Half-Positional Objectives Recognized by Deterministic Büchi Automata

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Abstract

A central question in the theory of two-player games over graphs is to understand which objectives are *half-positional*, that is, which are the objectives for which the protagonist does not need memory to implement winning strategies. Objectives for which *both* players do not need memory have already been characterized (both in finite and infinite graphs); however, less is known about half-positional objectives. In particular, no characterization of half-positionality is known for the central class of *ω-regular objectives*.

In this paper, we characterize objectives recognizable by deterministic Büchi automata (a class of ω-regular objectives) that are half-positional, in both finite and infinite graphs. Our characterization consists of three natural conditions linked to the language-theoretic notion of *right congruence*. Furthermore, this characterization yields a polynomial-time algorithm to decide half-positionality of an objective recognized by a given deterministic Büchi automaton.

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1 Introduction

Graph games and reactive synthesis. We study *zero-sum turn-based games on graphs* confronting two players (a protagonist and its opponent). They interact by moving a pebble in turns through the edges of a graph for an infinite amount of time. Each vertex belongs to a player, and the player controlling the current vertex decides on the next state of the game. Edges of the graph are labeled with *colors*, and the interaction of the two players therefore produces an infinite sequence of them. The objective of the game is specified by a subset of infinite sequences of colors, and the protagonist wins if the produced sequence belongs to this set. We are interested in finding a *winning strategy* for the protagonist, that is, a function indicating how the protagonist should move in any situation, guaranteeing the achievement of the objective.

This game-theoretic model is particularly fitted to study the *reactive synthesis problem* [7]: a system (the protagonist) wants to satisfy a specification (the objective) while interacting continuously with its environment (the opponent). The goal is to build a controller for the system satisfying the specification, whenever possible. This comes down to finding a winning strategy for the protagonist in the derived game.

Half-positionality. In order to obtain a controller for the system that is simple to implement, we are interested in finding the simplest possible winning strategy. Here, we focus on the amount of information that winning strategies have to remember. The simplest strategies are then arguably *positional* (also called *memoryless*) strategies, which do not remember anything about the past and base their decisions solely on the current state of the game. We intend to understand for which objectives positional strategies suffice for the protagonist to play optimally (i.e., to win whenever it is possible) — we call these objectives *half-positional*. We distinguish half-positionality from *bipositionality* (or *memoryless-determinacy*), which refers to objectives for which positional strategies suffice to play optimally for *both* players.

Many natural objectives have been shown to be bipositional over games on finite and sometimes infinite graphs: e.g., discounted sum [59], mean-payoff [29], parity [30], total payoff [33], energy [9], or average-energy games [11]. Bipositionality can be established using general criteria and characterizations, over games on both finite [33, 34, 3] and infinite [27] graphs. Yet, there exist many objectives and combinations thereof for which one player, but not both, has positional optimal strategies (Rabin conditions [38, 37], mean-payoff parity [23], energy parity [20], some window objectives [21, 14], energy mean-payoff [15]. . .), and to which these results do not apply.

Various attempts have been made to understand common underlying properties of half-positional objectives and provide sufficient conditions [39, 40, 41, 6], but little more was known until the recent work of Ohlmann [52] (discussed below). These conditions are not general enough to prove half-positionality of some very simple objectives, even in the well-studied class of *ω-regular objectives* [6, Lemma 13]. Furthermore, multiple questions concerning half-positionality remain open. For instance, in [41], Kopczyński conjectured that *prefix-independent* half-positional objectives are closed under finite union (this conjecture was recently refuted for games on finite graphs [42], but is still unsolved for games on infinite graphs). Also, Kopczyński showed that given a deterministic parity automaton recognizing a prefix-independent objective W, we can decide if W is half-positional [40]. However, the time complexity of his algorithm is $\mathcal{O}(n^{\mathcal{O}(n^2)})$, where n is the number of states of the automaton. It is unknown whether this can be done in polynomial time, and no algorithm exists in the non-prefix-independent case.

ω-regular objectives and deterministic Büchi automata. A central class of objectives, whose half-positionality is not yet completely understood, is the class of *ω-regular objectives*. There are multiple equivalent definitions for them: they are the objectives defined, e.g., by ω-regular expressions, by non-deterministic Büchi automata [49], and by deterministic parity automata [50]. These objectives coincide with the class of objectives defined by monadic second-order formulas [17], and they encompass linear-time temporal logic (LTL) specifications [54]. Part of their interest is due to the landmark result that finite-state machines are sufficient to implement optimal strategies in ω-regular games [16, 35], implying the decidability of the monadic second-order theory of natural numbers with the successor relation [17] and the decidability of the synthesis problem under LTL specifications [55].

In this paper, we focus on the subclass of ω-regular objectives recognized by *determin-*

istic Büchi automata (DBA), that we call *DBA-recognizable*. DBA-recognizable objectives correspond to the ω-regular objectives that can be written as a countable intersection of open objectives (for the Cantor topology, that is, that are G_δ-sets of the Borel hierarchy); or equivalently, that are the limit of a regular language of finite words [45, 53]. Deciding the winner of a game with a DBA-recognizable objective is doable in polynomial time in the size of the arena and the DBA (by solving a Büchi game on the product of the arena and the DBA [7]).

We now discuss two technical tools at the core of our approach: *universal graphs* and *right congruences*.

Universal graphs. One recent breakthrough in the study of half-positionality is the introduction of *well-monotonic universal graphs*, combinatorial structures that can be used to provide a witness of winning strategies in games with a half-positional objective. Recently, Ohlmann [52] has shown that the existence of a *well-monotonic universal graph* for an objective W exactly characterizes half-positionality (under minor technical assumptions on W). Moreover, under these assumptions, a wide class of algorithms, called *value iteration algorithms*, can be applied to solve any game with a half-positional objective [25, 52].

Although it brings insight on the structure of half-positional objectives, showing half-positionality through the use of universal graphs is not always straightforward, and has not yet been applied in a systematic way to ω-regular objectives.

Right congruence. Given an objective W, the *right congruence* $\sim_W$ *of* W is an equivalence relation on finite words: two finite words w_1 and w_2 are equivalent for $\sim_W$ if for all infinite continuations w, $w_1 w \in W$ if and only if $w_2 w \in W$. There is a natural automaton classifying the equivalence classes of the right congruence, which we refer to as the *prefix-classifier* [60, 48].

In the case of languages of *finite* words, a straightforward adaptation of the right congruence recovers the known Myhill-Nerode congruence. This equivalence relation characterizes the regular languages (a language is regular if and only if its congruence has finitely many equivalence classes), and the prefix-classifier is exactly the smallest deterministic finite automaton recognizing a language — this is the celebrated Myhill-Nerode theorem [51].

Objectives are languages of *infinite* words, for which the situation is not so clear-cut. In particular, an ω-regular objective may not always be recognized by its prefix-classifier along with a natural acceptance condition (Büchi, coBüchi, parity, Muller...) [48, 4].

Contributions. Our main contribution is a *characterization* of half-positionality for DBA-recognizable objectives through a conjunction of three easy-to-check conditions (Theorem 19).

(1) The equivalence classes of the right congruence are *totally* ordered w.r.t. inclusion of their winning continuations.

(2) Whenever the set of winning continuations of a finite word w_1 is a proper subset of the set of winning continuations of a concatenation $w_1 w_2$, the word $w_1(w_2)^\omega$ produced by repeating infinitely often w_2 is winning.

(3) The objective has to be recognizable by a DBA using the structure of its prefix-classifier.

A few examples of simple DBA-recognizable objectives that were not encompassed by previous half-positionality criteria [39, 6] are, e.g., reaching a color twice [6, Lemma 13] and weak parity [62]. We also refer to Example 16, which is half-positional but not bipositional, and whose half-positionality is straightforward using our characterization.

Various corollaries with practical and theoretical interest follow from our characterization.

- We obtain a painless path to show (by checking each of the three conditions) that given a deterministic Büchi automaton, the half-positionality of the objective it recognizes is decidable in time $\mathcal{O}(k^2 \cdot n^4)$, where k is the number of colors and n is the number of states of the DBA (Section 3.3).
- Prefix-independent DBA-recognizable half-positional objectives are exactly the very simple *Büchi conditions*, which consist of all the infinite words seeing infinitely many times some subset of the colors (Proposition 20). In particular, Kopczyński's conjecture trivializes for DBA-recognizable objectives (the union of Büchi conditions is a Büchi condition).
- We obtain a *finite-to-infinite* and *one-to-two-player* lift result (Proposition 23): in order to check that a DBA-recognizable objective is half-positional over arbitrary — possibly two-player and infinite — graphs, it suffices to check the existence of positional optimal strategies over *finite* graphs where all the vertices are controlled by the protagonist.

Technical overview. Condition (1) turns out to be equivalent to earlier properties used to study bipositionality and half-positionality [34, 6] (details in Appendix A). Condition (2), to the best of our knowledge, is a novel condition. Condition (3) has been studied multiple times in the language-theoretic literature, both for itself and for minimization and learning algorithms [60, 58, 48, 4]. As an example, all deterministic *weak* automata (a restriction on DBA) satisfy Condition (3) [60, 4].

Conditions (1) and (2) are necessary for respectively bipositionality and half-positionality of general objectives. Condition (3) is necessary for half-positionality of DBA-recognizable objectives, but not for all (even ω-regular) objectives in general (see Example 18). The proof of its necessity is more involved than for the first two conditions, and will build on automata-theoretic ideas introduced for good-for-games coBüchi automata [1, 2]. Together, the three conditions are sufficient for half-positionality of DBA-recognizable objectives: the proof of sufficiency uses the theory of universal graphs, and consists of building a family of well-monotonic universal graphs [52] for objectives satisfying the three properties.

Other related works. We have discussed the relevant literature on half-positionality [39, 40, 6, 52] and bipositionality [33, 34, 27, 3]. A more general quest is to understand *memory requirements* when positional strategies are not powerful enough: e.g., [46, 10, 12, 13].

Memory requirements have been precisely characterized for some classes of ω-regular objectives (not encompassing the class of DBA-recognizable objectives), such as Muller conditions [28, 64, 18, 19] and safety specifications, i.e., objectives that are closed for the Cantor topology [26]. The latter also uses the order of the equivalence classes of the right congruence as part of its characterization.

Recently, a link between the prefix-classifier, the memory requirements, and the recognizability of ω-regular objectives was established [13]. However, this result does not provide optimal bounds on the strategy complexity, and is thereby insufficient to study half-positionality.

Our article is an extended version (with complete proofs and additional examples and remarks) of a preceding conference version [8].

Structure of the paper. Notations and definitions are introduced in Section 2. Our main contributions are presented in Section 3: we introduce and discuss the three conditions used in our results, then we state our main characterization (Theorem 19) and some corollaries, and we end with an explanation on how to use the characterization to decide half-positionality of DBA-recognizable objectives in polynomial time. Section 4 and Section 5 contain the proof of Theorem 19: the former shows the necessity of the three conditions for half-positionality of DBA-recognizable objectives, and the latter shows their sufficiency through the use of universal graphs.

2 Preliminaries

In the whole article, letter C refers to a (finite or infinite) non-empty set of *colors*. Given a set A, we write respectively A^*, A^+, and A^ω for the set of finite, non-empty finite, and infinite sequences of elements of A. We denote by ε the empty word.

2.1 Games and positionality

Graphs. An *(edge-colored) graph* $\mathcal{G} = (V, E)$ is given by a non-empty set of *vertices* V (of any cardinality) and a set of *edges* $E \subseteq V \times C \times V$. We write $v \xrightarrow{c} v'$ if $(v, c, v') \in E$. We assume graphs to be *non-blocking*: for all $v \in V$, there exists $(v', c, v'') \in E$ such that $v = v'$. We allow graphs with infinite branching. For $v \in V$, an *infinite path of $\mathcal{G}$ from v* is an infinite sequence of edges $\pi = (v_0, c_1, v_1')(v_1, c_2, v_2') \ldots \in E^\omega$ such that $v_0 = v$ and for all $i \geq 1$, $v_i' = v_i$. A *finite path of $\mathcal{G}$ from v* is a finite prefix in E^* of an infinite path of $\mathcal{G}$ from v. For convenience, we assume that there is a distinct *empty path* λ_v for every $v \in V$. If $\gamma = (v_0, c_1, v_1) \ldots (v_{n-1}, c_n, v_n)$ is a non-empty finite path of $\mathcal{G}$, we define $\mathsf{last}(\gamma) = v_n$. For an empty path λ_v, we define $\mathsf{last}(\lambda_v) = v$. An infinite (resp. finite) path $(v_0, c_1, v_1)(v_1, c_2, v_2) \ldots$ (resp. $(v_0, c_1, v_1) \ldots (v_{n-1}, c_n, v_n)$) is sometimes represented as $v_0 \xrightarrow{c_1} v_1 \xrightarrow{c_2} \ldots$ (resp. $v_0 \xrightarrow{c_1} \ldots \xrightarrow{c_n} v_n$). A graph $\mathcal{G} = (V, E)$ is *finite* if both V and E are finite. A graph is *strongly connected* if for every pair of vertices $(v, v') \in V \times V$ there is a path from v to v'. A *strongly connected component* of $\mathcal{G}$ is a maximal strongly connected subgraph.

Arenas and strategies. We consider two players $\mathcal{P}_1$ and $\mathcal{P}_2$. An *arena* is a tuple $\mathcal{A} = (V, V_1, V_2, E)$ such that (V, E) is a graph and V is the disjoint union of V_1 and V_2. Intuitively, vertices in V_1 are controlled by $\mathcal{P}_1$ and vertices in V_2 are controlled by $\mathcal{P}_2$. An arena $\mathcal{A} = (V, V_1, V_2, E)$ is a *one-player arena of $\mathcal{P}_1$* (resp. *of $\mathcal{P}_2$*) if $V_2 = \emptyset$ (resp. $V_1 = \emptyset$). Finite paths of (V, E) are called *histories of $\mathcal{A}$*. For $i \in \{1, 2\}$, we denote by $\mathsf{Hists}_i(\mathcal{A})$ the set of histories γ of $\mathcal{A}$ such that $\mathsf{last}(\gamma) \in V_i$.

Let $i \in \{1, 2\}$. A *strategy of $\mathcal{P}_i$ on $\mathcal{A}$* is a function $\sigma_i \colon \mathsf{Hists}_i(\mathcal{A}) \to E$ such that for all $\gamma \in \mathsf{Hists}_i(\mathcal{A})$, the first component of $\sigma_i(\gamma)$ coincides with $\mathsf{last}(\gamma)$. Given a strategy σ_i of $\mathcal{P}_i$, we say that an infinite path $\pi = e_1 e_2 \ldots$ is *consistent with σ_i* if for all finite prefixes $\gamma = e_1 \ldots e_i$ of π such that $\mathsf{last}(\gamma) \in V_i$, $\sigma_i(\gamma) = e_{i+1}$. A strategy σ_i is *positional* (also called *memoryless* in the literature) if its outputs only depend on the current vertex and not on the whole history, i.e., if there exists a function $f \colon V_i \to E$ such that for $\gamma \in \mathsf{Hists}_i(\mathcal{A})$, $\sigma_i(\gamma) = f(\mathsf{last}(\gamma))$.

Objectives. An *objective* is a set $W \subseteq C^\omega$ (subsets of C^ω are sometimes also called *languages of infinite words*, *ω-languages*, or *winning conditions* in the literature). When an objective W is clear in the context, we say that an infinite word $w \in C^\omega$ is *winning* if $w \in W$, and *losing* if $w \notin W$. We write $\overline{W}$ for the complement $C^\omega \setminus W$ of an objective W. An objective W is *prefix-independent* if for all $w \in C^*$ and $w' \in C^\omega$, $w' \in W$ if and only if $ww' \in W$. An objective that we will often consider is the *Büchi condition*: given a subset $F \subseteq C$, we denote by $\mathsf{Büchi}(F)$ the set of infinite words seeing infinitely many times a color in F. Such an objective is prefix-independent. A *game* is a tuple $(\mathcal{A}, W)$ of an arena $\mathcal{A}$ and an objective W.

Optimality and half-positionality. Let $\mathcal{A} = (V, V_1, V_2, E)$ be an arena, $(\mathcal{A}, W)$ be a game, and $v \in V$. We say that *a strategy σ_1 of $\mathcal{P}_1$ is winning from v* if for all infinite paths $v_0 \xrightarrow{c_1} v_1 \xrightarrow{c_2} \ldots$ from v consistent with σ_1, $c_1c_2 \ldots \in W$.

A strategy of $\mathcal{P}_1$ is *optimal for $\mathcal{P}_1$ in $(\mathcal{A}, W)$* if it is winning from all the vertices from which $\mathcal{P}_1$ has a winning strategy. We often write *optimal for $\mathcal{P}_1$ in $\mathcal{A}$* if the objective W is clear from the context. We stress that this notion of optimality requires a *single* strategy to be winning from *all* the winning vertices (a property sometimes called *uniformity*).

An objective W is *half-positional* if for all arenas $\mathcal{A}$, there exists a positional strategy of $\mathcal{P}_1$ on $\mathcal{A}$ that is optimal for $\mathcal{P}_1$ in $\mathcal{A}$. We sometimes only consider half-positionality on a restricted set of arenas (typically, finite and/or one-player arenas). For a class of arenas $\mathcal{X}$, an objective W is *half-positional over $\mathcal{X}$* if for all arenas $\mathcal{A} \in \mathcal{X}$, there exists a positional strategy of $\mathcal{P}_1$ on $\mathcal{A}$ that is optimal for $\mathcal{P}_1$ in $\mathcal{A}$.

▶ Remark 1 (ε-edges). Sometimes, arenas are considered to be colored over the alphabet $C \cup \{\varepsilon\}$, adding the restriction that no cycle is entirely labeled by ε [28, 64, 40, 36, 18]. In that case, an infinite word in $(C \cup \{\varepsilon\})^\omega$ labeling a path is winning if the word obtained by removing the occurrences of ε belongs to W. In this paper, we consider arenas without ε-edges, but all our results apply to this other setting (cf. Remark 40). In general, allowing for arenas with ε-edges has an effect on strategy complexity [18]. ⌟

2.2 Büchi automata

Automaton structures and Büchi automata. A *non-deterministic automaton structure (on C)* is a tuple $\mathcal{S} = (Q, C, Q_{\mathsf{init}}, \Delta)$ such that Q is a finite set of *states*, $Q_{\mathsf{init}} \subseteq Q$ is a non-empty set of *initial states* and $\Delta \subseteq Q \times C \times Q$ is a set of *transitions*. We assume that all states of automaton structures are reachable from an initial state in Q_{init} by taking transitions in Δ.

A *(transition-based) non-deterministic Büchi automaton* (NBA) is an automaton structure $\mathcal{S}$ together with a set of transitions $\alpha \subseteq \Delta$. The transitions in α are called *Büchi transitions*.

Given an NBA $\mathcal{B} = (Q, C, Q_{\mathsf{init}}, \Delta, \alpha)$, a *(finite or infinite) run of $\mathcal{B}$ on a (finite or infinite) word $w = c_1c_2 \ldots \in C^* \cup C^\omega$* is a sequence $(q_0, c_1, q_1)(q_1, c_2, q_2) \ldots \in \Delta^* \cup \Delta^\omega$ such that $q_0 \in Q_{\mathsf{init}}$. An infinite run $(q_0, c_1, q_1)(q_1, c_2, q_2) \ldots \in \Delta^\omega$ of $\mathcal{B}$ is *accepting* if for infinitely many $i \geq 0$, $(q_i, c_{i+1}, q_{i+1}) \in \alpha$. A word $w \in C^\omega$ is *accepted* by $\mathcal{B}$ if there exists an accepting run of $\mathcal{B}$ on w — if not, it is *rejected*. We denote the set of infinite words accepted by $\mathcal{B}$ by $\mathcal{L}(\mathcal{B})$, and we then say that $\mathcal{L}(\mathcal{B})$ is the objective *recognized by $\mathcal{B}$*. Here, we take the definition of an *ω-regular objective* as an objective that can be recognized by an NBA (the classical definition uses *ω-regular expressions*, but our definition is well-known to be equivalent [49]). Given an automaton structure $\mathcal{S} = (Q, C, Q_{\mathsf{init}}, \Delta)$, we say that an NBA $\mathcal{B}$ is *built on top of $\mathcal{S}$* if there exists $\alpha \subseteq \Delta$ such that $\mathcal{B} = (Q, C, Q_{\mathsf{init}}, \Delta, \alpha)$.

▶ Remark 2. Notice that for generality, we allow the set of colors C to be infinite. This is uncommon for ω-regular objectives and automata, but has no real impact: when an objective is specified by an NBA, there will be at most finitely many equivalence classes of colors for the equivalence relation "inducing exactly the same transitions in the NBA". In Section 4.3.2, it will be helpful to consider infinite sets of colors. ⌟

Deterministic automata. An automaton structure $\mathcal{S} = (Q, C, Q_{\mathsf{init}}, \Delta)$ is *deterministic* if $|Q_{\mathsf{init}}| = 1$ and, for each $q \in Q$ and $c \in C$, there is exactly one $q' \in Q$ such that $(q, c, q') \in \Delta$ (we remark that, without loss of generality, we define deterministic automaton structures so that for each state and each color there is one outgoing transition — such automata are sometimes called *complete* or *total*). A *deterministic Büchi automaton* (DBA) is an NBA whose underlying automaton structure is deterministic. For a DBA $\mathcal{B} = (Q, C, \{q_{\mathsf{init}}\}, \Delta, \alpha)$, we denote by q_{init} the unique initial state (and we will drop the braces around q_{init} in the tuple), and by $\delta\colon Q \times C \to Q$ the *update function* that associates to $(q, c) \in Q \times C$ the only $q' \in Q$ such that $(q, c, q') \in \Delta$. We denote by δ^* the natural extension of δ to finite words — by induction, the function $\delta^*\colon Q \times C^* \to Q$ is such that $\delta^*(q, \varepsilon) = q$, and for $w \in C^*$, $c \in C$, $\delta^*(q, wc) = \delta(\delta^*(q, w), c)$. As transitions are uniquely determined by their first two components, we also assume for brevity that $\alpha \subseteq Q \times C$.

For a DBA $\mathcal{B}$, a state $q \in Q$ and a word $w = c_1c_2 \ldots \in C^* \cup C^\omega$, we denote by $\mathcal{B}(q, w) = (q, c_1, q_1)(q_1, c_2, q_2) \ldots \in \Delta^* \cup \Delta^\omega$ the only run on w starting from q.

An objective W is *DBA-recognizable* if there exists a DBA $\mathcal{B}$ such that $W = \mathcal{L}(\mathcal{B})$. For $F \subseteq C$, notice that $\mathsf{Büchi}(F)$ is DBA-recognizable: it is recognized by the DBA $(\{q_{\mathsf{init}}\}, C, q_{\mathsf{init}}, \Delta, \alpha)$ with a *single* state such that $(q_{\mathsf{init}}, c) \in \alpha$ if and only if $c \in F$.

▶ Remark 3. The fact that a single state suffices to recognize $\mathsf{Büchi}(F)$ relies on the assumption that our DBA are *transition-based* and not *state-based* (α is a set of transitions, not of states). Indeed, apart from the trivial cases $F = \emptyset$ and $F = C$, a state-based DBA recognizing $\mathsf{Büchi}(F)$ requires two states. The third condition of our upcoming characterization (Theorem 19) would therefore not apply to this simple example if we only considered state-based DBA. ⌟

In this paper, all automata will be deterministic, and the term "automaton" will stand for "deterministic automaton" by default.

▶ Remark 4. Deterministic Büchi automata recognize a proper subset of the ω-regular objectives. That is, not every non-deterministic Büchi automaton can be converted into a deterministic one recognizing the same objective [63]. ⌟

We restate a well-known lemma about ω-regular objectives: if two ω-regular objectives are not equal, then they are distinguished by an ultimately periodic word. Ultimately periodic words can easily be finitely represented, and this lemma will be used in Section 4 to force some behaviors to appear in *finite* arenas.

▶ **Lemma 5.** *Let W_1, W_2 be two ω-regular objectives. If $W_1 \neq W_2$, then there exist $w_1 \in C^*$ and $w_2 \in C^+$ such that either $w_1(w_2)^\omega \in W_1 \setminus W_2$, or $w_1(w_2)^\omega \in W_2 \setminus W_1$. In particular, if $W_1 \not\subseteq W_2$, then there exist $w_1 \in C^*$ and $w_2 \in C^+$ such that $w_1(w_2)^\omega \in W_1 \setminus W_2$.*

Proof. The first statement is standard and follows from McNaughton's theorem [49]. For the second statement, if $W_1 \not\subseteq W_2$, then $W_1 \setminus W_2 \neq \emptyset$. Objective $W_1 \setminus W_2 \neq \emptyset$ is ω-regular (as ω-regular objectives are closed by complement and intersection) and so is $\emptyset$. By the first statement, we take $w_1 \in C^*$ and $w_2 \in C^+$ such that either $w_1(w_2)^\omega \in (W_1 \setminus W_2) \setminus \emptyset$, or $w_1(w_2)^\omega \in \emptyset \setminus (W_1 \setminus W_2)$. As $\emptyset \setminus (W_1 \setminus W_2) = \emptyset$, we have $w_1(w_2)^\omega \in (W_1 \setminus W_2) \setminus \emptyset = W_1 \setminus W_2$. ◀

Right congruence. Let $W \subseteq C^\omega$ be an objective. For a finite word $w \in C^*$, we write $w^{-1}W = \{w' \in C^\omega \mid ww' \in W\}$ for the set of *winning continuations of* w. We define the *right congruence* $\sim_W \subseteq C^* \times C^*$ *of* W as $w_1 \sim_W w_2$ if $w_1^{-1}W = w_2^{-1}W$ (meaning that w_1 and w_2 have the same winning continuations). Relation $\sim_W$ is an equivalence relation. When W is clear from the context, we write $\sim$ for $\sim_W$. For $w \in C^*$, we denote by $[w] \subseteq C^*$ its equivalence class for $\sim$.

When $\sim$ has finitely many equivalence classes, we can associate a natural deterministic automaton structure $\mathcal{S}_\sim = (Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim)$ to $\sim$ such that $Q_\sim$ is the set of equivalence classes of $\sim$, $\widetilde{q}_{\mathsf{init}} = [\varepsilon]$, and $\delta_\sim([w], c) = [wc]$ [60, 48]. The transition function $\delta_\sim$ is well-defined since it follows from the definition of $\sim$ that if $w_1 \sim w_2$, then for all $c \in C$, $w_1c \sim w_2c$.

Hence, the choice of representatives for the equivalence classes does not have an impact on this definition. We call the automaton structure $\mathcal{S}_\sim$ the *prefix-classifier of W*.

▶ Remark 6. Equivalence relation $\sim_W$ has only one equivalence class if and only if W is prefix-independent. In particular, an objective has a prefix-classifier with a single state if and only if it is prefix-independent. ⌟

An important property of $\sim$ is the following.

▶ **Lemma 7.** *Let $w_1, w_2 \in C^*$. If $w_1 \sim w_2$, then for all $w \in C^*$, $w_1 w \sim w_2 w$.*

Proof. If w_1 and w_2 have the same winning continuations, they have in particular the same winning continuations starting with w. ◀

We define the *prefix preorder $\preceq_W$ of W*: for $w_1, w_2 \in C^*$, we write $w_1 \preceq_W w_2$ if $w_1^{-1}W \subseteq w_2^{-1}W$ (meaning that any continuation that is winning after w_1 is also winning after w_2). Intuitively, $w_1 \preceq_W w_2$ means that a game starting with w_2 is always preferable to a game starting with w_1 for $\mathcal{P}_1$, as there are more ways to win after w_2. When W is clear from the context, we write $\preceq$ for $\preceq_W$. Relation $\preceq \subseteq C^* \times C^*$ is a preorder. Notice that $\sim$ is equal to $\preceq \cap \succeq$. We also define the strict preorder $\prec = \preceq \setminus \sim$.

Given a DBA $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ recognizing the objective W, observe that for $w, w' \in C^*$ such that $\delta^*(q_{\mathsf{init}}, w) = \delta^*(q_{\mathsf{init}}, w')$, we have $w \sim w'$. In this case, equivalence relation $\sim$ has at most $|Q|$ equivalence classes. For $q \in Q$, we write abusively $q^{-1}W$ for the objective recognized by the DBA $(Q, C, q, \Delta, \alpha)$. Objective $q^{-1}W$ equals $w^{-1}W$ for any word $w \in C^*$ such that $\delta^*(q_{\mathsf{init}}, w) = q$. We extend the equivalence relation $\sim$ and preorder $\preceq$ to elements of Q (we sometimes write $\sim_{\mathcal{B}}$ and $\preceq_{\mathcal{B}}$ to avoid any ambiguity).

α-free words. Let $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ be a DBA. We say that a run $\varrho \in \Delta^*$ of $\mathcal{B}$ is *α-free* if it does not contain any transition from α. For $q \in Q$, we define

$$\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q) = \{w \in C^* \mid \mathcal{B}(q, w) \text{ is } \alpha\text{-free}\},$$
$$\alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q) = \{w \in C^* \mid w \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q) \text{ and } \delta^*(q, w) = q\}.$$

We call the words in the first set the *α-free words from q*, and the words in the second set the *α-free cycles from q*. We state an important property of α-free words.

▶ **Lemma 8.** *Let $q_1, q_2 \in Q$ be such that $\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_1) = \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_2)$. Then for all $w \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_1)$, $\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(\delta^*(q_1, w)) = \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(\delta^*(q_2, w))$.*

Proof. Let $w \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_1)$. Let $q_1' = \delta^*(q_1, w)$, $q_2' = \delta^*(q_2, w)$, and $w' \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_1')$. Since both runs $\mathcal{B}(q_1, w)$ and $\mathcal{B}(q_1', w')$ are α-free, we have $ww' \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_1) = \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_2)$. Therefore, the run $\mathcal{B}(q_2, ww')$ is α-free, so the run $\mathcal{B}(q_2', w')$ is α-free as well and $w' \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_2')$. ◀

Saturation of DBA with Büchi transitions. In what follows, we will make extensive use of a "normal form" of Büchi automata verifying that any α-free path can be extended to an α-free cycle. Such a normal form can be produced by saturating a given DBA $\mathcal{B}$ with Büchi transitions [44, 1, 2]. To do so, we add to α all transitions that do not appear in an α-free cycle of $\mathcal{B}$. Cycles that are α-free can be easily identified by decomposing in strongly connected components the structure obtained by removing the Büchi transitions from $\mathcal{B}$.

We say that $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ is *saturated* if for every $\alpha' \supsetneq \alpha$, the automaton obtained by replacing α with α' does not recognize $\mathcal{L}(\mathcal{B})$.

An *α-free component* of $\mathcal{B}$ is a strongly connected component of the graph obtained by removing the Büchi transitions from $\mathcal{B}$. That is, a strongly connected component of $(Q, \Delta_{\alpha\text{-free}})$, with $\Delta_{\alpha\text{-free}} = \Delta \setminus \alpha$.

▶ **Lemma 9.** *Let $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ be a DBA. There is a unique set $\alpha_{\mathsf{sat}} \subseteq \Delta$ such that the automaton $\mathcal{B}_{\mathsf{sat}} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha_{\mathsf{sat}})$ satisfies that:*
1. *$\mathcal{L}(\mathcal{B}_{\mathsf{sat}}) = \mathcal{L}(\mathcal{B})$.*
2. *$\mathcal{B}_{\mathsf{sat}}$ is saturated.*

Moreover, α_{sat} is the set of transitions not appearing in any α-free component of $\mathcal{B}$, and it can be computed in $\mathcal{O}(|C| \cdot |Q|)$ time when C is finite.

Proof. We first prove the existence of such an α_{sat}. Let $(Q_1, \Delta_1), \dots, (Q_k, \Delta_k)$ be the α-free components of $\mathcal{B}$. We consider the automaton $\mathcal{B}_{\mathsf{sat}}$ whose Büchi transitions are those that do not appear in any α-free component (Q_i, Δ_i), that is, we let

$$\alpha_{\mathsf{sat}} = \Delta \setminus \bigcup_{i=1}^{k} \Delta_i \quad \text{and} \quad \mathcal{B}_{\mathsf{sat}} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha_{\mathsf{sat}}).$$

We show that $\mathcal{L}(\mathcal{B}_{\mathsf{sat}}) = \mathcal{L}(\mathcal{B})$. Since $\alpha \subseteq \alpha_{\mathsf{sat}}$, it is verified that $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{B}_{\mathsf{sat}})$. For the other inclusion, let $w \notin \mathcal{L}(\mathcal{B})$. There are $w_0 \in C^*, w' \in C^\omega$ such that $w = w_0 w'$ and the run $\mathcal{B}(q_0, w')$ produced by reading w' from $q_0 = \delta^*(q_{\mathsf{init}}, w_0)$ does not visit any Büchi transition. In particular, $\mathcal{B}(q_0, w')$ is an infinite path in the finite graph $(Q, \Delta_{\alpha\text{-free}})$. This implies that eventually, run $\mathcal{B}(q_0, w')$ reaches and stays in the same strongly connected component of graph $(Q, \Delta_{\alpha\text{-free}})$. Formally, there are $w_1 \in C^*, w_2 \in C^\omega$ such that $w' = w_1 w_2$ and the run $\mathcal{B}(q_1, w_2)$ produced by reading w_2 from $q_1 = \delta^*(q_0, w_1)$ lies entirely in some α-free component (Q_i, Δ_i). Let Δ_{q_1, w_2} be the transitions appearing in $\mathcal{B}(q_1, w_2)$. We have $\Delta_{q_1, w_2} \subseteq \Delta_i$. Therefore, $\Delta_{q_1, w_2} \cap \alpha_{\mathsf{sat}} = \emptyset$, so $\mathcal{B}_{\mathsf{sat}}(q_{\mathsf{init}}, w)$ is also a rejecting run of $\mathcal{B}_{\mathsf{sat}}$.

We prove that $\mathcal{B}_{\mathsf{sat}}$ is saturated and the uniqueness of α_{sat} at the same time. Let α' be another acceptance set such that $\alpha' \nsubseteq \alpha_{\mathsf{sat}}$ and let $\mathcal{B}'$ be the automaton obtained by replacing α_{sat} with α' in $\mathcal{B}_{\mathsf{sat}}$. Let $(q, c, q') \in \alpha' \setminus \alpha_{\mathsf{sat}}$. Since $(q, c, q') \notin \alpha_{\mathsf{sat}}$, there is an α-free component (Q_i, Δ_i) such that $(q, c, q') \in \Delta_i$. We can therefore consider a word $w \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}_{\mathsf{sat}}}(q)$ labeling an α-free cycle including the transition (q, c, q'). Let $w_0 \in C^*$ such that $\delta^*(q_{\mathsf{init}}, w_0) = q$. Then, $w_0 w^\omega \notin \mathcal{L}(\mathcal{B}_{\mathsf{sat}})$, whereas $w_0 w^\omega \in \mathcal{L}(\mathcal{B}')$, so $\mathcal{B}'$ does not recognize the same objective as $\mathcal{B}_{\mathsf{sat}}$.

When C is finite, the set of transitions $\mathcal{B}_{\mathsf{sat}}$ can be computed in time $\mathcal{O}(|C| \cdot |Q|)$, as it consists of decomposing a graph with at most $|C| \cdot |Q|$ transitions into strongly connected components [61]. ◀

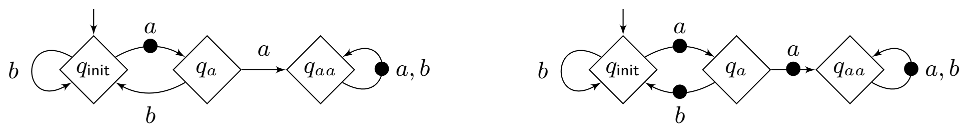

Figure 1 A DBA (left) and its unique saturation (right). Transitions labeled with a • symbol are the Büchi transitions. In figures, automaton states are depicted with diamonds.

In Figure 1, we show an example of the saturation process presented in the proof of Lemma 9.

The following simple lemma follows, which holds true in saturated DBA: every word that is α-free from a state can be completed into an α-free cycle from the same state. This is a key technical lemma used many times in the upcoming proofs.

▶ **Lemma 10.** *Suppose that* $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ *is a saturated DBA and let* $q \in Q$ *and* $w \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$*. There exists* $w' \in C^*$ *such that* $ww' \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q)$*.*

Proof. Let $q' = \delta^*(q, w)$. Thanks to the saturation property and by Lemma 9, α contains all transitions that do not belong to an α-free component. This implies that any two states connected by an α-free run are in the same α-free component. In particular, as q' is reachable from q through an α-free run, q' must belong to the same α-free component as q. Therefore, there exists an α-free run from q' to q. Taking the word w' labeling this run, we obtain the desired result. ◀

3 Half-positionality characterization for DBA-recognizable objectives

In this section, we present our main contribution in Theorem 19, by giving three conditions that exactly characterize half-positional DBA-recognizable objectives. These conditions are presented in Section 3.1. Theorem 19 and several consequences of it are stated in Section 3.2 (the proof of Theorem 19 is postponed to Sections 4 and 5). In Section 3.3, we use this characterization to show that we can decide the half-positionality of a DBA in polynomial time.

3.1 Three conditions for half-positionality

We define the three conditions on objectives at the core of our characterization.

▶ **Condition 1** (Total prefix preorder)**.** We say that an objective $W \subseteq C^\omega$ has a *total prefix preorder* if for all $w_1, w_2 \in C^*$, $w_1 \preceq_W w_2$ or $w_2 \preceq_W w_1$.

▶ **Example 11** (Not total prefix preorder)**.** Let $C = \{a, b\}$. We consider the objective W recognized by the DBA $\mathcal{B}$ depicted in Figure 2 (left). It consists of the infinite words starting with aa or bb. This objective does not have a total prefix preorder: words a and b are incomparable for $\preceq_W$. Indeed, a^ω is winning after a but not after b, and b^ω is winning after b but not after a. In terms of automaton states, we have that q_a and q_b are incomparable for $\preceq_{\mathcal{B}}$. This objective is not half-positional, as witnessed by the arena on the right of Figure 2. In this arena, $\mathcal{P}_1$ is able to win when the game starts in v_1 by playing a in v_3, and when the game starts in v_2 by playing b. However, no positional strategy wins from both v_1 and v_2. ⌟

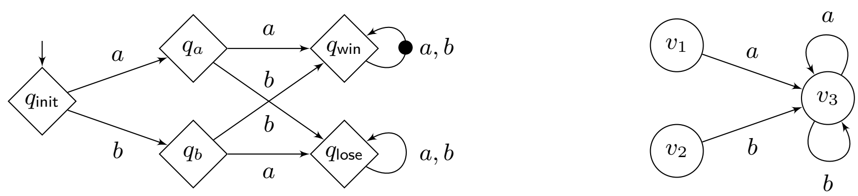

Figure 2 DBA $\mathcal{B}$ recognizing objective $W = (aa + bb)C^\omega$ (left), and an arena in which positional strategies do not suffice for $\mathcal{P}_1$ to play optimally for this objective (right). In figures, circles represent arena vertices controlled by $\mathcal{P}_1$.

▶ Remark 12. The prefix preorder of an objective W is total if and only if the prefix preorder of its complement $\overline{W}$ is total. ⌟

▶ Remark 13. Having a total prefix preorder is equivalent to the *strong monotony* notion [6] in general, and equivalent to *monotony* [34] for ω-regular objectives. We discuss in more depth the relation between the conditions appearing in the characterization and other properties from the literature studying half-positionality in Appendix A. ⌟

A straightforward result for an objective W recognized by a DBA $\mathcal{B}$ is that it has a total prefix preorder if and only if the (reachable) states of $\mathcal{B}$ are totally ordered for $\preceq_{\mathcal{B}}$. Moreover, transitions of $\mathcal{B}$ following a given word are non-decreasing w.r.t. states.

▶ **Lemma 14.** *If $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ recognizes an objective with a total prefix preorder, then for every $w \in C^*$, function $\delta^*(\cdot, w)$ is non-decreasing for $\preceq_{\mathcal{B}}$ (i.e., for $q_1, q_2 \in Q$, if $q_1 \preceq_{\mathcal{B}} q_2$, then $\delta^*(q_1, w) \preceq_{\mathcal{B}} \delta^*(q_2, w)$).*

Proof. Let W be the objective that $\mathcal{B}$ recognizes. Let $w \in C^*$ and $q_1, q_2 \in Q$ be such that $q_1 \preceq_{\mathcal{B}} q_2$. Let $q_1' = \delta^*(q_1, w)$ and $q_2' = \delta^*(q_2, w)$. We show that $q_1' \preceq_{\mathcal{B}} q_2'$, i.e., that $(q_1')^{-1}W \subseteq (q_2')^{-1}W$. Let $w' \in (q_1')^{-1}W$. This implies that $ww' \in q_1^{-1}W$. As $q_1 \preceq_{\mathcal{B}} q_2$, we also have that $ww' \in q_2^{-1}W$. This implies that $w' \in (q_2')^{-1}W$. ◀

▶ **Condition 2** (Progress-consistency)**.** We say that an objective W is *progress-consistent* if for all $w_1 \in C^*$ and $w_2 \in C^+$ such that $w_1 \prec w_1w_2$, we have $w_1(w_2)^\omega \in W$.

Intuitively, this means that whenever a word w_2 can be used to make progress after seeing a word w_1 (in the sense of getting to a position in which more continuations are winning), then repeating this word has to be winning.

▶ **Example 15** (Not progress-consistent)**.** Let $C = \{a, b\}$. We consider the objective $W = C^*aaC^\omega$ recognized by the DBA with three states in Figure 3 (left). This objective contains the words seeing, at some point, twice the color a in a row. Notice that the prefix preorder of this objective is total ($q_{\mathsf{init}} \prec q_a \prec q_{aa}$). This objective is not progress-consistent: we have $\varepsilon \prec ba$, but $(ba)^\omega \notin W$. This objective is not half-positional: if $\mathcal{P}_1$ plays in an arena with a choice among two cycles ba and ab depicted in Figure 3 (right), it is possible to win by playing ba and then ab, but a positional strategy can only achieve words $(ba)^\omega$ or $(ab)^\omega$, which are both losing. ⌟

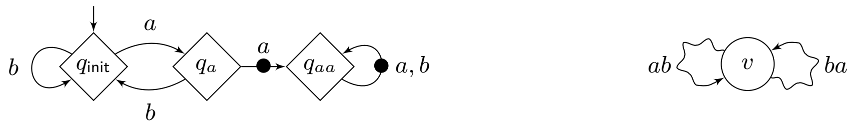

Figure 3 A DBA recognizing the set of words seeing aa at some point (left), an arena in which positional strategies do not suffice for $\mathcal{P}_1$ to play optimally for this objective (right). Squiggly arrows indicate a sequence of edges or transitions (here, a sequence of two edges).

▶ **Example 16** (Progress-consistent objective)**.** We consider a slight modification of the previous example by adding two Büchi transitions: see the DBA in Figure 4. The objective recognized by this DBA is $W = \mathsf{Büchi}(\{a\}) \cup C^*aaC^\omega$: W contains the words seeing a infinitely often, or that see a twice in a row at some point. The equivalence classes for $\sim_W$ are $q_{\mathsf{init}}^{-1}W = W$, $q_a^{-1}W = aC^\omega \cup W$ and $q_{aa}^{-1}W = C^\omega$. This objective is progress-consistent: any word reaching q_{aa} is straightforwardly accepted when repeated infinitely often, and any word w such that $\delta^*(q_{\mathsf{init}}, w) = q_a$ necessarily contains at least one a, and thus is accepted when repeated infinitely often. Objective W is half-positional, which will be readily shown with our upcoming characterization (Theorem 19).

Here, notice that the complement $\overline{W}$ of W is not progress-consistent. Indeed, $a \prec_{\overline{W}} a(bab)$, but $a(bab)^\omega \notin \overline{W}$. Unlike having a total prefix preorder, progress-consistency can hold for an objective but not its complement.

Note that half-positionality of W cannot be shown using existing half-positionality criteria [39, 6] (it is neither prefix-independent nor *concave*) nor bipositionality criteria, as it is simply not bipositional. ⌟

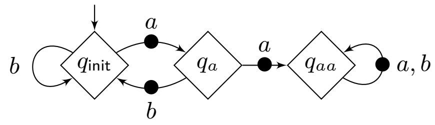

Figure 4 A DBA recognizing the set of words seeing a infinitely many times, or aa at some point.

▶ **Condition 3** (Recognizability by the prefix-classifier)**.** Being recognized by a Büchi automaton built on top of the prefix-classifier is our third condition. In other words, for a DBA-recognizable objective $W \subseteq C^\omega$ and its prefix-classifier $\mathcal{S}_\sim = (Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim)$, this condition requires that there exists $\alpha_\sim \subseteq Q_\sim \times C$ such that W is recognized by DBA $(Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim, \alpha_\sim)$.

We show an example of a DBA-recognizable objective that satisfies the first two conditions (having a total prefix preorder and progress-consistency), but not this third condition, and which is not half-positional.

▶ **Example 17** (Not recognizable by the prefix-classifier)**.** Let $C = \{a, b\}$. We consider the objective $W = \mathsf{Büchi}(\{a\}) \cap \mathsf{Büchi}(\{b\})$ recognized by the DBA in Figure 5. This objective is prefix-independent: as such (Remark 6), there is only one equivalence class for $\sim$. This implies that the prefix preorder is total, and that W is progress-consistent (the premise of the progress-consistency property can never be true). This objective is not half-positional, as witnessed by the arena in Figure 5 (right): $\mathcal{P}_1$ has a winning strategy from v, but it needs to take infinitely often both a and b.

Any DBA recognizing this objective has at least two states, but all their (reachable) states are equivalent for $\sim$ — no matter the state we choose as an initial state, the recognized objective is the same (by prefix-independence). As it is prefix-independent, its prefix-classifier $\mathcal{S}_\sim$ has only one state. ⌟

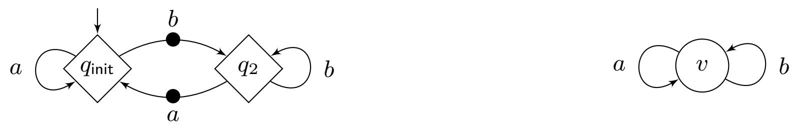

Figure 5 DBA recognizing the objective $\mathsf{Büchi}(\{a\}) \cap \mathsf{Büchi}(\{b\})$ (left), and an arena in which positional strategies do not suffice for $\mathcal{P}_1$ to play optimally for this objective (right).

As will be shown formally, being recognized by a DBA built on top of the prefix-classifier is necessary for half-positionality of *DBA-recognizable* objectives over finite one-player arenas. Unlike the two other conditions, it is in general not necessary for half-positionality of general objectives, including objectives recognized by other standard classes of ω-automata.

▶ **Example 18.** We consider the complement $\overline{W}$ of the objective $W = \mathsf{Büchi}(\{a\}) \cap \mathsf{Büchi}(\{b\})$ of Example 17, which consists of the words ending with a^ω or b^ω. Objective $\overline{W}$ is not DBA-recognizable (a close proof can be found in [5, Theorem 4.50]). Still, it is recognizable by a *deterministic coBüchi automaton* similar to the automaton in Figure 5, but which accepts infinite words that visit transitions labeled by • only finitely often. This objective is half-positional, which can be shown using [28, Theorem 6]. However, its prefix-classifier has just one state, and there is no way to recognize $\overline{W}$ by building a coBüchi (or even parity) automaton on top of it. ⌟

3.2 Characterization and corollaries

We have now defined the three conditions required for our characterization.

▶ **Theorem 19.** *Let $W \subseteq C^\omega$ be a DBA-recognizable objective. Objective W is half-positional (over all arenas) if and only if*

- *its prefix preorder $\preceq$ is total,*
- *it is progress-consistent, and*
- *it can be recognized by a Büchi automaton built on top of its prefix-classifier $\mathcal{S}_\sim$.*

Proof. The proof of the necessity of the three conditions can be found in Section 4, respectively in Propositions 26, 27, and 28. The proof of the sufficiency of the conjunction of the three conditions can be found in Section 5, Proposition 41. ◀

This characterization is valuable to prove (and disprove) half-positionality of DBA-recognizable objectives. Examples 11, 15, and 17 are all not half-positional, and each of them falsifies exactly one of the three conditions from the statement. On the other hand, Example 16 is half-positional. We have already discussed its progress-consistency, but it is also straightforward to verify that its prefix preorder is total and that it is recognizable by its prefix-classifier: the right congruence has three totally ordered equivalence classes corresponding to the states of the automaton of Figure 4.

We state two notable consequences of Theorem 19 and of its proof technique. The first one is the specialization of Theorem 19 to prefix-independent objectives. It states that all prefix-independent, DBA-recognizable objectives that are half-positional are of the kind $\mathsf{Büchi}(F)$ for some $F \subseteq C$. Prefix-independence of objectives is a frequent assumption in the literature [39, 27, 31, 25] — we show that under this assumption, half-positionality of DBA-recognizable objectives is very easy to understand and characterize.

▶ **Proposition 20.** *Let $W \subseteq C^\omega$ be a prefix-independent, DBA-recognizable objective. Objective W is half-positional if and only if there exists $F \subseteq C$ such that $W = \mathsf{Büchi}(F)$.*

Proof. The right-to-left implication follows from the known half-positionality of objectives of the kind $\mathsf{Büchi}(F)$ (this is a special case of a parity game [30]). For the left-to-right implication, let W be a prefix-independent, DBA-recognizable, half-positional objective. By Theorem 19, it is recognized by a DBA $\mathcal{B}$ built on top of $\mathcal{S}_\sim$. As W is prefix-independent (Remark 6), its prefix-classifier has just one state, and there is a single transition from and to this single state for each color. Hence, $W = \mathsf{Büchi}(F)$, where F is the set of colors whose only transition is a Büchi transition in $\mathcal{B}$. ◀

▶ Remark 21. A corollary of this result is that when W is prefix-independent, DBA-recognizable and half-positional, we also have that $\overline{W}$ is half-positional. Indeed, the complement of objective $W = \mathsf{Büchi}(F)$ is a so-called *coBüchi objective*, which is also known to be half-positional [30]. This statement does not hold in general when W is not prefix-independent, as was shown in Example 16. Moreover, the reciprocal of the statement also does not hold, as was shown in Example 18. ⌟

▶ Remark 22. A second corollary is that prefix-independent DBA-recognizable half-positional objectives are closed under finite union (since a finite union of Büchi conditions is a Büchi condition). This settles Kopczyński's conjecture for DBA-recognizable objectives. ⌟

A second consequence of Theorem 19 and its proof technique shows that half-positionality of DBA-recognizable objectives can be reduced to half-positionality over the restricted class of *finite, one-player arenas.* Results reducing strategy complexity in two-player arenas to the easier question of strategy complexity in one-player arenas are sometimes called *one-to-two-player lifts* and appear in multiple places in the literature [34, 10, 43, 13].

▶ **Proposition 23** (One-to-two-player and finite-to-infinite lift). *Let $W \subseteq C^\omega$ be a DBA-recognizable objective. If objective W is half-positional over finite one-player arenas, then it is half-positional over all arenas (of any cardinality).*

Proof. When showing the necessity of the three conditions for half-positionality of DBA-recognizable objectives in Section 4 (Propositions 26, 27, and 28), we actually show their necessity for half-positionality over *finite one-player* arenas. Hence, assuming half-positionality over finite one-player arenas, we have the three conditions from the characterization of Theorem 19, so we have half-positionality over all arenas. ◀

One-to-two-player lifts from the literature all require an assumption on the strategy complexity of *both* players, and are either stated solely over finite arenas, or solely over infinite arenas. Proposition 23, albeit set in the more restricted context of DBA-recognizable objectives, displays stronger properties than the known one-to-two-player lifts.

- It is *asymmetric* in the sense that we simply need a hypothesis on *one* player: *half*-positionality of DBA-recognizable objectives over one-player arenas implies their *half*-positionality over two-player arenas.
- It shows that half-positionality of DBA-recognizable objectives over *finite* arenas implies half-positionality over *infinite* arenas.

Both these properties do not hold for general objectives.

- Some objectives are half-positional over one-player but not over two-player arenas [32, Section 7] — we have that this is not possible for DBA-recognizable objectives.
- Some objectives are half-positional over finite but not over infinite arenas (see, e.g., the mean-payoff objective [29, 56]) — we have that this is not possible for DBA-recognizable objectives.

3.3 Deciding half-positionality in polynomial time

In this section, we assume that C is finite. We show that the problem of deciding, given a DBA $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ as an input, whether $\mathcal{L}(\mathcal{B})$ is half-positional can be solved in polynomial time, and more precisely in time $\mathcal{O}(|C|^2 \cdot |Q|^4)$.

We investigate how to verify each property used in the characterization of Theorem 19. Let $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ be a DBA (we assume w.l.o.g. that all states in Q are reachable from q_{init}) and $W = \mathcal{L}(\mathcal{B})$ be the objective it recognizes. Our algorithm first verifies that the prefix preorder is total and recognizability by $\mathcal{S}_\sim$, and then, under these first two assumptions, progress-consistency. For each condition, we sketch an algorithm to decide it, and we discuss the time complexity of this algorithm.

Total prefix preorder. To check that W has a total prefix preorder, it suffices to check that the states of $\mathcal{B}$ are totally preordered by $\preceq_{\mathcal{B}}$. We start by computing, for each pair of states $q, q' \in Q$, whether $q \preceq_{\mathcal{B}} q'$, $q' \preceq_{\mathcal{B}} q$, or none of these. This can be rephrased as an *inclusion problem* for two DBA-recognizable objectives: if $\mathcal{B}_q = (Q, C, q, \Delta, \alpha)$ and $\mathcal{B}_{q'} = (Q, C, q', \Delta, \alpha)$, we have that $q \preceq_{\mathcal{B}} q'$ if and only if $\mathcal{L}(\mathcal{B}_q) \subseteq \mathcal{L}(\mathcal{B}_{q'})$. Such a problem can be solved in time $\mathcal{O}(|C|^2 \cdot |Q|^2)$ [24]. We can therefore know for all $|Q|^2$ pairs $q, q' \in Q$ whether

$q \preceq_{\mathcal{B}} q'$, $q' \preceq_{\mathcal{B}} q$, $q' \sim_{\mathcal{B}} q$ (as $\sim_{\mathcal{B}} = \preceq_{\mathcal{B}} \cap \succeq_{\mathcal{B}}$), or none of these in time $\mathcal{O}(|Q|^2 \cdot (|C|^2 \cdot |Q|^2)) = \mathcal{O}(|C|^2 \cdot |Q|^4)$. In particular, the prefix preorder is total if and only if for all $q, q' \in Q$, we have $q \preceq_{\mathcal{B}} q'$ or $q' \preceq_{\mathcal{B}} q$.

Recognizability by the prefix-classifier. After all the relations $\preceq_{\mathcal{B}}$ and $\sim_{\mathcal{B}}$ between pairs of states are computed in the previous step, we can compute the states and transitions of the prefix-classifier $\mathcal{S}_\sim = (Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim)$ by merging all the equivalence classes for $\sim_{\mathcal{B}}$. We assume for simplicity that $Q_\sim = Q/_{\sim_{\mathcal{B}}}$.

We now wonder whether it is possible to recognize W by carefully selecting a set $\alpha_\sim$ of Büchi transitions in $\mathcal{S}_\sim$. We simplify the search for such a set with the following result, which shows that it suffices to try with one specific set $\alpha_\sim$. We can then simply check whether $W = \mathcal{L}((Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim, \alpha_\sim))$, an equivalence query which, as discussed above, can be performed in time $\mathcal{O}(|C|^2 \cdot |Q|^2)$.

▶ **Lemma 24.** *We assume that $\mathcal{B}$ is saturated and that W is recognized by a DBA built on top of the prefix-classifier $\mathcal{S}_\sim = (Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim)$. We define*

$$\alpha_\sim = \{([q], c) \in Q_\sim \times C \mid \forall q' \in [q], (q', c) \in \alpha\}.$$

Then, W is recognized by $(Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim, \alpha_\sim)$.

Proof. We assume that W is recognized by a DBA built on top of $\mathcal{S}_\sim$. We start by saturating this DBA, which yields a set of Büchi transitions α' such that W is also recognized by the saturated DBA $\mathcal{B}' = (Q_\sim, C, \widetilde{q}_{\mathsf{init}}, \Delta_\sim, \alpha')$ (Lemma 9). To prove the claim, we show that $\alpha' = \alpha_\sim$.

We first show that $\alpha' \subseteq \alpha_\sim$. Let $([q], c) \notin \alpha_\sim$ — we show that $([q], c) \notin \alpha'$. As $([q], c) \notin \alpha_\sim$, by definition of $\alpha_\sim$, there is $q' \in Q$ such that $(q', c) \notin \alpha$. As $\mathcal{B}$ is saturated, by Lemma 10, there exists $w' \in C^*$ such that $cw' \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q')$. By construction of the prefix-classifier, $\delta^*_\sim([q], cw') = [q]$. Also, as $W = \mathcal{L}(\mathcal{B}')$, word $(cw')^\omega$ must be rejected from $[q]$ in $\mathcal{B}'$. Therefore, $([q], c)$ cannot be a Büchi transition in $\mathcal{B}'$ and is not in α'.

We now show that $\alpha_\sim \subseteq \alpha'$. Let $([q], c) \notin \alpha'$ — we show that $([q], c) \notin \alpha_\sim$. As $\mathcal{B}'$ is saturated, by Lemma 10, there exists $w' \in C^*$ such that $cw' \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}'}([q])$. As $W = \mathcal{L}(\mathcal{B}')$, word $(cw')^\omega$ is rejected from any state in $[q]$ in $\mathcal{B}$. If for all $q' \in [q]$, (q', c) was in α, $(cw')^\omega$ would be accepted from all states in $[q]$ in $\mathcal{B}$. Hence, there exists $q' \in [q]$ such that $(q', c) \notin \alpha$. We conclude that $([q], c) \notin \alpha_\sim$. ◀

Progress-consistency. We assume that we have already checked that W is recognizable by a Büchi automaton built on top of $\mathcal{S}_\sim$, and that we know the (total) ordering of the states. We show that checking progress-consistency, under these two hypotheses, can be done in polynomial time. We prove a lemma reducing the search for words witnessing that W is not progress-consistent to a problem computationally easier to investigate.

▶ **Lemma 25.** *We assume that $\mathcal{B}$ is built on top of the prefix-classifier $\mathcal{S}_\sim$ and that the prefix preorder of W is total. Then, W is progress-consistent if and only if for all $q, q' \in Q$ with $q \prec_{\mathcal{B}} q'$,*

$$\{w \in C^+ \mid \delta^*(q, w) = q'\} \cap \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q') = \emptyset.$$

Proof. For the left-to-right implication, we assume by contrapositive that there exist $q, q' \in Q$ with $q \prec_{\mathcal{B}} q'$ and $w \in C^+$ such that $\delta^*(q, w) = q'$ and $w \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q')$. Let $w_q \in C^*$ be a word such that $\delta^*(q_{\mathsf{init}}, w_q) = q$. We have that $w_q \prec w_q w$, but $w_q w^\omega$ is not accepted by $\mathcal{B}$ as w is a cycle on q' that does not see any Büchi transition. Hence, W is not progress-consistent.

For the right-to-left implication, we assume by contrapositive that W is not progress-consistent. Thus, there exist $w' \in C^*$ and $w \in C^+$ such that $w' \prec w'w$ and $w'w^\omega \notin W$. Let $q_1 = \delta^*(q_{\mathsf{init}}, w')$ and $q_2 = \delta^*(q_1, w)$ — we have $q_1 \prec q_2$. As $q_1 \prec q_2$, by Lemma 14, we have $\delta^*(q_1, w) = q_2 \preceq \delta^*(q_2, w)$. We distinguish two cases, using the fact that there is exactly one state per equivalence class of $\sim_{\mathcal{B}}$. We represent what happens in Figure 6.

- If $q_2 = \delta^*(q_2, w)$, we then have that $w \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_2)$, and we have what we want with $q = q_1$ and $q' = q_2$.
- If not, we have that $q_2 \prec \delta^*(q_2, w)$. Let $q_3 = \delta^*(q_2, w)$. We can repeat the argument on q_2 and q_3: either $w \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_3)$ and we are done, or $q_3 \prec \delta^*(q_3, w)$. As there are finitely many states, this process necessarily ends with two states $q = q_n$ and $q' = q_{n+1}$ such that $\delta^*(q, w) = q'$ and $w \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q')$. ◀

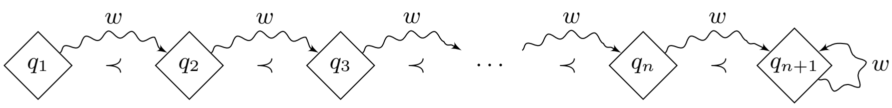

Figure 6 Situation in the proof of Lemma 25.

Notice that for each pair of states $q, q' \in Q$, the sets $\{w \in C^+ \mid \delta(q, w) = q'\}$ and $\alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q')$ are both regular languages recognized by deterministic finite automata with at most $|Q|$ states. The emptiness of their intersection can be decided in time $\mathcal{O}(|C|^2 \cdot |Q|^2)$ (by solving a reachability problem in the product of the two automata) [57]. Thanks to Lemma 25, we can therefore decide whether $\mathcal{B}$ is progress-consistent in time $\mathcal{O}(|Q|^2 \cdot (|C|^2 \cdot |Q|^2)) = \mathcal{O}(|C|^2 \cdot |Q|^4)$: for all $|Q|^2$ pairs of states $q, q' \in Q$, if $q \prec q'$, we test the emptiness of the intersection of these two regular languages.

Complexity wrap-up. By checking the three conditions as explained and in this order, the time complexities are respectively $\mathcal{O}(|C|^2 \cdot |Q|^4)$, $\mathcal{O}(|C|^2 \cdot |Q|^2)$, and $\mathcal{O}(|C|^2 \cdot |Q|^4)$. This yields a time complexity of $\mathcal{O}(|C|^2 \cdot |Q|^4)$ for the whole algorithm.

4 Necessity of the conditions

We prove that each of the three conditions introduced in Section 3.1 is necessary for half-positionality over finite one-player arenas of DBA-recognizable objectives. Each condition gets its own devoted subsection. For the first two conditions (having a total prefix preorder and progress-consistency), we also show for completeness that they are necessary for half-positionality of general objectives over countably infinite one-player arenas, and necessary for half-positionality of ω-regular objectives over finite one-player arenas. This distinction is worthwhile, as there exist half-positional objectives that are not ω-regular (see, e.g., *finitary Büchi* objectives [22]).

4.1 Total prefix preorder

Having a total prefix preorder is necessary in general for half-positionality over countably infinite arenas, and even over finite arenas for ω-regular objectives.

▶ **Proposition 26.** *Let $W \subseteq C^\omega$ be an objective. If W is half-positional over countably infinite one-player arenas, then its prefix preorder is total. If W is ω-regular and half-positional over finite one-player arenas, then its prefix preorder is total.*

Proof. By contrapositive, we assume that the prefix preorder of W is not total. Then, there exist two finite words $w_1, w_2 \in C^*$ such that $w_1 \not\preceq w_2$ and $w_2 \not\preceq w_1$. We can find two infinite continuations $w'_1, w'_2 \in C^\omega$ such that $w_1 w'_1 \in W$, $w_2 w'_1 \notin W$, $w_2 w'_2 \in W$, and $w_1 w'_2 \notin W$. Using these four words, we build a countably infinite one-player arena depicted in Figure 7 (left) for which $\mathcal{P}_1$ has no positional optimal strategy. Indeed, if the game started with w_1, $\mathcal{P}_1$ needs to reply with w'_1 in v_3 to win, but if the game started with w_2, $\mathcal{P}_1$ needs to reply with w'_2 in v_3 to win.

Moreover, if W is ω-regular, so are $w_1^{-1}W$ and $w_2^{-1}W$. By Lemma 5, we can therefore assume w.l.o.g. that $w'_1 = x_1(y_1)^\omega$ and $w'_2 = x_2(y_2)^\omega$ are ultimately periodic, and we can carry out similar arguments with the finite arena depicted in Figure 7 (right). ◀

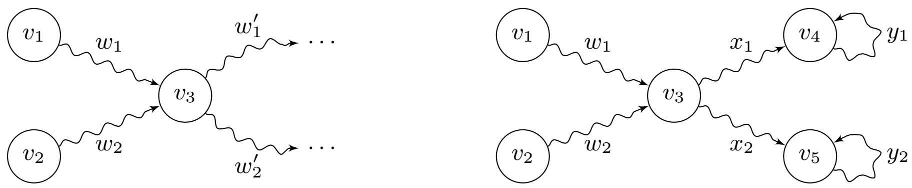

Figure 7 Arenas in which $\mathcal{P}_1$ cannot play optimally with a positional strategy used in the proof of Proposition 26. The arena on the right is used for the ω-regular case.

4.2 Progress-consistency

Progress-consistency is necessary in general for half-positionality in countably infinite arenas, and even in finite arenas for ω-regular objectives.

▶ **Proposition 27.** *Let $W \subseteq C^\omega$ be an objective. If W is half-positional over countably infinite one-player arenas, then it is progress-consistent. If W is ω-regular and half-positional even over finite one-player arenas, then it is progress-consistent.*

Proof. By contrapositive, we assume that W is not progress-consistent. Then there exist $w_1 \in C^*$ and $w_2 \in C^+$ such that $w_1 \prec w_1 w_2$, but $w_1(w_2)^\omega \notin W$. As $w_1 \prec w_1 w_2$, there exists an infinite continuation $w' \in C^\omega$ such that $w_1 w' \notin W$ and $w_1 w_2 w' \in W$. Using these three words, we build a countably infinite one-player arena depicted in Figure 8 (left). In this arena, from vertex v_1, a positional strategy can only achieve words $w_1(w_2)^\omega$ or $w_1 w'$, which are both losing. However, there is a (non-positional) winning strategy achieving word $w_1 w_2 w'$.

If W is additionally ω-regular, using Lemma 5, we can assume w.l.o.g. that $w' = xy^\omega$ is ultimately periodic, and we can carry out similar arguments with the finite arena depicted in Figure 8 (right). ◀

4.3 Recognizability by the prefix-classifier

We now prove that for a DBA-recognizable objective, being recognized by a Büchi automaton built on top of its prefix-classifier $\mathcal{S}_\sim$ is necessary for half-positionality.

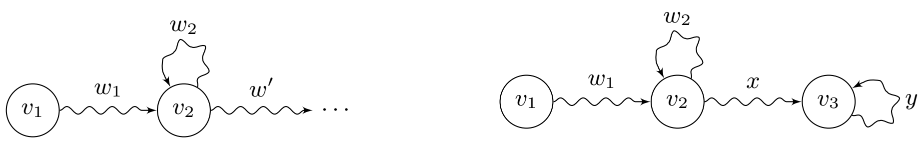

Figure 8 Arenas in which $\mathcal{P}_1$ cannot play optimally with a positional strategy used in the proof of Proposition 27. The arena on the right is used in the ω-regular case.

▶ **Proposition 28.** *Let $W \subseteq C^\omega$ be a DBA-recognizable objective that is half-positional over finite one-player arenas. Then, W is recognized by a Büchi automaton built on top of $\mathcal{S}_\sim$.*

The rest of Section 4.3 is devoted to the proof of this result, which is more involved than the proofs in Sections 4.1 and 4.2. We fix an objective $W \subseteq C^\omega$ recognized by a DBA $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$. We make the assumption that W is half-positional over finite one-player arenas. Our goal is to show that W can be defined by a Büchi automaton built on top of $\mathcal{S}_\sim$. We assume w.l.o.g. that $\mathcal{B}$ is saturated. Many upcoming arguments heavily rely on this assumption through the use of Lemma 10 (any α-free word can be completed into an α-free cycle).

Our proof first assumes in Section 4.3.1 that $\mathcal{B}$ recognizes a prefix-independent objective. We will then use build on this first case to conclude for the general case in Section 4.3.2. We provide a proof sketch at the start of each subsection.

4.3.1 Prefix-independent case

We assume that the objective W recognized by $\mathcal{B}$ is prefix-independent, so all the states of $\mathcal{B}$ are equivalent for $\sim$. We want to show that W can be recognized by a Büchi automaton built on top of $\mathcal{S}_\sim$, and in this case, the automaton structure $\mathcal{S}_\sim$ has just one state. Therefore, we want to find $F \subseteq C$ such that $W = \mathsf{Büchi}(F)$. We start with a high level description of the proof technique.

Proof sketch. The goal is to find a suitable definition for F. To do so, we exhibit a state q_{max} of $\mathcal{B}$ that is "the most rejecting state of the automaton": it satisfies that the set of α-free words from q_{max} contains the α-free words from all the other states (q_{max} is then called an *α-free-maximum*) and that the set of α-free cycles on q_{max} contains the α-free cycles on all the other states (it is also an *α-free-cycle-maximum*). We define F as the set of colors c such that $(q_{\mathsf{max}}, c) \in \alpha$.

We first show that if an α-free-maximum exists, we can assume w.l.o.g. that it is unique (Lemma 29). In Lemmas 31, 32 and 33, we show the existence of an α-free-cycle-maximum. This part of the proof relies on the half-positionality over finite one-player arenas of W. Finally, defining F using q_{max} as explained above, we prove that $W = \mathsf{Büchi}(F)$ (Lemma 34). ◀

We call a state $q_{\mathsf{max}} \in Q$ of $\mathcal{B}$ an *α-free-maximum* (resp. an *α-free-cycle-maximum*) if for all $q \in Q$, we have $\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q) \subseteq \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$ (resp. $\alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q) \subseteq \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_{\mathsf{max}})$). We remark that if $\mathcal{B}$ is saturated, an α-free-cycle-maximum is also an α-free-maximum (this can be shown using Lemma 10).

We first show that we can remove states from $\mathcal{B}$, while still recognizing the same objective, until it has at most one α-free-maximum.

▶ **Lemma 29.** *There exists a DBA $\mathcal{B}'$ recognizing W with at most one α-free-maximum.*

Proof. Assume that $q^1_{\mathsf{max}}, q^2_{\mathsf{max}} \in Q$ are distinct α-free-maxima. In particular, $\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q^1_{\mathsf{max}}) = \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q^2_{\mathsf{max}})$. We show that in such a situation, the objective recognized by $\mathcal{B}$ can be recognized by an automaton with one less state, in which we discard one of the two α-free-maxima. To simplify the upcoming arguments, we assume that $q_{\mathsf{init}} = q^1_{\mathsf{max}}$ (which is without loss of generality as all states of $\mathcal{B}$ are equivalent for $\sim$).

We define a new automaton in which we remove q^2_{max} and redirect all its incoming transitions to q^1_{max}. Formally, let $\mathcal{B}' = (Q', C, q'_{\mathsf{init}}, \Delta', \alpha')$ with update function δ' be such that

- $Q' = q \setminus \{q^2_{\mathsf{max}}\}$, $q'_{\mathsf{init}} = q^1_{\mathsf{max}}$,
- for $q \in Q'$ and $c \in C$, if $\delta(q, c) = q^2_{\mathsf{max}}$, then $\delta'(q, c) = q^1_{\mathsf{max}}$; otherwise, $\delta'(q, c) = \delta(q, c)$,
- for $q \in Q'$ and $c \in C$, $(q, c) \in \alpha'$ if and only if $(q, c) \in \alpha$.

We also assume that states that are not reachable from q'_{init} in $\mathcal{B}'$ are removed from Q'.

We show that this automaton with (at least) one less state recognizes the same objective as $\mathcal{B}$. Let $w = c_1c_2\ldots \in C^\omega$ be an infinite word. We show that w is accepted by $\mathcal{B}$ if and only if it is accepted by $\mathcal{B}'$.

Let $\Delta_{\not\to q_2} = \{(q, c, q^1_{\mathsf{max}}) \in Q' \times C \times \{q^1_{\mathsf{max}}\} \mid \delta(q, c) = q^2_{\mathsf{max}}\}$ be the transitions of $\mathcal{B}'$ that were directed to q^2_{max} in $\mathcal{B}$ but are now redirected to q^1_{max} in $\mathcal{B}'$. Let ϱ be the run of $\mathcal{B}$ on w, and $\varrho' = (q'_0, c_1, q'_1)(q'_1, c_2, q'_2)\ldots$ be the run of $\mathcal{B}'$ on w. The two runs start by taking corresponding transitions, but differ once a transition in $\Delta_{\not\to q_2}$ is taken.

We first assume that ϱ' uses transitions in $\Delta_{\not\to q_2}$ only finitely many times. Then, there exists $k \in \mathbb{N}$ such that $q'_k = q^1_{\mathsf{max}}$ and for all $l \geq k$, $(q'_l, c_{l+1}, q'_{l+1}) \notin \Delta_{\not\to q_2}$. Let $w_{>k} = c_{k+1}c_{k+2}\ldots$ be the infinite word consisting of the colors taken after the last occurrence of a transition in $\Delta_{\not\to q_2}$. We have that

$$\begin{aligned}
\mathcal{B} \text{ accepts } w &\iff \mathcal{B} \text{ accepts } w_{>k} && \text{as } \mathcal{B} \text{ recognizes a prefix-independent objective} \\
&\iff \mathcal{B}' \text{ accepts } w_{>k} && \text{as } w_{>k} \text{ visits exactly the same transitions as in } \mathcal{B} \\
&\iff \mathcal{B}' \text{ accepts } w && \text{as } c_1 \ldots c_k \text{ is a cycle on the initial state } q^1_{\mathsf{max}} \text{ of } \mathcal{B}'.
\end{aligned}$$

We now assume that ϱ' uses transitions in $\Delta_{\not\to q_2}$ infinitely many times. We decompose ϱ' into infinitely many finite runs $\varrho'_1, \varrho'_2, \ldots$ such that $\varrho' = \varrho'_1\varrho'_2\ldots$ and every run ϱ'_i sees exactly one transition in $\Delta_{\not\to q_2}$ as its last transition. This implies that all these finite runs start in state q^1_{max}. We represent run ϱ' in Figure 9. We define words $w_1, w_2, \ldots$ as the respective projection of runs $\varrho'_1, \varrho'_2, \ldots$ to their colors (we have $w = w_1w_2\ldots$). Notice that

$$\forall i \geq 1, w_i \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q^1_{\mathsf{max}}) \iff w_i \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}'}(q^1_{\mathsf{max}}), \tag{1}$$

as the transitions used by w_i from q^1_{max} in $\mathcal{B}'$ correspond to the transitions used by w_i from q^1_{max} in $\mathcal{B}$ (this property is not true for all words, but this is true for these words that read only one transition in $\Delta_{\not\to q_2}$ as their last transition). We also have by construction that

$$\forall i \geq 1, \delta^*(q^1_{\mathsf{max}}, w_i) = q^2_{\mathsf{max}}. \tag{2}$$

We distinguish whether w is accepted or rejected by $\mathcal{B}'$.

Assume w is accepted by $\mathcal{B}'$. Then, we know that for infinitely many $i \in \mathbb{N}$, $w_i \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}'}(q^1_{\mathsf{max}})$. This implies that for these indices i, $w_i \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q^1_{\mathsf{max}})$ by Equation (1). As q^1_{max} is an α-free-maximum, for all $q \in Q$, $w_i \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$ (this is simply the contrapositive of the definition of α-free-maximum). Hence, for infinitely many $i \in \mathbb{N}$, when w_i is read in $\mathcal{B}$ (no matter from where), a Büchi transition is seen, so w is accepted by $\mathcal{B}$.

Assume w is rejected by $\mathcal{B}'$. Then there exists $k \in \mathbb{N}$ such that for all $l \geq k$, $w_l \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}'}(q^1_{\mathsf{max}})$. As $\mathcal{B}$ is prefix-independent, up to removing the start of w, we assume w.l.o.g.

that $k = 1$. We show by induction that

$$\forall i \geq 1, \alpha\text{-Free}_{\mathcal{B}}(q^1_{\max}) = \alpha\text{-Free}_{\mathcal{B}}(\delta^*(q^1_{\max}, w_1 \dots w_i)).$$

This is true for $i = 1$, as $\delta^*(q^1_{\max}, w_1) = q^2_{\max}$ by Equation (2) and the fact that $q^1_{\max}$ and $q^2_{\max}$ are both α-free-maxima. Assume $\alpha\text{-Free}_{\mathcal{B}}(q^1_{\max}) = \alpha\text{-Free}_{\mathcal{B}}(\delta^*(q^1_{\max}, w_1 \dots w_{i-1}))$ for some $i \geq 2$. Then, by Lemma 8, as $w_i \in \alpha\text{-Free}_{\mathcal{B}}(q^1_{\max})$, we have $\alpha\text{-Free}_{\mathcal{B}}(\delta^*(q^1_{\max}, w_i)) = \alpha\text{-Free}_{\mathcal{B}}(\delta^*(q^1_{\max}, w_1 \dots w_{i-1}w_i))$. Equation (2) gives $\alpha\text{-Free}_{\mathcal{B}}(\delta^*(q^1_{\max}, w_i)) = \alpha\text{-Free}_{\mathcal{B}}(q^2_{\max})$, which is itself equal to $\alpha\text{-Free}_{\mathcal{B}}(q^1_{\max})$. We now know that for all $i \geq 1$, $w_i \in \alpha\text{-Free}_{\mathcal{B}}(q^1_{\max})$ by Equation (1). Therefore, we conclude that for all $i \geq 1$, $w_i \in \alpha\text{-Free}_{\mathcal{B}}(\delta^*(q^1_{\max}, w_1 \dots w_{i-1}))$. In particular, w sees no Büchi transition when read from $q^1_{\max}$ in $\mathcal{B}$ and is also rejected by $\mathcal{B}$.

We have shown that $\mathcal{B}'$ is a DBA with fewer states than $\mathcal{B}$ recognizing W. If $\mathcal{B}'$ still has two or more α-free-maxima, we repeat our construction until there is at most one left. ◀

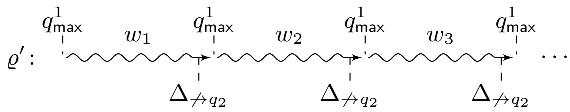

Figure 9 Features of run ϱ' when it takes infinitely many transitions in $\Delta_{\not\to q_2}$.

Thanks to Lemma 29, we now assume w.l.o.g. that $\mathcal{B}$ has at most one α-free-maximum. We intend to show that there exists an α-free-cycle-maximum. To do so, we exhibit an (infinite) arena in which $\mathcal{P}_1$ has no winning strategy, which we prove by using half-positionality of W over finite one-player arenas. We then prove that the non-existence of an α-free-cycle-maximum would imply that $\mathcal{P}_1$ has a winning strategy in this arena (Lemma 33).

Let $\mathcal{A}_{\mathcal{B}}$ be the infinite one-player arena of $\mathcal{P}_1$ depicted in Figure 10. This arena consists of one vertex v with a choice to make among all non-empty words that are α-free cycles from some state of $\mathcal{B}$. Vertex v is the only vertex with multiple outgoing edges. The goal of the next three short lemmas is to show that in this arena, $\mathcal{P}_1$ has no winning strategy.

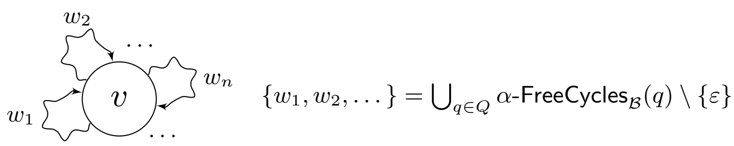

Figure 10 Infinite one-player arena $\mathcal{A}_{\mathcal{B}}$ of $\mathcal{P}_1$, with choices from v among every non-empty word in $\bigcup_{q \in Q} \alpha\text{-FreeCycles}_{\mathcal{B}}(q)$.

▶ **Lemma 30.** *If $\mathcal{P}_1$ has a winning strategy in $\mathcal{A}_{\mathcal{B}}$, then $\mathcal{P}_1$ has a positional winning strategy.*

Proof. Suppose that there is a winning strategy of $\mathcal{P}_1$ in $\mathcal{A}_{\mathcal{B}}$. Let $w = w_1w_2\dots$ be an infinite winning word such that for $i \geq 1$, $w_i \in \bigcup_{q \in Q} \alpha\text{-FreeCycles}_{\mathcal{B}}(q) \setminus \{\varepsilon\}$. Let $q_0 = q_{\mathsf{init}}$, and for $i \geq 1$, let $q_i = \delta^*(q_{\mathsf{init}}, w_1 \dots w_i)$ be the current automaton state after reading the first i finite words composing w. As there are only finitely many automaton states and w is winning, there are $k, l \geq 1$ with $k < l$ such that $q_k = q_l$ and $w_{k+1} \dots w_l \notin \alpha\text{-Free}_{\mathcal{B}}(q_k)$. Word $w_1 \dots w_k(w_{k+1} \dots w_l)^\omega$ is also a winning word and uses only finitely many different words in $\bigcup_{q \in Q} \alpha\text{-FreeCycles}_{\mathcal{B}}(q) \setminus \{\varepsilon\}$.

Hence, there is a finite restriction ("subarena") $\mathcal{A}'_{\mathcal{B}}$ of the arena $\mathcal{A}_{\mathcal{B}}$ with at most l choices in v in which $\mathcal{P}_1$ has a winning strategy. Arena $\mathcal{A}'_{\mathcal{B}}$ being finite and one-player, half-positionality of W over finite one-player arenas implies that $\mathcal{P}_1$ has a positional winning strategy in $\mathcal{A}'_{\mathcal{B}}$. This positional winning strategy can also be played in $\mathcal{A}_{\mathcal{B}}$ (as every choice available in $\mathcal{A}'_{\mathcal{B}}$ is also available in $\mathcal{A}_{\mathcal{B}}$). ◀

▶ **Lemma 31.** *No positional strategy of $\mathcal{P}_1$ is winning in $\mathcal{A}_{\mathcal{B}}$.*

Proof. Any positional strategy of $\mathcal{P}_1$ generates a word w^ω, where $w \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q) \setminus \{\varepsilon\}$ for some $q \in Q$. In particular, word w^ω is rejected when it is read from state q. As all the states in Q are equivalent for $\sim$ (as we assume that W is prefix-independent), we have $q \sim q_{\mathsf{init}}$, so w^ω is also rejected when read from the initial state q_{init} of the automaton. ◀

Using Lemmas 30 and 31, we deduce the desired result.

▶ **Lemma 32.** *No strategy of $\mathcal{P}_1$ is winning in $\mathcal{A}_{\mathcal{B}}$.*

We use the statement of Lemma 32 to show the existence of an α-free-cycle-maximum.

▶ **Lemma 33.** *There exists an α-free-maximum q_{max} in $\mathcal{B}$ that is a moreover an α-free-cycle-maximum: for all $q \in Q$, $\alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q) \subseteq \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_{\mathsf{max}})$.*

Proof. Let us assume by contradiction that there is no α-free-maximum, or if there is one, that it is not an α-free-cycle-maximum. We show how to build a winning strategy of $\mathcal{P}_1$ in $\mathcal{A}_{\mathcal{B}}$, contradicting Lemma 32. To do so, we build an infinite word accepted by $\mathcal{B}$ by combining finite words that are α-free cycles from some state.

We claim that for $q \in Q$ which is not an α-free-maximum, there exists

$$w_q \in \bigcup_{q' \in Q} \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q') \setminus \{\varepsilon\} \text{ such that } w_q \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q).$$

Let $q' \in Q$ be such that $\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q') \not\subseteq \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$, which exists as q is not an α-free-maximum. Let $w_1 \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q') \setminus \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$. By Lemma 10, there exists $w_2 \in C^*$ such that $w_1 w_2 \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q')$ (this holds as we have assumed w.l.o.g. that $\mathcal{B}$ is saturated). As $w_1 \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$, we also have $w_1 w_2 \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$. Taking $w_q = w_1 w_2$ proves the claim. For $q \in Q$ not an α-free-maximum, we fix $w_q \in C^+$ such that $w_q \in \bigcup_{q' \in Q} \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q') \setminus \{\varepsilon\}$ and $w_q \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$.

Let $q_{\mathsf{max}} \in Q$ be an α-free-maximum (that we suppose to be unique by Lemma 29), if it exists. We suppose by contradiction that q_{max} is not an α-free-cycle-maximum: there is $q \in Q$ such that $\alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q) \not\subseteq \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_{\mathsf{max}})$. Let $w_{\mathsf{max}} \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q) \setminus \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_{\mathsf{max}})$. Notice that as $w_{\mathsf{max}} \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$ and q_{max} is an α-free-maximum, $w_{\mathsf{max}} \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$. Therefore, w_{max} cannot be a cycle on q_{max}, i.e., $\delta^*(q_{\mathsf{max}}, w_{\mathsf{max}}) \neq q_{\mathsf{max}}$.

We build iteratively an infinite winning word that can be played by $\mathcal{P}_1$ in $\mathcal{A}_{\mathcal{B}}$. As $\mathcal{P}_1$ plays, we keep track in parallel of the current automaton state. The game starts in v, with current automaton state $q_0 = q_{\mathsf{init}}$. Let $n \geq 0$. We distinguish two cases.

- If q_n is not an α-free-maximum, then $\mathcal{P}_1$ plays word w_{q_n}. As $w_{q_n} \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_n)$, a Büchi transition is seen along the way. The current automaton state becomes $q_{n+1} = \delta^*(q_n, w_{q_n})$.
- If $q_n = q_{\mathsf{max}}$ is the α-free-maximum, then $\mathcal{P}_1$ plays w_{max}. The current automaton state becomes $q_{n+1} = \delta(q_{\mathsf{max}}, w_{\mathsf{max}})$, which is not equal to q_{max}.

For infinitely many $i \geq 0$, the automaton state q_i is not an α-free-maximum (as there is at most one α-free-maximum in $\mathcal{B}$, and it cannot appear twice in a row). Therefore, we have described a winning strategy for $\mathcal{P}_1$, since the corresponding run over $\mathcal{B}$ visits infinitely often a Büchi transition. ◀

We now know that there exists a unique α-free-maximum $q_{\mathsf{max}} \in Q$, and moreover, that it is an α-free-cycle-maximum. We show how to use the outgoing transitions of q_{max} in order to realize W as an objective of the kind $\mathsf{Büchi}(F)$ for some $F \subseteq C$, which is the goal of the current subsection.

▶ **Lemma 34.** *There exists $F \subseteq C$ such that $W = \mathsf{Büchi}(F)$.*

Proof. Let $F = \{c \in C \mid (q_{\mathsf{max}}, c) \in \alpha\}$ — equivalently, if we consider colors as words with one letter, F is the set of colors c such that $c \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$.

We first show that $\mathsf{Büchi}(F) \subseteq W$. Let $c \in F$. Then, $c \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$. As q_{max} is an α-free-maximum, for all $q \in Q$, $c \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$. Therefore, any word seeing infinitely many colors in F sees infinitely many Büchi transitions and is accepted by $\mathcal{B}$.

We now show that $W \subseteq \mathsf{Büchi}(F)$. By contrapositive, let $w = c_1 c_2 \ldots \notin \mathsf{Büchi}(F)$ be an infinite word with only finitely many colors in F. We show that $w \notin W$. As W is prefix-independent, we may assume w.l.o.g. that w has no color in F, i.e., that for all $i \geq 1$, $c_i \in C \setminus F$. We claim that when read from q_{max}, word w sees no Büchi transition and is thus rejected. This implies that $w \notin W$ as $q_{\mathsf{max}} \sim q_{\mathsf{init}}$.

Assume by contradiction that there is some Büchi transition when reading w from q_{max}, i.e., there exists $k \geq 0$ such that for $w_{\leq k} = c_1 \ldots c_k$, $w_{\leq k} \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$, but $w_{\leq k} c_{k+1} \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$. We will deduce that $(q_{\mathsf{max}}, c_{k+1})$ is a Büchi transition, contradicting that $c_{k+1} \in C \setminus F$.

We depict the situation in Figure 11. Let $q_1 = \delta^*(q_{\mathsf{max}}, w_{\leq k})$ (whether q_1 equals q_{max} or not does not matter). By Lemma 10, there exists $w' \in C^*$ such that $w_{\leq k} w' \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_{\mathsf{max}})$. By construction, we have $w' w_{\leq k} \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_1)$. As q_{max} is an α-free-cycle-maximum, we have $\alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_1) \subseteq \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_{\mathsf{max}})$, so we also have $w' w_{\leq k} \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q_{\mathsf{max}})$. Let $q_2 = \delta^*(q_{\mathsf{max}}, w')$. Notice that $q_{\mathsf{max}} = \delta^*(q_2, w_{\leq k})$ and $w_{\leq k} \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_2)$. As q_{max} is an α-free-maximum and $w_{\leq k} c_{k+1} \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$, we also have that $w_{\leq k} c_{k+1} \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_2)$. Therefore, transition $(q_{\mathsf{max}}, c_{k+1})$ must be a Büchi transition, which contradicts that $c_{k+1} \in C \setminus F$. ◀

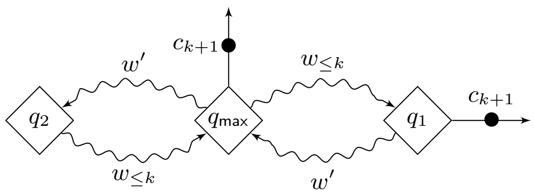

Figure 11 Situation in the proof of Lemma 34, with $w_{\leq k} \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$ but $w_{\leq k} c_{k+1} \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$.

4.3.2 General case

We now relax the prefix-independence assumption on W. We still assume that W is half-positional over finite one-player arenas, and show that W can be recognized by a Büchi automaton built on top of $\mathcal{S}_\sim$. If $\mathcal{B}$ has exactly one state per equivalence class of $\sim$, it means that it is built on top of $\mathcal{S}_\sim$, and we are done. If not, let $q_\sim \in Q$ be a state such that $|[q_\sim]| \geq 2$.

We briefly sketch the proof technique for this section.

Proof sketch. Our proof will show how to modify $\mathcal{B}$ by "merging" all states in equivalence class $[q_\sim]$ into a single state, while still recognizing the same objective W. The main technical argument is to build a variant $W_{[q_\sim]}$ of objective W on a new set of colors $C_{[q_\sim]}$, that turns out to also be half-positional over finite one-player arenas and DBA-recognizable, but which is *prefix-independent*. We can therefore use Lemma 34 from Section 4.3.1 and find $F_{[q_\sim]} \subseteq C_{[q_\sim]}$ such that $W_{[q_\sim]} = \mathsf{Büchi}(F_{[q_\sim]})$. Then, we exhibit a state $q_{\mathsf{max}} \in [q_\sim]$ whose α-free words are tightly linked to the elements of $F_{[q_\sim]}$ (Lemma 35 and Corollary 36). Finally, akin to the way we removed an α-free-maximum in Lemma 29, we show that it is still possible to recognize W while keeping only state q_{max} in $[q_\sim]$ (Lemma 37).

Once we know how to merge the equivalence class $[q_\sim]$ into a single state, we can simply repeat the operation for each equivalence class with multiple states, until we obtain a DBA built on top of $\mathcal{S}_\sim$. ◀

We define a new set of colors $C_{[q_\sim]}$ using finite words in C^+ such that

$$C_{[q_\sim]} = \{w \in C^+ \mid \delta^*(q_\sim, w) \sim q_\sim\}.$$

This set contains all the finite words that, read from $q_\sim$, come back to a state in $[q_\sim]$. By Lemma 7, for all $q \in [q_\sim]$, for all $w \in C_{[q_\sim]}$, we also have that $\delta^*(q, w) \sim \delta^*(q_\sim, w) \sim q_\sim$. The set $C_{[q_\sim]}$ therefore corresponds to the set of words with the seemingly stronger property that, when read from any state in $[q_\sim]$, come back to a state in $[q_\sim]$. We define an objective $W_{[q_\sim]}$ of infinite words on this new set of colors such that

$$W_{[q_\sim]} = \{w_1 w_2 \ldots \in C_{[q_\sim]}^\omega \mid w_1 w_2 \ldots \in q_\sim^{-1} W\}.$$

We show that $W_{[q_\sim]}$ has the three conditions allowing us to apply Lemma 34 to it.

- Objective $W_{[q_\sim]}$ is DBA-recognizable: we consider the DBA $\mathcal{B}_{[q_\sim]} = ([q_\sim], C_{[q_\sim]}, q_\sim, \Delta', \alpha')$, whose update function δ' is the restriction of δ^* to $[q_\sim] \times C_{[q_\sim]}$, and $\alpha' = \{(q, w) \in [q_\sim] \times C_{[q_\sim]} \mid w \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)\}$.
- Objective $W_{[q_\sim]}$ is prefix-independent, as adding or removing a finite number of cycles on $q_\sim$ does not affect the accepted status of a word in $q_\sim^{-1} W$.
- Half-positionality of $W_{[q_\sim]}$ over finite one-player arenas is implied by half-positionality of W over finite one-player arenas. Indeed, every (finite one-player) arena $\mathcal{A}_{[q_\sim]}$ using colors in $C_{[q_\sim]}$ can be transformed into a (finite one-player) arena $\mathcal{A}$ with similar properties using colors in C. Two transformations are applied: (i) we replace every $C_{[q_\sim]}$-colored edge in $\mathcal{A}_{[q_\sim]}$ by a corresponding finite chain of C-colored edges, and (ii) for every vertex v of $\mathcal{A}_{[q_\sim]}$, we prefix it with a chain of C-colored edges starting from a vertex v' reading a word $w_{q_\sim}$ such that $\delta^*(q_{\mathsf{init}}, w_{q_\sim}) = q_\sim$. We then have that $\mathcal{P}_1$ has a winning strategy from a vertex v in $\mathcal{A}_{[q_\sim]}$ if and only if $\mathcal{P}_1$ has a winning strategy from v' in $\mathcal{A}$, and a positional winning strategy from v' in $\mathcal{A}$ can be transformed into a positional winning strategy from v in $\mathcal{A}_{[q_\sim]}$.

By Lemma 34, there exists a set $F_{[q_\sim]} \subseteq C_{[q_\sim]}$ such that $W_{[q_\sim]} = \mathsf{Büchi}(F_{[q_\sim]})$.

We now show links between α-free words from states of $[q_\sim]$ and the words in $F_{[q_\sim]}$. The arguments once again rely on the saturation of $\mathcal{B}$.

▶ **Lemma 35.**

- *Let $w \in C_{[q_\sim]}$. If $w \in F_{[q_\sim]}$, then for all $q \in [q_\sim]$, $w \notin \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$.*
- *There exists $q \in [q_\sim]$ such that, for all $w \in C_{[q_\sim]} \setminus F_{[q_\sim]}$, $w \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$.*

Proof. For the first item, we assume by contrapositive that there exists a state $q \in [q_\sim]$ such that $w \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$. By Lemma 10, there is $w' \in C^*$ such that $ww' \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q)$.

In particular, $(ww')^\omega \notin q^{-1}W = q_\sim^{-1}W$. We can assume w.l.o.g. that $w' \in C^+$ (if $w' = \varepsilon$, then we can simply replace it with $w' = w$). Therefore, $(ww')^\omega$ is also an infinite word on $C_{[q_\sim]}$, and we have $(ww')^\omega \notin W_{[q_\sim]}$ since $(ww')^\omega \notin q_\sim^{-1}W$. As $W_{[q_\sim]} = \mathsf{Büchi}(F_{[q_\sim]})$, clearly $w \notin F_{[q_\sim]}$, which ends the proof of the first item.

For the second item, assume by contradiction that for all $q \in [q_\sim]$, there exists $w_q \in C_{[q_\sim]} \setminus F_{[q_\sim]}$ such that $w_q \notin \alpha\text{-}\mathsf{Free}_\mathcal{B}(q)$. As there are only finitely many states in $[q_\sim]$, it is then possible to build a word $w = w_{q_1} \dots w_{q_n}$ such that for all $1 \le i < n$, $\delta^*(q_i, w_{q_i}) = q_{i+1}$, $\delta^*(q_n, w_{q_n}) = q_1$, and for all $1 \le i \le n$, $w_{q_i} \in C_{[q_\sim]} \setminus F_{[q_\sim]}$ and $w_{q_i} \notin \alpha\text{-}\mathsf{Free}_\mathcal{B}(q_i)$. Word w^ω is accepted from q_1 as it sees infinitely many Büchi transitions, so it is in $(q_1)^{-1}W = q_\sim^{-1}W$. However, if we consider w^ω as an infinite word on $C_{[q_\sim]}$, then it is not in $W_{[q_\sim]} = \mathsf{Büchi}(F_{[q_\sim]})$ as every letter of the word is in $C_{[q_\sim]} \setminus F_{[q_\sim]}$. This yields a contradiction. ◀

We use the previous result in a straightforward way to exhibit a state q_{max} whose non-α-free words in $C_{[q_\sim]}$ are exactly the words in $F_{[q_\sim]}$. The reader may notice that, echoing the proof of the prefix-independent case (Section 4.3.1), the state q_{max} given by Corollary 36 is actually an α-free-maximum among states in $[q_\sim]$.

▶ **Corollary 36.** *There exists $q_{\mathsf{max}} \in [q_\sim]$ such that for all $w \in C_{[q_\sim]}$, $w \in F_{[q_\sim]}$ if and only if $w \notin \alpha\text{-}\mathsf{Free}_\mathcal{B}(q_{\mathsf{max}})$.*

Proof. By the second item of Lemma 35, we take $q_{\mathsf{max}} \in [q_\sim]$ such that, for all $w \in C_{[q_\sim]} \setminus F_{[q_\sim]}$, $w \in \alpha\text{-}\mathsf{Free}_\mathcal{B}(q_{\mathsf{max}})$. Let $w \in C_{[q_\sim]}$. The property we already have on q_{max} gives us by contrapositive that $w \notin \alpha\text{-}\mathsf{Free}_\mathcal{B}(q_{\mathsf{max}})$ implies that $w \in F_{[q_\sim]}$. Reciprocally, the first item of Lemma 35 gives us that if $w \in F_{[q_\sim]}$, then $w \notin \alpha\text{-}\mathsf{Free}_\mathcal{B}(q_{\mathsf{max}})$. ◀

From now on, we assume that $q_{\mathsf{max}} \in [q_\sim]$ is a state having the property of Corollary 36. We show that W can be recognized by a smaller DBA consisting of DBA $\mathcal{B}$ in which all the states in $[q_\sim]$ have been merged into the single state q_{max}, by redirecting all incoming transitions of $[q_\sim]$ to q_{max}. We assume w.l.o.g. that if $q_{\mathsf{init}} \in [q_\sim]$, then $q_{\mathsf{init}} = q_{\mathsf{max}}$ (this does not change the objective recognized by $\mathcal{B}$, and will be convenient in the upcoming construction). We consider DBA $\mathcal{B}' = (Q', C, q'_{\mathsf{init}}, \Delta', \alpha')$ with

- $Q' = (Q \setminus [q_\sim]) \cup \{q_{\mathsf{max}}\}$, $q'_{\mathsf{init}} = q_{\mathsf{init}}$,
- for $q \in Q'$ and $c \in C$, if $\delta(q, c) \in [q_\sim]$, then $\delta'(q, c) = q_{\mathsf{max}}$; otherwise, $\delta'(q, c) = \delta(q, c)$,
- for $q \in Q'$ and $c \in C$, $(q, c) \in \alpha'$ if and only if $(q, c) \in \alpha$.

We also assume that states that are not reachable from q'_{init} in $\mathcal{B}'$ are removed from Q'.

▶ **Lemma 37.** *The objective recognized by $\mathcal{B}'$ is also W.*

Proof. Let $w = c_1 c_2 \ldots \in C^\omega$, $\varrho = (q_0, c_1, q_1)(q_1, c_2, q_2) \ldots$ be the run of $\mathcal{B}$ on w, and $\varrho' = (q'_0, c_1, q'_1)(q'_1, c_2, q'_2) \ldots$ be the run of $\mathcal{B}'$ on w. We have that $q_0 = q'_0$, but states in both runs may not coincide after a state in $[q_\sim]$ has been reached. Yet, we show inductively that

$$\forall i \ge 0, q_i \sim_\mathcal{B} q'_i. \tag{3}$$

It is true for $i = 0$ (we even have equality in this case), and if $q_n \sim_\mathcal{B} q'_n$, then by construction of the transitions of $\mathcal{B}'$ and Lemma 7, we still have $q_{n+1} \sim_\mathcal{B} q'_{n+1}$.

We want to show that w is accepted by $\mathcal{B}$ if and only if it is accepted by $\mathcal{B}'$. This is clear if w never goes through a state in $[q_\sim]$ (as the same transitions are then taken in $\mathcal{B}$ and $\mathcal{B}'$).

We first assume that run ϱ visits $[q_\sim]$ finitely many times, and that the last visit to $[q_\sim]$ happens in q_n for some $n \ge 0$. Notice that run ϱ' also visits $[q_\sim]$ for the last time in q'_n by Equation (3). Therefore, word $c_{n+1}c_{n+2}\ldots$ is accepted from q'_n in $\mathcal{B}'$ if and only if it is accepted from q'_n in $\mathcal{B}$: all the subsequent transitions coincide. As $q_n \sim_\mathcal{B} q'_n$, we have moreover that $c_{n+1}c_{n+2}\dots$ is accepted from q'_n in $\mathcal{B}'$ if and only if it is accepted from q_n in $\mathcal{B}$. This implies that w is accepted by $\mathcal{B}$ if and only if it is accepted by $\mathcal{B}'$.

We now assume that run ϱ visits $[q_\sim]$ infinitely many times. We decompose w inductively into a prefix $w_{q_\sim}$ reaching $[q_\sim]$ followed by cycles $w_1, w_2, \dots$ on $[q_\sim]$. Formally, let $w_{q_\sim}$ be any finite prefix of w such that $\delta^*(q_{\mathsf{init}}, w_{q_\sim}) \in [q_\sim]$. By induction hypothesis, assume $w_{q_\sim}w_1\dots w_n$ is a prefix of w such that $\delta^*(q_{\mathsf{init}}, w_{q_\sim}w_1\dots w_n) \in [q_\sim]$. We define $w_{n+1} \in C^+$ as the shortest non-empty word such that $w_{q_\sim}w_1\dots w_n w_{n+1}$ is a prefix of w and $\delta^*(q_{\mathsf{init}}, w_{q_\sim}w_1\dots w_n w_{n+1}) \in [q_\sim]$. We have $w = w_{q_\sim}w_1w_2\dots$ by construction. By Equation (3), we also have that for all $n \geq 0$, $(\delta')^*(q'_{\mathsf{init}}, w_q w_1 \dots w_n) \sim_{\mathcal{B}} q_\sim$, so $(\delta')^*(q'_{\mathsf{init}}, w_q w_1 \dots w_n) = q_{\mathsf{max}}$ as this is the only state left in that class in $\mathcal{B}'$.

If w is accepted by $\mathcal{B}$, then for infinitely many $i \geq 1$, word w_i is in $F_{[q_\sim]}$. By Corollary 36, all these infinitely many words are not in $\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$ and therefore see a Büchi transition when read from q_{max}, so w is also accepted by $\mathcal{B}'$.

If w is rejected by $\mathcal{B}$, then there exists $n \geq 1$ such that for all $n' \geq n$, $w_{n'} \in C_{[q_\sim]} \setminus F_{[q_\sim]}$. By Corollary 36, for all $n' \geq n$, $w_{n'}$ is in $\alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q_{\mathsf{max}})$ and therefore does not see a Büchi transition when read from q_{max}, so w is also rejected by $\mathcal{B}'$. ◀

We have all the arguments to show our goal for the section (Proposition 28), that is, to show that W can be recognized by a Büchi automaton built on top of $\mathcal{S}_\sim$.

Proof of Proposition 28. We have shown in Lemma 37 how to merge an equivalence class of $\mathcal{B}$ into a single state, while still recognizing the same objective. Repeating this construction for each equivalence class with two or more states, we end up with a DBA with exactly one state per equivalence class of $\sim$ still recognizing W. By definition of $\mathcal{S}_\sim$, this DBA is necessarily built on top of (some automaton isomorphic to) $\mathcal{S}_\sim$. ◀

5 Sufficiency of the conditions

We show that a DBA $\mathcal{B}$ with the three conditions from Section 3.1 (recognizing a progress-consistent objective having a total prefix preorder and being recognizable by a Büchi automaton built on top of $\mathcal{S}_\sim$) recognizes a half-positional objective. As these three conditions have been shown to be necessary for the half-positionality of objectives recognized by a DBA, this will imply a characterization of half-positionality.

Our main technical tool is to construct, thanks to these three conditions, a family of *completely well-monotonic universal graphs*. The existence of such objects implies thanks to recent result [52] that $\mathcal{P}_1$ has positional optimal strategies, even in two-player arenas of arbitrary cardinality.

5.1 Completely well-monotonic universal graphs

We fix extra terminology about graphs only used in Section 5, and recall the relevant results from [52].

Extra preliminaries on graphs. Let $\mathcal{G} = (V, E)$ be a graph and $W \subseteq C^\omega$ be an objective. A vertex v of $\mathcal{G}$ *satisfies* W if for all infinite paths $v_0 \xrightarrow{c_1} v_1 \xrightarrow{c_2} \dots$ from v, we have $c_1c_2\dots \in W$.

Given two graphs $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$, a *(graph) morphism from $\mathcal{G}$ to $\mathcal{G}'$* is a function $\phi\colon V \to V'$ such that $(v_1, c, v_2) \in E$ implies $(\phi(v_1), c, \phi(v_2)) \in E'$.

A morphism ϕ from $\mathcal{G}$ to $\mathcal{G}'$ is W*-preserving* if for all $v \in V$, v satisfies W implies that $\phi(v)$ satisfies W. Notice that if $\phi(v)$ satisfies W, then v satisfies W, as any path $v \xrightarrow{c_1} v_1 \xrightarrow{c_2} \dots$ of

$\mathcal{G}$ implies the existence of a path $\phi(v) \xrightarrow{c_1} \phi(v_1) \xrightarrow{c_2} \ldots$ of $\mathcal{G}'$ — there are "more paths" in $\mathcal{G}'$. A graph $\mathcal{U}$ is (κ, W)-*universal* if for all graphs $\mathcal{G}$ of cardinality $\leq \kappa$, there is a W-preserving morphism from $\mathcal{G}$ to $\mathcal{U}$.

We consider a graph $\mathcal{G} = (V, E)$ along with a total order $\leq$ on its vertex set V. We say that $\mathcal{G}$ is *monotonic* if for all $v, v', v'' \in V$, for all $c \in C$,

- $(v \xrightarrow{c} v' \text{ and } v' \geq v'') \implies v \xrightarrow{c} v''$, and
- $(v \geq v' \text{ and } v' \xrightarrow{c} v'') \implies v \xrightarrow{c} v''$.

This means that (i) whenever there is an edge $v \xrightarrow{c} v'$, there is also an edge with color c from v to all states smaller than v' for $\leq$, and (ii) whenever $v \geq v'$, then v has at least the same outgoing edges as v'. Graph $\mathcal{G}$ is *well-monotonic* if it is monotonic and the total order $\leq$ is a well-order (i.e., any set of vertices has a minimum). Graph $\mathcal{G}$ is *completely well-monotonic* if it is well-monotonic and there exists a vertex $\top \in V$ maximum for $\leq$ such that for all $v \in V$, $c \in C$, $\top \xrightarrow{c} v$.

▶ **Example 38.** We provide an example illustrating these notions with $C = \{a, b\}$ and $W = \mathsf{Büchi}(\{a\})$. This is a special case of our upcoming construction, and it is already discussed in more depth in [52, Chapter 2]. For θ an ordinal, we define a graph $\mathcal{U}_\theta$ with vertices $U_\theta = \theta \cup \{\top\}$, and such that for all $\lambda, \lambda' < \theta$,

- $\lambda \xrightarrow{a} \lambda'$, and
- $\lambda \xrightarrow{b} \lambda'$ if and only if $\lambda' < \lambda$.

Moreover, for all $v \in U_\theta$, we define edges $\top \xrightarrow{a} v$ and $\top \xrightarrow{b} v$. We order vertices using the natural order on the ordinals, and with $\lambda < \top$ for all $\lambda < \theta$.

Vertex $\top$ does not satisfy W, as there is an infinite path $\top \xrightarrow{b} \top \xrightarrow{b} \ldots$, and $b^\omega \notin W$. All other vertices satisfy W by construction: they cannot reach $\top$, and as reading b decreases the current ordinal, a path cannot have an infinite suffix using only color b (there is no infinite decreasing sequence of ordinals).

This graph is (κ, W)-universal for $\kappa < |\theta|$. Intuitively, for any graph $\mathcal{G}$ with at most κ vertices, a W-preserving morphism from $\mathcal{G}$ to $\mathcal{U}_\theta$ can be defined by mapping vertices not satisfying W to $\top$, and vertices satisfying W to an ordinal λ that depends on how long it may take to guarantee seeing an a from them.

Graph $\mathcal{U}_\theta$ is monotonic (which can be quickly checked with the definition), and as $\leq$ is a well-order and there is a vertex $\top$ with the right properties, $\mathcal{U}_\theta$ is even completely well-monotonic. The properties of $\mathcal{U}_\theta$ imply half-positionality of $\mathsf{Büchi}(\{a\})$, thanks to the following theorem. ⌟

We state an important result linking half-positionality and completely well-monotonic universal graphs from [52].

▶ **Theorem 39** (Consequence of [52, Theorem 1.1])**.** *Let $W \subseteq C^\omega$ be an objective. If for all cardinals κ, there exists a completely well-monotonic (κ, W)-universal graph, then W is half-positional (over all arenas).*

The exact result [52, Theorem 1.1] can actually be instantiated on more precise classes of arenas. However, we use it to prove here half-positionality of a family of objectives over *all* arenas, so the above result turns out to be sufficient.

▶ Remark 40. Our approach also implies the half-positionality of W over arenas with ε-edges (cf. Remark 1). Indeed, we can add a fresh color e to C and define an objective $W_e \subseteq (C \cup \{e\})^\omega$ such that for $w \in (C \cup \{e\})^\omega$,

$$w \in W_e \iff \begin{cases} \text{the word obtained by removing } e \text{ is infinite and belongs to } W\text{, or} \\ w = w'e^\omega \text{ for some } w' \in (C \cup \{e\})^*. \end{cases}$$

Then, the existence of a family of completely well-monotonic (κ, W)-universal graphs for W implies that W_e is half-positional [52] (in the terminology of [52], e is a *strongly neutral letter*). Therefore, we can label the ε-edges of any arena with e and obtain an equivalent game with objective W_e. ⌟

5.2 Universal graphs for Büchi automata

We show that for a DBA-recognizable objective, the three conditions that were shown to be necessary for half-positionality in Section 4 are actually sufficient.

▶ **Proposition 41.** *Let $W \subseteq C^\omega$ be an objective that has a total prefix preorder, is progress-consistent, and is recognizable by a Büchi automaton built on top of $\mathcal{S}_\sim$. Then, W is half-positional.*

The rest of the section is devoted to the proof of this result, using Theorem 39. Let $W \subseteq C^\omega$ be an objective with a total prefix preorder, that is progress-consistent, and that is recognized by a DBA $\mathcal{B} = (Q, C, q_{\mathsf{init}}, \Delta, \alpha)$ built on top of $\mathcal{S}_\sim$ for the rest of this section. We assume as in previous sections that $\mathcal{B}$ is saturated (in particular, by Lemma 10, any α-free path can be extended to an α-free cycle). An implication of the fact that $\mathcal{B}$ is built on top of $\mathcal{S}_\sim$ that we will use numerous times in the upcoming arguments is that for $q, q' \in Q$, $q \sim q'$ if and only if $q = q'$.

For θ an ordinal, we build a graph $\mathcal{U}_{\mathcal{B},\theta}$ in the following way.

- We set the vertices as $U_{\mathcal{B},\theta} = \{(q, \lambda) \mid q \in Q, \lambda < \theta\} \cup \{\top\}$.
- For every transition $\delta(q, c) = q'$ of $\mathcal{B}$,
  - if $(q, c) \in \alpha$, then for all ordinals λ, λ', we define an edge $(q, \lambda) \xrightarrow{c} (q', \lambda')$;
  - if $(q, c) \notin \alpha$, then for all ordinals λ, λ' s.t. $\lambda' < \lambda$, we define an edge $(q, \lambda) \xrightarrow{c} (q', \lambda')$.
  - for $q'' \prec q'$, then for all ordinals λ, λ'', we define an edge $(q, \lambda) \xrightarrow{c} (q'', \lambda'')$.
- For all $c \in C$, $v \in U_{\mathcal{B},\theta}$, we define an edge $\top \xrightarrow{c} v$.

We order the vertices lexicographically: $(q, \lambda) \le (q', \lambda')$ if $q \prec q'$ or ($q = q'$ and $\lambda \le \lambda'$), and we define $\top$ as the maximum for $\le$ ($(q, \lambda) < \top$ for all $q \in Q$, $\lambda < \theta$).

Graph $\mathcal{U}_{\mathcal{B},\theta}$ is built such that on the one hand, it is sufficiently large and has sufficiently many edges so that there is a morphism from any graph $\mathcal{G}$ (of cardinality smaller than some function of $|\theta|$) to $\mathcal{U}_{\mathcal{B},\theta}$. On the other hand, for the morphism to be W-preserving, at least some vertices of $\mathcal{U}_{\mathcal{B},\theta}$ need to satisfy W, which imposes a restriction on the infinite paths from vertices. Graph $\mathcal{U}_{\mathcal{B},\theta}$ is actually built so that for any automaton state $q \in Q$ and ordinal $\lambda < \theta$, the vertex (q, λ) satisfies $q^{-1}W$ (see Lemma 46). The intuitive idea is that for a non-Büchi transition $(q, c) \notin \alpha$ of the automaton such that $\delta(q, c) = q'$, a c-colored edge from a vertex (q, λ) in the graph either (i) reaches a vertex with first component q', in which case the ordinal must decrease on the second component, or (ii) reaches a vertex with first component $q'' \prec q'$, with no restriction on the second component, but therefore with fewer winning continuations. Using progress-consistency and the fact that there is no infinitely decreasing sequence of ordinals, we can show that this implies that no infinite path in $\mathcal{U}_{\mathcal{B},\theta}$ corresponds to an infinite run in the automaton visiting only non-Büchi transitions.

We state two properties that directly follow from the definition of $\mathcal{U}_{\mathcal{B},\theta}$:

$$\text{if } (q, \lambda) \xrightarrow{c} (q', \lambda'), \text{ then } q' \preceq \delta(q, c); \tag{4}$$

$$\text{if } (q, \lambda) \xrightarrow{c} (q', \lambda') \text{ and } \lambda'' \le \lambda', \text{ then } (q, \lambda) \xrightarrow{c} (q', \lambda''). \tag{5}$$

▶ **Example 42.** We consider again the DBA $\mathcal{B}$ from Example 16, recognizing the words seeing a infinitely many times, or a twice in a row at some point. We represent the graph $\mathcal{U}_{\mathcal{B},\theta}$, with $\theta = \omega$ in Figure 12. ⌟

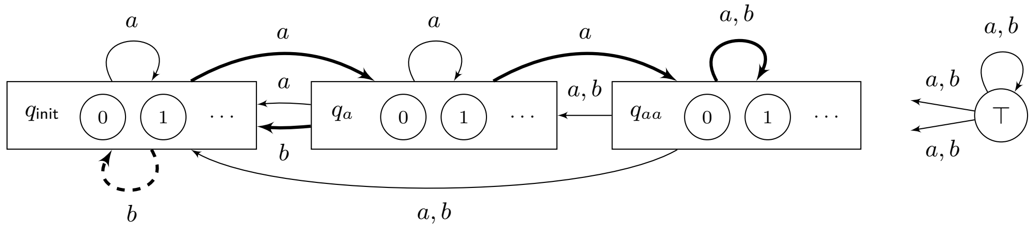

Figure 12 The graph $\mathcal{U}_{\mathcal{B},\omega}$, where $\mathcal{B}$ is the automaton from Example 16 ($\mathcal{L}(\mathcal{B}) = \mathsf{Büchi}(\{a\}) \cup C^*aaC^\omega$). The dashed edge with color b indicates that $(q_{\mathsf{init}}, \lambda) \xrightarrow{b} (q_{\mathsf{init}}, \lambda')$ if and only if $\lambda' < \lambda$ (it corresponds to the only non-Büchi transition in $\mathcal{B}$). Elsewhere, an edge between two rectangles labeled q, q' with color c means that for all ordinals λ, λ', $(q, \lambda) \xrightarrow{c} (q, \lambda')$. Thick edges correspond to the original transitions of $\mathcal{B}$. There are edges from $\top$ to all vertices of the graph with colors a and b. Vertices are totally ordered from left to right.

In order to use Theorem 39, we show that the graph $\mathcal{U}_{\mathcal{B},\theta}$ is completely well-monotonic (Lemma 43) and, for all cardinals κ, is (κ, W)-universal for sufficiently large θ (Proposition 47).

▶ **Lemma 43.** *Graph $\mathcal{U}_{\mathcal{B},\theta}$ is completely well-monotonic.*

Proof. The order $\leq$ on the vertices is a well-order, and there exists a vertex $\top \in U_{\mathcal{B},\theta}$ maximum for $\leq$ such that for all $v \in U_{\mathcal{B},\theta}$, $c \in C$, $\top \xrightarrow{c} v$. To show that $\mathcal{U}_{\mathcal{B},\theta}$ is completely well-monotonic, it now suffices to show that $\mathcal{U}_{\mathcal{B},\theta}$ is monotonic.

The first item of the monotonicity definition follows from the construction of the graph. We assume that $(q, \lambda) \xrightarrow{c} (q', \lambda')$ and $(q', \lambda') \geq (q'', \lambda'')$, and we show that $(q, \lambda) \xrightarrow{c} (q'', \lambda'')$. By Equation (4), we have $q' \preceq \delta(q, c)$. The inequality $(q', \lambda') \geq (q'', \lambda'')$ means by definition that $q'' \prec q'$ or ($q'' = q'$ and $\lambda'' \leq \lambda'$). If $q'' \prec q'$, we obtain that $q'' \prec \delta(q, c)$, so we also have $(q, \lambda) \xrightarrow{c} (q'', \lambda'')$. If $q'' = q'$ and $\lambda'' \leq \lambda'$, then we also have $(q, \lambda) \xrightarrow{c} (q', \lambda'')$ by Equation (5), which is what we want as $q' = q''$.

The second item of the monotonicity definition is slightly more involved and follows from progress-consistency, the fact that the prefix preorder is total, and the saturation of $\mathcal{B}$. We assume that $(q, \lambda) \geq (q', \lambda')$ and $(q', \lambda') \xrightarrow{c} (q'', \lambda'')$, and we show that $(q, \lambda) \xrightarrow{c} (q'', \lambda'')$. The assumption $(q, \lambda) \geq (q', \lambda')$ implies that $q \succeq q'$, so $\delta(q, c) \succeq \delta(q', c)$ by Lemma 14. Moreover, $(q', \lambda') \xrightarrow{c} (q'', \lambda'')$ implies that $\delta(q', c) \succeq q''$ by Equation (4). Hence, $\delta(q, c) \succeq q''$. If $\delta(q, c) \succ q''$, then $(q, \lambda) \xrightarrow{c} (q'', \lambda'')$ by definition of the graph. The same holds if $\delta(q, c) = q''$ and $(q, c) \in \alpha$.

It is left to discuss the case $\delta(q, c) = q''$ and $(q, c) \notin \alpha$. By the above inequalities, this implies that we also have $\delta(q', c) = q''$.

- If $q = q'$, then $\lambda \geq \lambda'$. Moreover, as $(q', c) = (q, c) \notin \alpha$, the existence of edge $(q', \lambda') \xrightarrow{c} (q'', \lambda'')$ implies that $\lambda' > \lambda''$. So $\lambda > \lambda''$ and we also have $(q, \lambda) \xrightarrow{c} (q'', \lambda'')$.
- We show that $q' \prec q$ is not possible — we assume it holds and draw a contradiction. As $(q, c) \notin \alpha$, we have $c \in \alpha\text{-}\mathsf{Free}_{\mathcal{B}}(q)$. By Lemma 10, there is $w \in C^*$ such that $cw \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q)$. As $\delta(q, c) = \delta(q', c)$ and $\delta^*(q, cw) = q$, we have $\delta^*(q', cw) = q$. As $q' \prec q$, by progress-consistency, the word $(cw)^\omega$ must be accepted by $\mathcal{B}$ when read from q'. It must therefore also be accepted when read from q (as $q' \prec q$), which contradicts that $cw \in \alpha\text{-}\mathsf{FreeCycles}_{\mathcal{B}}(q)$. ◀

We now intend to show (κ, W)-universality of some $\mathcal{U}_{\mathcal{B},\theta}$ for all cardinals κ. Lemmas 44 and 45 give insight on properties of the paths of $\mathcal{U}_{\mathcal{B},\theta}$, to then establish which vertices of $\mathcal{U}_{\mathcal{B},\theta}$ satisfy W (Lemma 46). Understanding which vertices satisfy W is useful to later define a W-preserving morphism into $\mathcal{U}_{\mathcal{B},\theta}$. We first show that paths in this graph "underapproximate" corresponding runs in the automaton: a finite path $\gamma = (q_0, \lambda_0) \xrightarrow{c_1} \dots \xrightarrow{c_n} (q_n, \lambda_n)$ in $\mathcal{U}_{\mathcal{B},\theta}$ visits vertices labeled by automaton states at most as large (for $\preceq$) as the corresponding states visited by the run from q_0 on $c_1 \dots c_n$.

▶ **Lemma 44.** *Let $\gamma = (q_0, \lambda_0) \xrightarrow{c_1} \dots \xrightarrow{c_n} (q_n, \lambda_n)$ be a finite path of $\mathcal{U}_{\mathcal{B},\theta}$ and $w = c_1 \dots c_n$. Then, $q_n \preceq \delta^*(q_0, w)$.*

Proof. We proceed by induction on the length n of γ. If $n = 0$, then $w = \varepsilon$, so $q_n = q_0 = \delta^*(q_0, w)$, and the result holds. If $n \geq 1$, we assume by induction hypothesis that $q_{n-1} \preceq \delta^*(q_0, c_1 \dots c_{n-1})$. By Lemma 14 and the fact that the prefix preorder is total, we have $\delta(q_{n-1}, c_n) \preceq \delta(\delta^*(q_0, c_1 \dots c_{n-1}), c_n) = \delta^*(q_0, w)$. By Equation (4), if $(q_{n-1}, \lambda_{n-1}) \xrightarrow{c_n} (q_n, \lambda_n)$, then $q_n \preceq \delta(q_{n-1}, c_n)$. By transitivity, $q_n \preceq \delta^*(q_0, w)$. ◀

We now show that in $\mathcal{U}_{\mathcal{B},\theta}$, a finite path that goes back to its initial value w.r.t. the first component without decreasing the ordinal necessarily induces an accepted word when repeated.

▶ **Lemma 45.** *Let $\gamma = (q_0, \lambda_0) \xrightarrow{c_1} \dots \xrightarrow{c_n} (q_n, \lambda_n)$ be a finite path of $\mathcal{U}_{\mathcal{B},\theta}$ with $n \geq 1$, $q_0 = q_n$, and $\lambda_0 \leq \lambda_n$. Let $w = c_1 \dots c_n$. Then, $w^\omega \in q_0^{-1}W$.*

Proof. Let $\varrho = \mathcal{B}(q_0, w) = (q'_0, c_1, q'_1) \dots (q'_{n-1}, c_n, q'_n)$ be the finite run of $\mathcal{B}$ obtained by reading w from q_0. States $q_0, \dots, q_n$ correspond to the first component of the vertices visited by γ in $\mathcal{U}_{\mathcal{B},\theta}$, whereas $q'_0, \dots, q'_n$ are the states visited by the finite word w in $\mathcal{B}$. We have that $q_0 = q'_0$, but the subsequent states may or may not correspond. By Lemma 44, we still know that for all i, $0 \leq i \leq n$, $q_i \preceq q'_i$. In particular, $q_0 = q_n \preceq \delta^*(q_0, w) = q'_n$. We distinguish three cases, depending on whether $q_0 \prec \delta^*(q_0, w)$ or $q_0 \sim \delta^*(q_0, w)$ (which implies $q_0 = \delta^*(q_0, w)$ as $\mathcal{B}$ is built on top of its prefix-classifier), and depending on whether $q_i = q'_i$ for all $0 \leq i \leq n$ or not.

- If $q_0 \prec \delta^*(q_0, w)$, by progress-consistency, $w^\omega \in q_0^{-1}W$.
- If $q_0 = \delta^*(q_0, w)$ and for all i, $0 \leq i \leq n$, $q_i = q'_i$ (i.e., γ only uses edges that directly correspond to transitions of the automaton $\mathcal{B}$), then for the ordinal to be greater than or equal to its starting value, some Büchi transition has to be taken since non-Büchi transitions strictly decrease the ordinal on the second component. Hence, w is not an α-free cycle from q_0, so $w^\omega \in q_0^{-1}W$.
- If $q_0 = \delta^*(q_0, w)$ and for some index i, $1 \leq i < n$, we have $q_i \prec q'_i$ (i.e., γ takes at least one edge that does not correspond to a transition of the automaton). We represent the situation in Figure 13. We know that $\delta^*(q'_i, c_{i+1} \dots c_n) = q'_n \sim q_0$. By Lemma 14, as $q_i \prec q'_i$, this implies that $\delta^*(q_i, c_{i+1} \dots c_n) \preceq q_0$. Also, in the graph, there is a path from q_i to q_0 with colors $c_{i+1} \dots c_n$. Therefore, by Lemma 44, $q_0 \preceq \delta^*(q_i, c_{i+1} \dots c_n)$. Thus, $q_0 \sim \delta^*(q_i, c_{i+1} \dots c_n)$, and as $\mathcal{B}$ is built on top of $\mathcal{S}_\sim$, $q_0 = \delta^*(q_i, c_{i+1} \dots c_n)$. Therefore, $q_i \prec \delta^*(q_i, c_{i+1} \dots c_n c_1 \dots c_i) = q'_i$. By progress-consistency, $(c_{i+1} \dots c_n c_1 \dots c_i)^\omega \in q_i^{-1}W$. As $\delta^*(q_i, c_{i+1} \dots c_n) = q_0$, this implies that $(c_1 \dots c_n)^\omega = w^\omega \in q_0^{-1}W$. ◀

We show that every vertex (q_0, λ_0) such that $q_0 \preceq q_{\mathsf{init}}$ satisfies W.

▶ **Lemma 46.** *Let $q_0 \in Q$. For all ordinals $\lambda_0 < \theta$, the vertex (q_0, λ_0) of $\mathcal{U}_{\mathcal{B},\theta}$ satisfies $q_0^{-1}W$. In particular, if $q_0 \preceq q_{\mathsf{init}}$, (q_0, λ_0) satisfies W.*

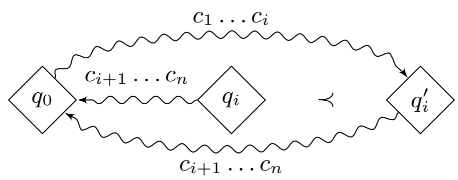

Figure 13 Situation in the proof of Lemma 45.

Proof. Let $\pi = (q_0, \lambda_0) \xrightarrow{c_1} (q_1, \lambda_1) \xrightarrow{c_2} \dots$ be an infinite path of $\mathcal{U}_{\mathcal{B},\theta}$ from (q_0, λ_0) and $w = c_1 c_2 \dots$ be the sequence of colors along its edges. We show that $w \in q_0^{-1}W$.

Let $q'_i = \delta^*(q_0, c_1 \dots c_i)$ be the state of $\mathcal{B}$ reached after reading the first i colors of w. We claim that there are two states $q, q' \in Q$ occurring infinitely often in the sequences $(q_i)_{i\geq 0}$, $(q'_i)_{i\geq 0}$, respectively, and an increasing sequence of indices $(i_k)_{k\geq 1}$ verifying that for all $k \geq 1$,

- $q_{i_k} = q$, $q'_{i_k} = q'$, and
- $\lambda_{i_k} \leq \lambda_{i_{k+1}}$.

Indeed, we can first choose a state q appearing infinitely often in $(q_i)_{i\geq 0}$ and pick a sequence $(i_j)_{j\geq 1}$ such that $q_{i_j} = q$ and $(\lambda_{i_j})_{j\geq 1}$ is not decreasing (this is possible since there is no infinite decreasing sequence of ordinals). Then, we can just pick q' appearing infinitely often in $(q'_{i_j})_{j\geq 1}$ and extract the subsequence corresponding to its occurrences.

Let $q, q' \in Q$, $(i_k)_{k\geq 1}$ verifying the above properties. Let $w_0 = c_1 \dots c_{i_1}$ and for $k \geq 1$, let $w_k = c_{i_k+1} \dots c_{i_{k+1}} \in C^+$ be the colors over the edges in π from (q_{i_k}, λ_{i_k}) to $(q_{i_{k+1}}, \lambda_{i_{k+1}})$.

By Lemma 44, it is verified that $q \preceq q' = \delta^*(q_0, w_0)$. Moreover, since $\lambda_{i_k} \leq \lambda_{i_{k+1}}$, by Lemma 45 we have that $w_k^\omega \in q^{-1}W \subseteq (q')^{-1}W$ for all $k \geq 1$. We conclude that for all $k \geq 1$, the word w_k labels a cycle over q' in $\mathcal{B}$ visiting some Büchi transition, and therefore $w = w_0 w_1 w_2 \dots \in q_0^{-1}W$. ◀

We now have all the tools to show, for all cardinals κ, (κ, W)-universality of $\mathcal{U}_{\mathcal{B},\theta}$ for sufficiently large θ.

▶ **Proposition 47.** *Let κ be a cardinal, and θ' be an ordinal such that $\kappa < |\theta'|$. Let $\theta = |Q| \cdot \theta'$. Graph $\mathcal{U}_{\mathcal{B},\theta}$ is (κ, W)-universal.*

Proof. Let $\mathcal{G} = (V, E)$ be a graph such that $|V| \leq \kappa$. For $v \in V$, let $q_v \in Q \cup \{\top\}$ be the smallest automaton state (for $\preceq$) such that v satisfies $q_v^{-1}W$, or $\top$ if it satisfies none of them. We remark that $q_v \preceq q$ if and only if v satisfies $q^{-1}W$. To show that there is a W-preserving morphism from $\mathcal{G}$ to $\mathcal{U}_{\mathcal{B},\theta}$, we follow the six steps outlined in [52, Lemma 2.4].

(i) In this first step, we classify and order vertices of $\mathcal{G}$ in an inductive way, which will later be used to map them to vertices of $\mathcal{U}_{\mathcal{B},\theta}$. For $q \in Q$ and λ an ordinal, we define by transfinite induction

$$V_\lambda^q = \left\{ v \in V \mid q_v \preceq q, \text{ and } \forall c \in C \left(v \xrightarrow{c} v' \implies [(q,c) \in \alpha \text{ or } \exists \eta < \lambda, v' \in V_\eta^{\delta(q,c)}] \right) \right\}.$$

Intuitively, for v to be in V_λ^q, it has to satisfy $q^{-1}W$ and to guarantee that a Büchi transition is seen "soon" when colors of paths from v are read from q in $\mathcal{B}$ (how soon depends on the value of λ). We remark that for each state $q \in Q$, the sequence $(V_\lambda^q)_\lambda$ is non-decreasing: for $\lambda \leq \lambda'$, $V_\lambda^q \subseteq V_{\lambda'}^q$. We illustrate this induction on a concrete case in Example 48. The subsequent steps mostly follow from this definition.

(ii) Let $V^q = \bigcup_\lambda V_\lambda^q$. We show that if v satisfies $q^{-1}W$, then it is in V^q. Assume that $v \notin V^q$. If $q \prec q_v$, then we immediately have that v does not satisfy $q^{-1}W$. If $q_v \preceq q$, then v has an outgoing edge $v \xrightarrow{c} v'$ such that $(q,c) \notin \alpha$ and $v' \notin \bigcup_\lambda V_\lambda^{\delta(q,c)}$. By induction, we build an infinite path from v whose projection in $\mathcal{B}$ only sees non-Büchi transitions, so v does not satisfy $q^{-1}W$.

(iii) In this step and the next one, we show that there is no use in considering ordinals beyond θ in our construction. We first show that if for all $q \in Q$, $V_\lambda^q = V_{\lambda+1}^q$, then for all $q \in Q$ and all $\lambda' \geq \lambda$, $V_\lambda^q = V_{\lambda'}^q$. For $\lambda \leq \lambda'$, we always have $V_\lambda^q \subseteq V_{\lambda'}^q$. For the other inclusion, we assume by transfinite induction that $V_\lambda^{q'} = V_\eta^{q'}$ for all $q' \in Q$ and for all $\lambda \leq \eta < \lambda'$. Let $v \in V_{\lambda'}^q$. Every edge $v \xrightarrow{c} v'$ either satisfies $(q,c) \in \alpha$, or there exists $\eta < \lambda'$ such that $v' \in V_\eta^{\delta(q,c)}$. Since $V_\eta^{\delta(q,c)} \subseteq V_\lambda^{\delta(q,c)}$ by induction hypothesis, we have $v' \in V_\lambda^{\delta(q,c)}$. Hence, $v \in V_{\lambda+1}^q = V_\lambda^q$.

(iv) We prove that there exists $\lambda < \theta$ such that for all $q \in Q$, $V_\lambda^q = V_{\lambda+1}^q$. If not, using the axiom of choice, we can build a map $\psi \colon \theta \to Q \times V$ such that for $\lambda < \theta$, $\psi(\lambda) = (q, v)$ for some $q \in Q$ and $v \in V_{\lambda+1}^q \setminus V_\lambda^q$. This map is injective, as any pair (q,v) can be chosen at most once (as $(V_\lambda^q)_\lambda$ is non-decreasing). This implies that $|\theta| = |Q| \cdot |\theta'| \leq |Q| \cdot |V|$, a contradiction since $|V| < |\theta'|$.
Using additionally Item (iii), we deduce that there exists $\lambda < \theta$ such that for all $\lambda' \geq \lambda$, $V_\lambda^q = V_{\lambda'}^q$.

(v) Let $\phi \colon V \to U_{\mathcal{B},\theta}$ be such that

$$\phi(v) = \begin{cases} (q_v, \min\{\lambda \mid v \in V_\lambda^{q_v}\}) & \text{if } q_v \prec \top, \\ \top & \text{if } q_v = \top. \end{cases}$$

By Item (ii), for all $v \in V$, there exists λ such that $v \in V_\lambda^{q_v}$, so $\{\lambda \mid v \in V_\lambda^{q_v}\}$ is non-empty. By Item (iv), we have that if $\phi(v) = (q,\lambda)$, then $\lambda < \theta$, so the image of ϕ is indeed in $U_{\mathcal{B},\theta}$. We show that ϕ is W-preserving: if v satisfies W, then $q_v \preceq q_{\mathsf{init}}$, so by Lemma 46, $\phi(v)$ also satisfies W.

(vi) We show that ϕ is a graph morphism. Let $v \xrightarrow{c} v'$ be an edge of $\mathcal{G}$ — we need to show that $\phi(v) \xrightarrow{c} \phi(v')$ is an edge of $\mathcal{U}_{\mathcal{B},\theta}$. If $\phi(v) = \top$, this is clear as there are all possible outgoing edges from $\top$. If not, we denote $\phi(v) = (q,\lambda)$ and $\phi(v') = (q',\lambda')$. We have that v satisfies $q^{-1}W$. Thus, v' must satisfy $\delta(q,c)^{-1}W$. This implies that $q' \preceq \delta(q,c)$. We distinguish two cases.
- If $(q,c) \in \alpha$, then by construction of $\mathcal{U}_{\mathcal{B},\theta}$, there are c-colored edges from (q,λ) to $(\delta(q,c),\lambda'')$ for all ordinals λ''. By monotonicity, as $q' \preceq \delta(q,c)$, there is also a c-colored edge from (q,λ) to (q',λ').
- We assume that $(q,c) \notin \alpha$. If $q' = \delta(q,c)$, this means that there exists $\eta < \lambda$ such that $v' \in V_\eta^{q'}$. Therefore, $\lambda' \leq \eta < \lambda$. So the edge $(q,\lambda) \xrightarrow{c} (q',\lambda')$ exists by construction of $\mathcal{U}_{\mathcal{B},\theta}$. If $q' \prec \delta(q,c)$, then by construction of $\mathcal{U}_{\mathcal{B},\theta}$, there are c-colored edges from (q,λ) to (q',λ'') for all ordinals λ''.

We have shown in Item (v) and Item (vi) that ϕ is a W-preserving morphism from $\mathcal{G}$ to $\mathcal{U}_{\mathcal{B},\theta}$. ◀

▶ **Example 48.** We consider the objective $W = \mathsf{Büchi}(\{a\}) \cup C^*aaC^\omega$ recognized by the DBA $\mathcal{B}$ from Example 16, for which graph $\mathcal{U}_{\mathcal{B},\omega}$ was shown in Example 42. We discuss how our construction maps the states of graph $\mathcal{G}$ in Figure 14. Notice that v_1, v_2 and v_3 satisfy W, but not v_4 and v_5.

We build explicitly the sets V_λ^q from Item (i) of the proof of Proposition 47. All five vertices are in $V_\lambda^{q_{aa}}$ for every ordinal λ, as all vertices satisfy $q_{aa}^{-1}W = C^\omega$, and any transition from q_{aa} is a Büchi transition. For the same reasons, v_1, v_2, v_3 and v_4 are in $V_\lambda^{q_a}$ for all λ. We determine the sets $V_\lambda^{q_{\mathsf{init}}}$ using the inductive definition. First, v_3 is in $V_\lambda^{q_{\mathsf{init}}}$ for all λ, since $(q_{\mathsf{init}}, a) \in \alpha$. Therefore, $v_2 \in V_\lambda^{q_{\mathsf{init}}}$ for $\lambda \geq 1$ (the b-edge from v_2 leads to v_3, and $\delta(q_{\mathsf{init}}, b) = q_{\mathsf{init}}$). Finally, $v_1 \in V_\lambda^{q_{\mathsf{init}}}$ for $\lambda \geq 2$ for the same reason. The morphism ϕ from Item (v) assigns $\phi(v_1) = (q_{\mathsf{init}}, 2)$, $\phi(v_2) = (q_{\mathsf{init}}, 1)$, $\phi(v_3) = (q_{\mathsf{init}}, 0)$, $\phi(v_4) = (q_a, 0)$, $\phi(v_5) = (q_{aa}, 0)$. ⌟

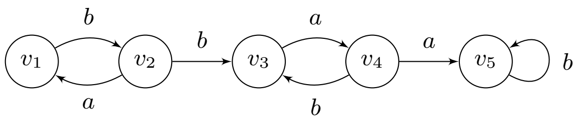

Figure 14 Graph $\mathcal{G}$ used in Example 48.

We conclude this section by proving Proposition 41, showing that W is half-positional under the three conditions from Theorem 19.

Proof of Proposition 41. Using Lemma 43 and Proposition 47, we have for all cardinal κ that there exists a completely well-monotonic (κ, W)-universal graph. By Theorem 39, this implies that W is half-positional. ◀

6 Conclusion and future work

We have provided a characterization of half-positionality for DBA-recognizable objectives. While half-positionality of ω-regular objectives is still not completely understood, this is a novel step in this direction.

Another interesting extension is to characterize the memory requirements of DBA-recognizable objectives. An intermediate, and already seemingly difficult step would be to characterize the memory requirements of objectives recognized by deterministic *weak* automata [63, 60, 47], generalizing the characterization for safety specifications [26].

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A Relation with other properties from the literature

We show precise links between the first two conditions we have defined in Section 3.1 and the *(strong) monotony* and *(strong) selectivity* notions from [34, 6].

Let $W \subseteq C^\omega$ be an objective. For $M \subseteq C^*$ a set of *finite* words, we write $[M]$ for the set of infinite words whose every finite prefix is a prefix of a finite word in M. For $w \in C^*$, we write wW for set of infinite words ww' with $w' \in W$. For two sets of infinite words $M, N \subseteq C^\omega$, we write $M \sqsubseteq_W N$ if $M \cap W \neq \emptyset$ implies $N \cap W \neq \emptyset$ (in other words, either all words in M are losing, or there exists a winning word in N). We write $M \sqsubset_W N$ if $M \cap W = \emptyset$ and $N \cap W \neq \emptyset$ (in other words, all words in M are losing and there exists a winning word in N).

Properties of prefixes. Objective W is *strongly monotone* if for all languages of infinite words $M, N \subseteq C^\omega$, if there exists $w \in C^*$ such that $wM \sqsubset_W wN$, then for all $w' \in C^*$, $w'M \sqsubseteq_W w'N$. Objective W is *monotone* if for all *regular* languages (of finite words) $M, N \subseteq C^*$, if there exists $w \in C^*$ such that $w[M] \sqsubset_W w[N]$, then for all $w' \in C^*$, $w'[M] \sqsubseteq_W w'[N]$.

We show that having a total prefix preorder is in general equivalent to strong monotony as defined in [6], and is even equivalent to monotony as defined in [34] for ω-regular objectives. In particular, strong monotony and monotony coincide for ω-regular objectives.

▶ **Lemma 49.** *Let $W \subseteq C^\omega$ be an objective. Objective W has a total prefix preorder if and only if W is strongly monotone. If W is ω-regular, W has a total prefix preorder if and only if W is monotone.*

Proof. We first prove the first equivalence, with no restriction on W.

We assume that W has a total prefix preorder and we show that it is strongly monotone. Let $w \in C^*$, $M, N \subseteq C^\omega$ such that $wM \sqsubset_W wN$. Let $w' \in C^*$. As the prefix-preorder is total, we have that $w \preceq w'$ or $w' \preceq w$. If $w \preceq w'$, as w has a winning continuation in N, this continuation is also winning for w' — so $w'M \sqsubseteq_W w'N$. If $w' \preceq w$, then all the infinite words in $w'M$ are also losing, thus $w'M \sqsubseteq_W w'N$.

We assume that W is strongly monotone and we show that the prefix preorder is total. Let $x, y \in C^*$. We show that x and y are comparable for $\preceq$: we assume w.l.o.g. that $x \not\preceq y$, and we show that $y \preceq x$, i.e., that $y^{-1}W \subseteq x^{-1}W$. Let $y' \in y^{-1}W$. As $x \not\preceq y$, there is $x' \in C^\omega$ such that $xx' \in W$ but $yx' \notin W$. By taking $M = \{x'\}$ and $N = \{y'\}$, we have $yM \sqsubset_W yN$. Hence, $xM \sqsubseteq_W xN$ by strong monotony. As xM contains a winning word, so does xN, so $y' \in x^{-1}W$.

We now assume that W is ω-regular. The left-to-right implication still holds, as monotony is a weaker notion than strong monotony. We reprove the right-to-left implication with this extra assumption. We assume that W is monotone and we show that the prefix preorder is total. Let $x, y \in C^*$. We show that x and y are comparable for $\preceq$: we assume w.l.o.g. that $x \not\preceq y$, and we show that $y \preceq x$, i.e., that $y^{-1}W \subseteq x^{-1}W$. As $x^{-1}W$ and $y^{-1}W$ are also ω-regular, by Lemma 5, it suffices to show that all ultimately periodic words in $y^{-1}W$ are in $x^{-1}W$. Let $u \in C^*, v \in C^+$ such that $uv^\omega \in y^{-1}W$. As $x \not\preceq y$, once again by Lemma 5, there is $u' \in C^*, v' \in C^+$ such that $xu'(v')^\omega \in W$ but $yu'(v')^\omega \notin W$. By taking $M = u'(v')^*$ and $N = uv^*$ (which are regular languages), we have $[M] = \{u'(v')^\omega\}$ and $[N] = \{uv^\omega\}$. Therefore, $y[M] \sqsubset_W y[N]$. Hence, $xM \sqsubseteq_W xN$ by monotony. As xM contains a winning word, so does xN, so $uv^\omega \in x^{-1}W$. ◀

As having a total prefix preorder holds symmetrically for an objective and its complement, we deduce that so does strong monotony, and so does monotony for ω-regular objectives. The latter is especially interesting for the study of ω-regular objectives, as monotony is not symmetric in general (there are objectives that are monotone but whose complement is not).

Properties of cycles. Objective W is *strongly selective* if for all languages of finite words $M, N, K \subseteq C^*$, for all $w \in C^*$, $w[(M \cup N)^* K] \sqsubseteq_W w[M^*] \cup w[N^*] \cup w[K]$. Selectivity is the same definition with M, N, K being restricted to being *regular* languages.

As for having a total prefix preorder and (strong) monotony, we can link progress-consistency and (strong) selectivity. We have that strong selectivity implies progress-consistency, and that selectivity implies progress-consistency when W is ω-regular.

▶ **Lemma 50.** *Let $W \subseteq C^\omega$ be an objective. If W is strongly selective, then it is progress-consistent. If W is ω-regular and selective, then it is progress-consistent.*

Proof. We prove the first implication. We assume by contrapositive that W is not progress-consistent, i.e., there exist $w_1 \in C^*$ and $w_2 \in C^+$ such that $w_1 \prec w_1 w_2$ and $w_1(w_2)^\omega \notin W$. Let $w_3 \in C^\omega$ be such that $w_1 w_3 \notin W$ and $w_1 w_2 w_3 \in W$, which exists as $w_1 \prec w_1 w_2$. Let $w = w_1$, $M = \{w_2\}$, $N = \emptyset$, and $K = \{w_3' \in C^* \mid w_3' \text{ is a prefix of } w_3\}$. We have that $w[(M \cup N)^* K]$ contains the winning word $w_1 w_2 w_3$. However, we have $w[M^*] = \{w_1(w_2)^\omega\}$, $w[N^*] = \emptyset$, and $w[K] = \{w_1 w_3\}$: the set $w[M^*] \cup w[N^*] \cup w[K]$ contains only losing words. We do not have $w[(M \cup N)^* K] \sqsubseteq_W w[M^*] \cup w[N^*] \cup w[K]$.

For the second implication, we assume that W is ω-regular. The proof proceeds in the exact same way, except that we have to make sure that M, N, and K are regular. This is the case for M and N. To make K regular, we use Lemma 5 and obtain that there is an ultimately periodic continuation uv^ω witnessing that $w_1 \prec w_1 w_2$. We can set $K = uv^*$ (which is regular) and the proof ends in the same way. ◀