Half-Positional Objectives Recognized by Deterministic Büchi Automata

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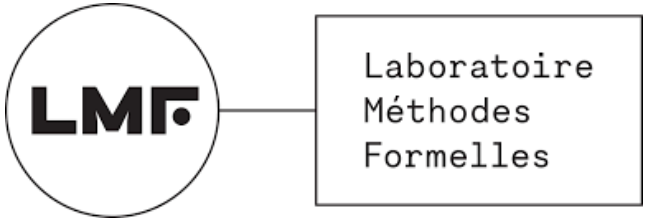
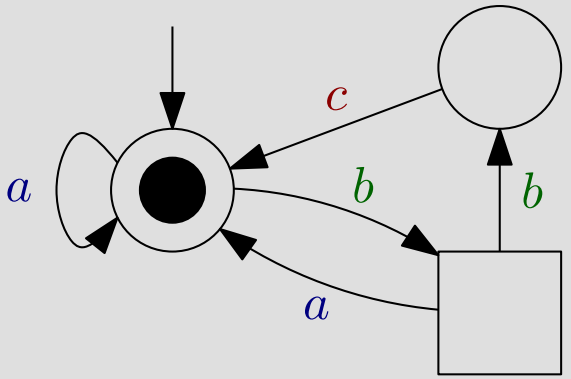
June 29, 2022 – Highlights of Logic, Games and Automata



Games on graphs

Zero-sum turn-based games on graphs



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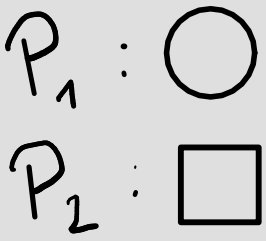

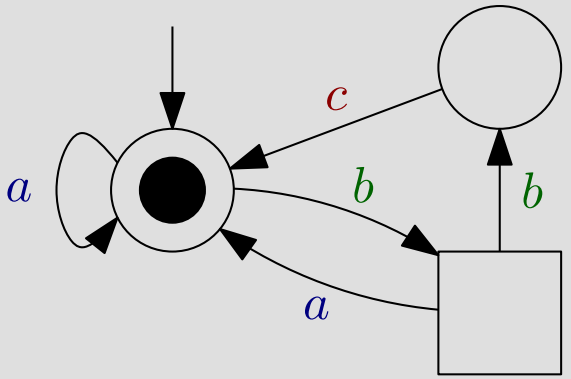

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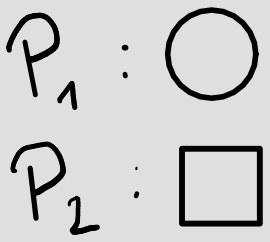

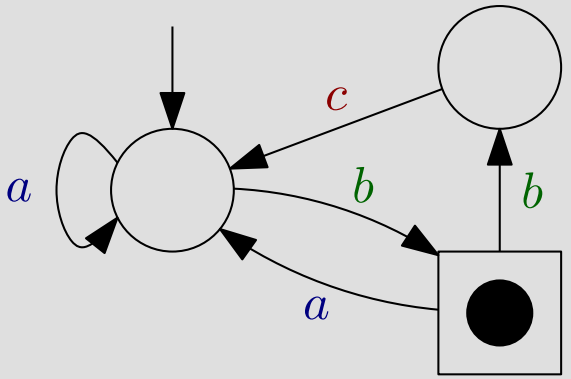

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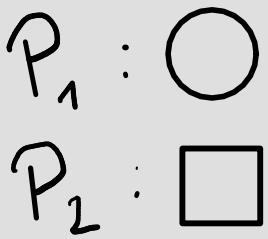

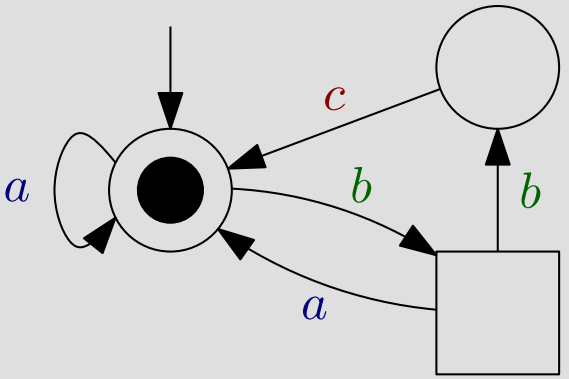

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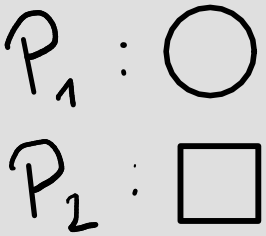

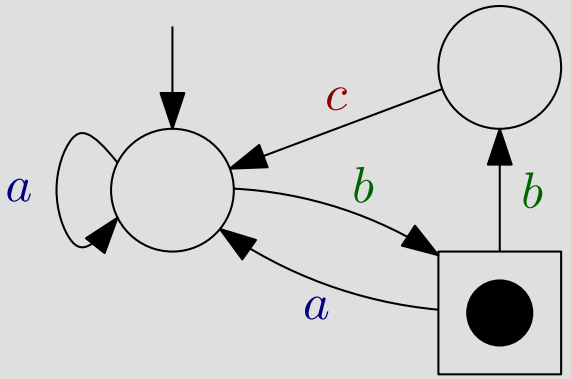

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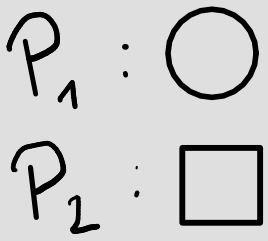

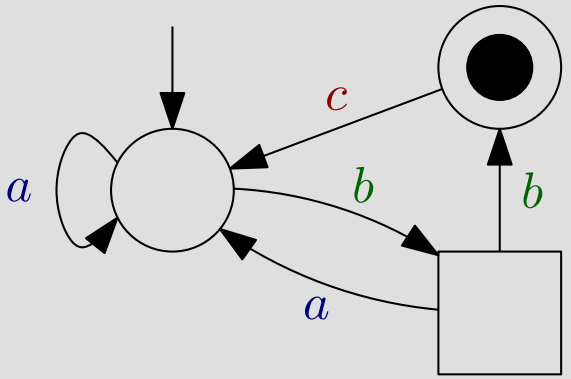

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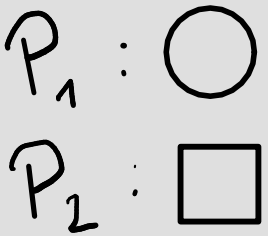

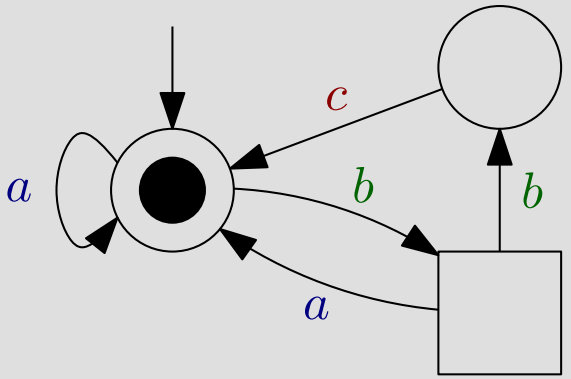

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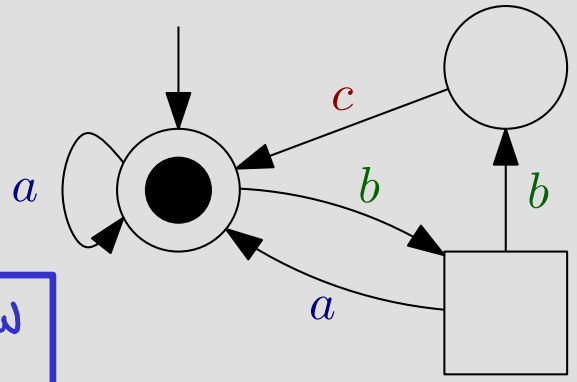
P_1 : ○

P_2 : □

Objectives : $W \subseteq C^\omega$



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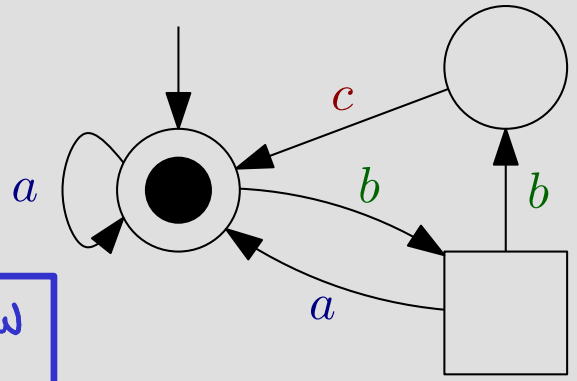

$w = babbc\ldots$

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Objectives : $W \subseteq C^w$

Motivation

Understand the **objectives** for which **simple** strategies suffice to win.

Half-positionality

Strategies

A **strategy** of $\mathcal{P}_1$ is a function $\sigma : E^* \to E$.
It is **positional** if the choices only depend on the **current** vertex.

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Existing results

- **Sufficient** conditions for half-positionality.[1,2]
- Structural **characterization**![3]

[1] Kopczyński, "Half-Positional Determinacy of Infinite Games", 2006.

[2] Bianco et al., "Exploring the boundary of half-positionality", 2011.

[3] Ohlmann, "Characterizing Positionality in Games of Infinite Duration over Infinite Graphs", 2022.

Objectives

Common class of objectives finitely representable:
ω-regular objectives.

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Open problem

Half-positionality **not** completely understood for ω-regular objectives!

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Here

Effective characterization of **half-positional** objectives recognized by **deterministic Büchi automata** (DBA).

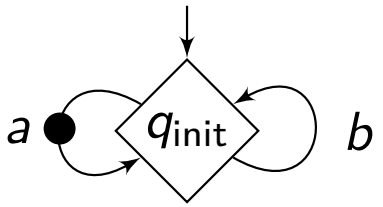
DBA recognize a **sub**class of the ω-regular objectives.

Examples

"Büchi transitions"

$C = \{a, b\}$.

- $W = \text{Büchi}(a)$ = "seeing a infinitely often": half-positional.[4]



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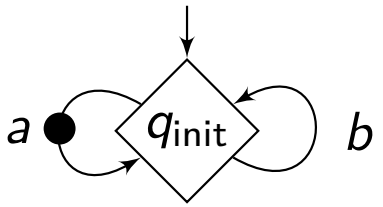

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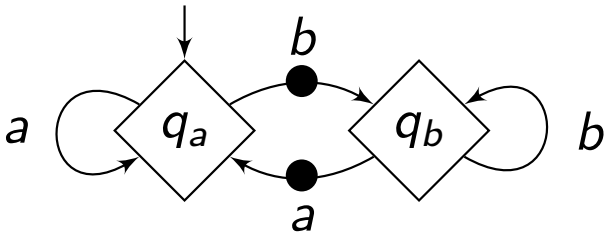

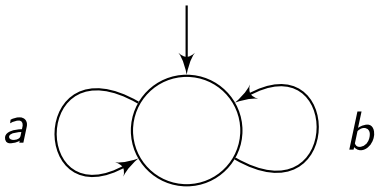

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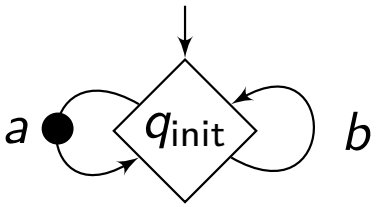

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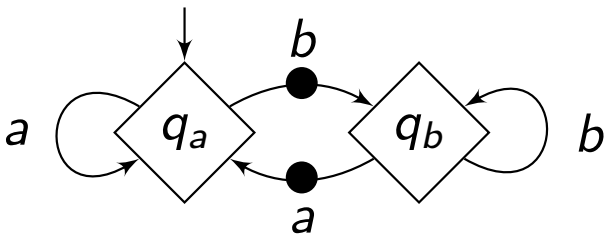

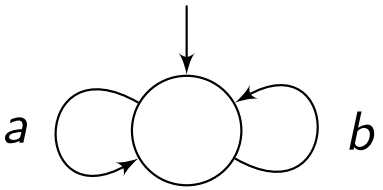

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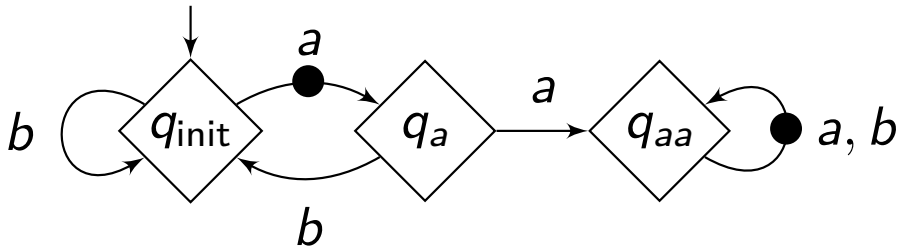

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Let W be recognized by a DBA $\mathcal{B}$.

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Characterization of half-positionality of W with a conjunction of **three** conditions.

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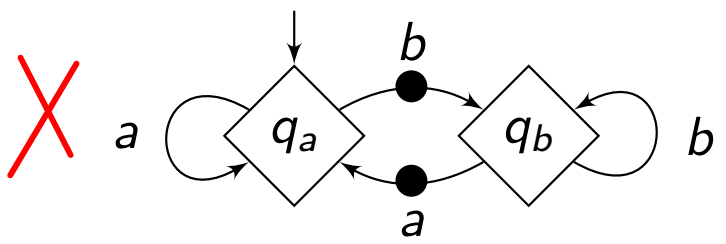
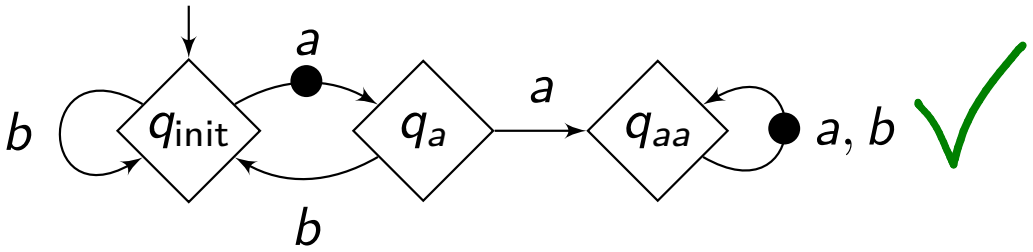
Intuition for **one** condition: what distinguishes these two DBA?





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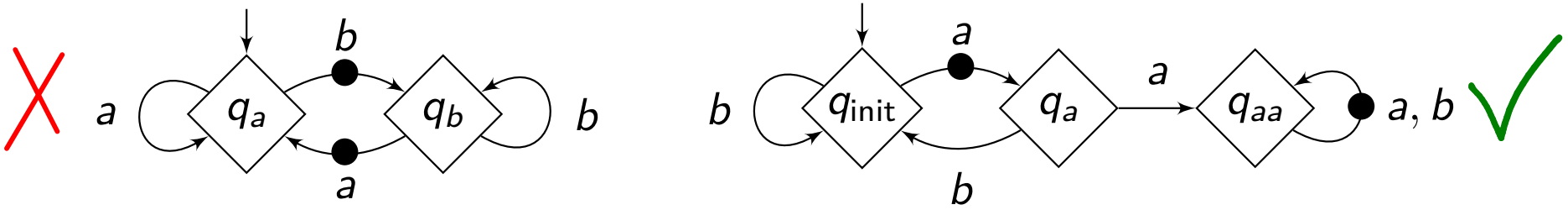

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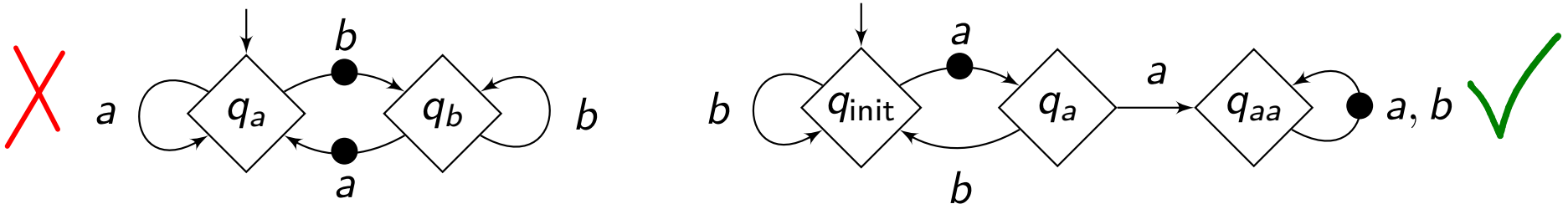

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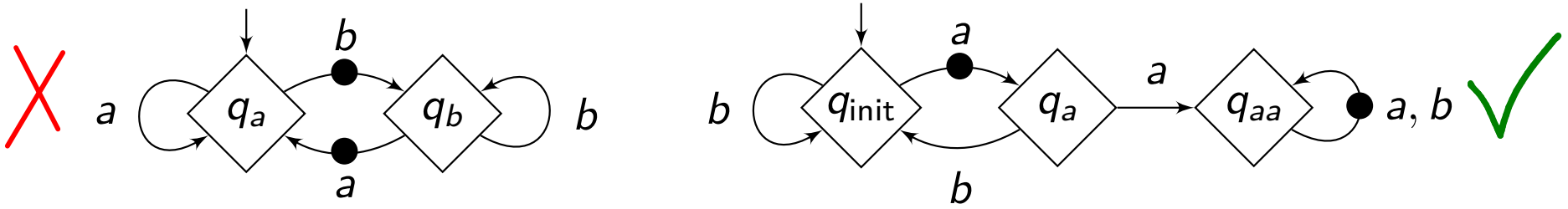

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Being "**Myhill-Nerode**-like" is necessary for half-positionality.

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