Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs

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ABSTRACT

We consider zero-sum games on infinite graphs, with objectives specified as sets of infinite words over some alphabet of *colors*. A well-studied class of objectives is the one of *ω-regular objectives*, due to its relation to many natural problems in theoretical computer science. We focus on the strategy complexity question: given an objective, how much memory does each player require to play as well as possible? A classical result is that finite-memory strategies suffice for both players when the objective is ω-regular. We show a reciprocal of that statement: when both players can play optimally with a *chromatic* finite-memory structure (i.e., whose updates can only observe colors) in all infinite game graphs, then the objective must be ω-regular. This provides a game-theoretic characterization of ω-regular objectives, and this characterization can help in obtaining memory bounds. Moreover, a by-product of our characterization is a new *one-to-two-player lift*: to show that chromatic finite-memory structures suffice to play optimally in two-player games on infinite graphs, it suffices to show it in the simpler case of one-player games on infinite graphs.

KEYWORDS

two-player games on graphs, infinite arenas, finite-memory determinacy, optimal strategies, ω-regular languages

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Context. In this work,[1] we study *zero-sum turn-based games on infinite graphs.* In such games, two players, $\mathcal{P}_1$ and $\mathcal{P}_2$, interact for an infinite duration on a graph, called an *arena*, whose state space is partitioned into states controlled by $\mathcal{P}_1$ and states controlled by $\mathcal{P}_2$. The game starts in some state of the arena, and the player controlling the current state may choose the next state following an edge of the arena. Moves of the players in the game are prescribed by their *strategy*, which can use information about the past of the play. Edges of the arena are labeled with a (possibly infinite) alphabet of *colors*, and the interaction of the players in the arena generates an *infinite word* over this alphabet of colors. These infinite words can be used to specify the players' objectives: a *winning condition* is a set of infinite words, and $\mathcal{P}_1$ wins a game on a graph if the infinite word generated by its interaction with $\mathcal{P}_2$ on the game graph belongs to this winning condition — otherwise, $\mathcal{P}_2$ wins.

This game-theoretic model has applications to the *reactive synthesis* problem [3]: a system (modeled as $\mathcal{P}_1$) wants to guarantee some specification (the winning condition) against an uncontrollable environment (modeled as $\mathcal{P}_2$). Finding a *winning strategy* in the game for $\mathcal{P}_1$ corresponds to building a controller for the system achieving the specification against all behaviors of the environment.

Strategy complexity. We are interested in the *strategy complexity* question: given a winning condition, how *complex* must winning strategies be, and how *simple* can they be? We are interested in establishing the sufficient and necessary amount of memory to play *optimally*. We consider in this work that an *optimal strategy* in an arena must be winning from any state from which winning is possible (a property sometimes called *uniformity* in the literature). The amount of memory relates to how much information about the past is needed to play in an optimal way. With regard to reactive synthesis, this has an impact in practice on the resources required for an optimal controller.

Three classes of strategies are often distinguished, depending on the number of states of memory they use: memoryless, finite-memory, and infinite-memory strategies. A notable subclass of finite-memory strategies is the class of strategies that can be implemented with finite-memory structures that only observe the sequences of colors (and not the sequences of states nor edges). Such memory structures are called *chromatic* [19]. By contrast, finite-memory structures having access to states and edges of arenas are called *general.* Chromatic memory structures are syntactically less powerful and may require more states than general ones [8], but have the benefit to be definable independently of arenas.

We intend to characterize the winning conditions for which chromatic-finite-memory strategies suffice to play optimally against arbitrarily complex strategies, for both players, in all finite and infinite arenas. We call this property *chromatic-finite-memory determinacy.* This property generalizes *memoryless determinacy*, which describes winning conditions for which memoryless strategies suffice to play optimally for both players in all arenas. Our work

[1] This short abstract is based on paper [7] of the same name, published in the proceedings of STACS 2022.

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follows a line of research [4, 6] giving various characterizations of chromatic-finite-memory determinacy for games on *finite* arenas.

ω-regular languages. A class of winning conditions commonly arising as natural specifications for reactive systems (it encompasses, e.g., linear temporal logic specifications [27]) consists of the *ω-regular languages*. They are, among other characterizations, the languages of infinite words that can be described by a *finite parity automaton* [25]. It is known that all ω-regular languages are chromatic-finite-memory determined, which is due to the facts that an ω-regular language is expressible with a parity automaton, and that *parity conditions* admit memoryless optimal strategies [16, 29]. Multiple works study the strategy complexity of ω-regular languages, giving, e.g., precise general memory requirements for all Muller conditions [11] or a characterization of the chromatic memory requirements of Muller conditions [8, Theorem 28].

A result in the other direction is given by Colcombet and Niwiński [10]: they showed that if a *prefix-independent* winning condition is memoryless-determined in infinite arenas, then this winning condition must be a parity condition. As parity conditions are memoryless-determined, this provides an elegant characterization of parity conditions from a strategic perspective, under prefix-independence assumption.

Congruence. A well-known tool to study a language L of finite (resp. infinite) words is its *right congruence relation* $\sim_L$: for two finite words w_1 and w_2, we write $w_1 \sim_L w_2$ if for all finite (resp. infinite) words w, $w_1 w \in L$ if and only if $w_2 w \in L$. There is a natural deterministic (potentially infinite) automaton recognizing the equivalence classes of the right congruence, called the *minimal-state automaton of* $\sim_L$ [24, 28].

The relation between a regular language of *finite* words and its right congruence is given by the Myhill-Nerode theorem [26], which provides a natural bijection between the states of the minimal deterministic automaton recognizing a regular language and the equivalence classes of its right congruence relation. Consequences of this theorem are that a language is regular if and only if its right congruence has finitely many equivalence classes, and a regular language can be recognized by the minimal-state automaton of its right congruence.

For the theory of languages of *infinite* words, the situation is not so simple: ω-regular languages have a right congruence with finitely many equivalence classes, but having finitely many equivalence classes does not guarantee ω-regularity (for example, a language is *prefix-independent* if and only if its right congruence has exactly one equivalence class, but this does not imply ω-regularity). Moreover, ω-regular languages cannot necessarily be recognized by adding a natural acceptance condition (parity, Rabin, Muller…) to the minimal-state automaton of their right congruence [1]. There has been multiple works about the links between a language of infinite words and the minimal-state automaton of its right congruence; one relevant question is to understand when a language can be recognized by this minimal-state automaton [1, 24, 28].

Contributions. We characterize the ω-regularity of a language of infinite words W through the strategy complexity of the zero-sum turn-based games on infinite graphs with winning condition W: the ω-regular languages are *exactly* the chromatic-finite-memory determined languages (seen as winning conditions). As discussed earlier, it is well-known that ω-regular languages admit chromatic-finite-memory optimal strategies [8, 25, 29] — our results yield the other implication. This therefore provides a characterization of ω-regular languages through a game-theoretic and strategic lens.

Our technical arguments consist in providing a precise connection between the representation of W as a parity automaton and a chromatic memory structure sufficient to play optimally. If strategies based on a chromatic finite-memory structure are sufficient to play optimally for both players, then W is recognized by a parity automaton built on top of the direct product of the *minimal-state automaton of the right congruence* and this *chromatic memory structure*. This result generalizes the work from Colcombet and Niwiński [10] in two ways: by relaxing the prefix-independence assumption about the winning condition, and by generalizing the class of strategies considered from memoryless to chromatic-finite-memory strategies. We recover their result as a special case.

Moreover, we actually show that chromatic-finite-memory determinacy in *one-player* games of both players is sufficient to show ω-regularity of a language. As ω-regular languages are chromatic-finite-memory determined in two-player games, we can reduce the problem of chromatic-finite-memory determinacy of a winning condition in *two-player* games to the easier chromatic-finite-memory determinacy in *one-player* games. Such a *one-to-two-player lift* holds in multiple classes of zero-sum games, such as deterministic games on finite arenas [4, 13, 20] and stochastic games on finite arenas [6, 14]. The proofs for finite arenas all rely on an *edge-induction technique* (also used in other works about strategy complexity in finite arenas [9, 12, 17]) that appears unfit to deal with infinite arenas. Although not mentioned by Colcombet and Niwiński, it was already noticed [19] that for prefix-independent winning conditions in games on infinite graphs, a one-to-two-player lift for *memoryless* determinacy follows from [10].

Related works. We have already mentioned [8, 10, 11, 18, 29] for fundamental results on the memory requirements of ω-regular conditions, [4, 6, 13, 14] for characterizations of "low" memory requirements in finite (deterministic and stochastic) arenas, and [1, 24, 28] for links between an ω-regular language and the minimal-state automaton of its right congruence.

One stance of our work is that our assumptions about strategy complexity affect *both* players. Another intriguing question is to understand when the memory requirements of only *one* player are finite. In finite arenas, a few results in this direction are sufficient conditions for the existence of memoryless optimal strategies for one player [2, 17], and a proof by Kopczyński that the chromatic memory requirements of prefix-independent ω-regular conditions are computable [18, 19].

We mention other works on finite-memory determinacy in different contexts: finite arenas [23], non-zero-sum games [22], countable one-player stochastic games [15], concurrent games [5, 21].

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