# A thesis with Yves. . .

## Or hairy black holes, boson stars and non-minimal couplings from Einstein to teleparallel gravity

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# Plan


1 Introduction
- Context
- No hair theorem(s)

2 Horndeski gravity

3 Hairy black holes, boson stars and non-minimal coupling to curvature invariants
- Results previously known
- Original results

4 Teleparallel gravity

5 Scalarized Black Holes in Teleparallel Gravity

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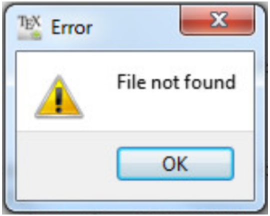


Y. Brihaye[†], L. Ducobu [†]

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Master thesis

1. Black Holes with Scalar Hairs in Einstein-Gauss-Bonnet Gravity

PhD thesis

2. Nutty black holes in galileon scalar-tensor gravity
3. Spinning-Charged-Hairy Black Holes in 5-d Einstein gravity
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■ Today's talk

# Introduction : *Why* should we modify general relativity ?

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Despite consequential successes . . .

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- Allow to explain many phenomena :
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**Not all of them** reduces to quantum correction problems !

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- Challenge
- Keep

# Introduction : *How* should we modify general relativity ?

One interesting way to modify GR is to consider that the unrated phenomena are due to unknown degrees of freedom (that can be interpreted as new particles or as a new component in the description of gravity).

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- Important element of many models :
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  - ...
- Also experimentally motivated since the Brout-Englert-Higgs boson's discovery (CERN 2012)

# Introduction : Why not considering the simplest case ?

Why not just using $\mathscr{L}_{\mathsf{EKG}} = \kappa \left(R - 2\Lambda\right) - \frac{1}{2}\nabla_{\mu}\phi\nabla^{\mu}\phi - V(\phi)$ ?

# Introduction : Why not considering the simplest case ?

## No Scalar-Hair Theorem (*Schematically*)

*Consider an asymptotically flat black hole spacetime*

**Hypothesis 1 : (Symmetries of spacetime)**

**Hypothesis 2 : (Symmetries of the scalar field)**

**Hypothesis 3 : (Coupling condition)**

**Hypothesis 4 : (Energetic condition)**

***Then***, *the scalar field must be trivial :* $\phi\left(x^{\mu}\right) = \phi_0, \forall x^{\mu}$.

See [Herdeiro and Radu, 2015] for a review.

# Introduction : Why not considering the simplest case ?

## No Scalar-Hair Theorem (*Example; due to Bekenstein*)

*Consider an asymptotically flat black hole spacetime*

**Hypothesis 1 : (Symmetries of spacetime)**
The spacetime is stationary
**Hypothesis 2 : (Symmetries of the scalar field)**
The scalar field shares the spacetime symmetries.
**Hypothesis 3 : (Coupling condition)**

$$S = \int_{\mathcal{M}} \left[ F(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \dots) - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right] \sqrt{-g} \, \mathrm{d}^n x$$

**Hypothesis 4 : (Energetic condition)**
Ex : $\phi V'(\phi) \geq 0 \;\; \forall \phi$, with $V'(\phi) = \mathrm{d}V/\mathrm{d}\phi$, & $\phi V'(\phi) = 0$ for some discrete values of $\phi$, say $\phi_i$.

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***Then**, the scalar field must be trivial :* $\phi(x^\mu) = \phi_0, \forall x^\mu$.

Note : In general, the proof makes **no use** of the Einstein's equations. It just uses the scalar field equation defined thanks to hypothesis 3.

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$$S = \int_{\mathcal{M}} \mathscr{L} \sqrt{-g}\, \mathrm{d}^4 x$$

# Horndeski

$$\begin{aligned}\mathscr{L} =& K(\phi,\rho) - G_3(\phi,\rho)\Box\phi + G_4(\phi,\rho)R + G_{4,\rho}(\phi,\rho)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ & + G_5(\phi,\rho)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ & - \frac{1}{6}G_{5,\rho}(\phi,\rho)\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right],\end{aligned}$$

where

$$\rho = \nabla_\mu\phi\nabla^\mu\phi,$$

and where the functions $G_i(\phi,\rho)$ $(i \in \{3,4,5\})$ & $K(\phi,\rho)$ are **arbitrary** functions.

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With $G_3 = 0 = G_5, G_4 = \kappa = c^4/16\pi\mathscr{G}$ and $K(\phi,\rho) = -\frac{1}{2}\rho - V(\phi) - 2\kappa\Lambda$, one gets

$$\mathscr{L}_{\text{EKG}} = \kappa\left(R - 2\Lambda\right) - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)$$

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If $G_i(\phi,\rho) = G_i(\rho), \forall i \in \{3,4,5\}$ and $K(\phi,\rho) = K(\rho)$, the system possesses an invariance under $\phi \to \phi + c$ (shift-symmetry), with $c \in \mathbb{R}$, and the field equation for $\phi$ reduces to a conservation law (Nœther) :

$$\nabla_\mu J^\mu = 0.$$

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# Coupling to the Gauss-Bonnet invariant

A first interesting subclass of the Horndeski Lagrangian is given by

$$\mathscr{L} = R - \frac{1}{2}\, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \mathcal{F}(\phi)\mathscr{L}_{\mathsf{GB}}, \tag{1}$$

where

$$\mathscr{L}_{\mathsf{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$$

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- (1) can be obtained from the Horndeski Lagrangian via specific choice of the arbitrary functions and some integration by parts. This has been established in [Kobayashi et al., 2011].
- In 4D, it is well known that $\mathscr{L}_{\text{GB}} = \nabla_\mu\mathcal{G}^\mu$.

# Coupling to the Gauss-Bonnet invariant

→ An interesting feature of this model is that the curvature of spacetime will source the scalar field and (almost certainly) force it to be non-trivial :

$$\Box\phi = -\mathcal{F}'(\phi)\mathscr{L}_{\mathsf{GB}}.$$

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→ In the following, we will focus our review on asymptotically flat spherically symmetric black hole solutions.

# Ansatz

Since we will focus on spherically symmetric spacetime, we will assume to work within a coordinate system $\{\, x^\mu \,\} = \{\, t, r, \theta, \varphi \,\}$ in which the metric takes the form

$$\mathrm{d}s^2 = -N(r)\sigma^2(r)\mathrm{d}t^2 + \frac{1}{N(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \quad .$$

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For simplicity, we will further assume our real scalar field to also respect spherical symmetry; that is

$$\phi\left(x^\mu\right) = \phi(r).$$

# Results previously known

## Linear coupling ($\mathcal{F}(\phi) = \gamma_1 \phi$)

The first explicit construction of asymptotically flat, spherically symmetric black hole solutions presenting a shift-symmetry in the context of Horndeski gravity has been provided in
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- Regularity of the scalar field derivative at the event horizon, $\phi'(r_h)$, require to fix $\phi'(r_h)$ as the solution of a quadratic polynomial equation.
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- This fixes a maximal value for the coupling constant $\gamma_1 \leq \gamma_{1,\max}$.
- Scalarized solutions can be numerically constructed for all $\gamma_1 \in [0, \gamma_{1,\max}]$.
- There are no excited solutions.

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## Quadratic coupling ($\mathcal{F}(\phi) = \gamma_2 \phi^2$)

Asymptotically flat, spherically symmetric black hole solutions have also been studied in [Silva et al., 2018] under the assumption of a quadratic non-minimal coupling to the Gauss-Bonnet invariant.

In this case, the spectrum of solutions is drastically different from the former case :

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  But this time, one should also have that $\phi(r_h) \neq 0$ and $\gamma_2 \neq 0$.
- Solutions can only be found if $\gamma_2$ lies in a band $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,\max}]$ with $\gamma_{2,c} > 0$.
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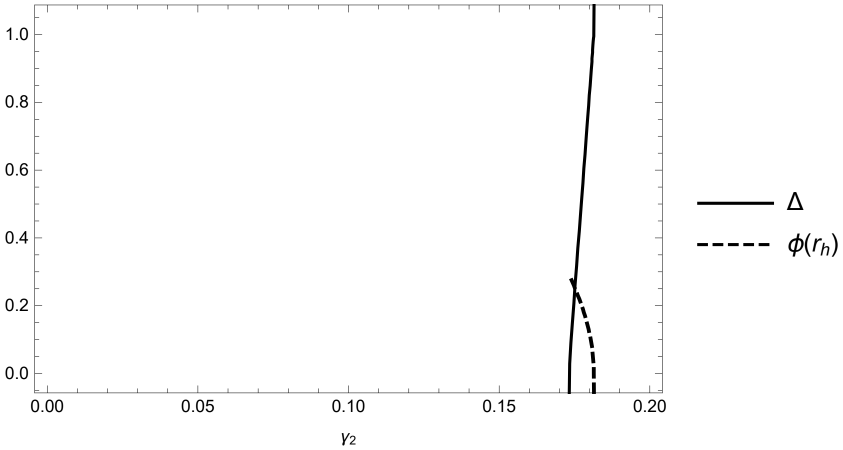
- Excited solutions exist.

# Results previously known

## Quadratic coupling ($\mathcal{F}(\phi) = \gamma_2 \phi^2$)

Schematically, the existence of solutions is then limited by the following pattern



# Generic linear + quadratic coupling ($\mathcal{F}(\phi) = \gamma_1\phi + \gamma_2\phi^2$)

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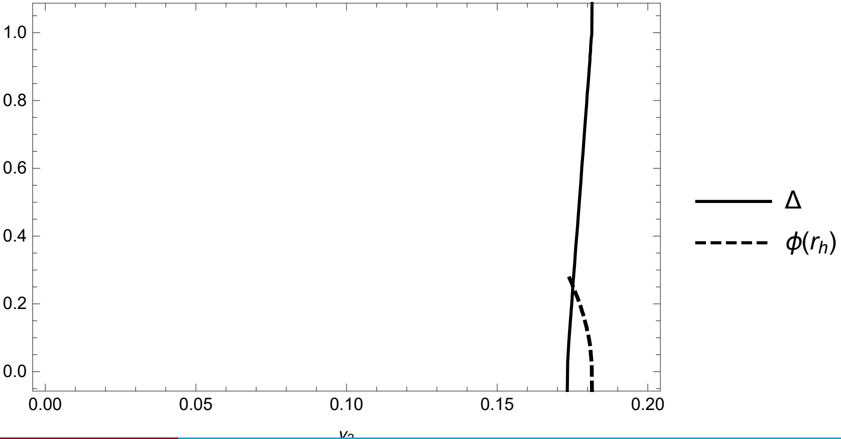

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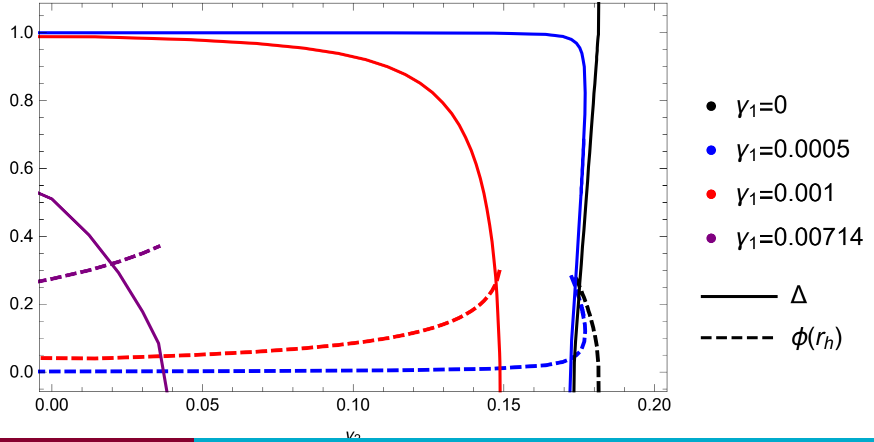

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- Context
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# A fresh look at General Relativity

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Nevertheless, from the point of view of differential geometry, connection and metric serve (*a priori*) independent purposes.

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  - $\hookrightarrow$ Tangent vectors $T_p\mathcal{M}$, Cotangent vectors $T_p^*\mathcal{M}$, Generic tensors $T_p\mathcal{M}^{(m,r)}$, Vector fields $\Gamma(T\mathcal{M})$, Coordinate basis $\{ \partial_\mu \}$ & $\{ \mathrm{d}x^\mu \}$, $\cdots$

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  - ↪ Causal structure : time-, space-, light-like vectors
  - ↪ Ability to lower and raise indices (musical isomorphisms), Tetrads $\boldsymbol{g}\left(\vec{e}_{(a)}, \vec{e}_{(b)}\right) = \eta_{ab}$ with $(\eta_{ab}) = \mathrm{diag}\,(+,-,-,-)$, Line element, $\cdots$
  - ↪ Local Lorentz transformations $\Lambda(p) \in \mathrm{O}(1,3)$.

# A fresh look at General Relativity

## "Interactions" between the geometric layers

If we have both a linear connection ${\omega^a}_b$ and a metric $\boldsymbol{g}$ defined on our manifold $\mathcal{M}$, we can characterise their compatibility by means of the non-metricity tensor

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Given a metric $\boldsymbol{g}$, there is a unique torsion-free $(\mathbf{T} \equiv 0)$ and metric-compatible $(\mathbf{Q} \equiv 0)$ connection :

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Actually, any linear connection can be related to the Levi-Civita connection via its disformation tensor $\mathbf{D}[\mathbf{Q}, \boldsymbol{g}]$ and contorsion tensor $\mathbf{K}[\mathbf{T}, \boldsymbol{g}]$,

$$\omega^b_{ac} = \overset{\circ}{\omega}{}^b_{ac} + {D^b}_{ac} + {K^b}_{ac},$$

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## Trinity of gravity

From this relation, one can get (for the Ricci scalar)

$$\overset{\circ}{R} = R - 2\left(D^{a}{}_{k[a|}K^{kb}{}_{|b]} + K^{a}{}_{k[a|}D^{kb}{}_{|b]}\right) - 2\left(D^{a}{}_{k[a|}D^{kb}{}_{|b]} + \overset{\circ}{\nabla}_{[a|}D^{ab}{}_{|b]}\right) - 2\left(K^{a}{}_{k[a|}K^{kb}{}_{|b]} + \overset{\circ}{\nabla}_{[a|}K^{ab}{}_{|b]}\right).$$

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# Teleparallel Equivalent of General Relativity

In the following, we will focus our discussion on TEGR (and its extensions by a scalar field).
In this case the connection – called a Weitzenböck connection – is such that $\mathbf{R} \equiv 0$ and $\mathbf{Q} \equiv 0$.

A manifold $\mathcal{M}$ equipped with a Lorentzian metric $\boldsymbol{g}$ and a Weitzenböck connection ${\omega^a}_b$ is called a Weitzenböck spacetime.

# Teleparallel Equivalent of General Relativity

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On a Weitzenböck spacetime, the geometry can be entirely specified (locally) by means of a couple $\left(\{\,\vec{e}_{(a)}\,\}\,,\Lambda(p)\right)$, where

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where the tetrad $\{\,\vec{e}_{(a)}\,\}$ is written as $\vec{e}_{(a)}=e_a{}^{\mu}\partial_\mu$ and $e_a{}^{\mu}e^a{}_{\nu}=\delta^{\mu}_{\nu}$, $e_a{}^{\mu}e^b{}_{\mu}=\delta^b_a$ and $(\eta_{ab})=\mathrm{diag}(+,-,-,-)$.

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2. $\Lambda(p) = \left(\Lambda_a{}^b(p)\right) \in \mathrm{O}(1,3)$ is a local Lorentz transformation which gives the connection coefficients (related to the tetrad $\{\, \vec{e}_{(a)} \,\}$) via

$$\omega^b_{a\mu} = -\Lambda_a{}^l\; \partial_\mu \Lambda^b{}_l.$$

# Teleparallel Equivalent of General Relativity

For a Weitzenböck connection, we thus had that

$$\overset{\circ}{R} = -T + B,$$

where

$$T \coloneqq 2\ K^{a}{}_{k[a|}K^{kb}{}_{|b]} = \frac{1}{4}T_{abc}T^{abc} + \frac{1}{2}T_{abc}T^{cba} - T^{a}{}_{ak}T^{b}{}_{b}{}^{k}$$

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The theory with $\mathscr{L}_{\text{TEGR}} = -T/(2\kappa)$ is called the Teleparallel Equivalent of General Relativity.
This theory provides a theory of gravity dynamically equivalent to GR but formulated in terms of the torsion of a Weitzenböck connection. See [Bahamonde et al., 2021] for a review.

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# Action

Such a “reformulation” of GR suggests new types of couplings. See [Hohmann, 2018a, Hohmann and Pfeifer, 2018, Hohmann, 2018b].

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$$\mathscr{L} = \frac{1}{2\kappa}\overset{\circ}{R} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - \mathcal{V}(\psi)$$

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In the following, we will consider solutions obtained within this ansatz for the theory whose Lagrangian density is given by

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In this case, the choice of an ansatz requires to fix the form of the scalar field $\psi$ and of a tetrad $\{ \vec{e}_{(a)} \}$ by means of the components $e^a{}_\mu$. In spherical symmetry two forms are possible for the tetrad.

# Ansatz

The first tetrad is real and it is described by

$$e^{(1)a}{}_{\mu} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & CA^{-1}\cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ 0 & CA^{-1}\sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ 0 & CA^{-1}\cos\theta & -r\sin\theta & 0 \end{pmatrix},$$

whereas the second one is complex and is given by

$$e^{(2)a}{}_{\mu} = \begin{pmatrix} 0 & \frac{iC}{A} & 0 & 0 \\ iA\sin\theta\cos\phi & 0 & -r\sin\phi & -r\sin\theta\cos\theta\cos\phi \\ iA\sin\theta\sin\phi & 0 & r\cos\phi & -r\sin\theta\cos\theta\sin\phi \\ iA\cos\theta & 0 & 0 & r\sin^2\theta \end{pmatrix},$$

where $A = A(r)$ and $C = C(r)$. In both cases, the metric takes the usual form

$$\mathrm{d}s^2 = A(r)^2\mathrm{d}t^2 - \frac{C(r)^2}{A(r)^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\varphi^2\right).$$

# Ansatz

For the scalar field, we simply chose $\psi(t, r, \theta, \varphi) = \psi(r)$.

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  1. $\mathcal{A}(\psi) = -\beta\psi^2/8\,, \quad \mathcal{V}(\psi) = 0\,,$

$$\mathrm{d}s^2 = \left(1 - \frac{K}{r}\right)^2 \mathrm{d}t^2 - \left(1 - \frac{K}{r}\right)^{-2} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2\,, \quad \psi(r) = -\frac{2\psi_0\sqrt{r}}{K\sqrt{r-K}}\,.$$

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2. $\mathcal{A}(\psi) = 3\beta\psi^2/8\,, \quad \mathcal{V}(\psi) = 0\,,$

$$\mathrm{d}s^2 = \left(2 - \frac{r}{2K} + \frac{\sqrt{r(r-4K)}}{2K}\right)^2 \mathrm{d}t^2 - \left(2 - \frac{r}{2K} + \frac{\sqrt{r(r-4K)}}{2K}\right)^{-2} \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,,$$

$$\psi(r) = \frac{\psi_0\left(\sqrt{r(r-4K)} - 4K + r\right)\sqrt[4]{\sqrt{r(r-4K)} - 2K + r}}{3Kr^{3/4}\sqrt{r-4K}}\,.$$

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## No Scalar-Hair Theorem

*Consider a scalar-torsion theory of gravity defined by the Lagrangian density*

$$\mathscr{L} = -\mathcal{A}(\psi)T - \tilde{\mathcal{C}}(\psi)B - \beta g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - \mathcal{V}(\psi),$$

*the spherically symmetric the tetrads (displayed above) and a scalar field* $\psi = \psi(r)$. ***There exist no spherically symmetric asymptotically flat scalarized black holes for the following couplings and potentials** :*

1. $\mathcal{A} = \alpha\psi^m$, $\tilde{\mathcal{C}} = 0$ *and* $\frac{2}{\beta(m-2)}\left(2m\mathcal{V} - \psi\mathcal{V}'\right) \leq 0$;
2. $\mathcal{A} = \alpha\psi^2$, $\tilde{\mathcal{C}} = \frac{c_1}{2}\psi^2 + c_2$ *and either* $\psi\mathcal{V}' > 4\mathcal{V}$ *or* $\psi\mathcal{V}' < 4\mathcal{V}$;
3. $\mathcal{A} = \alpha$, $\tilde{\mathcal{C}} = c_1\ln(\psi) + c_2$ *and* $\frac{\psi\mathcal{V}'}{\beta} \leq 0$;
4. $\mathcal{A} = \alpha$, $\tilde{\mathcal{C}} = \frac{\gamma}{m+1}\psi^{m+1}$ *and* $\frac{1}{\beta}\left(\psi\mathcal{V}' - (m+1)\gamma\psi^m T^r\psi'\right) \leq 0$ *or* $\frac{(m+1)}{m-1}\frac{1}{\beta}\left(\alpha\overset{\circ}{R} + \kappa^2(\psi\mathcal{V}' - 4\mathcal{V})\right) \leq 0$.

# Plan

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- Context
- No hair theorem(s)

2 Horndeski gravity

3 Hairy black holes, boson stars and non-minimal coupling to curvature invariants
- Results previously known
- Original results

4 Teleparallel gravity

5 Scalarized Black Holes in Teleparallel Gravity

6 Conclusion

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- → The present developments of both black hole and gravitational wave astronomy raise the need for as many models as possible to compare with experimental data.
- → For instance, unifying the two directions presented in this talk, it would be interesting to further investigate the formulation of **non-minimal couplings to the Gauss-Bonnet invariant in the context of teleparallel theories of gravity**.

## Conclusion

# Thank you Yves
# for this incredible experience !!!

I hope that we will continue on this track...


*Thank you for your attention !*

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*Annexes*

# (No-scalar-hair theorem) Skeleton of the proof.

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1. Construct a positively defined integral.

$$\int_{\mathfrak{E}} \mathfrak{F}\left(\phi, \nabla_\mu \phi\right) \sqrt{-g}\, \mathrm{d}^4 x \geq 0,$$

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$$\mathcal{E}_\phi \approx 0 \Longrightarrow \int_{\mathfrak{E}} \mathfrak{F}\left(\phi, \nabla_\mu \phi\right) \sqrt{-g}\, \mathrm{d}^4 x \approx 0.$$

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3. Use the form of the integrand to conclude that the scalar field must be trivial.

$$\int_{\mathfrak{E}} \mathfrak{F}\left(\phi, \nabla_\mu \phi\right) \sqrt{-g}\ \mathrm{d}^4 x = 0 \Longrightarrow \phi(x^\mu) = \phi_0, \forall x^\mu \in \mathfrak{E}.$$

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# Horndeski

## Construction : schematically

Let us briefly discuss the steps in the discovery/construction of this Lagrangian density from the (more recent) point of view of Galileon theory.

"*What is the most general theory including a single real scalar field, and giving second order equation ?*"

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1. Figure out how one can avoid higher-order derivatives in the EEL for a Lagrangian density polynomial in the $\partial_\mu \partial_\nu \phi$'s.
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(He directly asked the question of the most general Lagrangian density [in $4D$] presenting at most second-order equations for $g_{\mu\nu}$ and $\phi$)

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↑ ↑ ↑ ↑ ↑

This result is non-trivial.

Even though the generalised Galileon provided the most general Lagrangian density with second order field equation for $\phi$ on flat spacetime there was *a priori* no reasons why its covariant extension should still be the most general possibility on curved spacetime !

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$$\begin{aligned}\mathscr{L} =& K(\phi,\rho) - G_3(\phi,\rho)\Box\phi + G_4(\phi,\rho)R + G_{4,\rho}(\phi,\rho)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ & + G_5(\phi,\rho)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ & - \frac{1}{6}G_{5,\rho}(\phi,\rho)\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right],\end{aligned}$$

where

$$\rho = \nabla_\mu\phi\nabla^\mu\phi,$$

and where the functions $G_i(\phi,\rho)$ $(i \in \{3,4,5\})$ & $K(\phi,\rho)$ are **arbitrary** functions.

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# Results previously known

## Quadratic coupling ($\mathcal{F}(\phi) = \gamma_2 \phi^2$)

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When studying the perturbative regime (on a fixed Schwarzschild background), the equation reduces to an eigenvalue equation

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where $\hat{D}_{|\mathsf{Sch}}$ stands for $\hat{D}$ formulated on Schwarzschild spacetime and $\delta\phi$ the scalar field perturbation.
$\Longrightarrow$ In this limit, $\gamma_2$ corresponds to an eigenvalue of $\hat{D}_{|\mathsf{Sch}}$. This will correspond to the values of $\gamma_{2,\max}$.

# Levi-Civita connection, Disformation and Contorsion tensor

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left( \partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu} \right),$$

$$D^{a}{}_{bc} \coloneqq -\frac{1}{2} g^{ak} \left( Q_{kbc} + Q_{ckb} - Q_{bck} \right),$$

$$K^{a}{}_{bc} \coloneqq -\frac{1}{2} \left( T^{a}{}_{bc} - g_{ck}\ g^{al}\ T^{k}{}_{lb} + g_{bk}\ g^{al}\ T^{k}{}_{cl} \right).$$

# Weitzenböck geometry

In the following, we will focus our discussion on TEGR (and its extensions by a scalar field).
In this case the connection – called a Weitzenböck connection – is such that $\mathbf{R} \equiv 0$ and $\mathbf{Q} \equiv 0$.

A manifold $\mathcal{M}$ equipped with a Lorentzian metric $\boldsymbol{g}$ and a Weitzenböck connection ${\omega^a}_b$ is called a Weitzenböck spacetime.

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From these two conditions, one can get that there exist bases of vector fields $\{ \vec{e}_{(a)} \}$ such that

1. In that basis $\omega^b_{ac} \equiv 0$ *i.e.* $\nabla_{\vec{e}_{(c)}} \vec{e}_{(a)} \equiv 0$ (Weitzenböck basis)
2. $\boldsymbol{g}\left(\vec{e}_{(a)}, \vec{e}_{(b)}\right) = \eta_{ab}$ (tetrad)

Such a basis is called a Weitzenböck tetrad.

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1. Since two tetrads are always related by a local Lorentz transformation $\Lambda(p) = \left(\Lambda_a{}^b(p)\right) \in \mathrm{O}(1,3)$, in a generic tetrad one has

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2. From the definition of a tetrad, one has that

$$g_{\mu\nu} = e^a{}_\mu\, \eta_{ab}\, e^b{}_\nu,$$

where the tetrad $\{\, \vec{e}_{(a)} \,\}$ is written as $\vec{e}_{(a)} = e_a{}^\mu \partial_\mu$ and $e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$, $e_a{}^\mu e^b{}_\mu = \delta^b_a$.

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Consequently, on a Weitzenböck spacetime, the geometry can be entirely specified (locally) by means of a couple $\left(\{ \vec{e}_{(a)} \}, \Lambda(p)\right)$.
It is then also possible to work with Weitzenböck tetrads.

# Digression : Motion of pointwise particle

In GR : [Spacetime : $\left(\mathcal{M}, \boldsymbol{g}, \overset{\circ}{\omega}{}^a{}_b\right)$]

In Teleparallel Gravity : [Spacetime : $(\mathcal{M}, \boldsymbol{g}, \omega^a{}_b)$]

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In GR : [Spacetime : $\left(\mathcal{M}, \boldsymbol{g}, \overset{\circ}{\omega}{}^{a}{}_{b}\right)$]
Particles move along geodesic

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Particle motion $\neq$ geodesic. Gravity acts as a (universal) force.

# Field equations in teleparallel gravity

See [Bahamonde et al., 2021] for a review.
Action of the form $S = S_{\mathsf{g}}[\underline{\theta}^{(a)}, \omega^{a}{}_{b}] + S_{\mathsf{m}}[\underline{\theta}^{(a)}, \psi]$.

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Field equations for matter fields

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Local Lorentz invariance

$+$ Diffeomorphism invariance

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Local Lorentz invariance $\leftrightarrow$ action built from tensorial quantities

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$\Rightarrow$ The field equations for the connection become redundant with the antisymmetric part of the tetrad equations.

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$+$ Diffeomorphism invariance $\leftrightarrow$ invariance under coordinate changes

$$\overset{\circ}{\nabla}_\mu E^{(\mu\nu)} = 0,$$
$$\overset{\circ}{\nabla}_\nu E^{[\mu\nu]} = E^{[\rho\nu]} K_{\rho\nu}{}^{\mu},$$
$$\overset{\circ}{\nabla}_\mu \Theta^{\mu\nu} \approx 0.$$