Z_N-balls: Solitons from Z_N-symmetric scalar field theory

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We discuss the conditions under which static, finite-energy, configurations of a complex scalar field ϕ with constant phase and spherically-symmetric norm exist in a potential of the form $V(\phi^*\phi, \phi^N + \phi^{*N})$ with $N \in \mathbb{N}$ and $N \geq 2$, i.e., a potential with a Z_N-symmetry. Such configurations are called Z_N-balls. We build explicit solutions in $(3+1)$-dimensions from a model mimicking effective field theories based on the Polyakov loop in finite-temperature $SU(N)$ Yang-Mills theory. We find Z_N-balls for $N = 3, 4, 6, 8, 10$, and show that only static solutions with zero radial nodes exist for N odd, while solutions with radial nodes may exist for N even.

I. INTRODUCTION

Q-balls, originally introduced in [1], are nontopological solitons with finite energy and size, typically obtained by solving the equations of motion of a complex Klein-Gordon field with U(1) invariance: $\mathcal{L} = \eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi^* - v(\phi^*\phi)$ with $\eta = \text{diag}(+---)$.

The U(1) symmetry allows for the ansatz

$$\phi(\mathrm{x}) = f(r)\mathrm{e}^{i\omega t}, \qquad (1.1)$$

with x the position 4-vector, r the radial coordinate and $\omega \in \mathbb{R}$. Boundary conditions $f(0) \neq 0$ and $f(r \rightarrow \infty) = 0$ are chosen. The equations of motion then reduce to

$$\Delta f + \omega^2 f - \frac{1}{2}\partial_f v(f^2) = 0. \qquad (1.2)$$

This field equation allows for solitons called Q-balls in the absence of coupling to gravity, see e.g., Refs. [1–6] provided that a nonquadratic part is present in v. The potential is often chosen to be of the form $v(\phi^*\phi) = m^2\phi^*\phi - \lambda_4(\phi^*\phi)^2 + \lambda_6(\phi^*\phi)^3$ with λ_4 and $\lambda_6 > 0$. Due to the U(1) symmetry of the Lagrangian, Q-balls have a charge $Q \sim \omega$. The solitons are called boson star in the presence of gravity, see e.g., the review [7]. More general solutions than (1.1) can be found in Minkowski spacetime for U(1)-symmetric potentials:

(i) Spinning Q-balls: $\phi(\mathrm{x}) = f(r,\theta)\mathrm{e}^{i\omega t + iJ\varphi}$ [4], where (r, θ, φ) are the spherical coordinates.

(ii) Multipolar solutions: $\phi(\mathrm{x}) = f(r,\theta,\varphi)\mathrm{e}^{i\omega t}$ [8].

(iii) Coupled Q-balls [9–11].

In the present paper we propose to break the continuous U(1) symmetry of potential $v(\phi^*\phi)$ by allowing for terms depending on $\phi^N + \phi^{*N}$ with an integer number $N \geq 2$, i.e., by studying a complex scalar field theory with Z_N-symmetry. To our knowledge, the influence of such a breaking of U(1) symmetry on the solutions has been poorly studied in general, but it is worth quoting the Montonen-Sarker-Trullinger-Bishop (MSTB) model [12–16] in which the potential can be written under the form $(1 - \phi^*\phi)^2 + \frac{\kappa}{2}\phi^*\phi + \frac{\kappa}{4}(\phi^2 + \phi^{*2})$ with $\kappa \in \mathbb{R}$. The solutions of this Z_2-symmetric model have been well studied in $(1+1)$-dimensions, including generalizations of the potential, see e.g., the recent work [17]. Here the potential under study will be of the form $V(\phi^*\phi, \phi^N + \phi^{*N})$, and the field equations will be formulated in $(3+1)$-dimensions. As argued in the next paragraph, such a form may be of interest in the field on finite-temperature gauge theories.

Solutions built from Z_3-symmetric models have indeed been studied in several works, see e.g., Refs. [18–22], in the context of an effective model of SU(3) Yang-Mills theory at nonzero temperature T. In the latter case, the scalar field ϕ is identified with the mean value of the Polyakov loop, defined as $\phi(T, \vec{x}) = \langle P\mathrm{e}^{ig\int_0^{1/T} d\tau A_0(\tau, \vec{x})}\rangle$, with A_0 the

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temporal component of the Yang-Mills field and $(\tau, \vec{x})$ the temporal and spatial coordinates, respectively. P is the path ordering, g is the strong coupling constant and units where $\hbar = c = k_B = 1$ are used. The gauge group $SU(N)$ may be considered to better outline our motivations. On the one hand, under gauge transformations belonging to the centre of the gauge group, i.e., Z_N, the Polyakov loop is multiplied by an overall factor. The Z_N symmetry of Yang-Mills theory is therefore broken (present) if $\phi \neq 0 \, (= 0)$. On the other hand, it is known that $\phi \neq 0 \, (= 0)$ above (below) the deconfinement temperature T_c [23–25]. Hence, it has been conjectured that the confinement/deconfinement phase transition might be driven by the breaking of a global Z_N symmetry [26,27]. In that framework, the shape $V(\phi^*\phi, \phi^N + \phi^{*N})$ should be typical of any effective model of Yang-Mills theory based on the Polyakov loop [28]. It is worth pointing out that such effective models are able to accurately reproduce the Yang-Mills equation of state provided potential parameters are temperature-dependent and fitted on lattice QCD data [29,30]. Effective models are still worthy of interest nowadays, e.g., in glueball dark matter models [31,32] or in the phenomenology of ultra-relativistic heavy ion collisions [33,34].

Our work is structured as follows. In Sec. II we discuss the existence of static, spherically-symmetric, solitons with finite energy in generic Z_N-symmetric potentials. Such solitons are called "Z_N-balls" in the following. Then in Sec. III we propose a model mimicking features of Yang-Mills theory above deconfinement temperature. Numerical solutions are constructed for several values of N in Sec. IV. After a discussion of solutions going beyond Z_N-balls in Sec. V, concluding comments, including a remark on solutions' instability, are given in Sec. VI.

II. Z_N-BALLS: DEFINITION AND EXISTENCE

As proposed in the Introduction, let us consider a complex Klein-Gordon field ϕ with Z_N-symmetric potential $V(\phi^*\phi, \phi^N + \phi^{*N})$, such that $N \in \mathbb{N}$ and $N \geq 2$. The Lagrangian reads

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(\phi^*\phi, \phi^N + \phi^{*N}). \quad (2.1)$$

Using a polar decomposition of the complex field, i.e., $\phi = f e^{i\delta}$, the equations of motion can be set in the form

$$\Box f - f \partial_\mu \delta \partial^\mu \delta + V^{(1,0)} f + V^{(0,1)} N \cos(N\delta) f^{N-1} = 0,$$
$$(2.2a)$$

$$f\Box\delta + 2\partial_\mu f \partial^\mu \delta - V^{(0,1)} N \sin(N\delta) f^{N-1} = 0, \quad (2.2b)$$

with $V^{(1,0)} = \partial_a V(a, 2f^N \cos(N\delta))|_{a=f^2}$ and $V^{(0,1)} = \partial_a V(f^2, a)|_{a=2f^N \cos(N\delta)}$.

Setting $\delta = \frac{2\pi k}{N}$ with $k \in \mathbb{Z}$, i.e.,

$$\phi = \phi_k(\mathrm{x}) = f(\mathrm{x}) e^{i\frac{2\pi k}{N}} \quad \text{with} \quad k \in \mathbb{Z}, \quad f \in \mathbb{R}, \quad (2.3)$$

reduces Eqs. (2.2) to a single equation for f, reading

$$\Box f + V^{(1,0)} f + V^{(0,1)} N f^{N-1} = 0. \quad (2.4)$$

We define the Z_N-ball ansatz as a solution with spherical symmetry, that is

$$\phi = \phi_k(r) = f(r) e^{i\frac{2\pi k}{N}}, \quad (2.5)$$

with r the radial coordinate. The conditions for a localized, regular, solution imply the following boundary conditions

$$f(0) = f_0 > 0, \quad f'(0) = 0, \quad f(r \to \infty) \sim \frac{e^{-mr}}{r} \to 0. \quad (2.6)$$

m is the mass term in the Lagrangian: $V(\phi^*\phi, \phi^N + \phi^{*N}) \sim m^2 \phi^* \phi$ in a power expansion in ϕ. The constraint $f_0 > 0$ is necessary to avoid the trivial solution $f(r) = 0$ and Eq. (2.4) with the ansatz (2.5) reads

$$f'' + \frac{2}{r} f' - \partial_f U(f) = 0, \quad \text{with} \quad U(f) = \frac{1}{2} V(f^2, 2f^N). \quad (2.7)$$

By comparison with Eq. (1.2), one sees that the Z_N-ball equation (2.7) is equivalent to that of a Q-ball with $\omega = 0$ and potential $U(f)$. According to the study [4], $\omega = 0$ Q-balls, hence Z_N-balls in our study, may exist if

$$U''(0) > 0 \quad \text{and} \quad \min_f \left(\frac{2U(f)}{f^2} \right) < 0. \quad (2.8)$$

In the Z_N-ball case, no continuum of solutions (parametrized by ω) is possible, basically because the phase of ϕ_k is "locked", see Eq. (2.3). This phase-locking also forbids spinning Z_N-balls, whose phase should be proportional to $e^{iJ\varphi}$. For the same reason, multipole solutions as those studied in [8] are not expected. The best that can be found beyond the ground state is a discrete set of solutions distinguished by the phase $k = 0, \ldots, N-1$ in (2.3) and, if radially excited states exist, by the number of nodes of the scalar field.

Radially excited Z_N-balls may exist but not necessarily. The classical particle analogy sheds light on their existence. We recall indeed that Eq. (2.7) can be interpreted as Newton's equation for a particle whose position if f and for which r plays the role of time in the potential $-U(f)$. This particle experiences a friction term $\frac{2}{r} f'$. The interested reader will find a detailed description of this classical
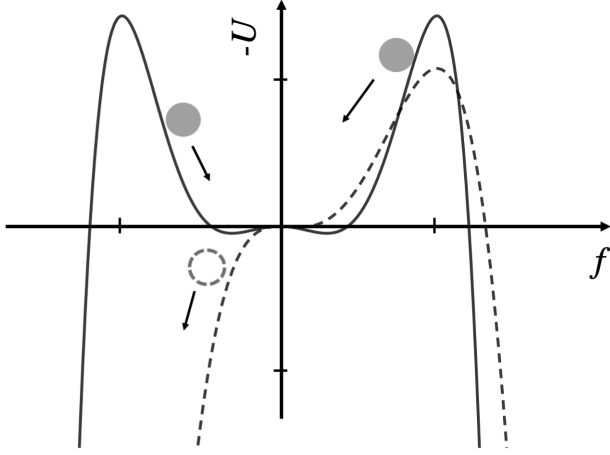
FIG. 1. Mechanical analogy in $(3 + 1)$-dimensions for Z_N-balls with the ansatz (2.5) and the effective potential (2.7). Typical plots corresponding to N even (sold line) and odd (dashed line) are shown, and arrows indicate the trajectory of the equivalent classical particle. More precisely, the case N even is obtained by plotting (4.4) and the case N odd is obtained by plotting (4.3), with the horizontal axis in units of l and the vertical axis in units of $m^2 l^2/2$.

analogy for Q-balls, Q-holes, and Q-bulges in Ref. [6]. Figure 1 shows typical classical trajectories in effective potentials with different N. When N is even, $U(f) = U(-f)$: Radially excited states *a priori* exist since the particle may oscillate between the two maxima before reaching $f = 0$. When N is odd, $U(f)$ may have no definite parity, and potentials can be found where the particle cannot reach an equilibrium position in $f = 0$. In this case, radially excited states will not exist. In [21] for example, only a solution with zero node was found for a Z_3-symmetric potential. It is worth mentioning that potentials with shapes similar to the presented odd-N case are also found in some hairy black hole models [35].

Having in mind Eq. (2.7) and effective potentials of the shape given in Fig. 1, it can be noted the Z_N − ball equation is—*mutatis mutandis*—similar to the bounce equation of Coleman [36,37]. The bounce is a classical solution in four-dimensional Euclidean space extrapolating between two inequivalent vacua of a potential; the matching between our equation and the bounce equation can be made by replacing $2/r$ by $3/r$ and by replacing r by $\rho = \sqrt{\tau^2 + r^2}$, τ being the Euclidean time and ρ being the radial coordinate in four-dimensional spherical coordinates. After fixation of the complex field phase in Z_N − ball case, the physical interpretation of f and of the Coleman bounce are obviously similar.

All the Z_N-symmetric potentials reducing to the same effective potential on the real axis will have the same radial profiles. As an example we consider the $N = 4$ potential

$$V = m^2 \phi^* \phi - \frac{\gamma}{2}(\phi^4 + \phi^{*4}) + (\phi^* \phi)^3 \qquad (2.9)$$

and the $N = 6$ potential

$$V = m^2 \phi^* \phi - \gamma(\phi^* \phi)^2 + \frac{1}{2}(\phi^6 + \phi^{*6}). \qquad (2.10)$$

Both are equivalent to a standard U(1)-symmetric potential $m^2 \phi^* \phi - \gamma(\phi^* \phi)^2 + (\phi^* \phi)^3$ on the real axis. The latter potential may admit $\omega = 0$ Q-balls; the effective potential reads $U(f) = m^2 f^2 - \gamma f^4 + f^6$, and criterion (2.8) states that $\omega = 0$ Q-balls, hence Z_4-balls and Z_6-balls, exist for $\gamma^2 > 4m^2$.

In the following, solitons will be further characterized by their mass M and mean radius $\langle R \rangle$

$$M = 4\pi \int_0^\infty \epsilon \, dr, \qquad \epsilon(r) = f'^2 + U(f), \qquad (2.11)$$

$$\langle R \rangle = \frac{M_1}{M_0}, \qquad M_j = \int_0^\infty f^2(r) r^{2+j} dr. \qquad (2.12)$$

In the above equations $\epsilon(r)$ is the local energy density.

III. THE MODEL

SU(N) Yang-Mills theories at finite temperature are fields where Z_N-symmetric effective theories seem particularly relevant. We therefore propose a potential mimicking key features of such effective theories and numerically solve the field equation for the latter potential. We previously made a proposal of U(1)-symmetric potential focusing on the $N \to \infty$ limit of Yang-Mills theories [38]; here we propose a potential relevant for finite values, i.e., $N \geq 3$. We refer the reader to the aforementioned MSTB model in the case $N = 2$.

The potential under study will be the Z_N-symmetric power-law potential

$$
\begin{aligned}
&V(\phi^* \phi, \phi^N + \phi^{*N}) \\
&= \frac{m^2}{2} \left[\phi^* \phi - (1 + \beta)(ml)^{2-N}(\phi^N + \phi^{*N}) \right. \\
&\quad \left. + \frac{(ml)^{4-2N}}{N-1}(N(1 + \beta) - 1)(\phi^* \phi)^{N-1} \right], \\
&\text{with} \quad m, \beta, l \in \mathbb{R}_0^+, \qquad N \in \mathbb{N}, \qquad N \geq 3. \quad (3.1)
\end{aligned}
$$

m has the dimension of energy and β, l are dimensionless. According to Eq. (2.7), it corresponds to the effective potential

$$
\begin{aligned}
U(f) = \frac{m^2}{2} \left[f^2 - 2(1 + \beta)(ml)^{2-N} f^N \right. \\
\left. + \frac{(ml)^{4-2N}}{N-1}(N(1 + \beta) - 1) f^{2N-2} \right]. \quad (3.2)
\end{aligned}
$$

The latter potential is such that:

(i) $f = 0$ is a local minimum: $U(0) = U'(0) = 0$ and $U''(0) = m^2 > 0$;

(ii) $f = ml$ is a global minimum: $U(ml) = \frac{m^4 l^2}{2} \times \beta \frac{2-N}{N-1} < 0$, $U'(ml) = 0$ and $U''(ml) = m^2(N-2) \times (N(1+2\beta) - 2) > 0$;

(iii) $\min_{f>0}\left(\frac{2U}{f^2}\right) = m^2\beta \frac{\beta(1-N)+(2-N)}{N(1+\beta)-1} < 0.$

Hence, it admits Z_N-balls according to the criterion (2.8).

The shape of potential (3.1) has the generic features of Polyakov-loop effective models of SU(N) Yang-Mills theory at finite temperature [28]. In our potential, l can be related to the nonzero average norm of the Polyakov loop at a given temperature $T > T_c$, showing the breaking of Z_N-symmetry and deconfinement. In this picture, f/m is the norm of the Polyakov loop. Note that if β goes to a small but negative value, the absolute minimum becomes $f = 0$. The case $\beta < 0$ would correspond to $T < T_c$. A U(1)-symmetric potential similar to (3.1), i.e., $V(\phi^*\phi, (\phi^*\phi)^{N/2})$, has been investigated in [39], where analytical formulas for solitons in $(1 + 1)$-dimensions have been found. The replacement $N \to n + 2$ transforms our potential into that of [39].

It is known that the pressure of Yang-Mills matter is given by $p = -\min(V)$, V being the considered effective potential. In our case,

$$p = -\min(U) = \frac{m^4 l^2}{2}\beta\frac{N-2}{N-1}. \tag{3.3}$$

The value of m^4 can be matched with the typical pressure scale, that is the Stefan-Boltzmann pressure for a gas of massless gluons. The equality

$$m^4 = p_{SB} = \frac{\pi^2(N^2-1)}{45}T^4 \tag{3.4}$$

leads to the pressure

$$p = p_{SB}\frac{N-2}{2(N-1)}l^2\beta. \tag{3.5}$$

As shown by lattice simulations, $p < p_{SB}$ for $T \gtrsim T_c$ In the limit $T \to +\infty$, $p \to p_{SB}$ [40] and $l \to 1$ by definition. It can be concluded that the "physical" values for β are bounded in the range

$$0 < \beta \le \frac{2(N-1)}{N-2}, \tag{3.6}$$

or $0 < \beta \le 4$ at most, the maximal upper bound being reached when $N = 3$.

The functions $l(T)$, $\beta(T)$ could be found by fitting the Polyakov loop and pressure computed in lattice QCD, see e.g., Refs. [41] but it is out of the scope of our paper, where we aim at a general discussion of Z_N-balls.

IV. NUMERICAL RESULTS

We first perform the scalings

$$f \to mlf, \qquad r \to \frac{r}{m}, \tag{4.1}$$

so that Eq. (2.7) becomes

$$f'' + \frac{2}{r}f' - f + (1+\beta)Nf^{N-1} - (N(1+\beta)-1)f^{2N-3} = 0. \tag{4.2}$$

β and N are the remaining parameters. Boundary conditions (2.6) are searched for. The three boundary conditions exceed the two conditions associated with the second-order field equation. The possibility to get nontrivial solutions will then consist to fine-tune one of the coupling constants (in our case β) in function of the control parameter f_0. A numerical resolution of Eq. (4.2) can then be performed. We use a collocation method for boundary-value ordinary differential equations, equipped with an adaptive mesh-selection procedure [42].

The Z_3-ball is displayed in Fig. 2 for various values of $\beta \in [0, 4]$ according to the upper bound (3.6). We notice that $f(0) \to 1$ when $\beta \to 0$: In this case the effective potential is proportional to $f^2(1-f)^2$, namely with two degenerate minima in $f = 0$ and $f = 1$. The more β is small, the more the mean radius is large. At $\beta = 0.5$, $M = 1.6$ and $\langle R \rangle = 2.7$. The typical energy scale associated with Yang-Mills theory is the deconfinement temperature, $T_c \sim 0.3$ GeV [43]. Setting $T = 0.3$ GeV and $N = 3$ in (3.4) leads to $m = 345$ MeV and to $M \sim 0.552$ GeV and $\langle R \rangle \sim 7.83$ GeV$^{-1} = 1.54$ fm. The latter radius is of the same order of magnitude than the profile found in [22] with a potential containing the same powers of ϕ, and the mass is typical of what is observed in [21] for temperatures between 1.1 and 1.2 T_c. The effective potential reads, for $N = 3$ and with the rescaled parameters,

$$U_{N=3}(f) = \frac{m^4 l^2}{2}\left[f^2 - 2(1+\beta)f^3 + \frac{2+3\beta}{2}f^4\right]. \tag{4.3}$$

No solution with radial node was found, in agreement with the picture of Fig. 1: The shape of the effective potential when $f < 0$ makes impossible for the soliton to reach a zero value at infinity.

Z_4-balls are displayed in Fig. 3 for various values of β. As for $N = 3$, we notice that $f(0) \to 1$ when $\beta \to 0$: In this case the effective potential is proportional to $f^2(1-f^2)^2$, again with two degenerate minima in $f = 0$ and $f = 1$. The mass and mean radius have similar trends as in the $N = 3$ case for the state with zero node. This time, a solution with one radial node has been found. The effective potential reads, for $N = 4$ and with the rescaled parameters,
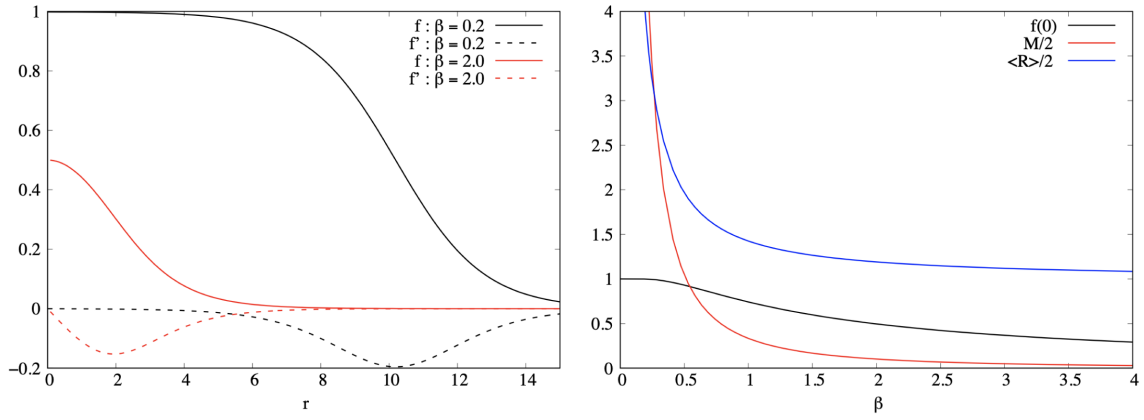
FIG. 2. Left panel: Z_3-ball profile computed from Eq. (4.2) with $N = 3$, $\beta = 0.2$ (black lines) or $\beta = 2.0$ (red lines). The profile f is displayed (solid lines) as well as its derivative (dashed lines). Right panel: Mass (red line), mean radius (blue line) and value at origin (black line) of the Z_3-ball for various values of β. f, r and M are expressed in units of ml, $1/m$ and m respectively.
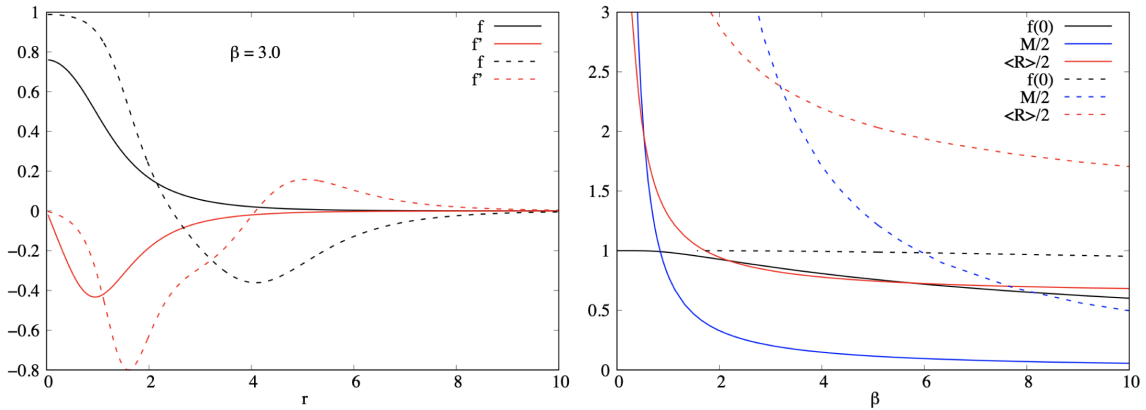

FIG. 3. Left panel: Z_4-balls profiles computed from Eq. (4.2) with $N = 4$ and $\beta = 3$. The profile with 0 node and its derivative are displayed (solid lines) as well as the profile with 1 node (dashed lines). Right panel: Mass (blue line), mean radius (red line) and value at origin (black line) of the Z_4-ball for various values of β. f, r and M are expressed in units of ml, $1/m$ and m respectively.
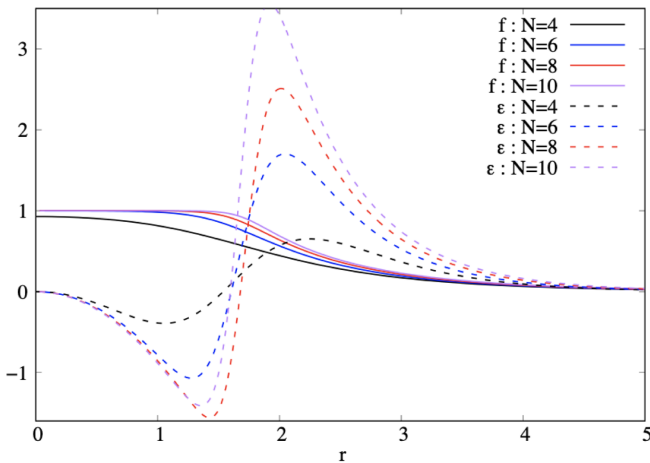

FIG. 4. Z_N-balls profiles computed from Eq. (4.2) with $\beta = 2$ for $N = 4$ (black), $N = 6$ (blue), $N = 8$ (red) and $N = 10$ (purple). The profiles with zero nodes are displayed (solid lines) as well as the local energy density (dashed lines). f, r, and ϵ are expressed in units of ml, $1/m$, and m^2, respectively.

$$U_{N=4}(f) = \frac{m^4 l^2}{2}\left[f^2 - 2(1 + \beta)f^4 + \frac{3 + 4\beta}{3}f^6\right]. \quad (4.4)$$

A comparison with Fig. 1 shows that the parity of $U(f)$ indeed allows for radially excited states. We searched for solutions with two radial nodes but did not find any.

The evolution of the zero-node Z_N-balls versus N is displayed in Fig. 4. It can be seen that the plateau of the soliton is larger at larger N, with an energy density more peaked around its maximal and minimal values.

V. FURTHER SOLUTIONS: NONCONSTANT PHASE

All solutions discussed so far trivialize Eq. (2.2b) and the question to deform the Z_N balls by means of a nonconstant phase $\delta(x)$ raises naturally. As a first step in this direction, it is instructive to study the linearized version of Eq. (2.2b) for a radial phase, i.e., for $\delta(r)$. With the potential (3.1) it reduces to

$$\delta' + 2\left(\frac{1}{r} + \frac{f'}{f}\right)\delta' - (1+\beta)N^2\delta f^{N-2} = 0. \tag{5.1}$$

Using the asymptotic decay (2.6) of $f(r)$ in rescaled units where $m = 1$, the dominant terms in the asymptotic region reduces to

$$\delta' - 2\delta' = 0 \rightarrow \delta(r \rightarrow +\infty) = \delta_0 + \delta_1 e^{2r}. \tag{5.2}$$

On the other hand the solutions of Eq. (5.1) can be expanded around the origin

$$\delta(r) = \tilde{\delta}_0\left(1 + \frac{N^2(1+\beta)f_0^{N-2}}{12}r^2 + \frac{N^2(1+\beta)f_0^{N-4}(N^2 f_0^N(1+\beta) + (6N-20)f_0 f''(0))}{480}r^4\right) + o(r^6), \tag{5.3}$$

where $\tilde{\delta}_0$ is an arbitrary constant. Note that only the presence of the last term in (5.1), which in due to the $U(1)$-breaking term, leads to the condition (5.3) and allows for a nonconstant and regular δ at the origin. The U(1) symmetry is recovered for $\beta = -1$ and only allows for a constant δ.

Our numerical integration gave strong evidence that solutions of Eq. (5.1) extrapolating between the behaviors (5.3) and (5.2) exist for generic values of N and β. Solutions of the full system (2.2), if they exist, would be specific to the Z_N symmetry of the potential and will be addressed in a forthcoming paper.

VI. CONCLUDING COMMENTS

We have shown that spherically-symmetric solitons with finite energy may exist in a complex Klein-Gordon model with Z_N-symmetric potential. The existence of such Z_N-balls in potential $V(\phi^*\phi, \phi^N + \phi^{*N})$ is guaranteed provided that Q-balls with zero charge exist in the potential $V(\phi^*\phi, 2(\phi\phi^*)^{N/2})$. As an illustration, we have built solutions in a potential inspired by finite-temperature $SU(N)$ Yang-Mills theory. We found solitons with zero radial node for all values of N, and states with one radial node for even N. In the latter case indeed, the parity of the potential allows for radially excited states.

Our solutions escape Derrick's nonexistence argument [44] because our effective potential is nonpositive definite for some values of the scalar field. However, the scaling argument of Derrick leads to an instability. The unstable mode under scaling of the radial variable can be isolated by perturbating our solution, say f_0, according to $f(r) = f_0(r) + \eta(r)$ and examining the quadratic term in η of the perturbated energy. It turns out that $\eta(r) = f'_0(r)$ constitutes an unstable mode with eigenvalue $\omega = -2$, independently of N and β. Keeping in mind the interpretation of Z_N-balls as possible "plasma bubbles" of deconfined Yang-Mills matter, their instability against fluctuations must not be regarded as unphysical. An illustration of bubble instability is given in [22], where profiles similar to our $Z_3 -$ balls are first obtained with an effective potential fitted on quark-gluon plasma lattice QCD equation of state. Then, the time evolution of a set of such bubbles, with randomly chosen initial phase, initially spatially separated, is computed. It is shown that bubbles tend to disappear at the confinement-deconfinement phase transition while Z_3-walls and strings appear. Even though the single-bubble case is not separately investigated in [22], the instability of one single bubble is clearly numerically demonstrated since there is no extra bubble-bubble interaction that could generate instability but only superposition principle between different bubbles.

As an outlook, we mention that more general solutions that those originating in the Z_N-ball ansatz we proposed may *a priori* be found to solve (2.2). Kink solutions have indeed been shown to exist in MSTB model (Z_2-symmetry) in $(1 + 1)$-dimensions since the pioneering works [12–15]. The case $N = 3$ has been recently studied [19] with the construction of bounce solutions in a Z_3-symmetric Polyakov-Nambu-Jona-Lasinio model in $(3 + 1)$-dimensions, i.e., where the Polyakov loop is coupled to a quark matter field. In that spirit, we hope to build solutions of the field equations (2.2) with position-dependent phase δ for arbitrary N in future works.

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