# Different strokes in randomised strategies: Revisiting Kuhn's theorem under finite-memory assumptions* 

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#### Abstract

Two-player (antagonistic) games on (possibly stochastic) graphs are a prevalent model in theoretical computer science, notably as a framework for reactive synthesis. Optimal strategies may require randomisation when dealing with inherently probabilistic goals, balancing multiple objectives, or in contexts of partial information. There is no unique way to define randomised strategies. For instance, one can use so-called mixed strategies or behavioural ones. In the most general settings, these two classes do not share the same expressiveness. A seminal result in game theory - Kuhn's theorem - asserts their equivalence in games of perfect recall. This result crucially relies on the possibility for strategies to use infinite memory, i.e., unlimited knowledge of all the past of a play. However, computer systems are finite in practice. Hence it is pertinent to restrict our attention to finite-memory strategies, defined as automata with outputs. Randomisation can be implemented in these in different ways: the initialisation, outputs or transitions can be randomised or deterministic respectively. Depending on which aspects are randomised, the expressiveness of the corresponding class of finite-memory strategies differs. In this work, we study two-player turn-based stochastic games and provide a complete taxonomy of the classes of finite-memory strategies obtained by varying which of the three aforementioned components are randomised. Our taxonomy holds both in settings of perfect and imperfect information.


Keywords: two-player games on graphs • stochastic games • Markov decision processes • finitememory strategies • randomised strategies

## 1 Introduction

Games on graphs. Games on (possibly stochastic) graphs have been studied for decades, both for their own interest (e.g., $[\mathrm{EM} 79, \mathrm{Con} 92, \mathrm{GZ} 55]$ ) and for their value as a framework for reactive synthesis (e.g., [GTW02, Ran13, $\mathrm{BCH}^{+} 16, \mathrm{BCJ18]}$ ). The core problem is almost always to find optimal strategies for the players: strategies that guarantee winning for Boolean winning conditions (e.g., [EJ88, Zie98, BHR16, BDOR19]), or strategies that achieve the best possible payoff in quantitative contexts (e.g., [EM79, $\mathrm{BMR}^{+} 18$, BHRR19]). In multi-objective settings, one is interested in Pareto-optimal strategies (e.g., [CRR14, VCD ${ }^{+} 15$, RRS17, DKQR20]), but the bottom line is the same: players are looking for strategies that guarantee the best possible results.

In reactive synthesis, we model the interaction between a system and its uncontrollable environment as a two-player antagonistic game, and we represent the specification to ensure as a winning objective. An optimal strategy for the system in this game then constitutes a formal blueprint for a controller to implement in the real world [BCJ18].
Randomness in strategies. In essence, a pure strategy is simply a function mapping histories (i.e., the past and present of a play) to an action deterministically.

Optimal strategies may require randomisation when dealing with inherently probabilistic goals, balancing multiple objectives, or in contexts of partial information: see, e.g., [CD12, RRS17, BRR17, DKQR20]. There are different ways of randomising strategies. For instance, a mixed strategy is essentially a probability distribution over a set of pure strategies. That is, the player randomly selects a pure strategy at the beginning of the game and then follows it for the entirety of the play without resorting to randomness ever again. By contrast, a behavioural strategy randomly selects an action at each step: it thus maps histories to probability distributions over actions.
Kuhn's theorem. In full generality, these two definitions yield different classes of strategies (e.g., [CDH10, OR94]). Nonetheless, Kuhn's theorem [Aum16] proves their equivalence under a mild hypothesis: in games of perfect recall, for any mixed strategy there is an equivalent behavioural strategy and vice-versa. A game

[^0]is said to be of perfect recall for a given player if said player never forgets their previous knowledge and the actions they have played (i.e., they can observe their own actions). Let us note that perfect recall and perfect information are two different notions: perfect information is not required to have perfect recall.

Let us highlight that Kuhn's theorem crucially relies on two elements. First, mixed strategies can be distributions over an infinite set of pure strategies. Second, strategies can use infinite memory, i.e., they are able to remember the past completely, however long it might be. Indeed, consider a game in which a player can choose one of two actions in each round. One could define a (memoryless) behavioural strategy that selects one of the two actions by flipping a coin each round. This strategy generates infinitely many sequences of actions, therefore any equivalent mixed strategy needs the ability to randomise between infinitely many different sequences, and thus, infinitely many pure strategies. Moreover, an infinite number of these sequences require infinite memory to be generated (due to their non-regularity).
Finite-memory strategies. From the point of view of reactive synthesis, infinite-memory strategies, along with randomised ones relying on infinite supports, cannot be realistically implemented. This is why a plethora of recent advances has focused on finite-memory strategies, usually represented as (a variation on) Mealy machines, i.e., finite automata with outputs. See, e.g., [GZ05, CRR14, BFRR17, DKQR20, $\mathrm{BLO}^{+}$20, BORV21]. Randomisation can be implemented in these finite-memory strategies in different ways: the initialisation, outputs or transitions can be randomised or deterministic respectively.

Depending on which aspects are randomised, the expressiveness of the corresponding class of finitememory strategies differs: in a nutshell, Kuhn's theorem crumbles when restricting ourselves to finite memory. For instance, we show that some finite-memory strategies with only randomised outputs (i.e., the natural equivalent of behavioural strategies) cannot be emulated by finite-memory strategies with only randomised initialisation (i.e., the natural equivalent of mixed strategies) - see Lemma 5.1. Similarly, it is known that some finite-memory strategies that are encoded by Mealy machines using randomisation in all three components admit no equivalent using randomisation only in outputs [dAHK07, CDH10].


Fig. 1.1. Lattice of strategy classes in terms of expressible probability distributions over plays. In the three-letter acronyms, the letters, in order, refer to the initialisation, outputs and updates of the Mealy machines: D and R respectively denote deterministic and randomised components.

Our contributions. We consider two-player zero-sum stochastic games (e.g., [Sha53, Con92, MS03, BORV21]), encompassing two-player (deterministic) games (e.g., $\left[\mathrm{BLO}^{+} 20\right]$ ) and Markov decision processes (e.g., [RRS17]) as particular subcases. We establish a Kuhn-like taxonomy of the classes of finite-memory strategies obtained by varying which of the three aforementioned components are randomised: we illustrate it in Figure 1.1, and describe it fully in Section 3.

Let us highlight a few elements. Naturally, the least expressive model corresponds to pure strategies. In contrast to Kuhn's theorem, and as noted in the previous paragraph, we see that mixed strategies are strictly less expressive than behavioural ones. We also observe that allowing randomness both in initialisation and in outputs (RRD strategies) yields an even more expressive class - and incomparable to
what is obtained by allowing randomness in updates only. Finally, the most expressive class is obviously obtained when allowing randomness in all components; yet it may be dropped in initialisation or in outputs without reducing the expressiveness - but not in both simultaneously.

To compare the expressiveness of strategy classes, we consider outcome-equivalence, as defined in Section 2. Intuitively, two strategies are outcome-equivalent if, against any strategy of the opponent, they yield identical probability distributions (i.e., they induce identical Markov chains). Hence we are agnostic with regard to the objective, winning condition, payoff function, or preference relation of the game, and with regard to how they are defined (e.g., colours on actions, states, transitions, etc).

Finally, let us note that in our setting of two-player stochastic games, the perfect recall hypothesis holds. Most importantly, we assume that actions are visible. Lifting this hypothesis drastically changes the relationships between the different models. While our main presentation considers perfect-information games for the sake of simplicity, we show in Section 6 that our results hold in games of imperfect information too, assuming visible actions.
Related work. There are three main axes of research related to our work.
The first one deals the various types of randomness one can inject in strategies and their consequences. Obviously, Kuhn's theorem [Aum16] is a major inspiration, as well as the examples of differences between strategy models presented in [CDH10]. On a different but related note, [CDGH15] studies when randomness is not helpful in games nor strategies (as it can be simulated by other means).

The second direction concentrates on trying to characterise the power of finite-memory strategies, with or without randomness. One can notably cite [GZ05] for memoryless strategies, and [LPR18, $\left.\mathrm{BLO}^{+} 20\right]$, [BORV21], and [BRV21] for finite-memory ones in deterministic, stochastic, and infinite-arena games respectively.

The third axis is interested in the use of randomness as a means to simplify strategies and/or reduce their memory requirements. Examples of this endeavour can be found in [CdAH04, CHP08, Hor09, CRR14, MPR20]. These are further motivations to understand randomised strategies even in contexts where randomness is not needed a priori to play optimally.
Outline. Section 2 summarises all preliminary notions. In Section 3, we present the taxonomy illustrated in Figure 1.1 and comment it. We divide its proofs into two sections: Section 4 establishes the inclusions, and Section 5 proves their strictness. Finally, Section 6 presents how we transfer our results to the richer setting of games of imperfect information.
Acknowledgements. Mickael Randour is a member of the TRAIL Institute.

## 2 Preliminaries

Probability. Given any finite or countable set $A$, we write $\mathcal{D}(A)$ of the set of probability distributions over $A$, i.e., the set of functions $p: A \rightarrow[0,1]$ such that $\sum_{a \in A} p(a)=1$. Similarly, given some set $A$ and some $\sigma$-algebra $\mathcal{F}$ over $A$, we denote by $\mathcal{D}(A, \mathcal{F})$ the set of probability distributions over the measurable space $(A, \mathcal{F})$.
Games. We consider two-player stochastic games of perfect information played on graphs. We denote the two players by $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$. In such a game, the set of states is partitioned between the two players. At the start of a play, a pebble is placed on some initial state and each round, the owner of the current state selects an action available in said state and the next state is chosen randomly following a distribution depending on the current state and chosen action. The game proceeds for an infinite number of rounds, yielding an infinite play.

Formally, a (two-player) stochastic game (of perfect information) is a tuple $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ where $S=S_{1} \uplus S_{2}$ is a non-empty finite set of states partitioned into a set $S_{1}$ of states of $\mathcal{P}_{1}$ and a set $S_{2}$ of states of $\mathcal{P}_{2}, A$ is a finite set of actions and $\delta: S \times A \rightarrow \mathcal{D}(S)$ is a (partial) probabilistic transition function. For any state $s \in S$, we write $A(s)$ for the set of actions available in $s$, which are the actions $a \in A$ such that $\delta(s, a)$ is defined. We assume that for all $s \in S, A(s)$ is non-empty, i.e., there are no deadlocks in the game.

A play of $\mathcal{G}$ is an infinite sequence $s_{0} a_{0} s_{1} \ldots \in(S A)^{\omega}$ such that for all $k \in \mathbb{N}, \delta\left(s_{k}, a_{k}\right)\left(s_{k+1}\right)>0$. A history is a finite prefix of a play ending in a state. Given a play $\pi=s_{0} a_{0} s_{1} a_{1} \ldots$ and $k \in \mathbb{N}$, we write $\pi_{\mid k}$ for the history $s_{0} a_{0} \ldots a_{k-1} s_{k}$. For any history $h=s_{0} a_{0} \ldots a_{k-1} s_{k}$, we let last $(h)=s_{k}$. We write Plays $(\mathcal{G})$ to denote the set of plays of $\mathcal{G}, \operatorname{Hist}(\mathcal{G})$ to denote the set of histories of $\mathcal{G}$ and $\operatorname{Hist}_{i}(\mathcal{G})=\operatorname{Hist}(\mathcal{G}) \cap(S A)^{*} S_{i}$ for the set of histories ending in states controlled by $\mathcal{P}_{i}$. Given some initial state $s_{\text {init }} \in S$, we write $\operatorname{Plays}\left(\mathcal{G}, s_{\text {init }}\right)$ and $\operatorname{Hist}\left(\mathcal{G}, s_{\text {init }}\right)$ for the set of plays and histories starting in state $s_{\text {init }}$ respectively.

An interesting class of stochastic games which has been extensively studied is that of deterministic games; a game $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ is a deterministic game if for all $s \in S$ and $a \in A(s), \delta(s, a)$ is a Dirac distribution. Another interesting class of games is that of one-player games. A game $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ is a one-player game of $\mathcal{P}_{i}$ if $S_{3-i}$ is empty, i.e., all states belong to $\mathcal{P}_{i}$. These one-player games are the equivalent of Markov decision processes in our context, and will be referred to as such.
Strategies and outcomes. A strategy is a function that describes how a player should act based on a history. Players need not act in a deterministic fashion: they can use randomisation to select an action. Formally, a (behavioural) strategy of $\mathcal{P}_{i}$ is a function $\sigma_{i}: \operatorname{Hist}_{i}(\mathcal{G}) \rightarrow \mathcal{D}(A)$ such that for all histories $h$ and all actions $a \in A, \sigma_{i}(h)(a)>0$ implies $a \in A(\operatorname{last}(h))$. In other words, a strategy assigns to any history ending in a state controlled by $\mathcal{P}_{i}$ a distribution over the actions available in this state.

When both players fix a strategy and an initial state is decided, the game becomes a purely stochastic process (a Markov chain). Let us recall the relevant $\sigma$-algebra for the definition of probabilities over plays. For any history $h$, we define

$$
\operatorname{Cyl}(h)=\{\pi \in \operatorname{Plays}(\mathcal{G}) \mid h \text { is a prefix of } \pi\}
$$

the cylinder of $h$, consisting of plays that extend $h$. Let us denote by $\mathcal{F}_{\mathcal{G}}$ the $\sigma$-algebra generated by all cylinder sets.

Let $\sigma_{1}$ and $\sigma_{2}$ be strategies of $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ respectively and $s_{\text {init }} \in S$ be an initial state. We define the probability measure $\left(\operatorname{over}\left(\operatorname{Plays}(\mathcal{G}), \mathcal{F}_{\mathcal{G}}\right)\right)$ induced by playing $\sigma_{1}$ and $\sigma_{2}$ from $s_{\text {init }}$ in $\mathcal{G}$, written $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}$, in the following way: for any history $h=s_{0} a_{0} \ldots s_{n} \in \operatorname{Hist}\left(\mathcal{G}, s_{\text {init }}\right)$, the probability assigned to $\mathrm{Cyl}(h)$ is given by the product

$$
\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h))=\prod_{k=0}^{n-1} \tau_{k}\left(s_{0} a_{0} \ldots s_{k}\right)\left(a_{k}\right) \cdot \delta\left(s_{k}, a_{k}, s_{k+1}\right)
$$

where $\tau_{k}=\sigma_{1}$ if $s_{k} \in S_{1}$ and $\tau_{k}=\sigma_{2}$ otherwise. For any history $h \in \operatorname{Hist}(\mathcal{G}) \backslash \operatorname{Hist}\left(\mathcal{G}, s_{\text {init }}\right)$, we set $\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\mathrm{Cyl}(h))=0$. By Carathéodory's extension theorem [Dur19, Theorem A.1.3], the measure described above can be extended in a unique fashion to $\left(\operatorname{Plays}(\mathcal{G}), \mathcal{F}_{\mathcal{G}}\right)$.

Let $\sigma_{1}$ be a strategy of $\mathcal{P}_{1}$. A play or prefix of play $s_{0} a_{0} s_{1} \ldots$ is said to be consistent with $\sigma_{1}$ if for all indices $k, s_{k} \in S_{1}$ implies $\sigma_{1}\left(s_{0} a_{0} \ldots s_{k}\right)\left(a_{k}\right)>0 .{ }^{1}$ Consistency with respect to strategies of $\mathcal{P}_{2}$ is defined analogously.
Outcome-equivalence of strategies. In later sections, we study the expressiveness of finite-memory strategy models depending on the type of randomisation allowed. Two strategies may yield the same outcomes despite being different: the actions suggested by a strategy in an inconsistent history can be changed without affecting the outcome. Therefore, instead of using the equality of strategies as a measure of equivalence, we consider some weaker notion of equivalence, referred to as outcome-equivalence.

We say that two strategies $\sigma_{1}$ and $\tau_{1}$ of $\mathcal{P}_{1}$ are outcome-equivalent if for any strategy $\sigma_{2}$ of $\mathcal{P}_{2}$ and for any initial state $s_{\text {init }}$, the probability distributions $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}$ and $\mathbb{P}_{s_{\text {init }}}^{\tau_{1}, \sigma_{2}}$ coincide.

We now provide a useful criterion to establish outcome-equivalence of strategies, that does not invoke the probability distributions they induce. When studying the outcome-equivalence of two strategies, we are only concerned with the suggestions these strategies provide in histories that are consistent with them. In other words, any deviation in unreachable histories does not affect the outcome. Hence, one could reformulate outcome-equivalence as having to suggest the same distributions over actions in histories that are consistent with (one of) the strategies. In the sequel, we prove that this reformulation is indeed equivalent to the definition of outcome-equivalence. We rely on this reformulation to prove the outcome-equivalence of two strategies.

Lemma 2.1 (Strategic criterion for outcome-equivalence). Let $\sigma_{1}$ and $\tau_{1}$ be two strategies of $\mathcal{P}_{1}$. These two strategies are outcome-equivalent if and only if for all histories $h \in \operatorname{Hist}_{1}(\mathcal{G})$, $h$ consistent with $\sigma_{1}$ implies $\sigma_{1}(h)=\tau_{1}(h)$.

Proof. First, we assume that $\sigma_{1}$ and $\tau_{1}$ are outcome-equivalent strategies. Let $h \in \operatorname{Hist}_{1}(\mathcal{G})$ be a history controlled by $\mathcal{P}_{1}$ that is consistent with $\sigma_{1}$. We must prove that $\sigma_{1}(h)=\tau_{1}(h)$ holds. Let $a \in A(\operatorname{last}(h))$ be an action enabled in the last state of $h$, we establish that $\sigma_{1}(h)(a)=\tau_{1}(h)(a)$.

Fix some $\mathcal{P}_{2}$ strategy $\sigma_{2}$ that is consistent with $h$ and some state $s \in S$ such that $\delta(\operatorname{last}(h), a)(s)>0$. Let us denote by $s_{\text {init }}$ the first state of $h$. The outcome-equivalence of $\sigma_{1}$ and $\tau_{1}$ ensures $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h))=$

[^1]$\mathbb{P}_{S_{\text {init }}}^{\tau_{1}, \sigma_{2}}(\operatorname{Cyl}(h))$ and $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h a s))=\mathbb{P}_{S_{\text {init }}}^{\tau_{1}, \sigma_{2}}(\operatorname{Cyl}(h a s))$. Furthermore, we have $\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h))>0$ because $h$ is assumed to be consistent with $\sigma_{1}$ and $\sigma_{2}$. We obtain from the definition of the probability of cylinders that, for $\lambda_{1} \in\left\{\sigma_{1}, \tau_{1}\right\}$,
$$
\mathbb{P}_{S_{\text {init }}}^{\lambda_{1}, \sigma_{2}}(\operatorname{Cyl}(\text { has }))=\mathbb{P}_{s_{\text {init }}}^{\lambda_{1}, \sigma_{2}}(\operatorname{Cyl}(h)) \cdot \lambda_{1}(h)(a) \cdot \delta(\operatorname{last}(h), a)(s)
$$

It follows from $\delta(\operatorname{last}(h), a)(s)>0$ and $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h))=\mathbb{P}_{s_{\text {init }}}^{\tau_{1}, \sigma_{2}}(\operatorname{Cyl}(h))>0$ and the equality above that $\sigma_{1}(h)(a)=\tau_{1}(h)(a)$. This concludes the proof of the first direction of the lemma.

Now, let us assume that for any history $h \in \operatorname{Hist}_{1}(\mathcal{G})$ controlled by $\mathcal{P}_{1}$, if $h$ is consistent with $\sigma_{1}$, then $\sigma_{1}(h)=\tau_{1}(h)$. We must prove that for any initial state $s_{\text {init }} \in S$ and any strategy $\sigma_{2}$ of $\mathcal{P}_{2}$, we have $\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}=\mathbb{P}_{s_{\text {init }}}^{\tau_{1}, \sigma_{2}}$.
${ }^{\text {inint }}$ Fix $s_{\text {init }} \in S$ and $\sigma_{2}$ a strategy of $\mathcal{P}_{2}$. It suffices to show that $\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}$ and $\mathbb{P}_{S_{\text {init }}}^{\tau_{1}, \sigma_{2}}$ coincide over cylinder sets by Carathéodory's extension theorem. We use an inductive argument as follows: we assume that the two probability measures coincide over all cylinders of histories with $k$ actions and deduce that the two probability measures coincide over cylinders of histories with $k+1$ actions.

The base case concerns histories with no actions. For any state $s \in S$, we have $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(s))=$ $\mathbb{P}_{s_{\text {init }}}^{\tau_{1}, \sigma_{2}}(\mathrm{Cyl}(s))=0$ if $s \neq s_{\text {init }}$ and 1 otherwise. Now, we assume that for all histories $h \in \operatorname{Hist}(\mathcal{G})$ with $k$


Fix a history $h^{\prime} \stackrel{s_{\text {int }}}{=} h a s \in \operatorname{Hist}(\mathcal{G})$ with $k+1$ actions. We will distinguish two cases based on the owner of the last state of the prefix $h$. First, let us assume that last $(h) \in S_{2}$, i.e., $\mathcal{P}_{2}$ controls the last state of $h$. For $\lambda_{1} \in\left\{\sigma_{1}, \tau_{1}\right\}$, we have

$$
\begin{aligned}
\mathbb{P}_{S_{\text {init }}}^{\lambda_{1}, \sigma_{2}}\left(\operatorname{Cyl}\left(h^{\prime}\right)\right) & =\mathbb{P}_{S_{\text {init }}}^{\lambda_{1}, \sigma_{2}}(\operatorname{Cyl}(h)) \cdot \sigma_{2}(h)(a) \cdot \delta(\operatorname{last}(h), a)(s) \\
& =\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h)) \cdot \sigma_{2}(h)(a) \cdot \delta(\operatorname{last}(h), a)(s),
\end{aligned}
$$

by the induction hypothesis. This proves that $\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}\left(\mathrm{Cyl}\left(h^{\prime}\right)\right)=\mathbb{P}_{S_{\text {init }}}^{\tau_{1}, \sigma_{2}}\left(\mathrm{Cyl}\left(h^{\prime}\right)\right)$.
Now, let us assume that last $(h) \in S_{1}$. Then we have, for $\lambda_{1} \in\left\{\sigma_{1}, \tau_{1}\right\}$,

$$
\begin{equation*}
\mathbb{P}_{S_{\text {init }}}^{\lambda_{1}, \sigma_{2}}\left(\operatorname{Cyl}\left(h^{\prime}\right)\right)=\mathbb{P}_{S_{\text {init }}}^{\lambda_{1}, \sigma_{2}}(\operatorname{Cyl}(h)) \cdot \lambda_{1}(h)(a) \cdot \delta(\operatorname{last}(h), a)(s) \tag{2.1}
\end{equation*}
$$

We distinguish two subcases. First, let us assume that the history $h$ is not consistent with $\sigma_{1}$. Then it follows from the inductive hypothesis that $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h))=\mathbb{P}_{s_{\text {init }}}^{\tau_{1}, \sigma_{2}}(\operatorname{Cyl}(h))=0$. From Equation (2.1), we conclude $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}\left(\operatorname{Cyl}\left(h^{\prime}\right)\right)=\mathbb{P}_{s_{\text {init }}}^{\tau_{1}, \sigma_{2}}\left(\operatorname{Cyl}\left(h^{\prime}\right)\right)=0$. If we now assume $h$ is consistent with $\sigma_{1}$, it follows from our hypothesis on the two strategies that $\sigma_{1}(h)(a)=\tau_{1}(h)(a)$. By combining the former with the inductive hypothesis and Equation (2.1), we obtain $\mathbb{P}_{s_{\text {init }}}^{\sigma_{1}, \sigma_{2}}\left(\operatorname{Cyl}\left(h^{\prime}\right)\right)=\mathbb{P}_{s_{\text {init }}}^{\tau_{1}, \sigma_{2}}\left(\operatorname{Cyl}\left(h^{\prime}\right)\right)$.

The inductive argument above shows that $\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}(\operatorname{Cyl}(h))=\mathbb{P}_{s_{\text {init }}^{\tau_{1}, \sigma_{2}}}^{s_{\text {init }}}(\operatorname{Cyl}(h))$ for any history $h \in \operatorname{Hist}(\mathcal{G})$, which is sufficient to ensure $\mathbb{P}_{S_{\text {init }}}^{\sigma_{1}, \sigma_{2}}=\mathbb{P}_{S_{\text {init }}}^{\tau_{1}, \sigma_{2}}$. This ends the proof of the second implication.

Subclasses of strategies. A strategy is called pure if it does not use randomisation; a pure strategy can be viewed as a function $\operatorname{Hist}_{i}(\mathcal{G}) \rightarrow A$. A strategy that only uses information on the current state of the play is called memoryless: a strategy $\sigma_{i}$ of $\mathcal{P}_{i}$ is memoryless if for all histories $h, h^{\prime} \in \operatorname{Hist}_{i}(\mathcal{G})$, last $(h)=\operatorname{last}\left(h^{\prime}\right)$ implies $\sigma_{i}(h)=\sigma_{i}\left(h^{\prime}\right)$. Memoryless strategies can be viewed as functions $S_{i} \rightarrow \mathcal{D}(A)$. Strategies that are both memoryless and pure can be viewed as functions $S_{i} \rightarrow A$.

A strategy $\sigma$ is said to be finite-memory (FM) if it can be encoded by a Mealy machine. A Mealy machine is an automaton with outputs along its edges. In the context of randomised strategies, we include randomisation in the initialisation, outputs and updates (i.e., transitions) of the Mealy machine. Formally, a stochastic Mealy machine of $\mathcal{P}_{i}$ is a tuple $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$, where $M$ is a finite set of memory states, $\mu_{\text {init }} \in \mathcal{D}(M)$ is an initial distribution, $\alpha_{\text {up }}: M \times S \times A \rightarrow \mathcal{D}(M)$ is the (stochastic) update function and $\alpha_{\text {next }}: M \times S_{i} \rightarrow \mathcal{D}(A)$ is the (stochastic) next-move function.

Before we explain how to define the strategy induced by a Mealy machine, let us first describe how these machines work. Fix a Mealy machine $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$. Let $s_{0} \in S$. At the start of a play, an initial memory state $m_{0}$ is selected randomly following $\mu_{\text {init }}$. Then, at each step of the play such that $s_{k} \in S_{i}$, an action $a_{k}$ is chosen following the distribution $\alpha_{\text {next }}\left(m_{k}, s_{k}\right)$, and otherwise an action is chosen following the other player's strategy. The memory state $m_{k+1}$ is then randomly selected following the distribution $\alpha_{\mathrm{up}}\left(m_{k}, s_{k}, a_{k}\right)$ and the game state $s_{k+1}$ is chosen following the distribution $\delta\left(s_{k}, a_{k}\right)$, both choices being made independently.

Let us now explain how a strategy can be derived from a Mealy machine. As explained previously, when in a certain memory state $m \in M$ and game state $s \in S_{i}$, the probability of an action $a \in A(s)$ being chosen is given by $\alpha_{\text {next }}(m, s)(a)$. Therefore, the probability of choosing the action $a \in A$ after some
history $h=w s$ (where $w \in(S A)^{*}$ and $s=\operatorname{last}(h)$ ) is given by the sum, for each memory state $m \in M$, of the probability that $m$ was reached after $\mathcal{M}$ processes $w$, multiplied by $\alpha_{\text {next }}(m, s)(a)$.

To provide a formal definition of the strategy induced by $\mathcal{M}$, we must first describe the distribution over memory states after $\mathcal{M}$ processes elements of $(S A)^{*}$. We formally define this distribution inductively. Details on how to derive the formulae for the update of these distributions, which use conditional probabilities, are relegated to Appendix A.

The distribution $\mu_{\varepsilon}$ over memory states after reading the empty word $\varepsilon$ is by definition $\mu_{\text {init }}$. Assume inductively we know the distribution $\mu_{w}$ for $w=s_{0} a_{0} \ldots s_{k-1} a_{k-1}$ and let us explain how one derives $\mu_{w s_{k} a_{k}}$ from $\mu_{w}$ for any state $s_{k} \in \operatorname{supp}\left(\delta\left(s_{k-1}, a_{k-1}\right)\right)$ and for any action $a_{k} \in A\left(s_{k}\right)$.

If $s_{k} \in S_{3-i}$, i.e., $s_{k}$ is not controlled by the owner of the strategy, the action $a_{k}$ does not introduce any conditions on the current memory state. Therefore, we set, for any memory state $m \in M$,

$$
\mu_{w s_{k} a_{k}}(m)=\sum_{m^{\prime} \in M} \mu_{w}\left(m^{\prime}\right) \cdot \alpha_{\mathrm{up}}\left(m^{\prime}, s_{k}, a_{k}\right)(m)
$$

which consists in checking for each predecessor state $m^{\prime}$, what the probability of moving to memory state $m$ is and weighing the sum by the probability of being in $m^{\prime}$.

If $s_{k} \in S_{i}$, i.e., $s_{k}$ is controlled by the owner of the strategy, then the choice of an action conditions what the predecessor memory states could be. If we have, for all memory states $m^{\prime} \in M$ such that $\mu_{w}\left(m^{\prime}\right)>0$, that $\alpha_{\text {next }}\left(m^{\prime}, s_{k}\right)\left(a_{k}\right)=0$, then the action $a_{k}$ is actually never chosen. In this case, to ensure a complete definition, we perform an update as in the previous case. Otherwise, we condition updates on the likelihood of being in a memory state knowing that the action $a_{k}$ was chosen. We set, for any memory state $m \in M$,

$$
\mu_{w s_{k} a_{k}}(m)=\frac{\sum_{m^{\prime} \in M} \mu_{w}\left(m^{\prime}\right) \cdot \alpha_{\mathrm{up}}\left(m^{\prime}, s_{k}, a_{k}\right)(m) \cdot \alpha_{\text {next }}\left(m^{\prime}, s_{k}\right)\left(a_{k}\right)}{\sum_{m^{\prime} \in M} \mu_{w}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime}, s_{k}\right)\left(a_{k}\right)}
$$

This quotient is not well-defined whenever for all $m^{\prime} \in \operatorname{supp}\left(\mu_{w}\right), \alpha_{\text {next }}\left(m^{\prime}, s_{k}\right)\left(a_{k}\right)=0$, justifying the distinction above.

Using these distributions, we formally define the strategy $\sigma_{i}^{\mathcal{M}}$ induced by the Mealy machine $\mathcal{M}=$ $\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ as the strategy $\sigma_{i}^{\mathcal{M}}: \operatorname{Hist}_{i}(\mathcal{G}) \rightarrow \mathcal{D}(A)$ such that for all histories $h=w s$, for all actions $a \in A(s)$,

$$
\sigma_{i}^{\mathcal{M}}(h)(a)=\sum_{m \in M} \mu_{w}(m) \cdot \alpha_{\text {next }}(m, s)(a)
$$

Classifying finite-memory strategies. In the sequel, we investigate the relationships between different classes of finite-memory strategies in terms of expressive power. We classify finite-memory strategies following the type of stochastic Mealy machines that can induce them. We introduce a concise notation for each class: we use three-letter acronyms of the form $X X X$ with $X \in\{D, R\}$, where the letters, in order, refer to the initialisation, outputs and updates of the Mealy machines, with D and R respectively denoting deterministic and randomised components. For instance, we will write RRD to denote the class of Mealy machines that have randomised initialisation and outputs, but deterministic updates. We also apply this terminology to FM strategies: we will say that an FM strategy is in the class XXX - i.e., it is an XXX strategy - if it is induced by an XXX Mealy machine.

Moreover, in the remainder of the paper, we will abusively identify Mealy machines and their induced FM strategies. For instance, we will say that $\mathcal{M}$ is an XXX strategy to mean that $\mathcal{M}$ is an XXX Mealy machine (thus inducing an XXX strategy). As a by-product of this identification, we apply the terminology introduced previously for strategies to Mealy machines, without explicitly referring to the strategy they induce. For instance, we may say a history is consistent with some Mealy machine, or that two Mealy machines are outcome-equivalent. Let us note however that we will not use a Mealy machine in lieu of its induced strategy whenever we are interested in the strategy itself as a function. This choice lightens notations; the strategy induced by a Mealy machine need not be introduced unless it is required as a function.

We close this section by commenting some of the classes, and discuss previous appearances in the literature, under different names. Pure strategies use no randomisation: hence, the class DDD corresponds to pure FM strategies, which can be represented by Mealy machines that do not rely on randomisation.

Strategies in the class DRD have been referred to as behavioural FM strategies in [CDH10]. The name comes from the randomised outputs, reminiscent of behavioural strategies that output a distribution over actions after a history. We note that stochastic Mealy machines that induce DRD strategies are such that their distributions over memory states are Dirac due to the deterministic initialisation and updates.

Similarly, RDD strategies have been referred to as mixed FM strategies [CDH10]. The general definition of a mixed strategy is a distribution over pure strategies: under a mixed strategy, a player randomly selects a pure strategy at the start of a play and plays according to it for the whole play. RDD strategies are similar in the way that the random initialisation can be viewed as randomly selecting some DDD strategy (i.e., a pure FM strategy) among a finite selection of such strategies.

The elements of RRR, the broadest class of FM strategies, have been referred to as general FM strategies [CDH10] and stochastic-update FM strategies [BBC+14, CKK17]. The latter name highlights the random nature of updates and insists on the difference with models that rely on deterministic updates, more common in the literature.

## 3 Taxonomy of finite-memory strategies

In this section, we present the relationships between the classes of finite-memory strategies in terms of expressiveness. We say that a class $\mathcal{C}_{1}$ of FM strategies is no less expressive than a class $\mathcal{C}_{2}$ if for all games $\mathcal{G}$, for all FM strategies $\mathcal{M} \in \mathcal{C}_{2}$ in $\mathcal{G}$, one can find some FM strategy $\mathcal{M}^{\prime} \in \mathcal{C}_{1}$ of $\mathcal{G}$ such that $\mathcal{M}$ and $\mathcal{M}^{\prime}$ are outcome-equivalent strategies. For the sake of brevity, we will say that $\mathcal{C}_{2}$ is included in $\mathcal{C}_{1}$, and write $\mathcal{C}_{2} \subseteq \mathcal{C}_{1}$.

Figure 1.1 summarises our results. In terms of set inclusion, each line in the figure indicates that the class below is strictly included in the class above. Each line is decorated with a reference to the relevant results. The strictness results hold in two-player deterministic games and in Markov decision processes, i.e., there are no collapses in the diagram in either of these settings.

Some relations follow purely from syntactic arguments. For instance, the inclusion DRD $\subseteq$ RRD follows from the fact that RRD Mealy machines have more randomisation power than DRD ones. Four other inclusions follow from the same argument: $\mathrm{DDD} \subseteq \mathrm{RDD}, \mathrm{DDD} \subseteq \mathrm{DDR}, \mathrm{DDR} \subseteq \mathrm{RRR}$ and $\mathrm{RRD} \subseteq \mathrm{RRR}$. We do not comment further on these trivial inclusions.

Pure strategies lie at the bottom of the diagram. It is easy to see that they are the least expressive: pure strategies cannot induce any non-Dirac distributions on plays in deterministic one-player games.

Now, let us comment the non-trivial inclusions covered in Section 4. Theorem 4.1 establishes the inclusion $\mathrm{RDD} \subseteq \mathrm{DRD}$. Theorems 4.2 and 4.3 respectively state that $\mathrm{RRR} \subseteq \mathrm{DRR}$ and $\mathrm{RRR} \subseteq \mathrm{RDR}$. The converse inclusions are purely syntactic, implying the equality on the uppermost level of Figure 1.1.

Let us move on to the strictness of inclusions, following from results of Section 5. Lemma 5.1 and Lemma 5.2 show that the classes RDD and DRD are strictly included in the classes DRD and RRD respectively. The two lemmata that imply the strict inclusion of classes DDR and RRD in the class RRR also explain why the class DDR is not comparable to the three classes RDD, DRD and RRD. On the one hand, Lemma 5.3 asserts the existence of a game in which there is some RDD strategy that has no outcome-equivalent counterpart in DDR. It follows that the classes RDD, DRD and RRD are not included in the class of DDR strategies, and that DDR is strictly included in the class RRR. On the other hand, Lemma 5.4 states that there is some game in which there is a DDR strategy such that there is no outcome-equivalent RRD strategy. This implies that the class DDR is not included in the class RRD, and it follows that the class DDR is incomparable with the classes RDD, DRD and RRD. Furthermore, it also entails that RRD is strictly included in the class RRR.

We close this section by comparing our results with Kuhn's theorem. Kuhn's theorem asserts that the classes of behavioural strategies and mixed strategies in games of perfect recall share the same expressiveness. Games of perfect recall have two traits: players never forget the sequence of histories controlled by them that have taken place and they can see their own actions. In particular, stochastic games of perfect information are a special case of games of perfect recall. Recall that mixed strategies are distributions over pure strategies. We comment briefly on the techniques used in the proof of Kuhn's theorem, and compare them with the finite-memory setting. Let us fix a game $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$.

On the one hand, the emulation of mixed strategies with behavioural strategies is performed as follows. Let $p_{i}$ be a mixed strategy of $\mathcal{P}_{i}$, i.e., a distribution over pure strategies of $\mathcal{G}$. An outcome-equivalent behavioural strategy $\sigma_{i}$ is constructed such that, for all histories $h \in \operatorname{Hist}_{i}(\mathcal{G})$ and actions $a \in A(\operatorname{last}(h))$, the probability $\sigma_{i}(h)(a)$ is defined as the probability

$$
\frac{p_{i}\left(\left\{\tau_{i} \text { pure strategy } \mid \tau_{i} \text { consistent with } h \text { and } \tau_{i}(h)=a\right\}\right)}{p_{i}\left(\left\{\tau_{i} \text { pure strategy } \mid \tau_{i} \text { consistent with } h\right\}\right)} .
$$

In the finite-memory case, similar ideas can be used to show that RDD $\subseteq$ DRD. In the proof of Theorem 4.1, from some RDD strategy (i.e., a so-called mixed FM strategy), we construct a DRD strategy (i.e., a
so-called behavioural FM strategy) that keeps track of the finitely many pure FM strategies that the RDD strategy mixes and that are consistent with the current history. An adaption of the quotient above is used in the next-move function of the DRD strategy.

On the other hand, the emulation of behavioural strategies by mixed strategies exploits the fact that mixed strategies may randomise over infinite sets. In a finite-memory setting, the same techniques cannot be applied. As a consequence, the class of RDD strategies is strictly included in the class of DRD strategies. In a certain sense, one could say that Kuhn's theorem only partially holds in the case of FM strategies.

## 4 Non-trivial inclusions

This section covers the non-trivial inclusions that are asserted in the lattice of Figure 1.1. The structure of this section is as follows. Section 4.1 contains the proof that $R D D \subseteq D R D$. The inclusion $R R R \subseteq D R R$ is presented in Section 4.2. Finally, we close this section by proving the inclusion RRR $\subseteq \operatorname{RDR}$ in Section 4.3.

### 4.1 Simulating RDD strategies with DRD ones

In this section, we focus on the classes RDD and DRD. We argue that for all RDD strategies in any game, one can find some outcome-equivalent DRD strategy (Theorem 4.1). Let us note that the converse inclusion is not true, and this discussion is relegated to Section 5.1. The construction provided in the proof of Theorem 4.1 yields a DRD strategy that has a state space of size exponential in the size of the state space of the original RDD strategy; we complement Theorem 4.1 by proving that there are some RDD strategies for which this exponential blow-up in the number of states is necessary for any outcome-equivalent DRD strategy (Lemma 4.1). We argue that this blow-up is unavoidable in both deterministic two-player games and Markov decision processes.

Let $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ be a game. Fix an RDD strategy $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ of $\mathcal{P}_{i}$. Let us sketch how to emulate $\mathcal{M}$ with a DRD strategy $\mathcal{B}=\left(B, b_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$ built with a subset construction-like approach. The memory states of $\mathcal{B}$ are functions $f: \operatorname{supp}\left(\mu_{\text {init }}\right) \rightarrow M \cup\{\perp\}$. A memory state $f$ is interpreted as follows. For all initial memory states $m_{0} \in \operatorname{supp}\left(\mu_{\text {init }}\right)$, we have $f\left(m_{0}\right)=\perp$ if the history seen up to now is not consistent with the pure FM strategy $\left(M, m_{0}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$, and otherwise $f\left(m_{0}\right)$ is the memory state reached in the same pure FM strategy after processing the current history. Updates are naturally derived from these semantics.

Using this state space and update scheme, we can compute the likelihood of each memory state of the mixed FM strategy $\mathcal{M}$ after some sequence $w \in(S A)^{*}$ has taken place. Indeed, we keep track of each initial memory state from which it was possible to be consistent with $w$, and, for each such initial memory state $m_{0}$, the memory state reached after $w$ was processed starting in $m_{0}$. Therefore, this likelihood can be inferred from $\mu_{\text {init }}$; the probability of $\mathcal{M}$ being in $m \in M$ after $w$ has been processed is given by the (normalised) sum of the probability of each initial memory state $m_{0} \in \operatorname{supp}\left(\mu_{\text {init }}\right)$ such that $f\left(m_{0}\right)=m$.

The definition of the next-move function of $\mathcal{B}$ is directly based on the distribution over states of $\mathcal{M}$ described in the previous paragraph, and ensures that the two strategies select actions with the same probabilities at any given state. For any action $a \in A(s)$, the probability of $a$ being chosen in game state $s$ and in memory state $f$ is determined by the probability of $\mathcal{M}$ being in some memory state $m$ such that $\alpha_{\text {next }}(m, s)=a$, where this probability is inferred from $f$.

Intuitively, we postpone the initial randomisation and instead randomise at each step in an attempt of replicating the initial distribution in the long run. In the sequel, we formalise the DRD strategy outlined above and prove its outcome-equivalence with the RDD strategy it is based on.

Theorem 4.1. Let $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ be a game. Let $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ be an $R D D$ strategy of $\mathcal{P}_{i}$. There exists a $D R D$ strategy $\mathcal{B}=\left(B, b_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$ such that $\mathcal{B}$ and $\mathcal{M}$ are outcome-equivalent.

Proof. We formalise the strategy described above. Let us write $M_{0}$ for the support of the initial distribution $\mu_{\text {init }}$ of $\mathcal{M}$. We define the set of memory states $B$ to be the set of functions $M_{0} \rightarrow M \cup\{\perp\}$. The initial memory state of $\mathcal{B}$ is given by the identity function $b_{\text {init }}: m_{0} \mapsto m_{0}$ over $M_{0}$. The update function $\beta_{\text {up }}$ is as follows: for any $f \in B$, any $s \in S$ and $a \in A(s)$, we let $\beta_{\text {up }}(f, s, a)$ be the function $f^{\prime}$ such that for all $m_{0} \in M_{0}$, if $s \in S_{i}$, we have

$$
f^{\prime}\left(m_{0}\right)= \begin{cases}\alpha_{\mathrm{up}}\left(f\left(m_{0}\right), s, a\right) & \text { if } f\left(m_{0}\right) \in M \text { and } \alpha_{\text {next }}\left(f\left(m_{0}\right), s\right)=a \\ \perp & \text { otherwise }\end{cases}
$$

and if $s \in S_{3-i}$, we have

$$
f^{\prime}\left(m_{0}\right)= \begin{cases}\alpha_{\mathrm{up}}\left(f\left(m_{0}\right), s, a\right) & \text { if } f\left(m_{0}\right) \in M \\ \perp & \text { otherwise }\end{cases}
$$

The asymmetry between $\mathcal{P}_{i}$-controlled states and $\mathcal{P}_{3-i}$-controlled states is explained as follows. Whenever the processed state is controlled by $\mathcal{P}_{i}$, we can refine our knowledge on which memory state we have possibly started from thanks to the action selected. We execute this by mapping to $\perp$ any initial memory states $m_{0}$ such that the played action would not have been selected in the memory state $f\left(m_{0}\right) \in M$, effectively removing $m_{0}$ from the set of initial memory states from which we could have started. However, if $\mathcal{P}_{3-i}$ controls the state, then we do not gain any information on the initial memory state, and perform updates without refining our knowledge.

The next-move function $\beta_{\text {next }}$ is defined as follows: for any memory state $f \in B$ and $s \in S_{i}$, we let $\beta_{\text {next }}(f, s)$ be arbitrary if $f$ maps $\perp$ to all memory states, and otherwise $\beta_{\text {next }}(f, s)$ is the distribution over $A$ such that, for all $a \in A$, we have

$$
\beta_{\text {next }}(f, s)(a)=\sum_{\substack{m_{0} \in M_{0} \\ \alpha_{\text {next }}\left(f\left(m_{0}\right), s\right)=a}} \frac{\mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0}^{\prime} \notin f^{-1}(\perp)} \mu_{\text {init }}\left(m_{0}^{\prime}\right)},
$$

where $f^{-1}(\perp)=\left\{m_{0} \in M_{0} \mid f\left(m_{0}\right)=\perp\right\}$.
We note that the memory state $f \in B$ mapping $\perp$ to all initial memory states is only reached whenever a history inconsistent with $\mathcal{M}$ has been processed. By Lemma 2.1, we need not take in account histories inconsistent with $\mathcal{M}$ to establish the outcome-equivalence of $\mathcal{M}$ and $\mathcal{B}$. This explains why the next-move function is left arbitrary in that case.

We will use Lemma 2.1 to establish the outcome-equivalence of $\mathcal{M}$ and $\mathcal{B}$. To this end, we first show a relation, for each $w \in(S A)^{*}$ consistent with $\mathcal{M}$, between the distribution $\mu_{w} \in \mathcal{D}(M)$ over states of memory $\mathcal{M}$ after processing $w$ and the function $f_{w}$ reached after $\mathcal{B}$ reads $w$ (recall that for a DRD strategy, the distribution over its states after processing $w$ is a Dirac distribution). Formally, this relation is as follows: for any $w \in(S A)^{*}$ consistent with $\mathcal{M}$ and any memory state $m \in M$, we have

$$
\begin{equation*}
\mu_{w}(m)=\frac{\sum_{m_{0} \in f_{w}^{-1}(m)} \mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0} \in M_{0}(w)} \mu_{\text {init }}\left(m_{0}\right)}, \tag{4.1}
\end{equation*}
$$

where $M_{0}(w)$ denotes the set of initial memory elements $m_{0} \in M_{0}$ of $\mathcal{M}$ such that $f_{w}\left(m_{0}\right) \neq \perp$, i.e., $M_{0}(w)$ is the set of potential initial memory states knowing that we have followed $w$, and $f_{w}^{-1}(m)=$ $\left\{m_{0} \in M_{0} \mid f_{w}\left(m_{0}\right)=m\right\}$.

This equation intuitively expresses that $\mathcal{B}$ accurately keeps track of the current distribution over memory states of $\mathcal{M}$ along a play. A corollary of the above is that whenever we follow histories consistent with $\mathcal{M}$, we are assured never to reach the memory state of $\mathcal{B}$ that assigns $\perp$ to all states in $M_{0}$.

We prove Equation (4.1) with an inductive argument. The case of $w=\varepsilon$ is trivial: by definition $\mu_{\varepsilon}=\mu_{\text {init }}$ and $f_{\varepsilon}$ is the identity function over $M_{0}$. Now, let us assume that Equation (4.1) holds for $w^{\prime} \in(S A)^{*}$ consistent with $\mathcal{M}$, and let us prove it for $w=w^{\prime}$ sa consistent with $\mathcal{M}$.

When writing relations between $\mu_{w^{\prime}}$ and $\mu_{w}$ in the sequel, we adopt notation slightly different to Section 2. In this case, the update function $\alpha_{\text {up }}$ and next-move $\alpha_{\text {next }}$ of $\mathcal{M}$ are deterministic. Thus, instead of weighing sums with Dirac distributions, we only sum over relevant states for clarity.

First, we proceed with the simpler case of $s \in S_{3-i}$. By definition, we have

$$
\mu_{w}(m)=\sum_{\substack{m^{\prime} \in M \\ \alpha_{\mathrm{up}}\left(m^{\prime}, s, a\right)=m}} \mu_{w^{\prime}}\left(m^{\prime}\right)
$$

From our inductive hypothesis, we obtain

$$
\begin{aligned}
\mu_{w}(m) & =\sum_{\substack{m^{\prime} \in M \\
\alpha_{\mathrm{up}}\left(m^{\prime}, s, a\right)=m}} \sum_{m_{0} \in f_{w^{\prime}}^{-1}\left(m^{\prime}\right)} \frac{\mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0}^{\prime} \in M_{0}\left(w^{\prime}\right)} \mu_{\text {init }}\left(m_{0}^{\prime}\right)} \\
& =\frac{1}{\sum_{m_{0}^{\prime} \in M_{0}\left(w^{\prime}\right)} \mu_{\text {init }}\left(m_{0}^{\prime}\right)} \cdot \sum_{\substack{m_{0} \in M_{0}\left(w^{\prime}\right) \\
\alpha_{\text {up }}\left(f_{w^{\prime}}\left(m_{0}\right), s, a\right)=m}} \mu_{\text {init }}\left(m_{0}\right) .
\end{aligned}
$$

It follows from the definition of $\beta_{\text {up }}$ that $M_{0}(w)=M_{0}\left(w^{\prime}\right)$. Thus, the denominator of the term above matches that of Equation (4.1). Furthermore, $f_{w}$ is the function that assigns $\alpha_{u p}\left(f_{w^{\prime}}\left(m_{0}\right), s, a\right)$ to any $m_{0} \in M_{0}\left(w^{\prime}\right)$. It follows that the rightmost sum in the last equality can be rewritten as the sum $\sum_{m_{0} \in f_{w}^{-1}(m)} \mu_{\text {init }}\left(m_{0}\right)$, matching the numerator of the fraction of Equation (4.1). This concludes the proof of the inductive case when $s \in S_{3-i}$.

Now, let us assume that $s \in S_{i}$. In this case, we may have $M_{0}(w) \neq M_{0}\left(w^{\prime}\right)$. In light of this, we must take care not to have $M_{0}(w)=\emptyset$, in which case the denominator of the right-hand side of Equation (4.1) evaluates to zero. From the definition of $\beta_{\text {up }}$, it follows that $M_{0}(w)$ is formed of the memory elements $m_{0} \in M_{0}\left(w^{\prime}\right)$ such that $\alpha_{\text {next }}\left(f_{w^{\prime}}\left(m_{0}\right), s\right)=a$. We know that $w=w^{\prime} s a$ is consistent with $\mathcal{M}$. This implies there is some $m \in M$ such that $\alpha_{\text {next }}(m, s)=a$ and $\mu_{w^{\prime}}(m)>0$. From the inductive hypothesis (Equation (4.1) with $w^{\prime}$ ), we obtain that there is some $m_{0} \in M_{0}\left(w^{\prime}\right)$ such that $f_{w^{\prime}}\left(m_{0}\right)=m$, otherwise the right-hand side of the equation would evaluate to zero. The equality $f_{w^{\prime}}\left(m_{0}\right)=m$ implies $m_{0} \in M_{0}(w)$, thus we have shown that $M_{0}(w)$ is non-empty.

Now that we have shown that Equation (4.1) is well-defined, we move on to its proof. Let us write $\alpha_{\text {next }}(\cdot, s)^{-1}(a)$ to denote the subset of $M$ consisting of memory states $m$ such that $\alpha_{\text {next }}(m, s)=a$. By definition, we have

$$
\begin{aligned}
& \sum_{m^{\prime} \in \alpha_{\text {next }(\cdot, s)^{-1}(a)}}^{\substack{\alpha_{\mathrm{up}}\left(m^{\prime}, s, a\right)=m}} \mu_{w^{\prime}}\left(m^{\prime}\right) \\
& \sum_{m^{\prime} \in \alpha_{\text {next }}(\cdot, s)^{-1}(a)} \mu_{w^{\prime}}\left(m^{\prime}\right)
\end{aligned} .
$$

For the numerator, we obtain from the inductive hypothesis that

$$
\begin{aligned}
\sum_{\substack{m^{\prime} \in \alpha_{\text {next }}(\cdot, s)^{-1}(a) \\
\alpha_{\text {up }}\left(m^{\prime}, s, a\right)=m}} \mu_{w^{\prime}}\left(m^{\prime}\right) & =\sum_{\substack{m^{\prime} \in \alpha_{\text {next }}(\cdot, s)^{-1}(a) \\
\alpha_{\text {up }}\left(m^{\prime}, s, a\right)=m}} \sum_{m_{0} \in f_{w^{\prime}}^{-1}\left(m^{\prime}\right)} \frac{\mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0}^{\prime} \in M_{0}\left(w^{\prime}\right)} \mu_{\text {init }}\left(m_{0}^{\prime}\right)} \\
& =\sum_{m_{0} \in f_{w}^{-1}(m)} \frac{\mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0}^{\prime} \in M_{0}\left(w^{\prime}\right)} \mu_{\text {init }}\left(m_{0}^{\prime}\right)} .
\end{aligned}
$$

We explain the passage from the double sum to the simple sum. It follows from the fact that $f_{w}\left(m_{0}\right)=m$ holds if and only if $f_{w^{\prime}}\left(m_{0}\right)$ is a memory state $m^{\prime}$ such that $\alpha_{\text {up }}\left(m^{\prime}, s, a\right)=m$ and $\alpha_{\text {next }}\left(m^{\prime}, s\right)=a$, by definition of $\beta_{\text {up }}$.

For the denominator, we obtain from the inductive hypothesis,

$$
\begin{aligned}
\sum_{m^{\prime} \in \alpha_{\text {next }}(\cdot, s)^{-1}(a)} \mu_{w^{\prime}}\left(m^{\prime}\right) & =\sum_{m^{\prime} \in \alpha_{\text {next }}(\cdot, s)^{-1}(a)} \sum_{m_{0} \in f_{w^{\prime}}^{-1}\left(m^{\prime}\right)} \frac{\mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0}^{\prime} \in M_{0}\left(w^{\prime}\right)} \mu_{\text {init }}\left(m_{0}^{\prime}\right)} \\
& =\sum_{m_{0} \in M_{0}(w)} \frac{\mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0}^{\prime} \in M_{0}\left(w^{\prime}\right)} \mu_{\text {init }}\left(m_{0}^{\prime}\right)} ;
\end{aligned}
$$

the last equality is a consequence of the definition of $\beta_{\mathrm{up}}$ : recall that $M_{0}(w)$ consists of the elements $m_{0}$ of $M_{0}\left(w^{\prime}\right)$ such that $\alpha_{\text {next }}\left(f_{w^{\prime}}\left(m_{0}\right), s\right)=a$. By combining the two equations above, we immediately obtain Equation (4.1), ending the inductive argument.

We now establish the outcome-equivalence of $\mathcal{M}$ and $\mathcal{B}$ via Lemma 2.1. Let $h=w s \in \operatorname{Hist}_{i}(\mathcal{G})$ be a history of $\mathcal{G}$ consistent with $\mathcal{M}$. Let $a \in A(s)$ be an action enabled in $s$. The probability of $a$ being played after $h$ under $\mathcal{M}$ is given by the weighted sum $\sum_{m \in \alpha_{\text {next }}(\cdot, s)^{-1}(a)} \mu_{w}(m)$. Under $\mathcal{B}$, the probability of $a$ being played is

$$
\sum_{\substack{m_{0} \in M_{0} \\ \text { next }\left(f_{w}\left(m_{0}\right), s\right)=a}} \frac{\mu_{\text {init }}\left(m_{0}\right)}{\sum_{m_{0}^{\prime} \in M_{0}(w)} \mu_{\text {init }}\left(m_{0}\right)}
$$

It follows from Equation (4.1) that these two probabilities coincide. Lemma 2.1 implies the outcomeequivalence of strategies $\mathcal{M}$ and $\mathcal{B}$, ending the proof.

The construction of a DRD strategy provided in the proof of Theorem 4.1 leads to an exponential blow-up of the memory state space. For an RDD strategy $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$, we have constructed an outcome-equivalent DRD strategy with a state space consisting of functions supp $\left(\mu_{\text {init }}\right) \rightarrow M \cup\{\perp\}$, therefore with a state space of size $(|M|+1)^{\left|\operatorname{supp}\left(\mu_{\text {init }}\right)\right|}$. In the upcoming lemma, we state that an exponential blow-up in the number of initial memory states cannot be avoided in general.

Lemma 4.1. For all $n \in \mathbb{N}$, there exists a two-player deterministic game (respectively a Markov decision process) $\mathcal{G}_{n}$ with $n+2$ states, $4 n+2$ transitions, $n+1$ actions, and an $R D D$ strategy $\mathcal{M}_{n}$ of $\mathcal{P}_{1}$ with $n$ states such that any outcome-equivalent $D R D$ strategy must have at least $2^{n}-1$ states.

Proof. Let $n \in \mathbb{N}$. We construct a two-player deterministic game $\mathcal{G}_{n}$ as follows. We let $S_{1}=\left\{s_{i} \mid 1 \leq i \leq\right.$ $n\} \cup\left\{s^{\star}\right\}$, and $S_{2}=\{t\}$. The set of action $A$ is $\left\{a_{i} \mid 1 \leq i \leq n\right\} \cup\{b\}$. We denote transitions with a function $\delta: S \times A \rightarrow S$ instead of probability measures over successor states. For each $i \in\{1, \ldots, n\}$, the actions $a_{i}$ and $b$ are the only actions enabled in $s_{i}$ and they both move to state $t$, i.e., $\delta\left(s_{i}, a_{i}\right)=\delta\left(s_{i}, b\right)=t$. In state $t$, all actions are enabled, and we set for all $i, \delta\left(t, a_{i}\right)=s_{i}$ and $\delta(t, b)=s^{\star}$. In state $s^{\star}$, all actions $a_{i}$ are enabled and label self-loops, i.e., for all $i$, we have $\delta\left(s^{\star}, a_{i}\right)=s^{\star}$. We illustrate the game $\mathcal{G}_{3}$ in Figure 4.1.


Fig. 4.1. The game $\mathcal{G}_{3}$ from the proof of Lemma 4.1.

We define the RDD strategy $\mathcal{M}_{n}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ as follows. We let $M=\{1, \ldots, n\}$, and $\mu_{\text {init }}$ is taken to be the uniform distribution over $M$. The memory update function is taken to be trivial: we set $\alpha_{\mathrm{up}}(m, s, a)=m$ for all $m \in M, s \in S$ and $a \in A$. For each memory state $m \in M$, we let $\alpha_{\text {next }}\left(m, s_{m}\right)=\alpha_{\text {next }}\left(m, s^{\star}\right)=a_{m}$ and whenever $k \neq m$, we let $\alpha_{\text {next }}\left(m, s_{k}\right)=b$. In $\mathcal{M}$, once the initial state is decided, it no longer changes. In the memory state $m \in\{1, \ldots, n\}$, the strategy prescribes action $a_{m}$ in the states $s_{m}$ and $s^{\star}$, and in states $s_{j}$ with $j \neq m$, the strategy prescribes action $b$.

We now argue that all DRD strategies that are outcome-equivalent to $\mathcal{M}$ must have at least $2^{n}-1$ memory states. Let $\mathcal{B}=\left(B, b_{\text {init }}, \beta_{\mathrm{up}}, \beta_{\mathrm{next}}\right)$ be one such FM strategy. We give a lower bound on $|B|$ by analysing the number of required next-move functions. In practice, we show that there must be at least $2^{n}-1$ distinct distributions of the form $\beta_{\text {next }}\left(\cdot, s^{\star}\right)$.

Let $E=\left\{k_{1}, \ldots, k_{\ell}\right\} \subsetneq M$ be a proper subset of $M$. Consider the history

$$
h_{E}=t a_{k_{1}} s_{k_{1}} b t a_{k_{2}} s_{k_{2}} b \ldots t a_{k_{\ell}} s_{k_{\ell}} b t b s^{\star} .
$$

Let $m \in E$. We see that along the history $h_{E}$, the action $b$ is used in state $s_{m}$. Therefore, $h_{E}$ is not consistent with the pure FM strategy ( $M, m, \alpha_{\text {up }}, \alpha_{\text {next }}$ ) derived from $\mathcal{M}$ by setting its initial state to $m$. Similarly, we see that for $m \notin E$, the history $h_{E}$ is consistent with the pure FM strategy ( $M, m, \alpha_{\text {up }}, \alpha_{\text {next }}$ ). Thus, the set of actions that can be played after $h_{E}$ when following $\mathcal{M}_{n}$ is exactly the set $\left\{a_{m} \mid m \in M \backslash E\right\} \neq \emptyset$. Due to the deterministic initialisation and updates of DRD strategies, there must be some $b_{E} \in B$ such that $\operatorname{supp}\left(\beta_{\text {next }}\left(b_{E}, s^{\star}\right)\right)=\left\{a_{m} \mid m \in M \backslash E\right\}$. Necessarily, we must have $\operatorname{supp}\left(\beta_{\text {up }}\left(b_{E}, s^{\star}\right)\right) \neq \operatorname{supp}\left(\beta_{\text {up }}\left(b_{E^{\prime}}, s^{\star}\right)\right)$ whenever $E \neq E^{\prime}$, hence $b_{E} \neq b_{E^{\prime}}$. Consequently, we must have at least one memory state in $\mathcal{B}$ per proper subset of $M$, i.e., $|B| \geq 2^{n}-1$.

The proof of the existence of a suitable Markov decision process remains. We explain how to adapt the deterministic game $\mathcal{G}_{n}$. To change $\mathcal{G}_{n}$ to a suitable Markov decision process, it suffices to give $\mathcal{P}_{1}$ the sole $\mathcal{P}_{2}$ state $t$, and change its outgoing transitions as follows. We only let a single action $b$ enabled in $t$ such that there is a uniform probability of reaching the states other than $t$ using this action. The strategy
needs to change to account for the change of ownership of $t$. Only the next-move function of the strategy is mended; it must now prescribe the action $b$ when in $t$.

By performing these two changes, it is possible to directly reuse the arguments used in the two-player case to find the same lower bound. This concludes our explanation of how to adapt the game and strategy above to the context of Markov decision processes, and ends the proof.

### 4.2 Simulating RRR strategies with DRR ones

In this section, we establish that DRR strategies are as expressive as RRR strategies, i.e., randomness in the initialisation can be removed. We outline the ideas behind the construction of a DRR strategy that is outcome-equivalent to a given RRR strategy. The rough idea behind the construction is to simulate the behaviour of the RRR strategy at the start of the play using a new initial memory state and then move back into the RRR strategy we simulate.

We substitute the random selection of an initial memory element in two stages. To ensure the first action is selected in the same way under both the supplied strategy and the strategy we construct, we rely on the randomised outputs. The probability of selecting an action $a$ in a given state $s$ of the game in our new initial memory state is given as the sum of selecting action $a$ in state $s$ in each memory state $m$ weighed by the initial probability of $m$.

We then leverage the stochastic updates to behave as though we had been using the supplied FM strategy from the start. If the first game state was controlled by the player who does not own the strategy, the probability of moving into a memory state $m$ is also described by a weighted sum similar in spirit to the case of the first action (albeit by considering the update function in place of the next-move function). Whenever the owner of the strategy controls the first state of the game, the chosen action conditions which possible initial memory states we could have found ourselves in. The reasoning in this case is similar to the one for the update of the distribution over memory states (denoted by $\mu_{w}$ in Section 2) after processing some sequence in $(S A)^{*}$.

We now state our expressiveness result and formalise the construction outlined above.
Theorem 4.2. Let $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ be a game. Let $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ be an RRR strategy owned by $\mathcal{P}_{i}$. There exists a $D R R$ strategy $\mathcal{B}=\left(B, b_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$ such that $\mathcal{B}$ and $\mathcal{M}$ are outcome-equivalent, and such that $|B|=|M|+1$.

Proof. Let us define $\mathcal{B}=\left(B, b_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$ as follows. Let $b_{\text {init }}$ be such that $b_{\text {init }} \notin M$. We set $B=$ $M \cup\left\{b_{\text {init }}\right\}$. We let $\beta_{\text {up }}$ and $\beta_{\text {next }}$ coincide with $\alpha_{\text {up }}$ and $\alpha_{\text {next }}$ over $M \times S \times A$ and $M \times S_{1}$ respectively (for the update function, we view distributions over $M$ as distributions over $B$ that assign probability zero to $\left.b_{\text {init }}\right)$. It remains to define these two functions over $\left\{b_{\text {init }}\right\} \times S \times A$ and $\left\{b_{\text {init }}\right\} \times S_{i}$ respectively.

First, we complete the definition of the memory update function $\beta_{\text {up }}$. Let $s \in S$ and $a \in A$. We let $\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)\left(b_{\text {init }}\right)=0$. For the remaining memory states, we distinguish two cases following whether $s \in S_{i}$ or $s \in S_{3-i}$. First, let us assume $s \in S_{i}$. We assume that there exists some $m_{0} \in M$ such that $\mu_{\text {init }}\left(m_{0}\right)>0$ and $\alpha_{\text {next }}\left(m_{0}, s\right)(a)>0$ (i.e., the action $a$ has a positive probability of being played in $s$ at the start of a play under the strategy $\mathcal{M})$. We set, for all $m \in M$,

$$
\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)(m)=\frac{\sum_{m^{\prime} \in M} \mu_{\text {init }}\left(m^{\prime}\right) \cdot \alpha_{\text {up }}\left(m^{\prime}, s, a\right)(m) \cdot \alpha_{\text {next }}\left(m^{\prime}, s\right)(a)}{\sum_{m^{\prime} \in M} \mu_{\text {init }}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime}, s\right)(a)} .
$$

Whenever we have $\alpha_{\text {next }}\left(m_{0}, s\right)(a)=0$ for all $m_{0} \in M$ such that $\mu_{\text {init }}\left(m_{0}\right)>0$, we let $\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)$ be arbitrary. Next, let us assume that $s \in S_{3-i}$. In this case, we set for all $m \in M$,

$$
\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)(m)=\sum_{m^{\prime} \in M} \mu_{\text {init }}\left(m^{\prime}\right) \cdot \alpha_{\text {up }}\left(m^{\prime}, s, a\right)(m) .
$$

For the next-move function $\beta_{\text {next }}$, we define, for all states $s \in S$ and actions $a \in A(s)$,

$$
\beta_{\text {next }}\left(b_{\text {init }}, s\right)(a)=\sum_{m \in M} \mu_{\text {init }}(m) \cdot \alpha_{\text {next }}(m, s)(a) .
$$

Now, it remains to prove that $\mathcal{M}$ and $\mathcal{B}$ are outcome-equivalent. By Lemma 2.1, it suffices to show that both strategies suggest the same distributions over actions along histories consistent with $\mathcal{M}$. We provide a proof in two steps. First, we consider histories with a single state and then all histories with a more than one state all at once.

Let $s \in S_{1}$ and $a \in A(s)$. On the one hand, the probability of the action $a$ being played after the history $s$ under $\mathcal{M}$ is given by

$$
\sum_{m \in M} \mu_{\text {init }}(m) \cdot \alpha_{\text {next }}(m, s)(a) .
$$

On the other hand, the probability of this same action $a$ being played after the history $s$ under $\mathcal{B}$ is given by $\beta_{\text {next }}\left(b_{\text {init }}, s\right)(a)$. These two probabilities coincide by construction.

Let us move on to histories $h=w s$ consisting of more than one state and consistent with $\mathcal{M}$. Because both $\alpha_{\text {next }}$ and $\beta_{\text {next }}$ coincide over $M \times S_{i}$, it suffices to show that the distributions over memory states attained after the strategies process $w$ coincide to deduce that the two strategies suggest the same distributions over actions after $h$. Furthermore, because both $\alpha_{\text {up }}$ and $\beta_{\text {up }}$ coincide over $M \times S \times A$, if we show that for all prefixes of plays of the form $s a \in S A$ consistent with $\mathcal{M}$, we have the same distribution over memory states in both FM strategies, then it follows that for all longer histories consistent with $\mathcal{M}$, we also have the same distributions over memory states in both strategies. Proving the previous claim ends this proof; we obtain that both strategies must suggest the same distributions over actions along histories consistent with $\mathcal{M}$.

Let $w=s a \in S A$ be consistent with $\mathcal{M}$. Let $\mu_{w}$ and $\beta_{w}$ denote the distribution over memory states after processing $w$ in $\mathcal{M}$ and $\mathcal{B}$ respectively. Fix some $m \in M$, and let us prove that $\mu_{w}(m)=\beta_{w}(m)$. If $s \in S_{3-i}$, we have

$$
\mu_{w}(m)=\sum_{m^{\prime} \in M} \mu_{\text {init }}\left(m^{\prime}\right) \alpha_{\text {up }}\left(m^{\prime}, s, a\right)(m)=\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)(m)=\beta_{w}(m) .
$$

Let us assume henceforth that $s \in S_{i}$. On the one hand, we have

$$
\mu_{w}(m)=\frac{\sum_{m^{\prime} \in M} \mu_{\text {init }}\left(m^{\prime}\right) \cdot \alpha_{\text {up }}\left(m^{\prime}, s, a\right)(m) \cdot \alpha_{\text {next }}\left(m^{\prime}, s\right)(a)}{\sum_{m^{\prime} \in M} \mu_{\text {init }}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime}, s\right)(a)}=\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)(m),
$$

and on the other hand, we have (because $b_{\text {init }}$ is the sole initial state of $\mathcal{B}$ ),

$$
\beta_{w}(m)=\frac{\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)(m) \cdot \beta_{\text {next }}\left(b_{\text {init }}, s\right)(a)}{\beta_{\text {next }}\left(b_{\text {init }}, s\right)(a)}=\beta_{\text {up }}\left(b_{\text {init }}, s, a\right)(m)
$$

ending the proof.

### 4.3 Simulating RRR strategies with RDR ones

We are concerned in this section with the simulation of RRR strategies by RDR strategies, i.e., with substituting randomised outputs with deterministic outputs. The idea behind the removal of randomisation in outputs is to simulate said randomisation by means of both stochastic initialisation and updates. These are used to preemptively perform the random selection of an action, simultaneously with the selection of an initial or successor memory state.

Let $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ be a stochastic game and let $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ be an RRR strategy of $\mathcal{P}_{i}$. We construct an RDR strategy $\mathcal{B}=\left(B, \beta_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$ that is outcome-equivalent to $\mathcal{M}$ and such that $|B| \leq|M| \cdot|S| \cdot|A|$. The state space of $\mathcal{B}$ consists of pairs ( $m, \sigma_{i}$ ) where $m \in M$ and $\sigma_{i}$ is a pure memoryless strategy of $\mathcal{P}_{i}$. To achieve our bound on the size of $B$, we cannot take all pure memoryless strategies of $\mathcal{P}_{i}$. To illustrate how we perform the selection of these pure memoryless strategies, we provide a simple example of the construction on a DRD strategy (which is a special case of RRR strategies) with a single memory state (i.e., a memoryless randomised strategy).

Example 4.1. We consider a game $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ where $S_{1}=\left\{s_{1}, s_{2}, s_{3}\right\}, S_{2}=\emptyset, A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and all actions are enabled in all states. We need not specify $\delta$ exactly for our purposes. For our construction, we fix an order on the actions of $\mathcal{G}: a_{1}<a_{2}<a_{3}$.

Let $\mathcal{M}=\left(\{m\}, m, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ be the DRD strategy such that $\alpha_{\text {next }}\left(m, s_{1}\right)$ and $\alpha_{\text {next }}\left(m, s_{2}\right)$ are uniform distributions over $\left\{a_{1}, a_{2}\right\}$ and $A$ respectively, and $\alpha_{\text {next }}\left(m, s_{3}\right)\left(a_{1}\right)=\frac{1}{3}, \alpha_{\text {next }}\left(m, s_{3}\right)\left(a_{2}\right)=\frac{1}{6}$ and $\alpha_{\text {next }}\left(m, s_{3}\right)\left(a_{3}\right)=\frac{1}{2}$.

Figure 4.2 illustrates the probability of each action being chosen in each state as the length of a segment. Let us write $0=x_{1}<x_{2}<x_{3}<x_{4}<x_{5}=1$ for all of the endpoints of the segments appearing in the illustration. For each index $i \in\{1, \ldots, 4\}$, we define a pure memoryless strategy $\sigma_{k}$ that assigns to each state the action lying in the segment above it in the figure. For instance, $\sigma_{2}$ is such that $\sigma_{2}\left(s_{1}\right)=a_{1}$

| $s_{1}$ | $a_{1}$ |  | $a_{2}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $s_{2}$ | $a_{1}$ | $a_{2}$ |  |  |
| $s_{3}$ | $a_{1}$ | $a_{2}$ | $a_{2}$ | $a_{3}$ |
| $\sigma_{i}$ | $\sigma_{1}$ | $\sigma_{2}$ | $\sigma_{3}$ | $\sigma_{4}$ |
| $x_{1}=0$ | $x_{2}=\frac{1}{3} \quad x_{3}=\frac{1}{2} \quad x_{4}=\frac{2}{3}$ | $x_{5}$ | $=1$ |  |

Fig. 4.2. Representation of cumulative probability of actions under strategy $\mathcal{M}$ and derived memoryless strategies.
and $\sigma_{2}\left(s_{2}\right)=\sigma_{2}\left(s_{3}\right)=a_{2}$. Furthermore, for all $i \in\{1, \ldots, 4\}$, the length $x_{k+1}-x_{k}$ of its corresponding interval denotes the probability of the strategy being chosen during stochastic updates.

We construct an $\operatorname{RDR}$ strategy $\mathcal{B}=\left(B, \beta_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$ that is outcome-equivalent to $\mathcal{M}$ in the following way. We let $B=\{m\} \times\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}\right\}$. The initial distribution is given by $\beta_{\text {init }}\left(m, \sigma_{k}\right)=x_{k+1}-x_{k}$, i.e., the probability of $\sigma_{k}$ in the illustration. We set, for any $j, k \in\{1, \ldots, 4\}, s \in S$ and $a \in A$, $\beta_{\text {up }}\left(\left(m, \sigma_{k}\right), s, a\right)\left(\left(m, \sigma_{j}\right)\right)=x_{j+1}-x_{j}$. Finally, we let $\beta_{\text {next }}\left(\left(m, \sigma_{k}\right), s\right)=\sigma_{k}(s)$ for all $k \in\{1, \ldots, 4\}$ and $s \in S$.

The argument for the outcome-equivalence of $\mathcal{B}$ and $\mathcal{M}$ is the following; for any state $s \in S_{1}$, the probability of moving into a memory state $\left(m, \sigma_{k}\right)$ such that $\sigma_{k}(s)=a$ is by construction the probability $\alpha_{\text {next }}(m, s)$.

In the previous example, we had a unique memory state $m$ and we defined some memoryless strategies from the next-move function partially evaluated in this state (i.e., from $\alpha_{\text {next }}(m, \cdot)$ ). In general, each memory state may have a different partially evaluated next-move function, and therefore we must define some memoryless strategies for each individual memory state. For each memory state, we can bound the number of derived memoryless strategies by $\left|S_{i}\right| \cdot|A|$; we look at cumulative probabilities over actions (of which there are at most $|A|$ ) for each state of $\mathcal{P}_{i}$. This explains our announced bound on $|B|$.

Furthermore, in general, the memory update function is not trivial. Generalising the construction above can be done in a straightforward manner to handle updates. Intuitively, the probability to move to some memory state of the form $(m, \sigma)$ will be given by the probability of moving into $m$ multiplied by the probability of $\sigma$ (in the sense of Figure 4.2).

We now formally state our result in the general setting and provide its proof.
Theorem 4.3. Let $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ be a game. Let $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ be an RRR strategy owned by $\mathcal{P}_{i}$. There exists an $R D R$ strategy $\mathcal{B}=\left(B, \beta_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$ such that $\mathcal{B}$ and $\mathcal{M}$ are outcome-equivalent, and such that $|B| \leq|M| \cdot\left|S_{i}\right| \cdot|A|$.

Proof. Let us fix a linear total order on the set of actions $A$, denoted by $<$. Fix some $m \in M$. We let $x_{1}^{m}<\ldots<x_{\ell(m)}^{m}$ denote the elements of the set

$$
\left\{\sum_{a^{\prime}<a} \alpha_{\text {next }}(m, s)\left(a^{\prime}\right) \mid s \in S_{i}, a \in A\right\}
$$

that are strictly inferior to 1 , and let $x_{\ell(m)+1}^{m}=1$. These $x_{j}^{m}$ represent the cumulative probability provided by $\alpha_{\text {next }}(m, \cdot)$ over actions taken in order, for each state of $\mathcal{G}$. For each $j \in\{1, \ldots, \ell(m)\}$, we define a memoryless strategy $\sigma_{j}^{m}: S_{i} \rightarrow A$ as follows: we have $\sigma_{j}^{m}(s)=a$ if $\sum_{a^{\prime}<a} \alpha_{\text {next }}(m, s)\left(a^{\prime}\right) \leq x_{j}^{m}<$ $\sum_{a^{\prime} \leq a} \alpha_{\text {next }}(m, s)\left(a^{\prime}\right)$. In other words, for any state $s \in S_{i}$, we have $\sigma_{j}^{m}(s)=a$ whenever $x_{j}^{m}$ is at least the cumulative probability of actions strictly inferior to $a$ in $\alpha_{\text {next }}(m, s)$ and at most the cumulative probability of actions up to action $a$ included. Refer to the Figure 4.2 of Example 4.1 for an explicit illustration. We refer to $x_{j+1}^{m}-x_{j}^{m}$ as the probability of $\sigma_{j}^{m}$ in the sequel.

Let $m \in M, s \in S_{i}$ and $a \in A(s)$. We observe that we can relate $\alpha_{\text {next }}(m, s)(a)$ and the sum of the probabilities of each $\sigma_{j}^{m}$ such that $\sigma_{j}^{m}(s)=a$ as follows. First, we introduce some notation. Let $I(m, s, a)$ denote the set of indices $j$ such that $\sigma_{j}^{m}(s)=a$, i.e., the indices such that the $j$ th strategy related to $m$ prescribes action $a$ in $s$. It holds that

$$
\begin{equation*}
\sum_{j \in I(m, s, a)}\left(x_{j+1}^{m}-x_{j}^{m}\right)=\alpha_{\text {next }}(m, s)(a) \tag{4.2}
\end{equation*}
$$

Let $s \in S_{i}$ and $a \in A(s)$. Equation (4.2) can be proven as follows. First, note that all indices $j$ appearing in the sum are consecutive by construction. Therefore, the sum above is telescoping and only $x_{j^{+}+1}^{m}-x_{j^{-}}^{m}$ remains, where $j^{+}$and $j^{-}$denote the largest and smallest indices in the sum respectively. By construction, we have $x_{j^{-}}^{m}=\sum_{a^{\prime}<a} \alpha_{\text {next }}(m, s)\left(a^{\prime}\right)$ and $x_{j^{+}+1}^{m}=\sum_{a^{\prime} \leq a} \alpha_{\text {next }}(m, s)\left(a^{\prime}\right)$. We conclude that $x_{j^{+}+1}^{m}-x_{j^{-}}^{m}=\alpha_{\text {next }}(m, s)(a)$, proving Equation (4.2). This observation will be useful to establish the outcome-equivalence of $\mathcal{M}$ with the strategy defined below.

We can now define our RDR strategy $\mathcal{B}=\left(B, \beta_{\text {init }}, \beta_{\text {up }}, \beta_{\text {next }}\right)$. We define

$$
B=\left\{\left(m, \sigma_{j}^{m}\right) \mid m \in M, 1 \leq j \leq \ell(m)\right\}
$$

The initial distribution and update function of $\mathcal{B}$ are derived from those of $\mathcal{M}$ multiplied with the probability of the memoryless strategy that appears in the second component of the memory state of $\mathcal{B}$ into which we move. The initial distribution $\beta_{\text {init }}$ is defined as $\beta_{\text {init }}\left(m, \sigma_{j}^{m}\right)=\mu_{\text {init }}(m) \cdot\left(x_{j+1}^{m}-x_{j}^{m}\right)$ for all $\left(m, \sigma_{j}^{m}\right) \in B$. The update function is defined as $\beta_{\text {up }}\left(\left(m, \sigma_{j}^{m}\right), s, a\right)\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right)\right)=\alpha_{\text {up }}(m, s, a)\left(m^{\prime}\right) \cdot\left(x_{k+1}^{m^{\prime}}-\right.$ $\left.x_{k}^{m^{\prime}}\right)$ for all $\left(m, \sigma_{j}^{m}\right),\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right) \in B, s \in S$ and $a \in A$. Finally, the deterministic next-move function of $\mathcal{B}$ is defined as $\beta_{\text {next }}\left(\left(m, \sigma_{j}^{m}\right), s\right)=\sigma_{j}^{m}(s)$ for all $\left(m, \sigma_{j}^{m}\right) \in B$ and all $s \in S_{i}$.

We now argue the outcome-equivalence of $\mathcal{M}$ and $\mathcal{B}$. For any $w \in(S A)^{*}$, let $\mu_{w}$ (resp. $\beta_{w}$ ) denote the distribution over $M$ (resp. $B$ ) after strategy $\mathcal{M}$ (resp. $\mathcal{B}$ ) has read $w$. It follows from Lemma 2.1 and the definition of strategies derived from FM strategies that it suffices to establish, for all histories $h=w s$ consistent with $\mathcal{M}$, that the following holds:

$$
\begin{equation*}
\sum_{m \in M} \mu_{w}(m) \cdot \alpha_{\text {next }}(m, s)(a)=\sum_{\substack{m \in M \\ j \in I(m, s, a)}} \beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right) \tag{4.3}
\end{equation*}
$$

To prove Equation (4.3), we first prove another statement inductively. We show that for any $w \in(S A)^{*}$ consistent with $\mathcal{M}$, we must have that $\mu_{w}(m)$ is proportional to $\beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right)$. To be precise, we establish that for any $w \in(S A)^{*}$ consistent with $\mathcal{M}$, we must have

$$
\begin{equation*}
\beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right)=\left(x_{j+1}^{m}-x_{j}^{m}\right) \cdot \mu_{w}(m) \tag{4.4}
\end{equation*}
$$

We proceed by induction. Consider the empty word $w=\varepsilon$. Because $\mu_{\text {init }}=\mu_{\varepsilon}$ and $\beta_{\text {init }}=\beta_{\varepsilon}$, Equation (4.4) follows from the definition of $\beta_{\text {init }}$. Let us now assume inductively that for $w^{\prime} \in(S A)^{*}$ consistent with $\mathcal{M}$, we have Equation (4.4) and let us prove it for $w=w^{\prime} s a$ consistent with $\mathcal{M}$. Fix $\left(m, \sigma_{j}^{m}\right) \in B$. We consider two cases, depending on the owner of $s$.

First, let us assume that $s \in S_{i}$. By definition, we have,

$$
\beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right)=\frac{\sum_{m^{\prime} \in M} \sum_{k \in I\left(m^{\prime}, s, a\right)} \beta_{w^{\prime}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right)\right) \cdot \beta_{\mathrm{up}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right), s, a\right)\left(\left(m, \sigma_{j}^{m}\right)\right)}{\sum_{m^{\prime} \in M} \sum_{k \in I\left(m^{\prime}, s, a\right)} \beta_{w^{\prime}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right)\right)}
$$

The numerator of the above can be rewritten as follows, by successively using the definition of $\beta_{\text {up }}$ followed by the inductive hypothesis and Equation (4.2):

$$
\begin{aligned}
& \sum_{m^{\prime} \in M} \sum_{k \in I\left(m^{\prime}, s, a\right)} \beta_{w^{\prime}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right)\right) \cdot \alpha_{\mathrm{up}}\left(m^{\prime}, s, a\right)(m) \cdot\left(x_{j+1}^{m}-x_{j}^{m}\right) \\
& \quad=\left(x_{j+1}^{m}-x_{j}^{m}\right) \cdot \sum_{m^{\prime} \in M}\left(\alpha_{\mathrm{up}}\left(m^{\prime}, s, a\right)(m) \cdot \mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \sum_{k \in I\left(m^{\prime}, s, a\right)}\left(x_{k+1}^{m^{\prime}}-x_{k}^{m^{\prime}}\right)\right) \\
& \quad=\left(x_{j+1}^{m}-x_{j}^{m}\right) \cdot \sum_{m^{\prime} \in M} \alpha_{\mathrm{up}}\left(m^{\prime}, s, a\right)(m) \cdot \mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime}, s\right)(a)
\end{aligned}
$$

Following the same reasoning, the denominator can be rewritten as

$$
\sum_{m^{\prime} \in M} \mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime}, s\right)(a)
$$

By combining the equations above and the formula for the update of $\mu_{w}$, we obtain $\beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right)=$ $\left(x_{j+1}^{m}-x_{j}^{m}\right) \cdot \mu_{w}(m)$.

We now move on to the case $s \in S_{3-i}$. By definition, we have

$$
\begin{aligned}
\beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right) & =\sum_{m^{\prime} \in M} \sum_{1 \leq k \leq \ell\left(m^{\prime}\right)} \beta_{w^{\prime}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right)\right) \cdot \beta_{\mathrm{up}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right), s, a\right)\left(\left(m, \sigma_{j}^{m}\right)\right) \\
& =\left(x_{j+1}^{m}-x_{j}^{m}\right) \cdot \sum_{m^{\prime} \in M}\left(\alpha_{\mathrm{up}}\left(m^{\prime}, s, a\right)(m) \cdot \sum_{1 \leq k \leq \ell\left(m^{\prime}\right)} \beta_{w^{\prime}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right)\right)\right)
\end{aligned}
$$

The innermost sum can be rewritten as follows by applying the inductive hypothesis

$$
\sum_{1 \leq k \leq \ell\left(m^{\prime}\right)} \beta_{w^{\prime}}\left(\left(m^{\prime}, \sigma_{k}^{m^{\prime}}\right)\right)=\mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \sum_{1 \leq k \leq \ell\left(m^{\prime}\right)}\left(x_{k+1}^{m^{\prime}}-x_{k}^{m^{\prime}}\right)=\mu_{w^{\prime}}\left(m^{\prime}\right)
$$

By substituting the latter equation in the equation of $\beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right)$ above, we see the definition of $\mu_{w}$ from $\mu_{w^{\prime}}$ appear in the outer sum. We obtain $\beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right)=\left(x_{j+1}^{m}-x_{j}^{m}\right) \cdot \mu_{w}\left(m^{\prime}\right)$, ending the proof of Equation (4.4).

We now show how Equation (4.4) implies Equation (4.3), which will prove that $\mathcal{M}$ and $\mathcal{B}$ are indeed outcome-equivalent. Let $h=w s \in \operatorname{Hist}_{i}(\mathcal{G})$ be a history consistent with $\mathcal{M}$. Let $a \in A(s)$. The probability that the action $a$ is chosen after history $h$ under $\mathcal{M}$ is given by $\sum_{m \in M} \mu_{w}(m) \cdot \alpha_{\text {next }}(m, s)(a)$. The probability that $a$ is selected after $h$ under $\mathcal{B}$, on the other hand, is given by

$$
\begin{aligned}
\sum_{m \in M} \sum_{j \in I(m, s, a)} \beta_{w}\left(\left(m, \sigma_{j}^{m}\right)\right) & =\sum_{m \in M}\left(\mu_{w}(m) \cdot \sum_{j \in I(m, s, a)}\left(x_{j+1}^{m}-x_{j}^{m}\right)\right) \\
& =\sum_{m \in M} \mu_{w}(m) \cdot \alpha_{\text {next }}(m, s)(a)
\end{aligned}
$$

In the above, the first equation is obtained from Equation (4.4) and the second equation follows from Equation (4.2). This concludes the argument for the outcome-equivalence of our two FM strategies.

To end the proof of this lemma, we prove the upper bound on $|B|$ given in the statement of the result. For any memory state $m \in M, \ell(m)$ is bounded by $\left|S_{i}\right| \cdot|A|$, by definition of the numbers $x_{j}^{m}$. Therefore, we have at most $\left|S_{i}\right| \cdot|A|$ pairs of the form $\left(m, \sigma_{j}^{m}\right)$ per memory state $m \in M$, i.e., $|B| \leq|M| \cdot\left|S_{i}\right| \cdot|A|$.
Remark 4.1. The choice of the order on the set of actions fixed at the start of the previous proof influences the size of the constructed strategy. We note that we do not require a uniform order for actions for all memory states. Indeed, the order is used to define all memoryless strategies of the form $\sigma_{j}^{m}$. Because these strategies do not interact with strategies associated to other memory states, it is possible to use different orderings on actions depending on the memory state $m$ that is considered.

## 5 Strictness of inclusions

We now discuss the strictness of inclusions in the lattice of Figure 1.1. Section 5.1 complements the previous Section 4.1 and presents a DRD strategy that has no outcome-equivalent RDD counterpart. The strict inclusion of the class DRD in the class of RRD strategies is covered in Section 5.2. Finally, we provide the necessary results to establish that the class DDR is incomparable to the classes of RDD, DRD and RRD strategies in Section 5.3.

### 5.1 DRD versus RDD strategies

The goal of this section is to show that there exists some (one-player deterministic) game where some DRD strategy cannot be emulated by any RDD strategy. Let us first explain some intuition behind this statement. Intuitively, an RDD strategy can only randomise once at the start between a finite number of pure FM (DDD) strategies. After this initial randomisation, the sequence of actions prescribed by the RDD strategy is fixed relatively to the play in progress. Any DRD strategy that chooses an action randomly at each step cannot be reproduced by an RDD strategy. Indeed, this randomisation generates an infinite number of patterns of actions. These patterns cannot all be captured by an RDD strategy due to the fact that its initial randomisation is over a finite set.

Lemma 5.1. There exist a one-player deterministic game $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ and some DRD strategy $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\mathrm{up}}, \alpha_{\text {next }}\right)$ of $\mathcal{P}_{1}$ such that there is no outcome-equivalent $R D D$ strategy.

Proof. Consider the one-player game $\mathcal{G}$ depicted in Figure 5.1. Consider the memoryless behavioural strategy $\sigma_{1}: S_{1} \rightarrow \mathcal{D}(\{a, b\})$ such that $\sigma_{1}(s)$ is the uniform distribution over $\{a, b\}$. This strategy can be represented by a DRD strategy with a single memory state $m$ with the next-move function $\alpha_{\text {next }}(m, s)=\sigma_{1}(s)$.


Fig. 5.1. A (one-player) game with a single state.

The strategy $\sigma_{1}$ induces a probability distribution over plays of $\mathcal{G}$ such that all plays have a probability of zero. Indeed, let $\pi$ be a play of $\mathcal{G}$. One can view the singleton $\{\pi\}$ as the decreasing intersection $\bigcap_{k \in \mathbb{N}} \mathrm{Cyl}\left(\pi_{\mid k}\right)$. Hence, the probability of $\{\pi\}$ is the limit of the probability of $\mathrm{Cyl}\left(\pi_{\mid k}\right)$ when $k$ goes to infinity. One can easily show that the probability under $\sigma_{1}$ of $\operatorname{Cyl}\left(\pi_{\mid k}\right)$ is $\frac{1}{2^{k}}$. It follows that the probability of $\{\pi\}$ is zero.

We now argue that there is no outcome-equivalent RDD strategy. First, let us recall that any RDD strategy can be presented as a distribution over a finite number of pure FM strategies. Given that there are no probabilities on the transitions of $\mathcal{G}$, for any pure strategy $\sigma_{1}^{\text {pure }}$, there is a single outcome under $\sigma_{1}^{\text {pure }}$. We can infer from the former that, for any RDD strategy of $\mathcal{G}$, there must be at least one play that has a non-zero probability, and therefore this strategy cannot be outcome-equivalent to $\sigma_{1}$, ending the proof.

### 5.2 RRD versus DRD strategies

In this section, we argue that there exists a game in which an RRD strategy has no outcome-equivalent DRD strategy. The example we provide is based on positive strategies for the snowball game of [KS81]. The snowball game is a concurrent safety game, i.e., a game in which both players act simultaneously each round and in which the goal of $\mathcal{P}_{1}$ is to avoid a given state. In this game, $\mathcal{P}_{1}$ has a single snowball and $\mathcal{P}_{2}$ can either hide or run to a safe spot. The goal of $\mathcal{P}_{1}$ is to hit $\mathcal{P}_{2}$ with the snowball, and $\mathcal{P}_{1}$ loses if either they throw the snowball while $\mathcal{P}_{2}$ hides or if they do not throw the snowball when $\mathcal{P}_{2}$ runs (i.e., if they miss the chance to hit $\mathcal{P}_{2}$ ).

There are no positive DRD strategies in this game [dAHK07], i.e., one cannot construct a DRD strategy that ensures $\mathcal{P}_{1}$ wins with a positive probability. However it is possible to construct a positive RRD strategy with two states [CDH10]. In the upcoming proof, we slightly change the snowball game to a turn-based game and exploit the strategy presented in [CDH10] to prove the existence of an RRD strategy with no outcome-equivalent DRD strategy in turn-based games.

Lemma 5.2. There exist a one-player deterministic game $\mathcal{G}$ and an $R R D$ strategy of $\mathcal{P}_{1}$ such that there is no outcome-equivalent $D R D$ strategy.

Proof. Consider the game depicted in Figure 5.2. In state start, $\mathcal{P}_{1}$ can either throw the snowball or wait, and in state finish, the snowball has been thrown and it is only possible to wait.


Fig. 5.2. A simplification of the snowball game.

The RRD strategy we consider has two states Never and Eventually. The initial memory distribution is uniform. The updates are simple; the memory state never changes after initialisation. The difference between the states lies in their next-move functions. In the memory state Never, the action wait is suggested with probability 1 in all game states. In the memory state Eventually, both actions have a
uniform probability of being chosen in game state start and in state finish, the action wait is prescribed with probability 1.

The RRD strategy presented above has two key characteristics: it has a non-zero probability of throwing the snowball at each step and has a non-zero probability of never throwing it. In the sequel, we show that for any DRD strategy, a non-zero probability of throwing the ball at each step implies that the ball is thrown almost-surely.

Let $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ denote a DRD strategy such that, after any history of the form (start throw) ${ }^{k}$ start, the action throw is selected with a non-zero probability. Let $m_{k}$ denote the memory element reached after $\mathcal{M}$ processes the word (start throw) ${ }^{k}$ for each $k \in \mathbb{N}$. There are finitely many different such memory states, and for all $k \in \mathbb{N}$, we have $\alpha_{\text {next }}\left(m_{k}\right.$, start) (throw) $>0$. In particular, there is a positive lower bound on the probability of the ball being thrown at each step, and therefore the ball is thrown almost-surely in state start under the strategy $\mathcal{M}$. Therefore, $\mathcal{M}$ cannot be outcome-equivalent to the RRD strategy described in the earlier part of the proof.

### 5.3 Comparing RRD and DDR strategies

We argue in this section that the classes RRD and DDR of finite-memory strategies are incomparable. While we have shown that RDR and DRR strategies are as powerful as RRR strategies, DDR strategies are not because they lack the ability to provide a random output at the first step of a game. Due to this trait, one can even construct some RDD strategy that cannot be emulated by any DDR strategy; any strategy that randomises between two pure FM strategies prescribing different actions for the first state of the game (assuming said state is controlled by the owner of the strategy and has at least two enabled actions) has no outcome-equivalent DDR strategy. The following result follows immediately.

Lemma 5.3. There exist a one-player deterministic game $\mathcal{G}$ and an $R D D$ strategy $\mathcal{M}$ such that there is no $D D R$ strategy that is outcome-equivalent to $\mathcal{M}$.

On the other hand, one can construct a DDR strategy that has no outcome-equivalent RRD strategy. Intuitively, the only randomisation in updates of the memory for RRD strategies is tied to the stochastic next-move function; an action is chosen randomly, and said action is used along with the current state of the game to select the next memory state. Randomised updates allow for more flexibility. In particular, it allows for randomisation in the updates of memory states even in the presence of deterministic actions, and this randomness is independent to some extent from the randomisation of actions. Furthermore, it also allows for stochastic updates even when the other player acts.

In the proof of the upcoming lemma, we provide an example of a two-player deterministic game and a DDR strategy with no outcome-equivalent RRD strategy. The main idea is as follows. When faced with histories containing only actions of the player that does not own the strategy, RRD strategies can at most suggest finitely many distributions over actions; the distribution over memory states after one such history assigns a (possibly empty) sum of initial probabilities to each memory state due to the deterministic updates. However, DDR strategies may have infinitely many possible distributions at hand thanks to the stochastic updates. In the proof, we also argue how the game can be adapted to a Markov decision process. Intuitively, we need simply replace the second player with randomised transitions.

Lemma 5.4. There exist a two-player deterministic game (respectively a Markov decision process) $\mathcal{G}$ and a $D D R$ strategy $\mathcal{M}$ such that there is no $R R D$ strategy that is outcome-equivalent to $\mathcal{M}$.

Proof. Consider the two-player deterministic game with two states $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$, where $S_{1}=\left\{s_{1}\right\}$, $S_{2}=\left\{s_{2}\right\}, A=\{a, b\}$, and in $\mathcal{P}_{1}$ 's state $s_{1}$, both actions are enabled and lead back into $s_{1}$, i.e., $\delta\left(s_{1}, a\right)\left(s_{1}\right)=\delta\left(s_{1}, b\right)\left(s_{1}\right)=1$, and in $\mathcal{P}_{2}$ 's state $s_{2}$, using action $a$ does not change the state and action $b$ moves to state $s_{1}$, i.e., $\delta\left(s_{2}, a\right)\left(s_{2}\right)=\delta\left(s_{2}, b\right)\left(s_{1}\right)=1$.

In this game, let us consider the DDR strategy $\mathcal{M}=\left(M, m_{a}, \alpha_{\mathrm{up}}, \alpha_{\text {next }}\right)$ of $\mathcal{P}_{1}$ defined as follows. There are two memory states $m_{a}$ and $m_{b}$, with $m_{a}$ as the initial state. The next-move function is defined as follows; in memory state $m_{a}$, the action $a$ is prescribed by the strategy, i.e., $\alpha_{\text {next }}\left(m_{a}, s_{1}\right)=a$ and in memory state $m_{b}$, the action $b$ is prescribed, i.e., $\alpha_{\text {next }}\left(m_{b}, s_{1}\right)=b$.

The updates of the memory are performed stochastically. When in memory state $m_{a}$, if we read the state $s_{2}$ of $\mathcal{P}_{2}$ with any action, then we move to memory state $m_{b}$ with probability $\frac{1}{2}$ and otherwise remain in memory state $m_{a}$. Formally, we set $\alpha_{\text {up }}\left(m_{a}, s_{2}, c\right)\left(m_{a}\right)=\alpha_{\text {up }}\left(m_{a}, s_{2}, c\right)\left(m_{b}\right)=\frac{1}{2}$ for any $c \in A$. If instead $\mathcal{P}_{1}$ 's state is processed in $m_{a}$, we do not change the memory state, i.e., $\alpha_{\text {up }}\left(m_{a}, s_{1}, c\right)\left(m_{a}\right)=1$ for $c \in A$. Once memory state $m_{b}$ is reached, it can no longer be left, i.e., $\alpha_{\mathrm{up}}\left(m_{b}, s, c\right)\left(m_{b}\right)=1$ for any $s \in S$ and $c \in A$.

The intuition of this strategy is as follows when starting in state $s_{2}$; the more turns $\mathcal{P}_{2}$ acts, the higher the likelihood of the action $b$ being played in $\mathcal{P}_{1}$ 's state is. Indeed, if we are not in memory state $m_{b}$ and we update the memory after $\mathcal{P}_{2}$ acts, there is a probability of $\frac{1}{2}$ of moving into $m_{b}$, and we try to move to $m_{b}$ at each step whenever we still are in memory state $m_{a}$.

To prove that there is no RRD strategy of $\mathcal{P}_{1}$ that is outcome-equivalent to $\mathcal{M}$, we proceed in two steps. Let $\sigma_{1}^{\mathcal{M}}$ denote the behavioural strategy induced by $\mathcal{M}$ (as a function of histories), and let $w_{k}$ denote $\left(s_{2} a\right)^{k} s_{2} b$ for any $k \in \mathbb{N}$. We first show that the set $D(\mathcal{M})=\left\{\sigma_{1}^{\mathcal{M}}\left(w_{k} s_{1}\right) \mid k \in \mathbb{N}\right\}$, containing distributions over $A$, is infinite. Then we show that the analogous set for an RRD strategy is necessarily finite to end the proof.

We now prove that $D(\mathcal{M})$ is infinite. For any $w \in(S A)^{*}$ consistent with $\mathcal{M}$, let $\mu_{w}^{\mathcal{M}}$ denote the distribution over memory states of $\mathcal{M}$ after processing $w$. By definition, we have $\sigma_{1}^{\mathcal{M}}\left(w_{k} s_{1}\right)(a)=\mu_{w_{k}}^{\mathcal{M}}\left(m_{a}\right)$ and $\sigma_{1}^{\mathcal{M}}\left(w_{k} s_{1}\right)(b)=\mu_{w_{k}}^{\mathcal{M}}\left(m_{b}\right)$. Hence, to prove that $D(\mathcal{M})$ is infinite, it suffices to show that all of the $\mu_{w_{k}}^{\mathcal{M}}$ are distinct from one another. It can easily be shown by induction that $\mu_{w_{k}}^{\mathcal{M}}\left(m_{a}\right)=\frac{1}{2^{k+1}}$ and $\mu_{w_{k}}^{\mathcal{M}}\left(m_{b}\right)=1-\frac{1}{2^{k+1}}$. This ends the argument for the infinity of $D(\mathcal{M})$.

We move on to the last part of the proof. Let $\mathcal{M}^{\prime}=\left(M^{\prime}, \mu_{\text {init }}^{\prime}, \alpha_{\text {up }}^{\prime}, \alpha_{\text {next }}^{\prime}\right)$ be some arbitrary RRD strategy, and let $\sigma_{1}^{\mathcal{M}^{\prime}}$ denote the strategy induced by $\mathcal{M}^{\prime}$. Let us prove that the set $D\left(\mathcal{M}^{\prime}\right)=\left\{\sigma_{1}^{\mathcal{M}^{\prime}}\left(w_{k} s_{1}\right) \mid\right.$ $k \in \mathbb{N}\}$ is finite. For any $w \in(S A)^{*}$ consistent with $\mathcal{M}^{\prime}$, let $\mu_{w}^{\mathcal{M}^{\prime}}$ denote the distribution over memory states of $\mathcal{M}^{\prime}$ after processing $w$. By definition, we have $\sigma_{1}^{\mathcal{M}^{\prime}}\left(w_{k} s_{1}\right)(a)=\sum_{m^{\prime} \in M^{\prime}} \mu_{w_{k}}^{\mathcal{M}^{\prime}}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}^{\prime}\left(m^{\prime}, s_{1}\right)(a)$, and $\sigma_{1}^{\mathcal{M}^{\prime}}\left(w_{k} s_{1}\right)(b)=1-\sigma_{1}^{\mathcal{M}^{\prime}}\left(w_{k} s_{1}\right)(a)$. It follows from the former that we need only show that there are finitely many different distributions $\mu_{w_{k}}^{\mathcal{N}^{\prime}}$ to end the proof, as these determine the distributions $\sigma_{1}^{\mathcal{M}^{\prime}}\left(w_{k} s_{1}\right)$ in $D\left(\mathcal{M}^{\prime}\right)$.

To prove that there are finitely many $\mu_{w_{k}}^{\mathcal{\mathcal { M } ^ { \prime }}}$, we show that each $\mu_{w_{k}}^{\mathcal{\mathcal { M } ^ { \prime }}}$ assigns to elements of $M^{\prime}$ a probability given by a sum of probabilities given by $\mu_{\text {init }}^{\prime}$. Given there are finitely many such sums, we obtain that there are indeed finitely many distributions $\mu_{w_{k}}^{\mathcal{M}^{\prime}}$. We proceed via an analysis of the update of the $\mu_{w}^{\mathcal{M}}{ }^{\prime}$ s.

For all $k \in \mathbb{N}$, let us write $v_{k}$ for $\left(s_{2} a\right)^{k}$, which is the prefix of $w_{k}$ without its last state-action pair. First, we show that for all $k \in \mathbb{N}, \mu_{v_{k}}^{\mathcal{N}^{\prime}}\left(m^{\prime}\right)$ is a sum of initial probabilities for all $m^{\prime} \in M^{\prime}$. This statement is proven by induction. The base case follows from the definition of $\mu_{\varepsilon}^{\mathcal{M}^{\prime}}=\mu_{\text {init }}^{\prime}$. Now assume the statement holds for some $k \in \mathbb{N}$. We have, by definition, for all $m^{\prime} \in M^{\prime}$,

$$
\mu_{v_{k+1}}^{\mathcal{M}^{\prime}}\left(m^{\prime}\right)=\sum_{\substack{m^{\prime \prime} \in M^{\prime} \\ \alpha_{\mathrm{up}}^{\prime}\left(m^{\prime \prime}, s, c\right)=m^{\prime}}} \mu_{v_{k}}\left(m^{\prime \prime}\right)
$$

due to the fact $\mathcal{P}_{2}$ controls $s_{2}$. This ends the inductive argument. Now, let $k \in \mathbb{N}$, and let us establish the announced result on the distributions $\mu_{w_{k}}$. Let $m^{\prime} \in M^{\prime}$. We have, by definition,

$$
\mu_{w_{k}}^{\mathcal{M}^{\prime}}\left(m^{\prime}\right)=\sum_{\substack{m^{\prime \prime} \in M^{\prime} \\ \alpha_{\text {up }}^{\prime}\left(m^{\prime \prime}, s, c\right)=m^{\prime}}} \mu_{v_{k}}\left(m^{\prime \prime}\right),
$$

and by the former this proves that $\mu_{w_{k}}^{\mathcal{M}^{\prime}}\left(m^{\prime}\right)$ is a sum of probabilities appearing in the image of $\mu_{\text {init }}^{\prime}$. This ends the argument that there are finitely many $\mu_{w_{k}}^{\mathcal{\mathcal { M } ^ { \prime }}}$ s.

We have established that $D\left(\mathcal{M}^{\prime}\right)$ must be finite. Therefore, there cannot be any RRD strategy that is outcome-equivalent to $\mathcal{M}$.

We now explain how the two-player game can be adapted to a suitable Markov decision process to end the proof. The state space of the game remains unchanged and the state $s_{2}$ of $\mathcal{P}_{2}$ is transferred to $\mathcal{P}_{1}$. The transitions of $s_{2}$ are altered as follows: only one action, the action $a$, is enabled in $s_{2}$ and we set the transition distribution from $s_{2}$ using action $a$ as the uniform distribution over $s_{1}$ and $s_{2}$. The next-move function of $\mathcal{M}$ is mended so that in all memory states, it suggests action $a$ in $s_{2}$.

The essence of the argument in the case of the two-player game was that $\mathcal{P}_{2}$ could remain for an arbitrarily long amount of rounds in their state $s_{2}$. Here, the random transitions play the same role. By adapting the calculations above, one can show that the mended DDR strategy described in the previous paragraph has no outcome-equivalent RRD counterpart in our Markov decision process. This ends the proof.

## 6 Games of imperfect information

We explain how to transfer the results presented in the previous sections to the setting of stochastic games of imperfect information in which players can observe their own actions. In Section 6.1, we introduce the relevant definitions of games of imperfect information and observation-based strategies. We then explain how to adapt our previous proofs to this context in Section 6.2.

### 6.1 Imperfect information and observation-based strategies

Imperfect information. We consider two-player stochastic games of imperfect information played on graphs. Unlike games of perfect information, the players are not fully informed of the current state of the play and the actions that are used along the play. Instead, they perceive an observation for each state and action, and this observation may be shared between different states and actions, making them indistinguishable. These observations are not shared between the players; each player perceives the ongoing play differently.

We now formalise this game model. A stochastic game of imperfect information is defined as a tuple $\Gamma=\left(S_{1}, S_{2}, A_{1}, A_{2}, \delta, \mathcal{Z}_{1}, \mathrm{Obs}_{1}, \mathcal{Z}_{2}\right.$, Obs ${ }_{2}$ ) where ( $\left.S_{1}, S_{2}, A=A_{1} \uplus A_{2}, \delta\right)$ is a stochastic game of perfect information such that for all $i \in\{1,2\}$ and $s \in S_{i}, A(s) \subseteq A_{i}$, and for $i \in\{1,2\}, \mathcal{Z}_{i}$ is a finite set of observations of $\mathcal{P}_{i}$ and $\mathrm{Obs}_{i}: S \cup A \rightarrow \mathcal{Z}_{i}$ is the observation function of $\mathcal{P}_{i}$, which assigns an observation to each state and action. We require that for any $i \in\{1,2\}$, for any two states $s$ and $s^{\prime} \in S_{i}, \mathrm{Obs}_{i}(s)=\mathrm{Obs}_{i}\left(s^{\prime}\right)$ implies $A(s)=A\left(s^{\prime}\right)$, i.e., in two indistinguishable states, the same actions are available. We fix $\Gamma$ for the remainder of the section and let $\mathcal{G}$ denote the underlying game of perfect information.

Plays and histories of $\Gamma$ are respectively defined as plays and histories of $\mathcal{G}$. We will reuse the notations $\operatorname{Plays}(\Gamma), \operatorname{Hist}(\Gamma)$ and $\operatorname{Hist}_{i}(\Gamma)$ for the sets of plays of $\Gamma$, histories of $\Gamma$ and histories of $\Gamma$ ending in a state of $\mathcal{P}_{i}$ respectively. We extend the observation functions to histories naturally: given some history $h=s_{0} a_{0} \ldots s_{n}$ of $\Gamma$, we let $\mathrm{Obs}_{i}(h)=\mathrm{Obs}_{i}\left(s_{0}\right) \mathrm{Obs}_{i}\left(a_{0}\right) \ldots \mathrm{Obs}_{i}\left(s_{n}\right)$. This extension will be used to define the relevant notion of strategies in games of imperfect information.

In the sequel, we place ourselves in the context of Kuhn's theorem and assume that a player's own actions are visible. Formally, we will require that for $\mathcal{P}_{i}$, the set of actions $A_{i}$ is included in the set $\mathcal{Z}_{i}$ and that for all $a \in A_{i}$ and $x \in S \cup A, \mathrm{Obs}_{i}(x)=a$ if and only if $x=a$.
Observation-based strategies. In $\Gamma$, players can only rely on the observations they perceive to select actions. Therefore, strategies in games of imperfect information are not defined over histories, but instead over sequences of observations induced by histories. These strategies are called observation-based strategies. Formally, an observation-based strategy of $\mathcal{P}_{i}$ is a function $\sigma_{i}: \mathrm{Obs}_{i}\left(\operatorname{Hist}_{i}(\Gamma)\right) \rightarrow \mathcal{D}\left(A_{i}\right)$ such that for any history $h \in \operatorname{Hist}_{i}(\Gamma), \operatorname{supp}\left(\sigma_{i}\left(\operatorname{Obs}_{i}(h)\right)\right) \subseteq A(\operatorname{last}(h))$, i.e., the actions suggested by $\sigma_{i}$ after perceiving the sequence of observations induced by $h$ are available in the last state of $h$. We will refer to strategies of the underlying game of perfect information $\mathcal{G}$ as history-based strategies to insist on the nuance between the two notions.

For any observation-based strategy $\sigma_{i}$ of $\mathcal{P}_{i}$, we can naturally derive a history-based strategy $\tau_{i}$; we define $\tau_{i}(h)=\sigma_{i}\left(\operatorname{Obs}_{i}(h)\right)$ for all $h \in \operatorname{Hist}_{i}(\mathcal{G})$. This allows us to easily define concepts we have introduced for games of perfect information in games of imperfect information.

First, we discuss the probability measure over plays induced by two given observation-based strategies $\sigma_{1}$ of $\mathcal{P}_{1}$ and $\sigma_{2}$ of $\mathcal{P}_{2}$ from a fixed initial state $s_{\text {init }}$ of $\Gamma$. This measure can be directly defined as the measure $\mathbb{P}_{S_{\text {init }}}^{\tau_{1}, \tau_{2}}$ over $\operatorname{Plays}(\mathcal{G})=\operatorname{Plays}(\Gamma)$, where $\tau_{i}$ denotes the history-based strategy derived from $\sigma_{i}$ for $i \in\{1,2\}$.

Then let us move on to the definition of consistency. We will say a history $h$ is consistent with an observation-based strategy $\sigma_{1}$ if it is consistent with the history-based strategy derived from $\sigma_{1}$. Finally, we will say that two observation-based strategies $\sigma_{1}$ and $\tau_{1}$ are outcome-equivalent if they induce the same distribution over plays from any initial state, i.e., if their respective derived history-based strategies are outcome-equivalent. As a direct consequence of Lemma 2.1, the following result holds.

Lemma 6.1. Let $\sigma_{1}$ and $\tau_{1}$ be two observation-based strategies of $\mathcal{P}_{1}$. These two strategies are outcomeequivalent if and only if for all histories $h \in \operatorname{Hist}_{1}(\mathcal{G}), h$ consistent with $\sigma_{1}$ implies $\sigma_{1}\left(\operatorname{Obs}_{1}(h)\right)=$ $\tau_{1}\left(\mathrm{Obs}_{1}(h)\right)$.

Finite-memory strategies. An observation-based strategy is finite-memory if it is induced by a (stochastic) Mealy machine that reads observations instead of states and actions. Formally, we define
an observation-based Mealy machine of $\mathcal{P}_{i}$ as a tuple $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\mathrm{up}}, \alpha_{\text {next }}\right)$ where $M$ is a finite set of memory states, $\mu_{\text {init }}$ is an initial distribution over $M, \alpha_{\text {up }}: M \times \mathcal{Z}_{i} \times \mathcal{Z}_{i} \rightarrow \mathcal{D}(M)$ is the update function and $\alpha_{\text {next }}: M \times \mathcal{Z}_{i} \rightarrow \mathcal{D}\left(A_{i}\right)$ is the next-move function.

The strategies induced by such Mealy machines are defined analogously to the strategies induced by Mealy machines in the context of perfect information. The only difference is that its updates and the actions it suggests are based on observations instead of states and actions themselves.

Let $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ be an observation-based stochastic Mealy machine of $\mathcal{P}_{i}$. We will formally derive an observation-based strategy from $\mathcal{M}$. The reasoning follows roughly the same lines as in Section 2: we will proceed based on the distribution over memory states attained after a sequence of states and actions has taken place. This distribution is defined in the exact same way, except that we replace states and actions with their respective observations in the next-move and update functions that appear in the relevant formulae.

To exploit this distribution to define an observation-based strategy, we must ensure the following: given two sequences $w$ and $v \in(S A)^{*}$ that are mapped to the same sequence of observations of $\mathcal{P}_{i}$, the distribution over the memory states of $\mathcal{M}$ after processing both strategies coincide. This property is not immediate; when we update this distribution over memory states when adding one step to a sequence in $(S A)^{*}$, we sometimes need to condition these updates on the actions themselves, rather than on associated observations. In other words, the distribution over memory states is not intrinsic to the sequence of observations that was perceived in general.

The visibility of actions is the key to this property. Intuitively, because we condition updates of the distribution over memory states in states of $S_{i}$ on the chosen action and we know this action, we can ensure that the update to the distribution over memory states is performed uniformly over all sequences in $(S A)^{*}$ that are mapped to the same sequence of observations.

Lemma 6.2. Let $w \in(S A)^{*}$. Let $\mu_{w}$ denote the distribution over memory states of $\mathcal{M}$ after the sequence $w$ has taken place. For all $v \in(S A)^{*}$ such that $w$ and $v$ are mapped to the same sequence of observations, we have $\mu_{v}=\mu_{w}$.

Proof. We will proceed by induction on the length of the considered sequence $w \in(S A)^{*}$. Recall that at the start of a play, an initial memory state is drawn following $\mu_{\text {init }}$. Hence the distribution over memory states after the empty word $\varepsilon$ is $\mu_{\varepsilon}=\mu_{\text {init }}$. In this case, there is nothing to show for the uniformity argument.

We now assume the following by induction: the sequence $w=s_{0} a_{0} \ldots s_{k} a_{k}$ has taken place and the distribution $\mu_{w}$ over $M$ coincides with any $\mu_{v}$ where $v=t_{0} b_{0} \ldots t_{k} b_{k}$ can be mapped to the same sequence of observations as $w$. We consider $w^{\prime}=w s_{k+1} a_{k+1}$. We describe $\mu_{w^{\prime}}$ and show that for any sequence $v^{\prime}$ that shares the same sequence of observations as $w^{\prime}$, we have $\mu_{w^{\prime}}=\mu_{v^{\prime}}$. Let us fix $v^{\prime}=v t_{k+1} b_{k+1}$ to prove the latter claim.

We distinguish two cases. If the state $s_{k+1}$ is not controlled by $\mathcal{P}_{i}$, we set, for any memory state $m$,

$$
\mu_{w^{\prime}}(m)=\sum_{m^{\prime} \in M} \mu_{w}\left(m^{\prime}\right) \cdot \alpha_{\text {up }}\left(m^{\prime}, \operatorname{Obs}_{i}\left(s_{k+1}\right), \operatorname{Obs}_{i}\left(a_{k+1}\right)\right)(m) .
$$

We now show that $\mu_{w^{\prime}}=\mu_{v^{\prime}}$. It suffices to show that $t_{k+1} \in S_{3-i}$. Indeed, that would imply that $\mu_{v^{\prime}}$ is computed from $\mu_{v}$ in the same manner that $\mu_{w^{\prime}}$ was computed from $\mu_{w}$, and $\mu_{w}=\mu_{v}$ holds by induction. We know that $\operatorname{Obs}_{i}\left(b_{k+1}\right)=\operatorname{Obs}_{i}\left(a_{k+1}\right) \notin A_{i}\left(\mathcal{P}_{i}\right.$ can perceive their own actions), and because $b_{k+1} \in A\left(t_{k+1}\right)$, we must have $t_{k+1} \in S_{3-i}$.

Now, let us assume that $s_{k+1}$ is controlled by $\mathcal{P}_{i}$. In this case, due to the visibility of actions, we have $a_{k+1}=\operatorname{Obs}_{i}\left(a_{k+1}\right)=b_{k+1}$. In particular, we have $t_{k+1} \in S_{i}$. Recall that we must condition updates on the action $a_{k+1}$. We distinguish two cases as before. If for all memory states $m \in M$ such that $\mu_{w}(m)>0$, we have $\alpha_{\text {next }}\left(m, \operatorname{Obs}_{i}\left(s_{k+1}\right)\right)\left(a_{k+1}\right)=0$, we let updates be performed as above. It follows immediately in this case that $\mu_{w^{\prime}}=\mu_{v^{\prime}} ; \mu_{v^{\prime}}$ is computed in the same way because $\mu_{v}=\mu_{w}$ and $\operatorname{Obs}_{i}\left(s_{k+1}\right)=\operatorname{Obs}_{i}\left(t_{k+1}\right)$.

Finally, let us assume that there is $m \in M$ such that $\mu_{w}(m)>0$ and $\alpha_{\text {next }}\left(m, \operatorname{Obs}_{i}\left(s_{k+1}\right)\right)\left(a_{k+1}\right)>0$. In this case, we set, for any memory state $m \in M$,

$$
\mu_{w^{\prime}}(m)=\frac{\sum_{m^{\prime} \in M} \mu_{w}\left(m^{\prime}\right) \cdot \alpha_{\text {up }}\left(m^{\prime}, \operatorname{Obs}_{i}\left(s_{k+1}\right), a_{k+1}\right)(m) \cdot \alpha_{\text {next }}\left(m^{\prime}, \operatorname{Obs}_{i}\left(s_{k+1}\right)\right)\left(a_{k+1}\right)}{\sum_{m^{\prime} \in M} \mu_{w}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m, \operatorname{Obs}_{i}\left(s_{k+1}\right)\right)\left(a_{k+1}\right)}
$$

The equation for $\mu_{v^{\prime}}$ is the same as above, except $s_{k+1}$ is replaced with $t_{k+1}$. Because these two states are such that $\mathrm{Obs}_{i}\left(s_{k+1}\right)=\mathrm{Obs}_{i}\left(t_{k+1}\right)$, it immediately follows that $\mu_{w^{\prime}}=\mu_{v^{\prime}}$, ending our uniformity argument.

Lemma 6.2 allows us to define a strategy from $\mathcal{M}$ as we had done in the perfect information setting. The observation-based strategy $\sigma_{i}^{\mathcal{M}}: \operatorname{Obs}_{i}\left(\operatorname{Hist}_{i}(\Gamma)\right) \rightarrow \mathcal{D}\left(A_{i}\right)$ induced by $\mathcal{M}$ is defined as follows. For any history $h=w s \in \operatorname{Hist}_{i}(\Gamma)$ and $a \in A(s)$, we set $\sigma_{i}^{\mathcal{M}}\left(\operatorname{Obs}_{i}(h)\right)(a)=\sum_{m \in M} \mu_{w}(m) \cdot \alpha_{\text {next }}\left(m, \operatorname{Obs}_{i}(s)\right)(a)$.
Classes of finite-memory strategies. In the sequel, we transfer our results on games of perfect information to games of imperfect information with visible actions. We will use the same classification of Mealy machines with three-letter acronyms for observation-based Mealy machines. As was the case in the earlier sections, we will abusively say, e.g., $\mathcal{M}$ is an RRR observation-based strategy to mean that $\mathcal{M}$ is an observation-based Mealy machine with stochastic initialisation, outputs and updates, and avoid referring to the observation-based strategy it induces in this way.

### 6.2 Transferring our taxonomy to imperfect information

In this section, we are concerned with transferring our taxonomy of finite-memory strategies in games of perfect information to games of imperfect information. We open this section by stating its main result.

Theorem 6.1. The taxonomy of Figure 1.1 established in the context of games of perfect information also holds for observation-based finite-memory strategies in the context of games of imperfect information in which a player's own actions are observable.

Games of perfect information are games of imperfect information in which the observation functions are the identity function. For this reason, all results that establish the strictness of inclusions or that two classes are not comparable carry over directly.

Therefore, we need only discuss the three following results: Theorems 4.1, 4.2 and 4.3. We comment each of these theorems and explain how their proofs can be adapted for observations instead of states and actions. All three proofs share one characteristic; the outcome-equivalence of the given and constructed strategies is shown using Lemma 2.1. Lemma 6.1 ensures the validity of this technique with imperfect information.

In Theorem 4.1, we simulate RDD strategies by means of DRD strategies. The approach is as follows: we keep track of a finite set of pure FM strategies and remove one whenever we perceive an action that is inconsistent with it. Using the visibility of actions, we can use this same approach in games of imperfect information. Furthermore, the RDD strategy that is simulated and all of the pure FM strategies encoded in the simulating DRD strategy all use exactly the same observation-based update scheme. Therefore, this construction is suitable to establish that any RDD strategy has an outcome-equivalent DRD counterpart in the context of imperfect information.

Theorem 4.2 claims that any RRR strategy admits some outcome-equivalent DRR strategy. The approach consists in adding a new initial memory state, and then leverage stochastic updates to enter the supplied RRR strategy from the second step of the game and proceed as though we had been using it from the start. We designed the updates from the new initial memory state so that, from the second step in the game, the distribution over memory states was the same in the RRR strategy and the constructed DRR one. More precisely, the update probability distribution from the new initial state is defined as the probability over the memory states of the RRR strategy after one step. Lemma 6.2 ensures that this distribution is robust to the passage to imperfect information, and justifies that the approach can be used.

Finally, Theorem 4.3, unlike the previous two, does not rely on the visibility of actions. This theorem states that for all RRR strategies, one can find an outcome-equivalent RDR strategy. The tactic in our proof was to preemptively draw actions before we entered new memory states. The updates in the RDR strategy we built directly used constants inferred from the next-move function (i.e., independent from states and actions themselves) and the updates of the RRR strategy. Therefore, the same construction can be directly adapted to games of partial information.

In light of the discussion above, we can conclude that Theorem 6.1 holds.

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## A Probability over memory states in stochastic-update strategies

## A. 1 Introduction

In this section, we explain the reasoning behind the formulae for the distribution over memory states of a Mealy machine after it processes a word in $(S A)^{*}$. Our arguments use conditional probabilities. We establish the result for $\mathcal{P}_{1}$; the reasoning for $\mathcal{P}_{2}$ is analogous. To this end, let us fix some game $\mathcal{G}=\left(S_{1}, S_{2}, A, \delta\right)$ and a Mealy machine $\mathcal{M}=\left(M, \mu_{\text {init }}, \alpha_{\text {up }}, \alpha_{\text {next }}\right)$ of $\mathcal{P}_{1}$.

Let $w=s_{0} a_{0} s_{1} a_{1} \ldots s_{k} a_{k} \in(S A)^{*}$. We are interested in the distribution over memory states in $M$ after $\mathcal{M}$ has processed $w$. We will need some hypotheses on $w$ : the probability of being in some memory state $m$ after processing $w$ is formally the conditional probability of being in $m$ at step $k+1$ given $w$. Thus, we will have to require $w$ to be of positive probability under $\mathcal{M}$ and (at least) one strategy of $\mathcal{P}_{2}$.

We will reuse the notation $\mu_{w}$ that is introduced in Section 2. The main goal of this section is to prove the inductive relations for $\mu_{w}$. We recall them hereafter. Assume $w$ is not the empty word and let $w^{\prime}=s_{0} a_{0} s_{1} a_{1} \ldots s_{k-1} a_{k-1}$. If $s_{k} \in S_{2}$, we have

$$
\begin{equation*}
\mu_{w}(m)=\sum_{m^{\prime} \in M} \mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \alpha_{\mathrm{up}}\left(m^{\prime}, s_{k}, a_{k}\right)(m), \tag{A.1}
\end{equation*}
$$

and if $s_{k} \in S_{1}$, we have

$$
\begin{equation*}
\mu_{w}(m)=\frac{\sum_{m^{\prime} \in M} \mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \alpha_{\mathrm{up}}\left(m^{\prime}, s_{k}, a_{k}\right)(m) \cdot \alpha_{\text {next }}\left(m^{\prime}, s_{k}\right)\left(a_{k}\right)}{\sum_{m^{\prime} \in M} \mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime}, s_{k}\right)\left(a_{k}\right)} \tag{A.2}
\end{equation*}
$$

We will derive these equations by studying the Markov chain induced by $\mathcal{M}$ and a strategy of $\mathcal{P}_{2}$. As indicated by the equations above, the strategy of $\mathcal{P}_{2}$ we consider has no impact on $\mu_{w}$. We introduce one such strategy so that the Markov chain is well-defined.

Let us fix some strategy $\sigma_{2}$ of $\mathcal{P}_{2}$ and some initial state $s_{\text {init }} \in S$. In the sequel, we prove that the equations above hold for any $w \in(S A)^{*}$ starting from $s_{\text {init }}$ and consistent with $\mathcal{M}$ and $\sigma_{2}$. By proving that the equations above hold, it will follow immediately that $\mathcal{P}_{2}$ 's strategy has no impact on the distributions of the form $\mu_{w}$.

## A. 2 Description of the Markov chain

First, let us describe the Markov chain induced by playing $\mathcal{M}$ and $\sigma_{2}$ from $s_{\text {init }}$ in $\mathcal{G}$. Formally, it is an infinite Markov chain where states are non-empty sequences $\left(s_{0}, m_{0}, a_{0}\right) \ldots\left(s_{k}, m_{k}, a_{k}\right)$ in $(S \times M \times A)^{*}$ where $s_{0} a_{0} \ldots a_{k-1} s_{k}$ is a history of $\mathcal{G}$ and $a_{k} \in A\left(s_{k}\right)$. The support of the initial distribution is $\left\{\left(s_{\text {init }}, m, a\right) \mid \mu_{\text {init }}(m)>0, \alpha_{\text {next }}\left(m, s_{\text {init }}\right)(a)>0\right\}$. The initial probability of a state $\left(s_{\text {init }}, m, a\right)$ is given as the product $\mu_{\text {init }}(m) \cdot \alpha_{\text {next }}\left(m, s_{\text {init }}\right)(a)$; it is the probability that $m$ is drawn as the initial memory state and that $a$ is selected in $s_{\text {init }}$ in memory state $m$.

Let $h=\left(s_{0}, m_{0}, a_{0}\right) \ldots\left(s_{k}, m_{k}, a_{k}\right)$ and $h^{\prime}=h\left(s_{k+1}, m_{k+1}, a_{k+1}\right)$ be two states of our Markov chain. The transition probability from $h$ to $h^{\prime}$ is defined as follows. We multiply three probabilities (recall that state transitions, memory updates and action draws are independent): the probability $\delta\left(s_{k}, a_{k}\right)\left(s_{k+1}\right)$ of moving into $s_{k+1}$ from $s_{k}$ using action $a_{k}$, the probability $\alpha_{\text {up }}\left(m_{k}, s_{k}, a_{k}\right)\left(m_{k+1}\right)$ that the memory is updated to $m_{k+1}$ from $m_{k}$ when reading $s_{k}$ and $a_{k}$, and the probability of the action $a_{k+1}$ being selected, which is computed through $\mathcal{M}$ if $s_{k+1} \in S_{1}$ and from $\sigma_{2}$ otherwise. Formally, the probability of the transition from $h$ to $h^{\prime}$ is $\delta\left(s_{k}, a_{k}\right)\left(s_{k+1}\right) \cdot \alpha_{\text {up }}\left(m_{k}, s_{k}, a_{k}\right)\left(m_{k+1}\right) \cdot \alpha_{\text {next }}\left(s_{k+1}, m_{k+1}\right)\left(a_{k+1}\right)$ if $s_{k+1} \in S_{1}$ and $\delta\left(s_{k}, a_{k}\right)\left(s_{k+1}\right) \cdot \alpha_{\text {up }}\left(m_{k}, s_{k}, a_{k}\right)\left(m_{k+1}\right) \cdot \sigma_{2}(h)\left(a_{k+1}\right)$ otherwise.

We define a probability measure over infinite sequences of states of the Markov chain described above in the standard way, using cylinders. Initial infinite sequences of this Markov chain belong in $\left((S \times M \times A)^{*}\right)^{\omega}$ and are of the form $t_{0}\left(t_{0} t_{1}\right)\left(t_{0} t_{1} t_{2}\right) \ldots$ where $t_{k} \in S \times M \times A$. In the sequel, we identify these infinite initial sequences to elements of $(S \times M \times A)^{\omega}$. We will write $\mathbb{P}$ for the probability distribution over $(S \times M \times A)^{\omega}$ obtained this way.

In the sequel, we will use random variables defined over $(S \times M \times A)^{\omega}$ to derive Equations (A.1) and (A.2). Let $B$ denote a set. For any random variable $X:(S \times M \times A)^{\omega} \rightarrow B$ and $b \in B$, we will write $\{X=b\}$ for $X^{-1}(\{b\})$ and omit the braces when evaluating $\mathbb{P}$ over such sets, e.g., we write $\mathbb{P}(X=b)$ for $\mathbb{P}(\{X=b\})$.

Now, let us define the relevant random variables. We will denote by $S_{k}$ (resp. $M_{k}, A_{k}$ ) the random variable that describes the state of the game (resp. memory state, action) at position $k$ of a sequence in ( $S \times$
$M \times A)^{\omega}$. We will write $W_{k}$ for the random variable describing the sequence $W_{k}=S_{0} A_{0} S_{1} A_{1} \ldots S_{k-1} A_{k-1}$ which is the sequence read by $\mathcal{M}$ prior to step $k$. Similarly, we write $H_{k}$ (resp. $\overline{M_{k}}$ ) for the random variable $H_{k}=W_{k} S_{k}$ (resp. $\overline{M_{k}}=M_{0} M_{1} \ldots M_{k}$ ) that describes the history at step $k$ (resp. the sequence of memory states up to step $k$ ).

We now list some useful properties of these random variables we will rely on. We will be concerned with conditional probabilities, and therefore all upcoming equations will assume that some event has a positive probability. The four aspects we will exploit are the following: (i) memory updates only depend on the latest memory state, game state and action; (ii) memory updates are independent from game state updates; (iii) the probability of an action depends only on the last game and memory states when the game state is controlled by $\mathcal{P}_{1}$; (iv) the probability of an action is independent from the sequence of memory states when the last game state is controlled by $\mathcal{P}_{2}$.

Recall that for memory states, the updates depend solely on the previous memory state, the previous game state and the previous action. Formally, let us take a non-empty sequence $w=s_{0} a_{0} \ldots s_{k-1} a_{k-1} \in$ $(S A)^{*}$ such that $\mathbb{P}\left(W_{k}=w\right)>0$. For any sequence of memory states $\bar{m}=m_{0} m_{1} \ldots m_{k-1} \in M^{k}$ such that $\mathbb{P}\left(\overline{M_{k-1}}=\bar{m} \mid W_{k}=w\right)>0$, we have, for every state $m \in M$,

$$
\begin{aligned}
& \mathbb{P}\left(M_{k}=m \mid\right.
\end{aligned} \begin{aligned}
& \left.W_{k}=w \wedge \overline{M_{k-1}}=\bar{m}\right) \\
& \quad=\mathbb{P}\left(M_{k}=m \mid S_{k-1}=s_{k-1} \wedge M_{k-1}=m_{k-1} \wedge A_{k-1}=a_{k-1}\right) \\
& \quad=\alpha_{\text {up }}\left(m_{k-1}, s_{k-1}, a_{k-1}\right)(m)
\end{aligned}
$$

Recall also that state transitions and memory updates are performed independently. In other words, for any history $h=s_{0} a_{0} \ldots s_{k} \in \operatorname{Hist}(\mathcal{G})$ such that $\mathbb{P}\left(H_{k}=h\right)>0$, we have for any memory state $m \in M$,

$$
\mathbb{P}\left(M_{k}=m \mid H_{k}=h\right)=\mathbb{P}\left(M_{k}=m \mid W_{k}=w\right)
$$

where $w$ denotes $s_{0} a_{0} \ldots s_{k-1} a_{k-1}$.
Now, let us move on to the probability of actions following a history. Let $h=s_{0} a_{0} \ldots s_{k} \in \operatorname{Hist}(\mathcal{G})$ such that $\mathbb{P}\left(H_{k}=h\right)>0$. First, let us assume $s_{k} \in S_{1}$. Whenever $\mathcal{P}_{1}$ controls the last state of a history, the probability of the next action depends only on the last state of the history and the last memory state. We have, for any sequence of memory states $\bar{m}=m_{0} m_{1} \ldots m_{k} \in M^{k+1}$ such that $\mathbb{P}\left(\overline{M_{k}}=\bar{m} \mid H_{k}=h\right)>0$ (i.e., any sequence of memory states likely to occur by processing $h$ ) and action $a \in A\left(s_{k}\right)$,

$$
\mathbb{P}\left(A_{k}=a \mid H_{k}=h \wedge \overline{M_{k}}=\bar{m}\right)=\mathbb{P}\left(A_{k}=a \mid S_{k}=s_{k} \wedge M_{k}=m_{k}\right)=\alpha_{\text {next }}\left(m_{k}, s_{k}\right)(a)
$$

Now, let us assume $s_{k} \in S_{2}$. In this case, the probability of the next action is given by $\sigma_{2}(h)$ and is independent of the sequence of memory states seen along $h$. Formally, we have, for any sequence of memory states $\bar{m}=m_{0} m_{1} \ldots m_{k} \in M^{k+1}$ such that $\mathbb{P}\left(\overline{M_{k}}=\bar{m} \mid H_{k}=h\right)>0$ and action $a \in A\left(s_{k}\right)$,

$$
\mathbb{P}\left(A_{k}=a \mid H_{k}=h \wedge \overline{M_{k}}=\bar{m}\right)=\mathbb{P}\left(A_{k}=a \mid H_{k}=h\right)=\sigma_{2}(h)(a)
$$

## A. 3 Proving the formulae for $\mu_{w}$

Let $w=s_{0} a_{0} s_{1} a_{1} \ldots s_{k} a_{k} \in(S A)^{*}$ such that $\mathbb{P}\left(W_{k+1}=w\right)>0$. For any $m \in M$, the so-called probability $\mu_{w}(m)$ is formalised by the conditional probability $\mathbb{P}\left(M_{k+1}=m \mid W_{k+1}=w\right)$. Henceforth, we assume that $w$ is non-empty. Let $w^{\prime}=s_{0} a_{0} \ldots s_{k-1} a_{k-1}$ be the prefix of $w$ without its last state and last action. To prove Equations (A.1) and (A.2), we must express $\mu_{w}$ as a function of $\mu_{w^{\prime}}$. Discussions following whether $s_{k} \in S_{1}$ or $s_{k} \in S_{2}$ are relegated to further in the proof, after preliminary work common to both cases.

We fix $m \in M$ for the remainder of the section. The first step in our approach is to consider all possible paths in $\mathcal{M}$ that reach $m$ and have a positive probability of occurring whenever $w$ is processed by $\mathcal{M}$. Considering these paths will allow us to exhibit terms in which $\alpha_{\text {up }}$ appears within Equations (A.1) and (A.2). We use the following notation for sets of paths: we write Paths ${ }_{w}^{m}$ for the set of sequences $m_{0} m_{1} \ldots m_{k}$ such that the path $m_{0} m_{1} \ldots m_{k} m$ in $\mathcal{M}$ is consistent with $w$, i.e., we let

$$
\operatorname{Paths}_{w}^{m}=\left\{m_{0} m_{1} \ldots m_{k} \in M^{k+1} \mid \mathbb{P}\left(\overline{M_{k+1}}=m_{0} \ldots m_{k} m \mid W_{k+1}=w\right)>0\right\}
$$

We define, for any memory state $m^{\prime} \in M$, the set Paths $_{w^{\prime}}^{m^{\prime}}$ as a subset of $M^{k}$ in the same fashion. It follows from the law of total probability (formulated for conditional probabilities), that

$$
\begin{aligned}
\mu_{w}(m) & =\mathbb{P}\left(M_{k+1}=m \mid W_{k+1}=w\right) \\
& =\sum_{\bar{m} m^{\prime} \in \text { Paths }_{w}^{m}} \mathbb{P}\left(M_{k+1}=m \mid W_{k+1}=w \wedge \overline{M_{k}}=\bar{m} m^{\prime}\right) \cdot \mathbb{P}\left(\overline{M_{k}}=\bar{m} m^{\prime} \mid W_{k+1}=w\right) \\
& =\sum_{\bar{m} m^{\prime} \in \text { Paths }_{w}^{m}} \alpha_{\mathrm{up}}\left(m^{\prime}, s_{k}, a_{k}\right)(m) \cdot \mathbb{P}\left(\overline{M_{k}}=\bar{m} m^{\prime} \mid W_{k+1}=w\right) \\
& =\sum_{\bar{m} \in \text { Paths }_{w^{\prime}}^{m^{\prime}}} \sum_{m^{\prime} \in M} \alpha_{\mathrm{up}}\left(m^{\prime}, s_{k}, a_{k}\right)(m) \cdot \mathbb{P}\left(\overline{M_{k}}=\bar{m} m^{\prime} \mid W_{k+1}=w\right) \\
& =\sum_{m^{\prime} \in M}\left(\alpha_{\text {up }}\left(m^{\prime}, s_{k}, a_{k}\right)(m) \cdot \sum_{\bar{m} \in \text { Paths }_{w^{\prime}}^{m^{\prime}}} \mathbb{P}\left(\overline{M_{k}}=\bar{m} m^{\prime} \mid W_{k+1}=w\right)\right) .
\end{aligned}
$$

We now shift our focus to the general term of the inner sum. Let us fix $m^{\prime} \in M$. This sum is indexed by all paths in $\mathcal{M}$ that reach $m^{\prime}$ and have positive probability. Therefore, it follows from the law of total probability that

$$
\sum_{\bar{m} \in \text { Paths }_{w^{\prime}}^{m^{\prime}}} \mathbb{P}\left(\overline{M_{k}}=\bar{m} m^{\prime} \mid W_{k+1}=w\right)=\mathbb{P}\left(M_{k}=m^{\prime} \mid W_{k+1}=w\right) .
$$

We underscore that this probability is not $\mu_{w^{\prime}}\left(m^{\prime}\right)=\mathbb{P}\left(M_{k}=m^{\prime} \mid W_{k}=w^{\prime}\right)$. Up to this point, we have established that

$$
\begin{equation*}
\mu_{w}(m)=\sum_{m^{\prime} \in M} \alpha_{\mathrm{up}}\left(m^{\prime}, s_{k}, a_{k}\right)(m) \cdot \mathbb{P}\left(M_{k}=m^{\prime} \mid W_{k+1}=w\right) . \tag{A.3}
\end{equation*}
$$

Using Bayes' theorem, we can show a relation between the probability $\mathbb{P}\left(M_{k}=m^{\prime} \mid W_{k+1}=w\right)$ and $\mu_{w^{\prime}}\left(m^{\prime}\right)$. Let us write $h^{\prime}$ in the following for the history $w^{\prime} s_{k}$ given by $w$ without its last action. Let us note that $\left\{W_{k+1}=w\right\}$ and $\left\{H_{k}=h^{\prime}\right\} \cap\left\{A_{k}=a_{k}\right\}$ both denote the same set. We have the following chain of equations.

$$
\begin{aligned}
\mathbb{P}\left(M_{k}=m^{\prime} \mid\right. & \left.W_{k+1}=w\right) \\
& =\mathbb{P}\left(M_{k}=m^{\prime} \wedge H_{k}=h^{\prime} \mid W_{k+1}=w\right) \\
& =\frac{\mathbb{P}\left(W_{k+1}=w \mid M_{k}=m^{\prime} \wedge H_{k}=h^{\prime}\right) \cdot \mathbb{P}\left(M_{k}=m^{\prime} \wedge H_{k}=h^{\prime}\right)}{\mathbb{P}\left(W_{k+1}=w\right)} \\
& =\frac{\mathbb{P}\left(A_{k}=a_{k} \mid M_{k}=m^{\prime} \wedge H_{k}=h^{\prime}\right) \cdot \mathbb{P}\left(M_{k}=m^{\prime} \mid H_{k}=h^{\prime}\right)}{\mathbb{P}\left(A_{k}=a_{k} \mid H_{k}=h^{\prime}\right)} .
\end{aligned}
$$

The first equality is a consequence of $W_{k+1}=w$ implying $H_{k}=h^{\prime}$. Bayes' theorem is used between lines two and three. To go from the third to the fourth line, both the numerator and denominator of the fraction have been multiplied by $\mathbb{P}\left(H_{k}=h^{\prime}\right)$ and the definition of conditional probabilities has been used to rewrite the denominator and the rightmost factor of the numerator.

We now analyse the three terms of the fraction above. The probability $\mathbb{P}\left(M_{k}=m^{\prime} \mid H_{k}=h^{\prime}\right)$ is equal to the probability $\mathbb{P}\left(M_{k}=m^{\prime} \mid W_{k}=w^{\prime}\right)$. This is exactly $\mu_{w^{\prime}}\left(m^{\prime}\right)$. For the two remaining probabilities, these differ following which player controls $s_{k}$.
First case $\left(\mathcal{P}_{2}\right.$ controls $\left.s_{k}\right)$. Assume that $s_{k} \in S_{2}$. It follows from the fact that the randomisation of $\sigma_{2}$ is done independently of the current memory state that

$$
\mathbb{P}\left(A_{k}=a_{k} \mid M_{k}=m^{\prime} \wedge H_{k}=h^{\prime}\right)=\mathbb{P}\left(A_{k}=a_{k} \mid H_{k}=h^{\prime}\right)=\sigma_{2}\left(h^{\prime}\right)\left(a_{k}\right) .
$$

Therefore, we have

$$
\mathbb{P}\left(M_{k}=m^{\prime} \mid W_{k+1}=w\right)=\mathbb{P}\left(M_{k}=m^{\prime} \mid H_{k}=h^{\prime}\right)=\mu_{w^{\prime}}\left(m^{\prime}\right) .
$$

By injecting the above in Equation (A.3), we directly obtain Equation (A.1).

Second case ( $\mathcal{P}_{1}$ controls $s_{k}$ ). Assume that $s_{k} \in S_{1}$. It follows that

$$
\mathbb{P}\left(A_{k}=a_{k} \mid M_{k}=m^{\prime} \wedge H_{k}=h^{\prime}\right)=\alpha_{\text {next }}\left(m_{k}, s_{k}\right)\left(a_{k}\right)
$$

As for the probability $\mathbb{P}\left(A_{k}=a_{k} \mid H_{k}=h^{\prime}\right)$ in the denominator, we use the law of total probability once more. We have:

$$
\begin{aligned}
\mathbb{P}\left(A_{k}=a_{k} \mid H_{k}\right. & \left.=h^{\prime}\right) \\
& =\sum_{\substack{m^{\prime \prime} \in M \\
\mathbb{P}\left(M_{k}=m^{\prime \prime} \mid H_{k}=h^{\prime}\right)>0}} \mathbb{P}\left(A_{k}=a_{k} \mid M_{k}=m^{\prime \prime} \wedge H_{k}=h^{\prime}\right) \cdot \mathbb{P}\left(M_{k}=m^{\prime \prime} \mid H_{k}=h^{\prime}\right) \\
& =\sum_{m^{\prime \prime} \in M} \alpha_{\text {next }}\left(m^{\prime \prime}, s_{k}\right)\left(a_{k}\right) \cdot \mu_{w^{\prime}}\left(m^{\prime \prime}\right) .
\end{aligned}
$$

By combining the equations above with Equation (A.3), we conclude that

$$
\mu_{w}(m)=\sum_{m^{\prime} \in M} \alpha_{\text {up }}\left(m^{\prime}, s_{k}, a_{k}\right)(m) \cdot \frac{\mu_{w^{\prime}}\left(m^{\prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime}, s_{k}\right)\left(a_{k}\right)}{\sum_{m^{\prime \prime} \in M} \mu_{w^{\prime}}\left(m^{\prime \prime}\right) \cdot \alpha_{\text {next }}\left(m^{\prime \prime}, s_{k}\right)\left(a_{k}\right)},
$$

which is equivalent to Equation (A.2).
This concludes the explanations on how to derive the formulae for the distribution over memory states whenever a consistent history is processed by $\mathcal{M}$.


[^0]:    * Mickael Randour is an F.R.S.-FNRS Research Associate and James C. A. Main is an F.R.S.-FNRS Research Fellow.

[^1]:    ${ }^{1}$ We use the terminology of consistency not only for plays and histories, but also for prefixes of plays that end with an action.

