynamics of each location. We show that this can be done for a quite general class of hybrid systems defined on o-minimal structures. In particular, we recover the main result of a paper by Lafferriere G., Pappas G.J. and Sastry S. on o-minimal hybrid systems.
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1 Introduction

Hybrid systems consist of finite state machines equipped with a continuous dynamics. This notion has been intensively studied [ACH+,HKPV,Hen95] (see [Hen96] for a survey), and is a generalization of timed automata [AD]. Hybrid systems encompass many interesting applications such as air traffic management [TPS] and highway systems [LGS].

Given a hybrid system, a natural question is to know whether the system can reach some prohibited states. This question is known as *the reachability problem*. Since the state space is usually uncountable it is necessary to have an algorithmic approach to this problem. The main difficulty is the richness of continuous dynamics and its interaction with a discrete dynamics. Several results on decidability and undecidability of the reachability problem have been developed in [ACH+,HKPV].

One approach to solve the reachability problem is to study equivalence relations preserving reachability and to find finite state systems equivalent to the original one. Building *bisimulations* is a way to achieve this goal. This is the

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point of view adopted in this paper. Bisimulations have many other interesting properties (e.g. they preserve CTL, [AHLP]).

In [LPS], the notion of *o-minimal hybrid system* is defined. This class of hybrid systems have a particularly rich continuous dynamics, in particular it may be non-linear. Through this paper, we adopt the conventions introduced in [LPS, p. 6] for the discrete transitions. This allows to decouple the discrete and continuous components of the hybrid system. Hence the problem to find a finite bisimulation of such a hybrid system is equivalent to find a finite bisimulation, on each location, which respects some initial partition induced by resets, guards, initial and final regions. In [LPS, p. 12], the continuous dynamics of an o-minimal hybrid system is given by a smooth complete vector field F from $\mathbb{R}^n$ to $\mathbb{R}^n$ and the flow is assumed to be definable in an o-minimal extension of $\langle \mathbb{R}, <, +, - \rangle$. In particular, the system is time-invariant, the flow is injective w.r.t. the time and thus the trajectories are non self-intersecting. We relax these assumptions by permitting the system to be time-varying and to have self-intersecting trajectories, which are natural features of many real systems. The continuous transition relation of such systems is therefore much richer (see Section 2.3). Moreover the generalization allows for general dynamics instead of flow, for an output space M^{k_2} distinct from the input space M^{k_1} and for linearly ordered structures over spaces other than the reals.

In Section 3 of this paper, we present a general construction to associate words with trajectories of a continuous dynamics w.r.t. an initial partition of the space. By using this general tool, a finite symbolic dynamical system is associated with any *o-minimal dynamical system*, the states of which are represented by words (see Section 4). Let us mention that this kind of idea already appears in the literature (see for example [ASY]).

Under the extra assumption that there is a unique trajectory passing through a point, we show that this finite symbolic dynamical system bisimulates the original one. As a byproduct of this result, we obtain a simple proof of the main result of [LPS] which asserts that every o-minimal hybrid system admits a finite bisimulation.

In the last section, we give an example of an o-minimal dynamical system which does not admit a finite bisimulation w.r.t. some initial partition, setting in this way some limits to our results.

We do not address the effectiveness of our constructions, this question will be studied in subsequent papers, in which the techniques developed here will be applied to a wider class of hybrid systems.

2 Preliminaries

In this section, we recall some basic definitions and results. However we do not recall classical definitions about hybrid systems, they can be found for example in [Hen96]. For o-minimal hybrid systems and their extensions treated in the paper, we refer to [LPS].

2.1 Transition systems and bisimulation

Definition 2.1. A *transition system* $T = (Q, \Sigma, \rightarrow)$ consists of a set of states Q (which may be uncountable), Σ an alphabet of events, and $\rightarrow \subseteq Q \times \Sigma \times Q$ a transition relation.

A transition $(q_1, a, q_2) \in \rightarrow$ is denoted by $q_1 \xrightarrow{a} q_2$. A transition system is finite if Q is finite. If the alphabet of events is reduced to a singleton, $\Sigma = \{a\}$, we will denote the transition system $(Q, \rightarrow)$ and omit the event a.

Definition 2.2. Given two transition systems on the same alphabet of events, $T_1 = (Q_1, \Sigma, \rightarrow_1)$ and $T_2 = (Q_2, \Sigma, \rightarrow_2)$, a *partial simulation of* T_1 *by* T_2 is a binary relation $\sim \subseteq Q_1 \times Q_2$ which satisfies the following condition:

$$\begin{aligned}&\forall q_1, q_1' \in Q_1,\ \forall q_2 \in Q_2,\ \forall a \in \Sigma,\\ &\big(q_1 \sim q_2 \text{ and } q_1 \xrightarrow{a}_1 q_1'\big) \Rightarrow \big(\exists q_2',\ q_1' \sim q_2' \text{ and } q_2 \xrightarrow{a}_2 q_2'\big)\end{aligned}$$

This condition is read T_2 *simulates* T_1.

Definition 2.3. Given $\sim$ a partial simulation of T_1 by T_2, we say that $\sim$ is a *simulation of* T_1 *by* T_2 if, for each $q_1 \in Q_1$, there exists $q_2 \in Q_2$ such that $q_1 \sim q_2$.

Definition 2.4. Given two transition systems on the same alphabet of events, $T_1 = (Q_1, \Sigma, \rightarrow_1)$ and $T_2 = (Q_2, \Sigma, \rightarrow_2)$, a *bisimulation between* T_1 *and* T_2 is a relation $\sim \subseteq Q_1 \times Q_2$ such that $\sim$ is a simulation of T_1 by T_2 and the *inverse relation*[1] $\sim^{-1}$ is a simulation of T_2 by T_1.

Definition 2.5. Given $\sim$ a bisimulation between T_1 and T_2 if $\sim$ is a function from Q_1 to Q_2, we call it a *functional bisimulation*.

Remarks 2.6.
- Given a transition system $T = (Q, \Sigma, \rightarrow)$, we can look at bisimulations on $Q \times Q$; they are called *bisimulations on* T.
- Given T_1, T_2 two transition systems and $\sim \subseteq Q_1 \times Q_2$ a bisimulation between T_1 and T_2, the kernel[2] $\mathrm{Ker}(\sim)$ is a bisimulation on T_1.
- Given $\sim$ a functional bisimulation between T_1 and T_2, we have that $\mathrm{Ker}(\sim)$ is an equivalence relation on Q_1; moreover there is a bisimulation between $T_1/\mathrm{Ker}(\sim)$ and T_2 (these statements and their proofs can be found in [Cau]).

Definition 2.7. Given T a transition system, $\mathcal{P}$ a partition of Q and $\sim \in Q \times Q$ a bisimulation which is an equivalence relation on Q, we say that the bisimulation $\sim$ *respects the partition* $\mathcal{P}$ if any $P \in \mathcal{P}$ is an union of equivalence classes for $\sim$. We will speak of *bisimulations w.r.t.* $\mathcal{P}$.

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[1] If $\sim = \{(q_1, q_2) \in Q_1 \times Q_2 | q_1 \sim q_2\}$, then $\sim^{-1} = \{(q_2, q_1) \in Q_2 \times Q_1 | q_1 \sim q_2\}$.
[2] $\mathrm{Ker}(\sim) = \sim \circ \sim^{-1} = \big\{(p, q) \in Q_1 \times Q_1 \ \big|\ \exists r \in Q_2,\ p \sim r \text{ and } q \sim r\big\}$.

2.2 O-minimality and definability

Let $\mathcal{M}$ be a structure. In this paper when we say that some relation, subset, function is definable, we mean it is first-order definable (possibly with parameters) in the sense of the structure $\mathcal{M}$. A general reference for first-order logic is [Ho]. All the notions related to o-minimality and an extensive bibliography can be found in [vdD98]. Let us recall the definition of an o-minimal structure:

Definition 2.8. An extension of an ordered structure $\mathcal{M} = \langle M, <, ...\rangle$ is *o-minimal* if every definable subset of M is a finite union of points and open intervals (possibly unbounded).

In other words the definable subsets of M are the simplest possible: the ones which are definable with parameters in $\langle M, <\rangle$. This assumption implies that definable subsets of M^n (in the sense of M) admit very nice structure theorems (like *Cell decomposition*) or Theorem 2.10. The following are examples of o-minimal structures.

Example 2.9. The field of reals $\langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$, the group of rationals $\langle \mathbb{Q}, <, +, \cdot, 0, 1\rangle$, the field of reals with exponential function, the field of reals expanded by restricted pfaffian functions and the exponential function, and many more interesting structures.

The main result we use on o-minimal structures is (see [vdD98, Corollary 3.6, p. 60]):

Theorem 2.10 (Uniform Finiteness). *Let $S \subseteq M^m \times M^n$ be definable, we denote by S_a the fiber $\{y \in M^n | (a, y) \in S\}$. Then there is a number $N_S \in \mathbb{N}$ such that for each $a \in M^m$ the set $S_a \subseteq M^n$ has at most N_S definably connected components.*

2.3 Dynamics

Definition 2.11. A *dynamical system* is a pair $(\mathcal{M}, \gamma)$ where:

- $\mathcal{M} = \langle M, <, ...\rangle$ is a totally ordered structure,
- $\gamma : M^{k_1} \times M \to M^{k_2}$ is a definable function of $\mathcal{M}$.

The function γ is called the *dynamics* of the dynamical system [3]. More generally, we can consider the case where γ is defined on definable subsets of $\mathcal{M}$ that is $\gamma : V_1 \times V \to V_2$ with $V_1 \subseteq M^{k_1}$, $V \subseteq M$ and $V_2 \subseteq M^{k_2}$.

Classically, when $M = \mathbb{R}$ is the field of the reals, we see M as the time, $M^{k_1} \times M$ as the space-time, M^{k_2} as the (output) space and M^{k_1} as the input space. We keep this terminology in the more general context of a structure $\mathcal{M}$.

[3] Since we do not assume that the dynamics is given by a flow, this allows quite more general behavior than in the vector field case.

Definition 2.12. If we fix a point $x \in M^{k_1}$, the set $\Gamma_x = \{\gamma(x,t) \mid t \in M\} \subseteq M^{k_2}$ is called the trajectory determined by x.

Definition 2.13. Given $(\mathcal{M}, \gamma)$ a dynamical system, we define a *transition system* $T_\gamma = (Q, \rightarrow_\gamma)$ *associated with the dynamical system* by:

- the set Q of states is M^{k_2};
- the transition relation $y_1 \rightarrow_\gamma y_2$ is defined by:

$$\exists x \in M^{k_1},\ \exists t_1, t_2 \in M,\ \big(t_1 \leqslant t_2 \text{ and } \gamma(x,t_1) = y_1 \text{ and } \gamma(x,t_2) = y_2\big)$$

Let us make an *important* observation. Given a transition $y_1 \rightarrow_\gamma y_2$, we denote the couple of instants of time corresponding to the positions y_1, y_2 by (t_1, t_2). If there exists a position y and different times $t < t'$ such that $\gamma(x,t) = \gamma(x,t') = y$ (see Figures 1 and 3 for example), then the transition relation $\rightarrow_\gamma$ allows the following sequence of transitions: $y_1 \rightarrow_\gamma y \rightarrow_\gamma y_2$ with couples of time (t_1, t') and (t, t_2). Let us look at a simple example of this behavior, in Figure 1, there clearly exists $t < t'$ such that $\gamma(x,t) = \gamma(x,t') = y$. The composition of transitions as explained above allows an arbitrary large number of passages in the loop.

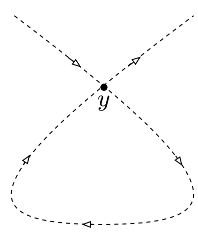

Fig. 1. A simple loop

3 Encoding trajectories by words

In this section, we describe the general tools that we use further on.

Given a dynamical system $(\mathcal{M}, \gamma)$ and $\mathcal{P}$ a finite definable partition of the space M^{k_2}, $\mathcal{P} = \{P_1, \ldots, P_s\}$, we want to encode the trajectories on M^{k_2} as words[4] on the finite alphabet $\mathcal{P}$.

Let us first remark that the partition $\mathcal{P}$ of the space M^{k_2} induces a partition $\tilde{\mathcal{P}}$ on the space-time $M^{k_1} \times M$ defined by the preimages of the P_i's under γ. The preimage of trajectory Γ_x is the line $\{x\} \times M$ in the space-time $M^{k_1} \times M$.

[4] In this general (possibly uncountable) context, a word is a function from M (or from a quotient of M induced by a partition on M) to $\mathcal{P}$.

This line crosses the regions $\tilde{P}_i$'s and looking to this crossing, when time is increasing, naturally gives a word on the alphabet $\tilde{\mathcal{P}}$. Replacing each letter $\tilde{P}_i$ by its corresponding letter P_i gives the word ω_x on the alphabet $\mathcal{P}$ we want to associate with Γ_x. For the sake of completeness, we mathematically formalize this idea.

Given $x \in M^{k_1}$, we consider the sets $\{t \mid \gamma(x,t) \in P_i\}$ for $i = 1, \ldots, s$. This gives a partition of the time M. We associate a word on $\mathcal{P}$ with the trajectory determined by x such that two consecutive letters are different. Let $\mathcal{F}_x$ be the set of intervals defined by:

$$\mathcal{F}_x = \big\{I \bigm| I \text{ is a time interval and is maximal for the property} \\ \exists i \in \{1, \ldots, s\},\ \forall t \in I,\ \gamma(x,t) \in P_i\big\}.$$

For each x, the set $\mathcal{F}_x$ is totally ordered by the order induced from M. By analogy with the work of [Tr], we introduce a family of functions of *coloration* $\mathcal{C}_x : \mathcal{F}_x \to \mathcal{P}$ defined by:

$$\mathcal{C}_x(I) = P_i \quad \Leftrightarrow \quad \exists t \in I,\ \gamma(x,t) \in P_i\ .$$

The word ω_x is defined by:

$$\omega_x \text{ is the sequence } \left(\mathcal{C}_x(I)\right)_{I \in \mathcal{F}_x}.$$

We denote by Ω the set of words associated with $(\mathcal{M}, \gamma)$ w.r.t. $\mathcal{P}$. In the sequel we will have to consider this construction w.r.t. different partitions.

Example 3.1. Consider the dynamical system and the partition $\mathcal{P} = \{A, B\}$ described in Figure 2. In this situation, we have $\Omega = \{A, ABA, ABABA\}$.

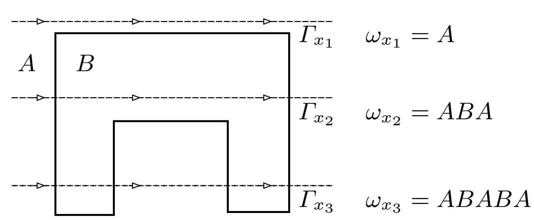

Fig. 2. Encoding trajectories by words

By encoding trajectories by words, we give a description of the "support" of the dynamical system. But, in order to recover the dynamics of a point in the trajectory, we need to encode more information: given a point (x,t) of the space-time, we want to know what the "position of $\gamma(x,t)$" in ω_x is. Given $(x,t) \in M^{k_1} \times M$, we associate a unique *dotted word* $\dot{\omega}_{(x,t)}$ in the following way: let $I \in \mathcal{F}_x$ be the unique interval such that $t \in I$, we add a dot on $\mathcal{C}_x(I)$ in ω_x. The set of dotted words associated with $(\mathcal{M}, \gamma)$ w.r.t. $\mathcal{P}$ is denoted by $\dot{\Omega}$.

Example 3.2. If we now consider the dotted words associated with Figure 2, we have $\dot{\Omega} = \{\dot{A}, \dot{A}BA, A\dot{B}A, AB\dot{A}, \dot{A}BABA, \ldots, ABAB\dot{A}\}$.

Remark 3.3. In general, γ is not injective and so a point y of the space M^{k_2} has more than one preimage (x,t). So several words ω_x and dotted words $\dot{\omega}_{(x,t)}$ are associated with y.

In view to describe this general situation, we introduce the notion of *dynamical type* W_y, for $y \in M^{k_2}$:

$$W_y = \left\{\dot{\omega}_{(x,t)} \;\middle|\; \exists (x,t) \in M^{k_1} \times M,\ \gamma(x,t) = y\right\}.$$

We denote by Δ the set of dynamical types associated with $(\mathcal{M}, \gamma)$ with respect to $\mathcal{P}$. Let us consider the partition given by the equivalence relation on the space M^{k_2} "*to have same dynamical type*". We can now repeat the previous construction w.r.t. this new partition (we will also denote it by Δ). So, we naturally obtain a set of words on Δ, denoted Ω_Δ. Let us notice that Δ is a refinement of $\mathcal{P}$. Given $x \in M^{k_1}$, we denote u_x the word on Δ associated with Γ_x, $\mathcal{F}_x^\Delta$ the ordered set of intervals induced on M and $\mathcal{C}_x^\Delta : \mathcal{F}_x^\Delta \to \Delta$ the coloration function.

Example 3.4. Figure 3 represents the trajectory Γ_x of some dynamical system through the partition $\mathcal{P} = \{A, B, C\}$, the word ω_x associated with the trajectory is $ABCBA$. For $y \in \Gamma_x$, there exists seven different dynamical types: $W_1 = \{\dot{A}BCBA\}, \ldots, W_5 = \{ABCB\dot{A}\}$, $W_6 = \{\dot{A}BCBA, ABCB\dot{A}\}$ and $W_7 = \{A\dot{B}CBA, ABC\dot{B}A\}$. The word u_x associated with the trajectory is $W_1W_6W_1$ $W_2W_7W_2W_3W_4W_7W_4W_5W_6W_5$.

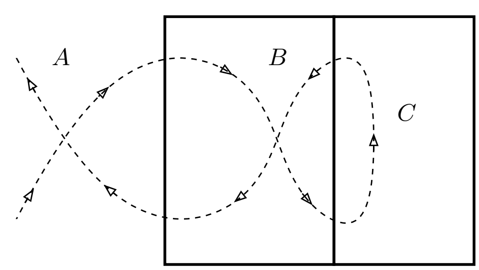

Fig. 3. Double loop

Given a trajectory Γ_x for some $x \in M^{k_1}$ and $y \in \Gamma_x$, we want to know "the position of y" in u_x. But by Remark 3.3 this position is not necessarily unique. We introduce a unique *multidotted word* $\ddot{u}_{(x,y)}$ in the following way: we add dots on $\mathcal{C}_x^\Delta(I)$ for all interval $I \in \mathcal{F}_x^\Delta$ such that there exists $t \in I$ with $\gamma(x,t) = y$.

We denote by $\ddot{\Omega}_\Delta$ the set of multidotted words associated with $(\mathcal{M}, \gamma)$ w.r.t. Δ.

4 O-minimal dynamical system

We have just described the general framework. Now we will be interested by *o-minimal dynamical systems.* In particular, we discuss two special and interesting cases in Sections 4.2 and 4.3. We freely use the notations introduced in the previous sections.

Definition 4.1. An *o-minimal dynamical system* $(\mathcal{M}, \gamma)$ is a dynamical system where $\mathcal{M}$ is an o-minimal structure.

4.1 Symbolic dynamical system

In Section 3 we gave a description of the trajectories of any dynamical system in term of words. In the case of an o-minimal dynamical system, finitely many finite words are enough to describe the trajectories. This will allow us to define *finite* transition systems on the words.

Lemma 4.2. *Given* $(\mathcal{M}, \gamma)$ *an o-minimal dynamical system and a finite definable partition* $\mathcal{P}$*, the set of words* Ω *is a finite set of finite words.*

Proof. Let us recall from Section 3 that the partition $\mathcal{P}$ of the space induces a definable partition of the space-time whose regions are the $\tilde{P}_i$'s. Given $x \in M^{k_1}$, we have that $\mathcal{F}_x$ exactly consists in the connected components of the fibers of the $\tilde{P}_i$'s: $(\tilde{P}_i)_x = \{t \in M \mid \gamma(x,t) \in P_i\}$. By the Uniform Finiteness Theorem 2.10, we have that the number of connected components of the $(\tilde{P}_i)_x$'s is uniformly finite w.r.t. x, this implies that the length of the ω_x's is uniformly bounded. So since the number of P_i's is finite, we have that Ω is finite. □

The next result is a trivial consequence of Lemma 4.2 and the definition of $\dot{\Omega}$.

Corollary 4.3. $\dot{\Omega}$ *is finite.*

Remark 4.4. Let us remark that in the proof of Lemma 4.2, we just used the Uniform Finiteness Theorem 2.10. So this result holds in all the structures admitting the Uniform Finiteness Theorem 2.10.

We define $T_{\dot{\Omega}}$, a finite transition system on the dotted words. In order to mathematically formalize $T_{\dot{\Omega}}$, we need to introduce two functions: $\textsc{undot} : \dot{\Omega} \to \Omega$ which gives the word ω corresponding to $\dot{\omega}$ without dot; $\textsc{dot} : \dot{\Omega} \to \mathbb{N}$ which gives the position of the dot on $\dot{\omega}$. Given $x \in M^{k_1}$, the set $\mathcal{F}_x$ can be described as a finite ordered sequence of intervals $I_0 < I_1 < \cdots < I_k$ with $k < N_S$. If we consider $\dot{\omega}_x$ a dotted word constructed from ω_x, we have the following relation: the dot of $\dot{\omega}_x$ is on $\mathcal{C}_x(I_i)$ with $I_i \in \mathcal{F}_x$ if and only if $\textsc{dot}(\dot{\omega}_x) = i$. We can now define $T_{\dot{\Omega}} = (\dot{\Omega}, \to_{\dot{\Omega}})$:

- the set of states Q is $\dot{\Omega}$

– the transition relation $\dot\omega_1 \rightarrow_{\dot\Omega} \dot\omega_2$ is defined by: the undotted words $\dot\omega_1$ and $\dot\omega_2$ are equal and the dot on $\dot\omega_2$ is on a righter (or on the same) position as the dot on $\dot\omega_1$. This can be formalized by:

$$\begin{gathered}\dot\omega_1 \rightarrow_{\dot\Omega} \dot\omega_2\\ \Updownarrow \\ \text{UNDOT}(\dot\omega_1) = \text{UNDOT}(\dot\omega_2) \text{ and } \text{DOT}(\dot\omega_1) \leqslant \text{DOT}(\dot\omega_2)\end{gathered}$$

Example 4.5. Here is an example of transition on the dotted words w.r.t. Figure 2: $A\dot{B}ABAB \rightarrow_{\dot\Omega} ABAB\dot{A}B$

Lemma 4.6. *Given* $(\mathcal{M}, \gamma)$ *an o-minimal dynamical system and a finite definable partition* $\mathcal{P}$*, the set of words* Ω_Δ *is a finite set of finite words.*

Proof. We first notice that the number of dynamical types is finite since $|\Delta| \leqslant 2^{|\dot\Omega|}$ and $\dot\Omega$ is finite by Corollary 4.3. Since *being of dynamical type* W_y for some $W_y \in \Delta$ is definable, this induces a finite definable partition Δ of the space M^{k_2} and so we can use the same argument as in the proof of Lemma 4.2. □

The next result is a trivial consequence of Lemma 4.6 and the definition of $\ddot\Omega_\Delta$.

Corollary 4.7. $\ddot\Omega_\Delta$ *is finite.*

We define also $T_{\ddot\Omega_\Delta}$, a finite transition system on the multidotted words. To mathematically formalize $T_{\ddot\Omega_\Delta}$, we need to introduce three functions: $\text{UNDOT} : \ddot\Omega_\Delta \rightarrow \Omega_\Delta$ gives the word u corresponding to $\ddot{u}$ without dot; $\text{MINDOT} : \ddot\Omega_\Delta \rightarrow \mathbb{N}$ gives the position of the left most dot on $\ddot{u}$ and $\text{MAXDOT} : \ddot\Omega_\Delta \rightarrow \mathbb{N}$ gives the position of the right most dot on $\ddot{u}$.

Given $x \in M^{k_1}$, the set $\mathcal{F}_x^\Delta$ can be described as a finite ordered sequence of intervals $I_0 < I_1 < \cdots < I_k$ with $k < N_S^\Delta$. If we consider a multidotted word $\ddot{u}_{(x,y)}$, constructed from u_x and y on the trajectory Γ_x, let W be the element of Δ such that $y \in W$. Those letters W correspond to some intervals $I_i \in \mathcal{F}_x^\Delta$ such that $\text{MINDOT}(\ddot{u}_{(x,y)}) \leqslant i \leqslant \text{MAXDOT}(\ddot{u}_{(x,y)})$. We can now define $T_{\ddot\Omega_\Delta} = (\ddot\Omega_\Delta, \rightarrow_{\ddot\Omega_\Delta})$:

– the set of states is $\ddot\Omega_\Delta$
– the transition relation $\ddot{u}_1 \rightarrow_{\ddot\Omega_\Delta} \ddot{u}_2$ is defined by: the undotted words $\dot{u}_1$ and $\dot{u}_2$ are equal and the right most dot on $\ddot{u}_2$ is on a righter (or the same) position than the left most dot on $\dot\omega_1$. This can be formalized by:

$$\begin{gathered}\ddot{u}_1 \rightarrow_{\ddot\Omega_\Delta} \ddot{u}_2\\ \Updownarrow \\ \text{UNDOT}(\ddot{u}_1) = \text{UNDOT}(\ddot{u}_2) \quad \text{and} \quad \text{MINDOT}(\ddot{u}_1) \leqslant \text{MAXDOT}(\ddot{u}_2)\end{gathered}$$

Example 4.8. Here is an example of transition on multidotted words w.r.t. Figure 3:

$$\begin{aligned}&W_1\dot{W}_6W_1W_2W_7W_2W_3W_4W_7W_4W_5\dot{W}_6W_5\\ &\qquad\rightarrow_{\ddot\Omega_\Delta} W_1W_6W_1W_2\dot{W}_7W_2W_3W_4\dot{W}_7W_4W_5W_6W_5\end{aligned}$$

4.2 The injective case

The first situation that we will be interested in is the following: we suppose that there is a unique trajectory going through each point of the space M^{k_2} and that each trajectory does not self-intersect. In this situation, given $y \in M^{k_2}$, there exists a unique $(x,t) \in M^{k_1} \times M$ such that $\gamma(x,t) = y$. So the dotted words will encode enough information; precisely we can state the following theorem.

Theorem 4.9. *Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let T_γ be the associated transition system on M^{k_2}, and let $\mathcal{P}$ be a finite definable partition of M^{k_2}. If from every $y \in M^{k_2}$ there exists a unique trajectory, which does not self-intersect, then there exists a finite bisimulation of T_γ that respects $\mathcal{P}$.*

Proof. To prove this theorem, we will show that there exists a bisimulation between the transition systems T_γ and $T_{\dot{\Omega}}$. Let us first recall that $T_{\dot{\Omega}}$ is a finite transition system by Corollary 4.3. We define a binary relation $\sim \subseteq M^{k_2} \times \dot{\Omega}$ as follow:

$$y \sim \dot{\omega} \quad \Leftrightarrow \quad \exists (x,t) \in M^{k_1} \times M,\ \big(\dot{\omega}_{(x,t)} = \dot{\omega} \text{ and } \gamma(x,t) = y\big).$$

Under the assumption of Theorem 4.9, given $y \in M^{k_2}$, there exists a unique $(x,t) \in M^{k_1} \times M$ such that $\gamma(x,t) = y$.

We begin by showing that $T_{\dot{\Omega}}$ simulates T_γ. Given y_1, $y_2 \in M^{k_2}$ and $\dot{\omega}_1 \in \dot{\Omega}$ such that $y_1 \to_\gamma y_2$ and $y_1 \sim \dot{\omega}_1$, we have to find $\dot{\omega}_2 \in \dot{\Omega}$ such that $\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2$ and $y_2 \sim \dot{\omega}_2$. By definition of $\to_\gamma$, there exists $x \in M^{k_1}$ and $t_1 \leqslant t_2 \in M$ such that $\gamma(x,t_1) = y_1$ and $\gamma(x,t_2) = y_2$. Since there exists a unique trajectory going though y_1, we have that $\dot{\omega}_1 = \dot{\omega}_{(x,t_1)}$. We set that $\dot{\omega}_2 = \dot{\omega}_{(x,t_2)}$. We have clearly that $y_2 \sim \dot{\omega}_2$. To prove that $\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2$, we first remark that $\textsc{undot}(\dot{\omega}_1) = \textsc{undot}(\dot{\omega}_2) = \omega_x$. Since $t_1 \leqslant t_2$, we have that $t_1 \in I_i$ an $t_2 \in I_j$, for some I_i, $I_j \in \mathcal{F}_x$, with $i \leqslant j$, so $\textsc{dot}(\dot{\omega}_1) \leqslant \textsc{dot}(\dot{\omega}_2)$.

Conversely let us prove that T_γ simulates $T_{\dot{\Omega}}$. Given $y_1 \in M^{k_2}$ and $\dot{\omega}_1$, $\dot{\omega}_2 \in \dot{\Omega}$ such that $\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2$ and $\dot{\omega}_1 \sim^{-1} y_1$, we have to find $y_2 \in M^{k_2}$ such that $y_1 \to_\gamma y_2$ and $\dot{\omega}_2 \sim^{-1} y_2$. Since $\dot{\omega}_1 \in \dot{\Omega}$, there exists $(x,t_1) \in M^{k_1} \times M$ such that $\dot{\omega}_1 = \dot{\omega}_{(x,t_1)}$ and $t_1 \in I_i$ for some $I_i \in \mathcal{F}_x$. We can find $I_j \in \mathcal{F}_x$ with $I_i \leqslant I_j$ such that if we add the dot corresponding to I_j on ω_x we obtain $\dot{\omega}_2$. We take $t_2 \in I_j$, and set $y_2 = \gamma(x,t_2)$, we clearly have that $y_1 \to_\gamma y_2$ and $\dot{\omega}_2 \sim^{-1} y_2$.

We have proved that $\sim \subseteq M^{k_2} \times \dot{\Omega}$ is a bisimulation. Since a unique word is associated with each $y \in M^{k_2}$, it is a functional bisimulation. By Remark 2.6, $\sim$ induces a finite bisimulation on $M^{k_2} \times M^{k_2}$ given by $\mathrm{Ker}(\sim)$; moreover, by definition of $\sim$ and $\mathrm{Ker}(\sim)$, this bisimulation is an equivalence relation which respects $\mathcal{P}$. $\square$

Remarks 4.10. By Theorem 4.9, we can recover the main result of [LPS, Theorem (4.3), p.11]. First by the argument that decouples the continuous and discrete components of the hybrid system given in [LPS, p. 6], we only need to prove that there exists a finite bisimulation on each location which respects the finite partition given by the resets, guards which are definable in the o-minimal structure we are working in, by assumption. In the assumptions of [LPS, Theorem (4.3), p.11], $\gamma(.,.)$ is the definable flow of a vector field $F:\mathbb{R}^n \to \mathbb{R}^n$ which does not depend of the time [LPS, p. 12], so in particular $\gamma(x,.)$ is injective [LPS, p. 13], therefore we are in situation of Theorem 4.9.

We can remark that in the proof of Theorem 4.9, we only use the Uniform Finiteness Theorem 2.10. In the proof of [LPS] *Cell decomposition* and the fact that *connectedness and arc-connectedness are equivalent* are used. If we were interested in bisimulations on the space-time, the proof of Theorem 4.9 shows that there always exists a finite bisimulation of $(\mathcal{M},\gamma)$ that respects $\mathcal{P}$.

4.3 Self-intersecting curves

In this section, we consider a second situation: an o-minimal dynamical system such that with each point of the space is associated a unique trajectory but the trajectory can self-intersect (Figure 3 is an example of this situation). Let us remark that the self intersection set can be an arbitrary definable set.

In this context, given $y \in M^{k_2}$, there are different $(x,t) \in M^{k_1} \times M$ such that $\gamma(x,t) = y$. So the simple dotted words are no longer sufficient to encode the whole information. We will need the multidotted words.

Theorem 4.11. *Let $(\mathcal{M},\gamma)$ be an o-minimal dynamical system, let T_γ be the associated transition system on M^{k_2}, and let $\mathcal{P}$ be a finite definable partition of M^{k_2}. If from every $y \in M^{k_2}$ there exists a unique trajectory then there exists a finite bisimulation of T_γ that respects $\mathcal{P}$.*

Proof. As in the proof of Theorem 4.9, we show that there exists a bisimulation between T_γ and $T_{\ddot{\Omega}_\Delta}$, which is a finite transition system by Corollary 4.7. We define a binary relation $\sim\,\subseteq M^{k_2} \times \ddot{\Omega}_\Delta$ in the following way:

$$y \sim \ddot{u} \quad\Leftrightarrow\quad \exists (x,t) \in M^{k_1} \times M,\ \big(\ddot{u}_{(x,y)} = \ddot{u} \text{ and } \gamma(x,t) = y\big).$$

Let us recall that there exists a unique multidotted word associated with each y (see last paragraph of Section 3).

First, we prove that $T_{\ddot{\Omega}_\Delta}$ simulates T_γ. Given $y_1,\, y_2 \in M^{k_2}$ and $\ddot{u}_1 \in \ddot{\Omega}_\Delta$ such that $y_1 \to_\gamma y_2$ and $y_1 \sim \ddot{u}_1$, we have to find $\ddot{u}_2 \in \ddot{\Omega}_\Delta$ such that $\ddot{u}_1 \to_{\ddot{\Omega}_\Delta} \ddot{u}_2$ and $y_2 \sim \ddot{u}_2$. By definition of $\to_\gamma$, there exists $x \in M^{k_1}$ and $t_1 \leqslant t_2 \in M$ such that $\gamma(x,t_1) = y_1$ and $\gamma(x,t_2) = y_2$. Since there is a unique trajectory going through y_1, we have that $\ddot{u}_1 = \ddot{u}_{(x,y_1)}$. By choosing $\ddot{u}_2 = \ddot{u}_{(x,y_2)}$, we have clearly that $y_2 \sim \ddot{u}_2$. Moreover we have that $\textsc{undot}(\ddot{u}_1) = \textsc{undot}(\ddot{u}_2)$. Since $t_1 \leqslant t_2$, $t_1 \in I_i$ and $t_2 \in I_j$ for some $I_i, I_j \in \mathcal{F}_x^\Delta$ with $i \leqslant j$ and so $\textsc{mindot}(\ddot{u}_1) \leqslant i \leqslant j \leqslant \textsc{maxdot}(\ddot{u}_2)$.

Conversely let us prove that T_γ simulates $T_{\ddot{\Omega}_\Delta}$. Given $y_1 \in M^{k_2}$ and $\ddot{u}_1$, $\ddot{u}_2 \in \ddot{\Omega}_\Delta$ such that $\ddot{u}_1 \to_{\ddot{\Omega}_\Delta} \ddot{u}_2$ and $\ddot{u}_1 \sim^{-1} y_1$, we have to find $y_2 \in M^{k_2}$ such that $y_1 \to_\gamma y_2$ and $\ddot{u}_2 \sim^{-1} y_2$. Since $\ddot{u}_1 \sim^{-1} y_1$, we have that $\ddot{u}_1 = \ddot{u}_{(x,y_1)}$ for some $x \in M^{k_1}$ and $y_1 = \gamma(x,t_1)$ for some $t_1 \in M$. We take $t_0 \in I_{\textsc{mindot}(\ddot{u}_1)} \in \mathcal{F}_x^\Delta$ such that $\gamma(x,t_0) = y_1$. Since $\textsc{mindot}(\ddot{u}_1) \leqslant \textsc{maxdot}(\ddot{u}_2)$, it is always possible to choose $t_2 \in I_{\text{MAXDOT}(\ddot{u}_2)} \in \mathcal{F}_x^{\Delta}$ such that $t_0 \leqslant t_2$. We now set $y_2 = \gamma(x, t_2)$. All this construction respects the rules given for the composition of transitions (see the observation mentioned after Definition 2.13) .

We have proved that $\sim \subseteq M^{k_2} \times \ddot{\Omega}_\Delta$ is a bisimulation. Since there exists a unique multidotted word associated with each y, it is a functional bisimulation. By Remark 2.6, $\sim$ induces a finite bisimulation on $M^{k_2} \times M^{k_2}$ given by $\mathrm{Ker}(\sim)$. Moreover this bisimulation is an equivalence and clearly respects Δ, and so $\mathcal{P}$ since Δ is finer than $\mathcal{P}$. □

Remark 4.12. In Theorems 4.9 and 4.11 the assumption that "there exists a unique trajectory going through $y \in M^{k_2}$" can be relaxed by requiring the uniqueness of the (multi)dotted word associated with each point y, as it can be seen by slight modifications of the proofs.

Remark 4.13. If we look at a different transition system on $(\mathcal{M}, \gamma)$ where the set of states Q is given by $M^{k_1} \times M^{k_2}$ and the transition relation $(x_1, y_1) \rightarrow_{\tilde{\gamma}} (x_2, y_2)$ is defined by: $(x_1 = x_2) \wedge \exists t_1 \leqslant t_2 \in M \ \ \big((\gamma(x_1, t_1) = y_1) \wedge (\gamma(x_2, t_2) = y_2)\big)$, the proof of Theorem 4.11 shows that any such o-minimal dynamical system admits a finite bisimulation which respects a given finite definable partition $\mathcal{P}$.

4.4 Counter-example on the torus

We proved that in particular situations (see Sections 4.2 and 4.3) we can obtain a finite bisimulation of the space w.r.t. a given partition. Unfortunately, we cannot hope to extend this result to any o-minimal dynamical system. This will be illustrated in this section by the study of a dynamical system on the torus. To establish the lack of finite bisimulation w.r.t. a given partition, it is sufficient to show the non-termination of the *bisimulation algorithm* appearing in [BFH,Hen95].

Given a transition system $T = (Q, \Sigma, \rightarrow)$ and $\mathcal{P}$ a finite transition of Q, the bisimulation algorithm iterates the computation of predecessors,[5] let us recall this algorithm:

Initialization: $Q/\!\sim\, := \mathcal{P}$
While $\exists P, P' \in Q/\!\sim$ such that $\varnothing \neq P \cap \mathrm{Pre}(P') \neq P$
  Set $P_1 = P \cap \mathrm{Pre}(P')$ and $P_2 = P \setminus \mathrm{Pre}(P')$
  Refine $Q/\!\sim\, := (Q/\!\sim \setminus \{P\}) \cup \{P_1, P_2\}$
End while

We work in the structure $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, 0, 1, \sin\restriction_{[0,4\pi]} \rangle$ which is o-minimal, as it can be seen from [vdD96]. A torus is a definable set of $\mathcal{M}$ since it is given by the following equations :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix} =: \varphi(u, v)$$

[5] Given T a transition system and $q \in Q$, the set of predecessors of q, denoted $\mathrm{Pre}(q)$, is defined by $\mathrm{Pre}(q) = \{q' \in Q | \exists a \in \Sigma,\ q' \xrightarrow{a} q\}$.

with $u, v \in [0, 2\pi[$.

We define a dynamics $\gamma : [0, 2\pi[^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ on the torus : for all $t \in [0, 2\pi[$,

$$\gamma(u_0, v_0, a, t) = \begin{cases} \varphi(u_0 + t, v_0 + t) & \text{if } a = 1, \\ \varphi(u_0 + t, v_0 + 2t) & \text{if } a = 2, \\ \varphi(u_0, v_0) & \text{otherwise.} \end{cases}$$

The dynamics is definable in $\mathcal{M}$, so $(\mathcal{M}, \gamma)$ is an o-minimal dynamical system and the transition relation is the one given in Definition 2.13. The torus can be represented by a *square of length* 2π *where the opposite sides are identified*. We adopt this description in order to study the dynamics on the torus. Therefore the trajectories on the torus are given by pieces of lines on the square. We note that trajectories are closed curves. In this context, the equation of the dynamics $\gamma : [0, 2\pi[^2 \times \mathbb{R} \times \mathbb{R} \to [0, 2\pi[^2$ becomes :

$$\gamma(u_0, v_0, a, t) = \begin{cases} (u_0 + t, v_0 + t) \bmod 2\pi & \text{if } a = 1 \text{ and } t \in [0, 2\pi[, \\ (u_0 + t, v_0 + 2t) \bmod 2\pi & \text{if } a = 2 \text{ and } t \in [0, 2\pi[, \\ (u_0, v_0) & \text{otherwise.} \end{cases}$$

Given a point $(u_0, v_0) \in [0, 2\pi[^2$, three behaviors of the dynamics are possible: it can follow a line of slope 1 or 2, or it can remain stationary (see Figure 4).

We consider the following initial partition of the square $\mathcal{P} = \{P_0, P_1, P_2, P_3\}$ where:

$$P_0 = \big\{(0,0)\big\}, \qquad P_1 = \big\{(0, v) \bigm| v \in]0, 2\pi[\big\},$$
$$P_2 = \big\{(u, 0) \bigm| u \in]0, 2\pi[\big\}, \qquad P_3 = [0, 2\pi[^2 \backslash (P_0 \cup P_1 \cup P_2).$$

This induces a definable (in the sense of the structure $\mathcal{M}$) partition of the torus.

We will now apply the bisimulation algorithm and show that it does not terminate when we take this initial partition.

To formalize the non-termination of the algorithm we need to compute the set of predecessors of a given point (y_1, y_2) of the space. By the previous observation, we have that :

$$\begin{aligned} \mathrm{Pre}(y_1, y_2) = \big\{(y_1 + t, y_2 + t) \bmod 2\pi \bigm| t \in [0, 2\pi]\big\} \cup \\ \big\{(y_1 + t, y_2 + 2t) \bmod 2\pi \bigm| t \in [0, 2\pi]\big\} \end{aligned}$$

We observe that the sets $\mathrm{Pre}(y_1, y_2) \cap P_1$ and $\mathrm{Pre}(y_1, y_2) \cap P_2$ are finite. The iterations of the **While** instruction of the bisimulation algorithm isolates[6] an infinite number of points. The next lemma formalizes this :

Lemma 4.14. *For each* $n \geqslant 0$*, there exists odd integers* k, k' *such that the algorithm isolates the points* $(k\pi/2^n, 0)$ *and* $(0, k'\pi/2^n)$*.*

[6] By "isolating a point q" we mean that the algorithm has constructed $P \in Q/{\sim}$ such that $P = \{q\}$.

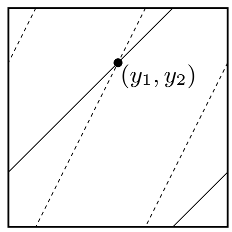

Fig. 4. $\mathrm{Pre}(y_1, y_2)$

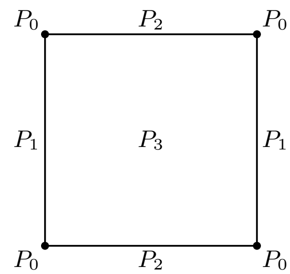

Fig. 5. The partition

Proof. We proceed by induction on n.

(1) In the case $n = 0$, we isolate $(\pi, 0)$ starting from $\{(0,0)\}$ and then we isolate $(0, \pi)$ by using the new isolated point $\{(\pi, 0)\}$, as shown on Figure 6.

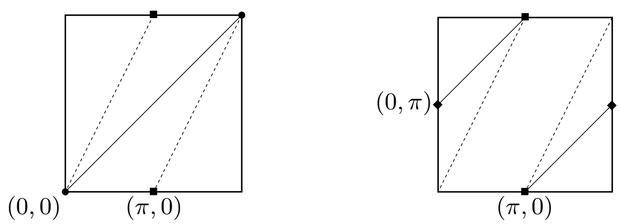

Fig. 6. Case $n = 0$

(2) Suppose now that we have isolated the points $(0, k\pi/2^n)$ and $(k'\pi/2^n, 0)$ with k, k' satisfying the required conditions, we show how to obtain the new isolated points:

- Consider first the intersection $A = \mathrm{Pre}(0, k\pi/2^n) \cap (X \times \{0\})$ where $X \times \{0\}$ is an element of a sub-partition of P_2; by the characterization of the predecessors above, we have that

$$\begin{aligned}
&(x, 0) \in A \\
&\Leftrightarrow \exists t \in [0, 2\pi],\ \big(t = x \bmod 2\pi \ \text{ and } \ k\pi/2^n = -t \bmod 2\pi\big) \text{ or} \\
&\qquad\qquad\qquad \big(t = x \bmod 2\pi \ \text{ and } \ k\pi/2^n = -2t \bmod 2\pi\big) \\
&\Leftrightarrow x = 2\pi - k\pi/2^n \ \text{ or } \ x = 2\pi - k\pi/2^{n+1}
\end{aligned}$$

  The second part of this disjunction permits to isolate the new point $(2^{n+2} - k)\pi/2^{n+1}$ with $2^{n+2} - k = 1 \pmod 2$.
- Using the same argument when considering $B = \mathrm{Pre}(k'n/2^n) \cap (\{0\} \times Y)$, we obtain the second isolated point of the lemma. □

Remark 4.15. Maybe the discussion above does not enlighten where the assumptions of Theorems 4.9 and 4.11 are not satisfied by the dynamics. In fact there are points y of the torus with several trajectories going through and even several multidotted words associated with y. For example the multidotted words $\dot{P}_0$, $\dot{P}_0 P_3$ and $\dot{P}_0 P_3 P_2$ are associated with $(0, 0)$.

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