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Higher dualisations of linearised gravity and the A_1^{+++} algebra

Nicolas Boulanger,[a] Paul P. Cook,[b] Josh A. O'Connor[a] and Peter West[b]

[a] *Physique de l'Univers, Champs et Gravitation, Université de Mons — UMONS, Place du Parc 20, 7000 Mons, Belgium*

[b] *Department of Mathematics, King's College London, Strand, London WC2R 2LS, U.K.*

E-mail: nicolas.boulanger@umons.ac.be, paul.cook@kcl.ac.uk, josh.o'connor@umons.ac.be, peter.west540@gmail.com

Abstract: The non-linear realisation based on A_1^{+++} is known to describe gravity in terms of both the graviton and the dual graviton. We extend this analysis at the linearised level to find the equations of motion for the first higher dual description of gravity that it contains. We also give a systematic method for finding the additional fields beyond those in the non-linear realisation that are required to construct actions for all of the possible dual descriptions of gravity in the non-linear realisation. We show that these additional fields are closely correlated with the second fundamental representation of A_1^{+++}.

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Contents

1 Introduction

It was shown that the conjectured [1, 2] non-linear realisation of the semi-direct product $E_{11} \ltimes \ell_1$ of E_{11} with its vector representation contains the fields and the equations of motion of every maximal supergravity theory [3, 4]. For a review, see [5]. As such, it contains the metric of gravity and the three form in eleven dimensions and there are very good reasons to believe that these are the only degrees of freedom that the non-linear realisation possesses [6, 7]. However, the non-linear realisation contains an infinite number of fields, of which only a few are the usual fields of the maximal supergravity theories.

It was conjectured in [8] and proven in [9] that many of these remaining fields represent equivalent descriptions of the degrees of freedom of the maximal supergravity theories. For example, in E_{11} at levels $1, 4, 7, \dots$, we find the fields $A_{a_1a_2a_3}$, $A_{a_1\dots a_9, b_1b_2b_3}$, $A_{a_1\dots a_9, b_1\dots b_9, c_1c_2c_3}$, and so on, which are related by an infinite set of duality relations. This ensures that the only degrees of freedom are those which are usually contained in the first field, the three form. However, any of these fields can be used to give an equivalent formulation of these degrees of freedom. At levels $2, 5, 8, \dots$, the story is similar except that the block of three indices $a_1a_2a_3$ in each field is replaced by a block of six indices $a_1 \dots a_6$. Then, at levels $0, 3, 6, 9, \dots$, we find fields associated with gravity. Indeed, at level zero, we find the usual description of gravity with the field h_{ab}. At level three, we find the field $A_{a_1\ldots a_8,b}$ which was proposed to provide a dual description of gravity, while at level six we have $A_{a_1\ldots a_9,b_1\ldots b_8,c}$, at level nine we find $A_{a_1\ldots a_9,b_1\ldots b_9,c_1\ldots c_8,d}, \ldots$, and so on. These fields also provide alternative descriptions of gravity and all the fields are related by a set of duality relations which ensure that the theory only propagates a single graviton. In fact, there are other fields in the non-linear realisation and some of these are required to account for the gauged supergravities.

It is useful to give an account of the history of the dual graviton field. It was first observed by Curtright that the field $A_{a_1a_2,b} = A_{[a_1a_2],b}$ could describe pure gravity in five dimensions [10]. It was then proposed that the field $A_{a_1\ldots a_{D-3},b}$ may describe pure gravity in D dimensions [11]. In order the show that the field $A_{a_1\ldots a_8,b}$ at level three in the non-linear realisation of $E_{11} \ltimes \ell_1$ did indeed describe gravity, a parent action in D dimensions was given in [1, 12]. By first linearising the parent action of [1], then varying the result with respect to one field or the other and finally substituting inside the linearised parent action, we obtain either the Fierz-Pauli action in the form where local Lorentz invariance holds, i.e. in terms of the field h_{ab} that is neither symmetric nor antisymmetric, or we obtain an action in terms of the dual field $A_{a_1\ldots a_{D-3},b}$. This result was fully explained and also extended to higher spin fields in reference [13]. As shown in [1, 12], the parent action of [1] also led to duality relations between the two fields. In this way, it was clear that the dual graviton field $A_{a_1\ldots a_{D-3},b}$ really did provide an equivalent formulation of gravity at the linearised level. Further connections were also established in [13] between [1], [10] and [11]. These developments are reviewed at the beginning of section 3.

It was also conjectured that the non-linear realisation of the semi-direct product $A_{D-3}^{+++} \ltimes \ell_1$ of the very-extended algebra A_{D-3}^{+++} with its vector representation, contains pure gravity in D dimensions [14]. Following early preparatory work in references [15] and [16], this was indeed shown to be the case in four and eleven dimensions [17] and [18] respectively. In four dimensions, at the lowest level, this non-linear realisation contains the usual field of gravity h_{ab}. At higher levels — indicated by numbers in brackets after each field — in addition to other fields, it contains

$$h_a{}^b\,(0)\,; \quad A_{(ab)}\,(1)\,; \quad A_{a_1a_2,(b_1b_2)}\,(2)\,; \quad A_{a_1a_2,b_1b_2,(c_1c_2)}\,(3)\,; \quad A_{a_1a_2,b_1b_2,c_1c_2,(d_1d_2)}\,(4)\,; \quad \ldots \tag{1.1}$$

where groups of indices are antisymmetric unless otherwise indicated by round brackets $(\cdots)$ in which case they are symmetric. We interpret these fields as being related to dual descriptions of gravity. The field at level one is called the dual graviton. We then find the first higher dual graviton at level two, the second higher dual graviton at level three, and so on. The equations of motion at the full non-linear level, as well as the duality relations, were found for $h_a{}^b$ and $A_{(ab)}$ in four dimensions [17] and in eleven dimensions [18].

The non-linear realisations of $E_{11} \ltimes \ell_1$ and $A_{D-3}^{+++} \ltimes \ell_1$ lead to an infinite number of duality relations which can then be used to derive the equations of motion of the fields. These field equations are constructed from fields that are irreducible representations of A_{D-1} and, as a result, they have more and more space-time derivatives for the fields at higher and higher levels. The equations of motion require only the fields in the non-linear realisation, and they correctly describe the relevant degrees of freedom. In [19] and [20], equations of this type which describe the irreducible representations of the Poincaré group were given precisely. As shown in [20] and reviewed in [21], one can also integrate these equations to find equations of motion that are second order in space-time derivatives provided that one makes a particular gauge choice that leads to the Labastida [22, 23] gauge transformations for arbitrary mixed-symmetry fields where the gauge parameters obey trace constraints. In fact, the duality relations derived from the non-linear realisation only hold modulo gauge transformations which can, as a matter of principle, be deduced from the non-linear realisation. See, for example, [4] or the review [24]. However, one must also introduce extra fields in order to have duality relations that hold as equations of motion in the usual sense and not just as equivalence relations [25].

A parent action containing the fields $A_{a_1a_2a_3}$, $A_{a_1\dots a_9,b_1b_2b_3}$, $\dots$ occurring at levels $1, 4, \dots$ in the $E_{11} \ltimes \ell_1$ non-linear realisation, also containing certain extra fields, was worked out in [25] along the lines of [9, 26]. Depending on which field was eliminated, one found an action only in terms of one field or the other. In this way, the authors of [25] found an action for the latter field which we call the first higher dual of the three from. The higher level fields were also discussed in [25], as were the infinite chain of duality relations and analogous results for the six form. Hence, using parent actions, one could find the additional fields required in order to write down an action, or duality relations, for the higher dual fields.

A similar strategy had previously been suggested for pure gravity in [9]. The method of parent actions was used to produce, for the first time, an infinite number of higher dual action principles, thereby proving the conjecture established in [8] on the equivalent dual descriptions of gravity. These parent actions involve extra fields in comparison to those that appear in the non-linear realisation of $A_1^{+++} \ltimes \ell_1$.

In this paper, we further pursue the approach set forth in [9] to higher dual descriptions of gravity, focusing on four space-time dimensions for the sake of concreteness. We provide an explicit procedure for constructing the parent actions that relate the different higher dual formulations of gravity. Using these parent actions, one can directly obtain action principles for each subsequent higher dual graviton. We find extra fields on top of those already in the non-linear realisation of $A_1^{+++} \ltimes \ell_1$. These extra fields are required to formulate actions for the dual fields as well as the duality relations between dual fields at adjacent levels.

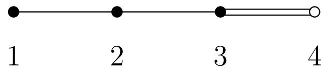
We will compare the type of additional fields required to form higher dual actions with those contained in the adjoint representation and the second fundamental representation, denoted l_2 , of A_1^{+++} . While there is a striking agreement between the GL(4)-irreducible symmetry types of the extra fields appearing in the higher dual actions and the ℓ_2 representation of A_1^{+++} , the number of times each type of extra field appears off-shell does not always coincide with their multiplicities in the ℓ_2 representation.

JHEP12(2022)152

2 The Kac-Moody algebra A_1^{+++}

2.1 The non-linear realisation of $A_1^{+++} \ltimes \ell_1$

Following earlier results [15, 16], the non-linear realisation of the semi-direct product $A_1^{+++} \ltimes \ell_1$ of A_1^{+++} with its vector representation was computed at low levels in [17]. This calculation will be reviewed in this section. The Dynkin diagram for A_1^{+++} is



While no complete description of the generators of any such Kac-Moody algebra exists, they can still be analysed by decomposing them with respect to certain subalgebras. Deleting node 4 from the Dynkin diagram of A_1^{+++} allows us to analyse the algebra in terms of its decomposition into GL(4) [15]. The resulting generators can be classified in terms of a level which, in this case, is the number of up minus down GL(4) indices divided by two. The positive low level generators $R^{\underline{\alpha}}$ are given, alongside the level zero generator, by

$$\begin{array}{llll} K^a{}_b\,(0)\,; \quad R^{(ab)}\,(1)\,; & R^{a_1a_2,(b_1b_2)}\,(2)\,; & R^{a_1a_2,b_1b_2,(c_1c_2)}\,(3)\,, & R^{a_1a_2a_3,b_1b_2,c}\,(3)\,; \\ R^{a_1a_2,b_1b_2,c_1c_2,(d_1d_2)}\,(4)\,, & R^{a_1a_2a_3,b_1b_2,(c_1c_2c_3)}\,(4)\,, & R^{a_1a_2a_3,b_1b_2,c_1c_2,d}_{(1)}\,(4)\,, & R^{a_1a_2a_3,b_1b_2,c_1c_2,d}_{(2)}\,(4)\,, \\ R^{a_1a_2a_3,b_1b_2b_3,(c_1c_2)}\,(4)\,, & R^{a_1a_2a_3a_4,b_1b_2,(c_1c_2)}\,(4)\,, & R^{a_1a_2a_3a_4,b_1b_2b_3,c}\,(4)\,; & \ldots \end{array} \tag{2.1}$$

The number in brackets corresponds to the level of the generators and the subscripts enumerate the generators when there is more than one with the same index structure. Groups of indices are antisymmetric except when shown to be symmetrised using round brackets. For example, the generator $R^{ab,cd}$ satisfies $R^{ab,cd} = R^{[ab],cd} = R^{ab,(cd)}$. The generators belong to irreducible representations of GL(4), i.e they all satisfy over-antisymmetrisation irreducibility conditions. For example, $R^{[ab,c]d} = R^{[ab|,c|d]} = 0$. Negative level generators have the same index structure with lowered indices. Commutation relations for these A_1^{+++} generators can be found in [17].

The generators in the vector representation of the A_1^{+++} are denoted by L_A and, when decomposed into representations of GL(4), the low level generators found in [17] are given by

$$\begin{array}{llll} P_a\,(0)\,; \quad Z^a\,(1)\,; & Z^{(a_1a_2a_3)}\,(2)\,, & Z^{a_1a_2,b}\,(2)\,; & Z^{a_1a_2,(b_1b_2b_3)}\,(3)\,, \\ Z^{a_1a_2,b_1b_2,c}_{(1)}\,(3)\,, & Z^{a_1a_2,b_1b_2,c}_{(2)}\,(3)\,, & Z^{a_1a_2a_3,(b_1b_2)}\,(3)\,, & Z^{a_1a_2a_3,b_1b_2}\,(3)\,, \\ Z^{a_1a_2,b_1b_2,(c_1c_2c_3)}_{(1)}\,(4)\,, & Z^{a_1a_2,b_1b_2,(c_1c_2c_3)}_{(2)}\,(4)\,, & \ldots\,, & \end{array} \tag{2.2}$$

where, as before, groups of indices are antisymmetric except for those in round brackets which are symmetric. Subscripts denote different generators when the multiplicity is greater than one. These generators satisfy the usual GL(4)-irreducibility conditions. For example, $Z^{[a_1a_2,b]} = 0$. Generators in the vector representation commute and their commutators with the generators of A_1^{+++} are given in [17].

The construction of the equations of motion follows the same pattern as that for $E_{11} \ltimes \ell_1$. See [5, 24] for reviews. For the non-linear realisation based on $A_1^{+++} \ltimes \ell_1$ in [17], we start from the group element of $A_1^{+++} \ltimes \ell_1$ denoted by $g = g_L g_A$, where g_A and g_L are group elements that are constructed in terms of non-negative level generators of the adjoint and vector representations, respectively, of A_1^{+++}. They take the form

$$g_A = e^{A_{\underline{\alpha}} R^{\underline{\alpha}}} := \cdots \exp(A_{ab,cd}\, R^{ab,cd}) \exp(A_{ab}\, R^{ab}) \exp(h_a{}^b\, K^a{}_b) \,, \tag{2.3}$$

$$g_L = e^{z^A L_A} := \exp(x^a P_a) \exp(z_a Z^a) \exp(z_{abc} Z^{abc} + z_{ab,c} Z^{ab,c}) \cdots \,. \tag{2.4}$$

Therefore, the theory is populated by a set of fields $A_{\underline{\alpha}}$ which contains the graviton $h_a{}^b$, the dual graviton A_{ab}, the first higher dual graviton $A_{ab,cd}$, and so on. We see from the list of generators in (2.1) that we have the generator $R^{a_1a_2,b_1b_2,c_1c_2,(d_1d_2)}$ at level four which results in the second higher dual graviton $A_{a_1a_2,b_1b_2,c_1c_2,(d_1d_2)}$. Indeed, the pattern continues so that one finds such fields at every level. This leads to an infinite tower of dual formulations of pure gravity with fields that depend on the generalised coordinates $z^A = \{x^a, z_a, z_{abc}, z_{ab,c}, \ldots\}$.

The non-linear realisation is invariant under rigid transformations $g_0 \in A_1^{+++} \ltimes \ell_1$ and local transformations $h \in I_c(A_1^{+++})$, where $I_c(A_1^{+++})$ is the Cartan involution invariant subalgebra of A_1^{+++}. This means that generic group elements $g = g_L g_A$ are invariant under

$$g \to g_0 g \qquad \text{and} \qquad g \to g h \,, \tag{2.5}$$

where g_0 is a rigid (i.e. constant) group element and h is a local transformation which can be used to set the coefficients of all negative level generators in g_A to zero [27]. The equations of motion are just those that are invariant under these transformations and, as for E_{11}, they are essentially unique.

The dynamics of the non-linear realisation is often constructed using Maurer-Cartan forms

$$\nu \equiv g^{-1} \mathrm{d} g = \nu_A + \nu_L \,, \tag{2.6}$$

where $\nu_A = g_A^{-1} \mathrm{d} g_A \equiv \mathrm{d}z^\Pi\, G_{\Pi,\underline{\alpha}}\, R^{\underline{\alpha}}$ and $\nu_L = g_A^{-1}(g_L^{-1}\mathrm{d} g_L) g_A = g_A^{-1}(\mathrm{d}z \cdot L)\, g_A \equiv \mathrm{d}z^\Pi E_\Pi{}^A L_A$. Here, $E_\Pi{}^A$ can be thought of as a vierbein on the generalised space-time. Its lowest component is the gravitational vierbein given by $e_\mu{}^a = (\exp(h))_\mu{}^a$. The $G_{\Pi,\underline{\alpha}}$ are the components of the Maurer-Cartan form where the index Π is a world-volume (derivative) index and $\underline{\alpha}$ is an index in the adjoint representation.

The low level Maurer-Cartan forms in the A_1^{+++} direction are given by

$$G_{\Pi,a}{}^b = e_a{}^\mu \partial_\Pi e_\mu{}^b \,, \qquad \overline{G}_{\Pi,bc} = e_b{}^\kappa e_c{}^\lambda \partial_\Pi A_{\kappa\lambda} \,, \tag{2.7}$$

$$\overline{\overline{G}}_{\Pi,a_1a_2,bc} = e_{a_1}{}^{\kappa_1} e_{a_2}{}^{\kappa_2} e_b{}^{\lambda_1} e_c{}^{\lambda_2} \left(\partial_\Pi A_{\kappa_1\kappa_2,\lambda_1\lambda_2} - A_{[\kappa_1|(\lambda_1} \partial_\Pi A_{\lambda_2)|\kappa_2]} \right) \,. \tag{2.8}$$

They are found as the coefficients of $K^a{}_b$, R^{bc} and $R^{a_1a_2,bc}$ in the Maurer-Cartan form, where ∂_Π is the derivative with respect to the coordinates z^Π. The dynamics is actually constructed from $G_{A,\underline{\alpha}} \equiv (E^{-1})_A{}^\Pi G_{\Pi,\underline{\alpha}} = e_a{}^\mu G_{\mu,\underline{\alpha}} + \cdots$, where "$\cdots$" corresponds to terms arising when higher contributions to the vierbein $E_\Pi{}^A$ are taken into account. These contributions contain derivatives with respect to the higher level coordinates. Working with $G_{A,\underline{\alpha}}$ has the advantage that it only transforms under $I_c(A_1^{+++})$.

Rather than deriving the equations of motion, one can derive a set of duality relations from which the equations of motion can be deduced, as explained in [3, 5, 24]. The duality relations between low level fields at the lowest level of generalised space-time derivatives are [17]

$$E_{a,b_1b_2} \equiv (\det e)^{1/2}\omega_{a,b_1b_2} + \tfrac{1}{2}\,\varepsilon_{b_1b_2}{}^{c_1c_2}\overline{G}_{c_1,c_2a} \doteq 0\,, \tag{2.9}$$

$$\overline{E}_{a,b_1b_2} \equiv \overline{G}_{a,b_1b_2} + \varepsilon_a{}^{c_1c_2c_3}\overline{\overline{G}}_{c_1,c_2c_3,b_1b_2} \doteq 0\,, \tag{2.10}$$

where ω_{a,b_1b_2} is the usual expression for the spin connection in terms of the vierbein which is given in terms of the low level Maurer-Cartan forms by

$$(\det e)^{1/2}\omega_{a,b_1b_2} = -\,G_{b_1,(b_2a)} + G_{b_2,(b_1a)} + G_{a,[b_1b_2]} \tag{2.11}$$

with $G_{a,bc} = e_a{}^\mu e_b{}^\nu \partial_\mu e_{\nu c}$.

Equation (2.9) relates the graviton field $h_a{}^b$ appearing at level zero to the dual graviton field A_{ab} at level one, while equation (2.10) is a duality relation between A_{ab} at level one and the first higher dual graviton field $A_{a_1a_2,bc}$ appearing at level two.

By combining equations (2.9) and (2.10), one derives a duality relation between the graviton and the first higher dual graviton:

$$E'_{a,b_1b_2} \equiv (\det e)^{1/2}\omega_{a,b_1b_2} + 3\,\overline{\overline{G}}_{[b_1,b_2c]}{}^c{}_{,a} \doteq 0\,. \tag{2.12}$$

The above duality relations only hold modulo certain gauge transformations, as indicated by the symbol "$\doteq$", so they really are equivalence relations. This is explained in [3, 4, 24, 27]. In order to further manipulate the above duality relations, one needs to know what the gauge transformations are. We will obtain them in the next section. As explained in [1], the duality relation (2.9) may be turned into a usual equation by adding an antisymmetric component to the symmetric dual graviton $A_{a_1a_2}$. This 2-form field will later be found inside the second fundamental representation of A_1^{+++}. In what follows we are only concerned with the linearised theory and so we drop, in particular, the $\det e$ factors.

2.2 Gauge transformations

It was proposed in [28] that a theory constructed from a non-linear realisation of $\mathfrak{g}^{+++} \ltimes \ell_1$, where $\mathfrak{g}^{+++}$ is any very-extended Kac-Moody algebra and ℓ_1 is its vector (first fundamental) representation, is invariant under a particular set of gauge transformations whose parameters are in a one-to-one correspondence with the spectrum of ℓ_1. For the linearised theory where base and fiber indices are identified, these gauge transformations take the form

$$\delta A_{\underline{\alpha}} = C^{-1}_{\underline{\alpha},\underline{\beta}}(D^{\underline{\beta}})_E{}^F\partial_F\Lambda^E\,. \tag{2.13}$$

In this equation, $C^{-1}_{\underline{\alpha},\underline{\beta}}$ is the inverse of the Cartan-Killing metric $C^{\underline{\alpha},\underline{\beta}}$ for $\mathfrak{g}^{+++}$. The matrix $(D^{\underline{\beta}})_E{}^F$ is that for the vector representation and, in particular, it occurs in the commutator

$$[R^{\underline{\alpha}}, L_A] = -(D^{\underline{\alpha}})_A{}^B L_B \,. \tag{2.14}$$

In addition, we have used the partial derivative in the linearised theory $\partial_F = \frac{\partial}{\partial z^F}$. The gauge parameters Λ^A correspond to elements in the vector representation.

Hence, in order to evaluate the gauge transformations, we require the inverse Cartan-Killing matrix and the analogous matrix for ℓ_1 at the corresponding level. The gauge transformations for the graviton and the dual graviton in the non-linear realisation of $A_1^{+++} \ltimes \ell_1$ were computed in [17] and we now extend these previous results to the main object of study in this paper: the first higher dual graviton.

We begin with the computation of the Cartan-Killing form which is determined by requiring that it is invariant. For our current purposes this means that it should satisfy

$$([R^{a_1a_2}, R^{b_1b_2}], R_{c_1c_2,d_1d_2}) + (R^{a_1a_2}, [R_{c_1c_2,d_1d_2}, R^{b_1b_2}]) = 0 \,, \tag{2.15}$$

where $(\cdot\,,\cdot)$ is the symmetric non-degenerate bilinear form on A_1^{+++} that generalises the Killing form for finite-dimensional semi-simple Lie algebras. One finds that

$$C^{a_1a_2,b_1b_2,}{}_{c_1c_2,d_1d_2} = \delta^{a_1a_2}_{c_1c_2}\delta^{b_1b_2}_{(d_1d_2)} + \delta^{[a_1|b_1}_{c_1c_2}\delta^{|a_2]b_2}_{(d_1d_2)} + \delta^{[a_1|b_2}_{c_1c_2}\delta^{|a_2]b_1}_{(d_1d_2)} \,, \tag{2.16}$$

where $\delta^{a_1a_2}_{(b_1b_2)} = \delta^{a_1}_{(b_1}\delta^{a_2}_{b_2)}$ in contrast to the usual symbol $\delta^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{[b_1}\delta^{a_2}_{b_2]}$.

Taking the previous results from [17], we find that the Cartan-Killing metric up to the level of the first higher dual graviton is given by

$$C_{\underline{\alpha},\underline{\beta}} = \begin{pmatrix} \delta^c_b\delta^a_d - \frac{1}{2}\delta^a_b\delta^c_d & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^{(a_1a_2)}_{b_1b_2} & 0 & 0 \\ 0 & \delta^{(a_1a_2)}_{b_1b_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C^{a_1a_2,b_1b_2,}{}_{c_1c_2,d_1d_2} \\ 0 & 0 & 0 & C^{c_1c_2,d_1d_2,}{}_{a_1a_2,b_1b_2} & 0 \end{pmatrix} . \tag{2.17}$$

where the basis is ordered to match the scalar products of $K^a{}_b$, $R^{a_1a_2}$, $R_{a_1a_2}$, $R^{a_1a_2,b_1b_2}$ and $R_{a_1a_2,b_1b_2}$ with $K^c{}_d$, $R^{b_1b_2}$, $R_{b_1b_2}$, $R^{c_1c_2,d_1d_2}$ and $R_{c_1c_2,d_1d_2}$. Note that the only non-zero entries of $C_{\underline{\alpha},\underline{\beta}}$ are found when the levels of $\underline{\alpha}$ and $\underline{\beta}$ sum to zero.

The inverse Cartan-Killing metric is given by

$$C^{-1}_{\underline{\alpha},\underline{\beta}} = \begin{pmatrix} \delta^e_c\delta^d_f - \frac{1}{2}\delta^d_c\delta^e_f & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^{b_1b_2}_{(e_1e_2)} & 0 & 0 \\ 0 & \delta^{b_1b_2}_{(e_1e_2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C^{e_1e_2,f_1f_2,}{}_{c_1c_2,d_1d_2} \\ 0 & 0 & 0 & C^{c_1c_2,d_1d_2,}{}_{e_1e_2,f_1f_2} & 0 \end{pmatrix} . \tag{2.18}$$

The vector representation appears in the commutators of (2.14) which were given in [17] at low levels. Omitting the commutators with $K^a{}_b$, they are given by

$$\begin{aligned}
[R^{ab},\, P_c] &= \delta_c^{(a}\, Z^{b)}\ , \qquad\qquad [R^{ab},\, Z^c] = Z^{abc} + Z^{c(a,b)}\ ,\\
[R^{ab,cd},\, P_e] &= -\,\delta_e^{[a}\, Z^{b]cd} + \tfrac{1}{4}\left(\delta_e^a\, Z^{b(c,d)} - \delta_e^b\, Z^{a(c,d)}\right) - \tfrac{3}{8}\left(\delta_e^c\, Z^{ab,d} + \delta_e^d\, Z^{ab,c}\right)\ ,\\
[R_{ab},\, P_c] &= 0\ , \qquad\qquad [R_{ab},\, Z^c] = 2\,\delta_{(a}^{c}\, P_{b)}\ ,\\
[R_{ab},\, Z^{cde}] &= \tfrac{2}{3}\left(\delta_{(ab)}^{cd}\, Z^e + \delta_{(ab)}^{de}\, Z^c + \delta_{(ab)}^{ec}\, Z^d\right)\ ,\\
[R_{ab},\, Z^{cd,e}] &= \tfrac{4}{3}\left(\delta_{(ab)}^{de}\, Z^c - \delta_{(ab)}^{ce}\, Z^d\right)\ .
\end{aligned} \tag{2.19}$$

To study the vector representation at the level of the first higher dual graviton, we need to compute certain commutators at higher levels. One finds that

$$[R_{a_1a_2,b_1b_2}, Z^{c_1c_2c_3}] = e_1\, \delta^{(c_1c_2c_3)}_{b_1b_2[a_1} P_{a_2]}\ , \quad e_1 = -4, \tag{2.20}$$

$$[R_{a_1a_2,b_1b_2}, Z^{c_1c_2,d}] = e_2 \left(\delta^{c_1c_2}_{a_1a_2}\delta^{d}_{(b_1}P_{b_2)} + \delta^{c_1c_2}_{a_1(b_1|}\delta^{d}_{a_2}P_{|b_2)} + \delta^{c_1c_2}_{a_1(b_1}\delta^{d}_{b_2)}P_{a_2}\right), \quad e_2 = 2, \tag{2.21}$$

where the last expression should be taken so that is anti-symmetric in a_1 and a_2.

Using the inverse Cartan-Killing metric (2.18) and reading off the analogous matrix for ℓ_1 from equations (2.19)–(2.21), we find that the gauge transformations with gauge parameters

$$\Lambda^A := \{\xi^a\,, \bar{\xi}_a\,, \Lambda_{abc}\,, \Lambda_{ab,c}\,, \ldots\} \tag{2.22}$$

are given, for the fields at low levels, by

$$\delta h_a{}^b = \partial_a \xi^b\ , \qquad \delta A_{ab} = -2\,\partial_{(a}\bar{\xi}_{b)}\ , \tag{2.23}$$

$$\delta A_{a_1a_2,b_1b_2} = -\,e_1\,\partial_{[a_1}\Lambda_{a_2]b_1b_2} - 3\,e_2\,\partial_{(b_1|}\Lambda_{a_1a_2,|b_2)} + 2\,e_2\,\partial_{[a_1}\Lambda_{a_2](b_1,b_2)}\ . \tag{2.24}$$

The parameters satisfy $\Lambda_{a_1a_2a_3} = \Lambda_{(a_1a_2a_3)}$ and $\Lambda_{a_1a_2,b} = \Lambda_{[a_1a_2],b}$ with the irreducibility condition $\Lambda_{[a_1a_2,b]} = 0$. In these equations, we have not written the gauge transformations that involve derivatives with respect to the higher level coordinates.

2.3 Linearised equations of motion

The duality relations given in (2.9)–(2.12) only hold modulo certain transformations which arise from the gauge transformations for the fields involved in the duality relations. As we computed these in the previous section, we can now compute the resulting transformations up to which the duality relations hold. Having done this, we can then compute the equations of motion from the duality relations at the linearised level.

We first consider the duality relation between gravity and dual gravity in (2.9). Using the gauge transformation (2.24) we find that, at the linearised level, it takes the form

$$E_{a,b_1b_2} \equiv \omega_{a,b_1b_2} + \tfrac{1}{2}\,\varepsilon_{b_1b_2}{}^{c_1c_2}\,\overline{G}_{c_1,c_2a} + \partial_a \xi_{b_1b_2} = 0\ , \tag{2.25}$$

where

$$\xi_{b_1b_2} := -\,\partial_{[b_1}\xi_{b_2]} - \varepsilon_{b_1b_2}{}^{c_1c_2}\,\partial_{c_1}\bar{\xi}_{c_2}\ . \tag{2.26}$$

In deriving this result, we have used local Lorentz symmetry to symmetrise the h_{ab} field, and so we obtain the variation $\delta h_{ab} = \partial_{(a}\xi_{b)}$. Had we not done this Lorentz gauge fixing, then the first term on the right-hand-side of (2.26) would have been replaced by a Lorentz transformation. We have removed the dot above the equals sign in (2.25) since it holds as a usual equation.

To find the equations of motion, we have to eliminate the gauge transformations from the duality relations by taking derivatives and, at the same time, eliminating one of the two fields involved. In the case of (2.25), we can eliminate the gauge parameter $\xi_{b_1b_2}$ by taking an exterior derivative of E_{a,b_1b_2} which produces

$$\partial_{[a_1}E_{a_2],b_1b_2} \equiv \partial_{[a_1}\omega_{a_2],b_1b_2} + \tfrac{1}{2}\,\varepsilon_{b_1b_2}{}^{c_1c_2}\,\partial_{[a_1|}\overline{G}_{c_1,c_2|a_2]} = 0\,. \tag{2.27}$$

By contracting a_1 with b_1 , we find that the term involving the dual graviton vanishes due to the fact that we have anti-symmetrised derivatives and A_{ab} is symmetric. Thus, we find that

$$\partial_{[a_1}\omega_{a_2],b_1b_2}\eta^{a_1b_1} = 0\,, \tag{2.28}$$

which is the equation of motion for linearised gravity.

We can also write (2.27) as

$$\tfrac{1}{2}\,\varepsilon_{b_1b_2c_1c_2}\,\partial_{[a_1}\omega_{a_2],}{}^{b_1b_2} - \partial_{[a_1}\overline{G}_{[c_1,c_2]a_2]} = 0\,. \tag{2.29}$$

The first term is $-\varepsilon_{b_1b_2c_1c_2}\partial_{[a_1}\partial^{b_1}h^{b_2}{}_{a_2]}$ and so it vanishes when we contract it with $\eta^{a_1c_1}$. As a result, we find that

$$\partial^{[a_1}\overline{G}_{[a_1,c_2]}{}^{a_2]} = 0\,, \tag{2.30}$$

which we recognise as the equation of motion for the dual graviton at the linearised level, which agrees with the results of [17] where the full non-linear equation of motion was found and its linearised version was also given.

We will now carry out the same procedure for the duality relation involving the dual graviton and the first higher dual graviton (2.10). Using the gauge transformations in (2.23) and (2.24), we find that the duality relation becomes

$$\overline{E}_{a,b_1b_2} \equiv \overline{G}_{a,b_1b_2} + \varepsilon_a{}^{c_1c_2c_3}\,\overline{\overline{G}}_{c_1,c_2c_3,b_1b_2} - 2\,\partial_a\partial_{(b_1}\overline{\xi}_{b_2)} - 3\,e_1\,\varepsilon_a{}^{e_1e_2e_3}\,\partial_{e_1}\partial_{(b_1|}\Lambda_{e_2e_3,|b_2)} = 0\,. \tag{2.31}$$

By taking two derivatives, we find that the gauge parameters disappear. We obtain

$$\partial^{[c_1}\partial_{[b_1|}\overline{E}_{a,|b_2]}{}^{c_2]} \equiv \partial^{[c_1}\partial_{[b_1|}\overline{G}_{a,|b_2]}{}^{c_2]} + \varepsilon_a{}^{d_1d_2d_3}\,\partial^{[c_1}\partial_{[b_1}\overline{\overline{G}}_{[d_1,d_2d_3],b_2]}{}^{c_2]} = 0\,, \tag{2.32}$$

which can also be written as

$$\tfrac{1}{3!}\,\varepsilon_{e_1e_2e_3}{}^{a}\partial^{[c_1}\partial_{[b_1|}\overline{G}_{a,|b_2]}{}^{c_2]} + \partial^{[c_1}\partial_{[b_1}\overline{\overline{G}}_{[e_1,e_2e_3],b_2]}{}^{c_2]} = 0\,. \tag{2.33}$$

The first term vanishes if we sum over e_1 and b_1 and also e_2 and b_2 . From this, we obtain

$$\partial^{[c_1}\partial_{[b_1}\overline{\overline{G}}^{[b_1,b_2e]}{}_{,b_2]}{}^{c_2]} = 0\,, \tag{2.34}$$

which is indeed the correct equation of motion for the first higher dual graviton in four spacetime dimensions [19].

Clearly, the duality equation (2.12) between the graviton and the first higher dual graviton will also lead to the same equations since it can be deduced from the above duality relations. However, it is instructive to treat this in the same way. Using the gauge transformation (2.24), we find that this duality relation is given by

$$E'^{\,a,}{}_{b_1b_2} \equiv \omega^{a,}{}_{b_1b_2} + 3\,\overline{\overline{G}}_{[b_1,b_2c],}{}^{ca} - 2\,\partial^a\partial_{[b_1}\xi_{b_2]} - 3\,e_2\,\partial^{(a}\partial_{[b_1}\Lambda_{b_2c],}{}^{c)} = 0\,. \tag{2.35}$$

The gauge parameter ξ^a is then eliminated by taking a derivative as follows:

$$\partial_{[a_1}E'_{a_2],}{}^{b_1b_2} \equiv \partial_{[a_1}\omega_{a_2],}{}^{b_1b_2} - 3\,\partial_{[a_1|}\overline{\overline{G}}^{[b_1,b_2c],}{}_{c|a_2]} - \tfrac{3}{2}\,e_2\,\partial_c\,\partial_{[a_1}\partial^{[b_1}\Lambda^{b_2c],}{}_{a_2]} = 0\,. \tag{2.36}$$

Contracting a_1 and b_1 allows us to discard the first term as it is the equation of motion for linearised gravity. We are left with the equation

$$3\,\partial^{[c|}\,\overline{\overline{G}}_{[c,bd],}{}^{d|a]} - \tfrac{3}{2}\,e_2\,\partial^d\partial^{[c}\partial_{[c}\Lambda_{bd],}{}^{a]} = 0\,. \tag{2.37}$$

Then, taking one more derivative, we can eliminate the last gauge parameter to arrive at

$$\partial^{[a_1}\partial^{[c}\,\overline{\overline{G}}_{[c,bd],}{}^{d]a_2]} = 0\,, \tag{2.38}$$

This is the correct equation of motion for the first higher dual graviton at the linearised level that we have also found in (2.34).

The equations of motion (2.28), (2.30) and (2.34) for the graviton, the dual graviton, and the first higher dual graviton, with their respective symmetry types $\ydiagram{2}$, $\ydiagram{2}$, and $\ydiagram{3,1}$, are tracelessness equations that may be written in the form

$$\mathrm{Tr}_{12}K_{a_1a_2,b_1b_2} = 0\,, \qquad \mathrm{Tr}_{12}\overline{K}_{a_1a_2,b_1b_2} = 0\,, \qquad \mathrm{Tr}^2_{12}\overline{\overline{K}}_{a_1a_2a_3,b_1b_2,c_1c_2} = 0\,, \tag{2.39}$$

where Tr_{ij} denotes a trace over columns i and j in a given Young diagram, and where we have introduced the curvature tensors for each field. They are given explicitly by

$$K_{a_1a_2,b_1b_2} \equiv \partial_{[a_1}\omega_{a_2],b_1b_2}\,, \qquad \overline{K}_{a_1a_2,b_1b_2} \equiv \partial_{[a_1}\overline{G}_{[b_1,b_2]a_2]}\,, \tag{2.40}$$

$$\overline{\overline{K}}_{a_1a_2a_3,b_1b_2,c_1c_2} \equiv \partial_{[c_1}\partial_{[b_1}\overline{\overline{G}}_{[a_1,a_2a_3],b_2]c_2]}\,. \tag{2.41}$$

As we have explained above, the duality relations only hold modulo gauge transformations, although the equations of motion derived from them hold exactly. One of the points of this paper is to obtain the extra fields that are required to have duality relations that also hold exactly. We will find evidence that these extra fields are contained in the second fundamental representation of A_1^{+++}, denoted ℓ_2. The content of this representation can be deduced by enlarging the A_1^{+++} algebra by attaching an additional node to the node labelled 2 in the A_1^{+++} Dynkin diagram, and then by taking only the generators of this enlarged algebra that have level one with respect to this new node. One may then deduce the commutation relations between generators in the adjoint and ℓ_2 representations of A_1^{+++} by using the fact that the level is preserved and that the Jacobi identities must

JHEP12(2022)152

hold. One can then add new fields corresponding to the ℓ_2 generators and deduce their A_1^{+++} transformations from their commutation relations. As the role of the new fields is to soak up the gauge transformations in the duality relations, the next step must be to propose their gauge transformations. This involves writing down the variation of the ℓ_2 fields in terms of the derivative, which belongs to ℓ_1, acting on the gauge parameters that also belong to the ℓ_1 representation. This transformation can be deduced using level matching and group theory. Given these transformations, one can then finally obtain new duality relations which hold as exact equations, at least in principle, and in detail at low levels. We leave this calculation to a future paper.

3 Higher dualisations of linearised gravity

In this section we give an action principle in four dimensions for the first higher dual graviton $A_{ab,cd}$ whose equations of motion and gauge transformations were obtained from the non-linear realisation of $A_1^{+++} \ltimes \ell_1$ in the previous section. We will only be concerned with free dynamics and we will build the action principle for the dual field $A_{ab,cd} \equiv A_{[ab],cd} \equiv A_{ab,(cd)}$ using the off-shell dualisation procedure proposed in [9]. In that paper, a field-theoretical interpretation was given for an infinite subset of E_{11} generators that transform in the GL(11)-irreducible representations whose Young tableaux are given in column notation as

$$\{\mathbb{Y}[8,1]\,,\mathbb{Y}[9,8,1]\,,\mathbb{Y}[9,9,8,1]\,,\dots\} \tag{3.1}$$

where the GL(11) Young diagram $\mathbb{Y}[8,1]$ of the dual graviton may have an unbounded number of columns of height nine glued to the left of it. It was argued in [9] that gauge fields transforming in these GL(11) representations enter higher and higher dual off-shell formulations of linearised gravity. This will now be made quantitative by working at the first few levels of dualisation with precise action principles.

In what follows, we first recall the basic ideas behind the parent action procedure to derive dual actions for linearised gravity in any dimension D, and then we will direct our attention to the four-dimensional case for which A_1^{+++} is the relevant Kac-Moody algebra.

3.1 From the graviton to the dual graviton

Off-shell dualisation of linearised gravity $h_a{}^b$ around D-dimensional Minkowski space-time was initiated in [1] and [12]. This was investigated further in [13] where the authors made contact with the Curtright action [10] and generalised this duality to higher-spin fields with spin $s > 2$. Although the analysis of [1] began with the fully non-linear Einstein-Hilbert action, it is only for its linearisation that one can make the dual graviton and all of its higher dual generalisations appear off-shell [9]. Following the original idea [1], consider the second order Einstein-Hilbert action based on the vielbein $e_\mu{}^a$:

$$S_{\mathrm{EH}}[e_\mu{}^a] \;=\; -\int \mathrm{d}^D x\, e\left[\Omega^{ab,c}(e)\,\Omega_{ab,c}(e) + 2\,\Omega^{ab,c}(e)\,\Omega_{ac,b}(e) - 4\,\Omega_{ab,}{}^{b}(e)\,\Omega^{ac,}{}_{c}(e)\right]\,, \tag{3.2}$$

where $e := \det(e_\mu{}^a)$ and $\Omega_{ab,}{}^c(e) := 2\, e_a{}^\mu\, e_b{}^\nu\, \partial_{[\mu} e_{\nu]}{}^c$. This form of the Einstein-Hilbert action can be recast into first-order form by introducing an auxiliary field $Y_{ab;c} = Y_{[ab];c}$ and then by considering the parent action [1]

$$S[Y_{ab;c}\,, e_\mu{}^a] \;=\; -2 \int \mathrm{d}^D x\; e \left[\Omega_{ab,c}(e)\, Y^{ab;c} - \tfrac{1}{2}\, Y_{ab;c}\, Y^{ac;b} + \tfrac{1}{2(D-2)}\, Y_{ab;}{}^b\, Y^{ac;}{}_c\right] . \tag{3.3}$$

Indeed, the field equation of $Y_{ab;c}$ can solved for $Y_{ab;c}$ in terms of $\Omega(e)$ which yields

$$Y_{ab;c}(e) \;=\; \Omega_{ab,c} - 2\, \Omega_{c[a,b]} + 4\, \eta_{c[a} \Omega_{b]d,}{}^d\, . \tag{3.4}$$

After inserting (3.4) into (3.3), one recovers the Einstein-Hilbert action (3.2). In fact, the action in (3.3) coincides with the standard first order action for gravity where the spin connection is an independent field, up to a field redefinition which replaces the spin connection by the $Y_{ab;c}$ field. The parent action in (3.3) is manifestly invariant under diffeomorphisms and local Lorentz transformations. In terms of the Hodge dual field

$$Y_{c_1 \ldots c_{D-2};}{}^d \;:=\; -\tfrac{1}{2}\, \varepsilon_{abc_1 \cdots c_{D-2}}\, Y^{ab;d}\, , \tag{3.5}$$

the parent action linearised around Minkowski spacetime, where $e_\mu{}^a = \delta_\mu{}^a + h_\mu{}^a$, reads

$$\begin{aligned} & S[Y_{a_1 \ldots a_{D-2};}{}^b\,,\, h_{ab}] \\ & = -\tfrac{2}{(D-2)!} \int \mathrm{d}^D x \Big[\varepsilon^{abc_1 \ldots c_{D-2}}\, Y_{c_1 \ldots c_{D-2};}{}^c\, \Omega_{ab,c}(h) + \tfrac{D-3}{2(D-2)}\, Y^{c_1 \ldots c_{D-2};b}\, Y_{c_1 \ldots c_{D-2};b} \\ & \quad - \tfrac{D-2}{2}\, Y^{c_1 \ldots c_{D-3}a;}{}_a\, Y_{c_1 \ldots c_{D-3}b;}{}^b + \tfrac{1}{2}\, Y^{c_1 \ldots c_{D-3}a;b}\, Y_{c_1 \ldots c_{D-3}b;a}\Big]\, , \end{aligned} \tag{3.6}$$

where $\Omega_{ab,c}(h) := 2\, \partial_{[a} h_{b]c}$ and the field h_{ab} has no symmetry on its two indices. The equation of motion for h_{ab} yields

$$\partial_{[a_1} Y_{a_2 \ldots a_{D-1}];b} \;=\; 0\, . \tag{3.7}$$

The Poincaré lemma implies that the dual field $Y_{a_1 \ldots\, a_{D-2};b}$ is the curl of a potential $C_{a_1 \ldots\, a_{D-3};b}$. This new field is completely antisymmetric in its first $D-3$ indices but it has no definite $\mathrm{GL}(D)$ symmetry otherwise:

$$Y_{a_1 \ldots a_{D-2};b} \;=\; \partial_{[a_1} C_{a_2 \ldots a_{D-2}];b}\, . \tag{3.8}$$

Inserting this back into the linearisation of (3.6) produces a consistent quadratic action $S[C]$ that describes linearised gravity by construction. Note that the field h_{ab} acted as a Lagrange multiplier for the constraint (3.7). It is not an auxiliary field like $Y_{a_1 \ldots a_{D-2};b}$ is, but the dual action obtained by substituting (3.8) inside the parent action (3.6) is classically equivalent to the original linearised Einstein-Hilbert action. The reader might want to see [29] for more comments on this issue.

Until now, the dual field $C_{a_1 \ldots a_{D-3};b}$ as defined in (3.8) does not transform in any irreducible $\mathrm{GL}(D)$ representation since $Y_{a_1 \ldots a_{D-2};b}$ does not have any irreducible $\mathrm{GL}(D)$ symmetry property. However, one may check [12, 13] that, after inserting (3.8) into (3.6),

JHEP12(2022)152

the resulting action $S[C]$ is invariant under a shift symmetry inherited from the local Lorentz symmetry

$$\delta_\Lambda C_{a_1\dots a_{D-3};b} \;=\; -\Lambda_{a_1\dots a_{D-3}b}\,, \tag{3.9}$$

with a completely antisymmetric $(D-2)$-form gauge parameter that is nothing but the Hodge dual of the local Lorentz parameter Λ_{ab}. In particular, in $D=11$ dimensions, the 9-form component of the field $C_{a_1\dots a_8;b}$ drops out from the action due to the above gauge symmetry. This gives rise to a dual action in terms of the other component of $C_{a_1\dots a_8;b}$ denoted by $A_{a_1\dots a_8,b}$ that we call the dual graviton [1, 12, 13]. In the antisymmetric convention for Young tableaux, the GL(11) irreducibility condition of the dual graviton is the over-antisymmetrisation identity

$$A_{[a_1\dots a_8,b]} \equiv 0\,. \tag{3.10}$$

To summarise, the dual graviton field $A_{a_1\dots a_{D-3},b}$ in D dimensions is antisymmetric in its first $D-3$ indices and it obeys the irreducibility constraint $A_{[a_1\dots a_{D-3},b]}\equiv 0$. The dual graviton is a GL(D)-irreducible tensor of type $\mathbb{Y}[D-3,1]$.

It is important to stress the fact that the dynamics of linearised gravity around Minkowski space-time, as given by the variational principle based on the original Fierz-Pauli action, can equivalently be described from the dual action principle $S[A_{a_1\dots a_{D-3},b}]$ given in [13]. The reason is that both the Fierz-Pauli action and the dual action appear upon elimination of different fields from the same parent action. Moreover, as explained in [13], the dual graviton in four dimensions is a symmetric field $A_{ab}=A_{(ab)}$ and the dual action $S[A_{ab}]$ reproduces the standard Fierz-Pauli action. In $D=4$, one concludes that "Fierz-Pauli is dual to Fierz-Pauli" [13].

In the next part, we review the dualisation procedure first explained in [9], which takes the dual action $S[A_{a_1\dots a_{D-3},b}]$ and produces a dual action featuring the first higher dual graviton $A_{a_1\dots a_{D-2},b_1\dots b_{D-3},c}$ as well as an extra field that cannot be eliminated from the action. In four dimensions, the first higher dual graviton $A_{a_1a_2,bc}$ corresponds to the A_1^{+++} generator $R^{a_1a_2,(bc)}$ at level 2. Therefore, this approach makes direct contact with the previous section where the non-linear realisation of $A_1^{+++}\ltimes \ell_1$ was reviewed. The extra fields that enter each higher dual action principle will then be shown to be closely correlated with the ℓ_2 representation of A_1^{+++}. Although they are not needed in order to write down self-duality equations, they are necessary for the off-shell formulation of various generations of higher dual graviton fields.

3.2 The first higher dual graviton in four dimensions

Action principle. As explained in [13], around Minkowski spacetime of dimension $D=4$, the dual graviton $A_{a_1\dots a_{D-3},b}$ is a symmetric rank-2 tensor $A_{ab}\equiv A_{(ab)}$ and the dual action is just the Fierz-Pauli action given as follows, up to boundary terms that we neglect:

$$S_{\rm FP}[A_{ab}] = \int \mathrm{d}^4x\,\Big[-\tfrac{1}{2}\,\partial_a A_{bc}\,\partial^a A^{bc} + \tfrac{1}{2}\,\partial_a A_b{}^b\,\partial^a A_c{}^c - \partial_a A^{ab}\,\partial_b A + \partial_a A^{ab}\,\partial^c A_{cb}\Big]\,. \tag{3.11}$$

We stress that the curl $\Omega_{ab,c}(A) := 2\,\partial_{[a}A_{b]c}$ is *not* featured in this formulation of the Fierz-Pauli action. Instead, it features the full gradient $G_{a;bc}(A) := \partial_a A_{bc}$ without any antisymmetrisation over indices. As proposed in [9], we define the following parent action $S[G_{a;bc}\,, D_{ab;}{}^{cd}]$:

$$S = \int \mathrm{d}^4x \Big[-\tfrac{1}{2}\,G_{a;bc}\,G^{a;bc} + \tfrac{1}{2}\,G_{a;c}{}^{c}\,G^{a;b}{}_{b} - G_{a;}{}^{ab}\,G_{b;c}{}^{c} + G_{a;}{}^{ab}\,G^{c;}{}_{cb} + G_{a;bc}\,\partial_d D^{da;bc} \Big] \tag{3.12}$$

featuring the two independent fields $G_{a;bc} = G_{a;(bc)}$ and $D_{ab;}{}^{cd} = D_{[ab];}{}^{cd} = D_{ab;}{}^{(cd)}$. The latter of these two fields is defined up to a gauge transformation

$$\delta_\Theta D_{ab;}{}^{cd} = \partial^e \Theta_{eab;}{}^{cd}\ , \quad \Theta_{eab;}{}^{cd} \equiv \Theta_{[eab];}{}^{cd} \equiv \Theta_{eab;}{}^{(cd)}\ , \tag{3.13}$$

which preserves the parent action. In fact, since the original Fierz-Pauli action (3.11) is invariant under the gauge transformation

$$\delta A_{ab} = 2\,\partial_{(a}\epsilon_{b)}\ , \tag{3.14}$$

it is easy to see that the parent action (3.12) is invariant under the combined transformations

$$\delta G_{a;bc} = 2\,\partial_a \partial_{(b}\epsilon_{c)}\ , \tag{3.15}$$

$$\delta D_{ab;}{}^{cd} = \partial^e \Theta_{eab;}{}^{cd} + 2\,\eta^{cd}\partial_{[a}\epsilon_{b]} + 4\,\delta_{[a}{}^{(c}\partial_{b]}\epsilon^{d)}\ . \tag{3.16}$$

On the one hand, one can vary the parent action (3.12) with respect to the GL(4)-reducible field $D^{da;bc}$ that acts as a Lagrange multiplier for the constraint $\partial_{[a}G_{d];bc} = 0$. This constraint is identically solved by

$$G_{a;bc} = G_{a;bc}(A) := \partial_a A_{bc}\ , \tag{3.17}$$

for some symmetric tensor A_{bc}. Substituting $G_{a;bc}(A)$ for $G_{a;bc}$ inside the parent action (3.12) reproduces the original Fierz-Pauli action (3.11).

On the other hand, in the parent action (3.12), the independent field $G_{a;bc}$ can be considered to be an auxiliary field. Its equation of motion

$$0 = \partial^e D_{ea;bc} - G_{a;bc} + \eta_{bc}\,G_{a;e}{}^{e} - \eta_{a(b}\,G_{c);e}{}^{e} - \eta_{bc}\,G_{e;a}{}^{e} + 2\,\eta_{a(b}\,G^{e;}{}_{c)e} \tag{3.18}$$

can be solved algebraically to express $G_{a;bc}$ in terms of $D_{ab;}{}^{cd}$ as follows:

$$\frac{\delta S[G,D]}{\delta G^{a;bc}} = 0 \quad \Longrightarrow \quad G_{a;bc} = \partial^e D_{ea;bc} + \tfrac{1}{2}\,\eta_{bc}\,\partial^d D_{ad;c}{}^{c} + \tfrac{2}{3}\,\partial^d D_{ed;}{}^{e}{}_{(b}\,\eta_{c)a}\ . \tag{3.19}$$

Upon substituting this expression for $G_{a;bc}$ into the parent action (3.12), we obtain the following alternative description of linearised gravity around four-dimensional Minkowski spacetime:

$$S[D_{ab;}{}^{cd}] = \int \mathrm{d}^4x\, \Big[\tfrac{1}{2}\,\partial^a D_{ab;}{}^{cd}\,\partial_e D^{eb;}{}_{cd} - \tfrac{1}{3}\,\partial^a D_{ea;}{}^{eb}\,\partial_c D^{dc;}{}_{db} + \tfrac{1}{4}\,\partial^a D_{ab;}{}^{c}{}_{c}\,\partial_e D^{be;}{}_{d}{}^{d} \Big]\ . \tag{3.20}$$

This action is invariant under the gauge transformation (3.16). We emphasise that (3.20) describes the same free graviton dynamics as the Fierz-Pauli action (3.11). The reason is that both action principles arise from the same parent action $S[G,D]$ when it is extremised with respect to one field or the other. Note that the spectrum of fields is in one-to-one correspondence with those that are obtained by taking the tensor product of a 2-form with a symmetric rank-2 tensor. This is depicted in terms of Young tableaux as follows:

$$D_{ab;}{}^{cd} \quad \sim \quad \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \quad \otimes \quad \begin{array}{|c|c|}\hline c & d \\ \hline \end{array} \tag{3.21}$$

Relation to A_1^{+++}. In what follows, we decompose the dual field $D_{ab;}{}^{cd}$ into GL(4)-irreducible components to allow for direct contact with the A_1^{+++} algebra. In particular, we will show how the first higher dual graviton $A_{ab,cd}$ that transforms in the GL(4)-irreducible representation of type $\mathbb{Y}[2,1,1]$ associated with the A_1^{+++} generator $R^{ab,(cd)}$ is contained in the GL(4)-reducible field $D_{ab;}{}^{cd}$ in (3.21). This will differ from the first section where we were concerned with duality relations modulo certain gauge transformations. In this section, we find an off-shell formulation of linearised gravity in terms of $A_{ab,cd}$ and an extra field $\widehat{Z}^{abc,d}$ which are both contained inside the $D_{ab;}{}^{cd}$ field given previously.

The GL(4)-irreducible decomposition of $D_{ab;}{}^{cd}$ reads

$$D_{ab;}{}^{cd} = X_{ab;}{}^{cd} + 4\,\delta_{[a}{}^{(c}\,Z_{b];}{}^{d)}\ , \qquad X_{ab;}{}^{cb} \equiv 0 \equiv Z_{a;}{}^{a}\ , \tag{3.22}$$

with inverse formulas

$$X_{ab;}{}^{cd} = D_{ab;}{}^{cd} + \delta_{[a}{}^{(c}D_{b]e;}{}^{d)e}\ , \qquad Z_{a;}{}^{b} = -\tfrac{1}{4}D_{ac;}{}^{bc}\ . \tag{3.23}$$

In terms these fields, the dual gravity action (3.20) becomes

$$\begin{aligned} S[X_{ab;}{}^{cd}, Z_{a;}{}^{c}] = \int \mathrm{d}^4x \, \Big[& \tfrac{1}{2}\,\partial_a X^{ab;}{}_{cd}\,\partial^e X_{eb;}{}^{cd} - \tfrac{1}{4}\,\partial^a X_{ab;}{}^{c}{}_{c}\,\partial_e X^{eb;}{}_{d}{}^{d} \\ & + 2\,\partial_a Z^{a;b}\partial_c Z_{b;}{}^{c} - \tfrac{10}{3}\,\partial_a\,Z^{a;b}\partial^c Z_{c;b} + \partial_c Z_{a;b}\,\partial^c Z^{a;b} \\ & + \,2\,\partial_b Z^{[a;b]}\,\partial^c X_{ac;}{}^{d}{}_{d} - 2\,\partial^c Z^{a;b}\,\partial^e X_{ae;bc}\Big]\ . \end{aligned} \tag{3.24}$$

Hodge dualising $X_{ab;}{}^{cd}$ and $Z_{a;}{}^{c}$ on their lower indices produces the GL(4)-irreducible fields

$$A^{ab,cd} := -\tfrac{1}{2}\,\varepsilon^{abij}\,X_{ij;}{}^{cd}\ , \qquad \widehat{Z}^{abc,d} := \varepsilon^{abce}\,Z_{e;}{}^{d}\ , \tag{3.25}$$

with inverse relations

$$X_{ab;}{}^{cd} = \tfrac{1}{2}\,\varepsilon_{abij}\,A^{ij,cd}\ , \qquad Z_{a;}{}^{e} = \tfrac{1}{6}\,\varepsilon_{abcd}\,\widehat{Z}^{bcd,e}\ . \tag{3.26}$$

These fields satisfy GL(4) irreducibility conditions in the antisymmetric convention for Young tableaux. That is to say, they satisfy the over-antisymmetrisation identities:

$$A^{[ab,c]d} \equiv 0\ , \qquad\qquad \widehat{Z}^{[abc,d]} \equiv 0\ , \tag{3.27}$$

where $A^{ab,cd} = A^{[ab],cd} = A^{ab,(cd)}$ and $\widehat{Z}^{abc,d} = \widehat{Z}^{[abc],d}$. The reader will recognise that the field $A^{ab,cd}$ possesses all the symmetries of the first higher dual graviton defined in the previous section. It corresponds to the generator $R^{ab,cd}$ of A_1^{+++}. We also see that the field $\widehat{Z}^{abc,d}$ is required for the action principle to exist. In terms of the two GL(4)-irreducible fields $A^{ab,cd}$ and $\widehat{Z}^{abc,d}$, the dual gravity action (3.20) now reads

$$
\begin{aligned}
S[A^{ab,cd},\widehat{Z}^{abc,d}] = \int \mathrm{d}^4x \Big[& -\tfrac{3}{4}\,\partial_e A_{ab,cd}\,\partial^{[e}A^{ab],cd} + \tfrac{3}{8}\,\partial_d A_{ab,c}{}^{c}\,\partial^{[d}A^{ab],e}{}_{e} \\
& - \tfrac{3}{2}\,\partial_d A_{ab,c}{}^{c}\,\partial^{[d}\widehat{Z}^{ab]e,}{}_{e} + \partial_e A_{ab,cd}\,\partial^{d}\widehat{Z}^{eab,c} \\
& + \partial_d \widehat{Z}_{abc,}{}^{c}\,\partial_e \widehat{Z}^{abd,e} - \tfrac{1}{3}\,\partial_e \widehat{Z}_{abc,d}\,\partial^{d}\widehat{Z}^{abc,e} - \tfrac{5}{3}\,\partial_e \widehat{Z}_{abc,d}\,\partial^{c}\widehat{Z}^{abe,d} + \tfrac{7}{18}\,\partial_e \widehat{Z}_{abc,d}\,\partial^{e}\widehat{Z}^{abc,d}\Big] \,.
\end{aligned}
\tag{3.28}
$$

This action is invariant under the gauge transformations

$$
\delta A^{ab,cd} = -4\partial^{[a}\lambda^{b]cd} + \partial^{[a|}\mu^{cd,|b]} + 2\partial^{(c}\mu^{d)[a,b]} - \varepsilon^{ij[a(c}\eta^{d)b]}\partial_i\epsilon_j - \tfrac{1}{2}\varepsilon^{ijab}\eta^{cd}\partial_i\epsilon_j \,, \tag{3.29}
$$

$$
\delta\widehat{Z}^{abc,d} = 3\,\partial^{[a|}\mu^{d|b,c]} + \tfrac{1}{4}\,\varepsilon^{abce}\,\partial^{d}\epsilon_e + \tfrac{3}{4}\,\varepsilon^{abce}\,\partial_e\epsilon^{d} - \tfrac{1}{4}\,\varepsilon^{abcd}\,\partial^{e}\epsilon_e \,, \tag{3.30}
$$

where the gauge parameters λ^{abc} and $\mu^{ab,c}$ are GL(4)-irreducible:

$$
\lambda^{abc} = \lambda^{(abc)} \quad \sim \quad \boxed{a\,|\,b\,|\,c}\,, \qquad \mu^{ab,c} = \mu^{(ab),c} \quad \sim \quad \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array}\,, \quad \mu^{(ab,c)} \equiv 0 \,. \tag{3.31}
$$

One may, of course, equivalently use the manifestly antisymmetric convention for Young tableaux in expressing the mixed-symmetric gauge parameter by taking

$$
m^{ab,c} := 2\,\mu^{c[a,b]} \;\sim\; \begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array} \;, \qquad m^{[ab,c]} \equiv 0 \qquad \Leftrightarrow \qquad \mu^{ab,c} = -\tfrac{2}{3}\,m^{c(a,b)} \,. \tag{3.32}
$$

In terms of this equivalent representation for the gauge parameter, one has

$$
\delta A^{ab,cd} = -4\partial^{[a}\lambda^{b]cd} - \tfrac{2}{3}\,\partial^{[a}m^{b](c,d)} + \partial^{(c|}m^{ab,|d)} - \varepsilon^{ij[a(c}\eta^{d)b]}\partial_i\epsilon_j - \tfrac{1}{2}\,\varepsilon^{ijab}\eta^{cd}\partial_i\epsilon_j \,, \tag{3.33}
$$

$$
\delta\widehat{Z}^{abc,d} = \tfrac{3}{2}\,\partial^{[a}m^{bc],d} + \tfrac{1}{4}\,\varepsilon^{abce}\,\partial^{d}\epsilon_e + \tfrac{3}{4}\,\varepsilon^{abce}\,\partial_e\epsilon^{d} - \tfrac{1}{4}\,\varepsilon^{abcd}\,\partial^{e}\epsilon_e \,. \tag{3.34}
$$

Notice that λ^{abc} and $m^{ab,c} \sim \mu^{ab,c}$ match the gauge parameters $\Lambda_{a_1a_2a_3}$ and $\Lambda_{a_1a_2,b}$ at level 2 in the ℓ_1 representation of A_1^{+++} given in (2.22). Up to trivial gauge parameter redefinitions, the transformation law of $A^{ab,cd}$ with respect to $m^{ab,c}$ and λ^{abc} fully agrees with (2.24). Note that off-shell dualisation is a different approach to the A_1^{+++} non-linear realisation presented in the previous section. The gauge transformations found here contain extra terms compared with (2.24) which are due to the extra field $\widehat{Z}^{abc,d}$ in (3.28). We will soon see that the ℓ_2 representation of A_1^{+++} is closely related to extra fields that appear during off-shell dualisation. Therefore, we expect to obtain (3.33) and (3.34) from the non-linear realisation by modifying it in a suitable way that incorporates the ℓ_2 representation.

The gauge parameters λ^{abc} and $\mu^{ab,c}$ arise from the decomposition

$$
\Theta_{abc;}{}^{de} = 2\,\varepsilon_{abci}\,(-\lambda^{dei} + \mu^{de,i}) \,, \tag{3.35}
$$

so that the $\Theta_{abc;}{}^{de}$ part of the gauge transformation in (3.16) reads

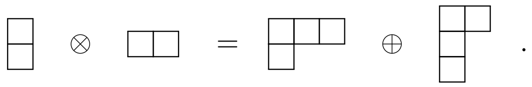
$$
\delta_\Theta D_{ab;}{}^{ef} = -2\,\varepsilon_{abcd}\,(\partial^c\lambda^{def} - \partial^c\mu^{ef,d}) \,. \tag{3.36}
$$

We stress that this dual action principle (3.28)–(3.30) describes equivalent dynamics to the Fierz-Pauli action principle. Namely, it propagates a single graviton in four-dimensional

JHEP12(2022)152

Minkowski spacetime. It is an alternative off-shell description of linearised gravity and we will further analyse this action principle in section 4. In that section, in order to make contact with the Labastida formulation for a gauge field with the symmetries of the first higher-dual graviton, we need to change convention for Young tableau. We refer to appendix A for this change of convention.

Note that the field content of the theory $\{A^{ab,cd}\,,\widehat{Z}^{abc,d}\}$ is in one-to-one correspondence with the set of Young diagrams obtained in the tensor product


(3.37)

This depicts the set of GL(4)-irreducible tensors that are contained in the reducible tensor

$$\widetilde{D}^{ab;cd} := \tfrac{1}{2}\,\varepsilon^{abij}\,D_{ij;}{}^{cd}\,. \tag{3.38}$$

In section 4, we will build two gauge invariant curvature tensors that do not vanish on-shell. Anticipating this (more technical) result, the curvature for the gauge field $A_{ab,cd}$ starts like

$$\widehat{G}_{a_1a_2a_3,b_2b_2,c_1c_2} = \partial_{a_1}\partial_{b_1}\partial_{c_1}A_{a_2a_3,(b_2c_2)} + \dots\,, \tag{3.39}$$

where "…" is used to denote terms that involve the field $\widehat{Z}_{abc,d}$ and where it is understood that indices with the same letters are antisymmetrised. For example, $\partial_{a_1}V_{a_2} \equiv \frac{1}{2}\,(\partial_{a_1}V_{a_2} - \partial_{a_2}V_{a_1})$. Then, in that same section, we show that the field equations for $A_{ab,cd}$ are equivalent to

$$\widehat{G}^{\,abc,}{}_{ab,de} = 0\,, \qquad\qquad \widehat{G}_{a_1a_2a_3,bc,}{}^{b}{}_{d} = 0\,. \tag{3.40}$$

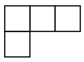
As demonstrated in [19, 20], this form of field equation is precisely what one should have for a mixed-symmetric gauge field $A_{ab,cd}$ that propagates non-trivially in four dimensional Minkowski spacetime. It is of higher-derivative type for a gauge field with more than two columns in its Young tableau representation, but a partial gauge-fixing procedure was found in [20] that brings such higher-derivative field equations down to the two-derivative equations (for bosonic fields) postulated in [22, 23].

The first field equation in (3.40) is precisely of the form we have seen before in (2.38), except that now, since we have an action principle for the first higher dual graviton, all the gauge invariant quantities involve both $A_{ab,cd}$ and $\widehat{Z}_{abc,d}$ which duly reflects the fact that the gauge transformations (3.33) and (3.34) are both expressed in terms of the parameters $m^{ab,c}$ and ϵ_a. The gauge transformations are entangled as is typical when performing higher off-shell dualisations [30]. Now, not only do we have the first higher dual graviton field $A_{ab,cd}$, but also the extra field $\widehat{Z}_{abc,d}$ that is required for our dual action principle to exist. Together, this pair of fields describes a single propagating graviton. The extra field $\widehat{Z}_{abc,d}$ is not in the adjoint representation of A_1^{+++} but we will later see that it belongs to the ℓ_2 representation of A_1^{+++} at level 1 in the decomposition of A_1^{+++} with respect to its GL(4) subalgebra (see table 2).

JHEP12(2022)152

3.3 Field theoretical analysis at higher levels

After having discussed, in great detail, off-shell dualisation from the dual graviton $A_{ab} = A_{(ab)}$ to the first higher dual graviton $A_{ab,cd}$, we may now proceed to the next step in the off-shell dualisation procedure. Recall that the dualisation procedure at level one transformed our set of fields from a symmetric tensor A_{ab} to the GL(4)-reducible field $D_{ab;}{}^{cd}$ whose Hodge dual (3.38) in four dimensions, $\widetilde{D}^{ab;cd}$, contains the GL(4)-irreducible fields $A^{ab,cd}$ and $\widehat{Z}^{abc,d}$ with symmetry types $\mathbb{Y}[2,1,1]$ and $\mathbb{Y}[3,1]$, respectively, with Young tableaux given in (3.37).

In order to dualise the field $A^{ab,cd} \sim$  , we build a parent action $S[G^{e;ab,cd}, \widehat{Z}^{abc,d}]$ from $S[A^{ab,cd}, \widehat{Z}^{abc,d}]$ by treating $G^{e;ab,cd}$ as an independent field that will be equal to the gradient $G^{e;ab,cd}(A) := \partial^e A^{ab,cd}$ after varying the parent action with respect to the Lagrange multiplier field $D_{ab;}{}^{cd,ef}$ that implements the constraint $\partial^{[e} G^{f];ab,cd} = 0$. Therefore, the parent action at the next level of dualisation is given schematically by

$$S[D_{ab;}{}^{cd,ef}, G^{e;ab,cd}, \widehat{Z}^{abc,d}] := S[G^{e;ab,cd}, \widehat{Z}^{abc,d}] + \int \mathrm{d}^4x \; G^{b;}{}_{cd,ef}\, \partial^a D_{ab;}{}^{cd,ef} \,. \tag{3.41}$$

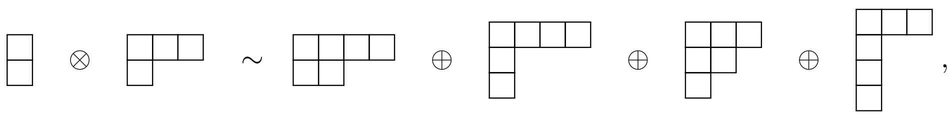
Both $G^{e;ab,cd}$ and $D_{ab;}{}^{cd,ef}$ have the GL(4)-irreducible symmetries of $A^{cd,ef}$ in their final four indices, and $D_{ab;}{}^{cd,ef} = D_{[ab];}{}^{cd,ef}$.

As with (3.13), the Lagrange multiplier field $D_{ab;}{}^{cd,ef}$ is defined up to

$$\delta_\Theta D_{ab;}{}^{cd,ef} = \partial^i \Theta_{iab;}{}^{cd,ef} \,, \qquad \Theta_{iab;}{}^{cd,ef} = \Theta_{[iab];}{}^{cd,ef} \,, \tag{3.42}$$

where $\Theta_{iab;}{}^{cd,ef}$ also shares the GL(4)-irreducible symmetries of $A^{cd,ef}$ in its final four indices.

The equation of motion for $G^{e;ab,cd}$ can be solved algebraically for $G^{e;ab,cd}$ in terms of $\widehat{Z}^{abc,d}$ and $D_{ab;}{}^{cd,ef}$. Then, as before, this expression may be substituted back into the parent action $S[D_{ab;}{}^{cd,ef}, G^{e;ab,cd}, \widehat{Z}^{abc,d}]$ to produce a new dual action $S[D_{ab;}{}^{cd,ef}, \widehat{Z}^{abc,d}]$ that we will not write explicitly here. The GL(4)-irreducible field content of the new field $D_{ab;}{}^{cd,ef}$ can be read off from its Hodge dual $\widetilde{D}^{ab;cd,ef} = \frac{1}{2}\varepsilon^{abij} D_{ij;}{}^{cd,ef}$ and the decomposition of its Young diagram:



(3.43)

$$\Leftrightarrow \quad \mathbb{Y}[2] \otimes \mathbb{Y}[2,1,1] \sim \mathbb{Y}[2,2,1,1] \oplus \mathbb{Y}[3,1,1,1] \oplus \mathbb{Y}[3,2,1] \oplus \mathbb{Y}[4,1,1] \,, \tag{3.44}$$

$$\Leftrightarrow \quad \mathbb{Y}(1,1) \otimes \mathbb{Y}(3,1) \sim \mathbb{Y}(4,2) \oplus \mathbb{Y}(4,1,1) \oplus \mathbb{Y}(3,2,1) \oplus \mathbb{Y}(3,1,1,1) \,, \tag{3.45}$$

$$\Leftrightarrow \quad D \sim A \qquad \hat{Y} \qquad \hat{Z} \qquad \widehat{W} \,. \tag{3.46}$$

This demonstrates how to label Young tableaux either by the heights of their columns as in (3.44), or by the lengths of their rows as in (3.45).

Before explicitly performing this decomposition, we switch the convention for Young tableaux to that where the final four indices of $D_{ab;}{}^{cd,ef}$ in the antisymmetric convention will be traded for $D_{ab;}{}^{cde,f} := -\frac{3}{2} D_{ab;}{}^{c(d,ef)}$ in the manifestly symmetric convention.[1] with the over-symmetrisation identity $D_{ab;}{}^{(cde,f)} \equiv 0$. This decomposition into GL(4)-irreducible components becomes

$$D_{a_1 a_2;}{}^{c_1 c_2 c_3,d} = X_{a_1 a_2;}{}^{c_1 c_2 c_3,d} + \delta_{[a_1}{}^{\langle c_1} Y_{a_2];}{}^{c_2 c_3 d\rangle} + \delta_{[a_1}{}^{\langle c_1} Z_{a_2];}{}^{c_2 c_3,d\rangle} + \delta_{[a_1}{}^{\langle c_1} \delta_{a_2]}{}^{c_2} W^{c_3 d\rangle} \,, \tag{3.47}$$

where $\langle \cdots \rangle$ denotes projection onto irreducible components. Indices $a_1 a_2$ are antisymmetric and indices $c_1 c_2 c_3$ are symmetric. The GL(4) irreducibility conditions are

$$X_{a_1 a_2;}{}^{(c_1 c_2 c_3,d)} = 0 \,, \qquad Z_{a;}{}^{(c_1 c_2,d)} = 0 \,, \tag{3.48}$$

together with the tracelessness constraints

$$0 \equiv X_{a_1 b;}{}^{c_1 c_2 b,d} \equiv X_{a_1 b;}{}^{c_1 c_2 c_3,b} \equiv Y_{b;}{}^{c_1 c_2 b} \equiv Z_{b;}{}^{c_1 b,d} \equiv Z_{b;}{}^{c_1 c_2,b} \,. \tag{3.49}$$

Projecting onto the symmetry of the final four indices, we find

$$\begin{aligned} D_{a_1 a_2;}{}^{c_1 c_2 c_3,d} = {}& X_{a_1 a_2;}{}^{c_1 c_2 c_3,d} + \delta_{[a_1}{}^{(c_1} Y_{a_2];}{}^{c_2 c_3)d} - \delta_{[a_1}{}^{d} Y_{a_2];}{}^{c_1 c_2 c_3} \\ & + \delta_{[a_1}{}^{(c_1} Z_{a_2];}{}^{c_2 c_3),d} + \delta_{[a_1}{}^{d} \delta_{a_2]}{}^{(c_1} W^{c_2 c_3)} \,, \end{aligned} \tag{3.50}$$

with inverse formulas

$$\begin{aligned} X_{a_1 a_2;}{}^{c_1 c_2 c_3,d} = {}& D_{a_1 a_2;}{}^{c_1 c_2 c_3,d} + \delta_{[a_1}{}^{d} D_{a_2]i;}{}^{c_1 c_2 c_3,i} - \tfrac{3}{5} \delta_{[a_1}{}^{(c_1} D_{a_2]i;}{}^{c_2 c_3)d,i} \\ & + \tfrac{6}{5} \delta_{[a_1}{}^{(c_1} D_{a_2]i;}{}^{c_2 c_3)i,d} - \tfrac{3}{5} \delta_{[a_1}{}^{d} \delta_{a_2]}{}^{(c_1} D_{ij;}{}^{c_2 c_3)i,j} \,, \end{aligned} \tag{3.51}$$

$$Y_{a;}{}^{c_1 c_2 c_3} = D_{ai;}{}^{c_1 c_2 c_3,i} - \tfrac{1}{2} \delta_a{}^{(c_1} D_{ij;}{}^{c_2 c_3)i,j} \,, \tag{3.52}$$

$$Z_{a;}{}^{c_1 c_2,d} = -\tfrac{6}{5} D_{ai;}{}^{c_1 c_2 i,d} - \tfrac{2}{5} D_{ai;}{}^{c_1 c_2 d,i} + \tfrac{4}{15} \delta_a{}^{(c_i} D_{ij;}{}^{c_2)di,j} - \tfrac{4}{15} \delta_a{}^{d} D_{ai;}{}^{c_1 c_2 i,j} \,, \tag{3.53}$$

$$W^{c_1 c_2} = -\tfrac{1}{3} D_{ij;}{}^{c_1 c_2 i,j} \,. \tag{3.54}$$

Previously, in order to dualise A_{ab} off-shell, we decomposed $D_{ab;}{}^{cd}$ into traceless components $\{X, Z\}$, whose Hodge duals are the irreducible components of the Hodge dual $\widetilde{D}^{ij;cd}$ of $D_{ab;}{}^{cd}$. In order to make contact with the E-theory literature expressed using fields and generators in the antisymmetric convention, we will do something similar to $\{X, Y, Z, W\}$. Hodge dualising all of them on their first blocks of indices creates GL(4)-irreducible fields in the symmetric convention, which may then be written in the antisymmetric convention with fields labelled $\{A, \widehat{Y}, \widehat{Z}, \widehat{W}\}$. The full calculation is given in appendix A, from which we find

$$X_{a_1 a_2;}{}^{c_1 c_2 c_3,d} := -\tfrac{6}{5}\, \varepsilon_{a_1 a_2 b_1 b_2} A^{(b_1|(b_2,d)|c_1,c_2,c_3)} \,, \tag{3.55}$$

$$Y_{a;}{}^{c_1 c_2 c_3} := \varepsilon_{a_1 b_1 b_2 b_3} \widehat{Y}^{b_2 b_3 (b_1,c_1,c_2,c_3)} \,, \tag{3.56}$$

$$Z_{a;}{}^{c_1 c_2,d} := \tfrac{8}{5}\, \varepsilon_{a b_1 b_2 b_3} \widehat{Z}^{(b_1|b_3(b_2,d)|c_1,c_2)} \,, \tag{3.57}$$

$$W^{c_1 c_2} := \varepsilon_{b_1 b_2 b_3 b_4} \widehat{W}^{b_4 b_3 b_2 (b_1,c_1,c_2)} \,. \tag{3.58}$$

[1]The reason is purely technical: it comes from the fact that we use the Mathematica package xTras [31] of the suite of Mathematica packages xAct that is able to implement tracelessness constraints more easily than mixed Young tableaux irreducibility constraints. The manifestly symmetric convention for Young tableaux is explained in appendix A.

with inverse relations

$$
\begin{aligned}
A^{a_1a_2,b_1b_2,c_1,c_2} &:= -\tfrac{1}{10}\,\varepsilon^{ija_1a_2}X_{ij;}{}^{c_1c_2[b_1,b_2]} + \tfrac{1}{10}\,\varepsilon^{ij[a_1[b_1|}X_{ij;}{}^{c_1c_2|b_2],a_2]} \\
&\quad -\tfrac{1}{5}\,\varepsilon^{ij[a_1(c_1}X_{ij;}{}^{c_2)a_2][b_1,b_2]} + (a_1a_2 \leftrightarrow b_1b_2)\ , && (3.59)\\
\widehat{Y}^{a_1a_2a_3,c_1,c_2,c_3} &:= -\tfrac{1}{6}\,\varepsilon^{ia_1a_2a_3}Y_{i;}{}^{c_1c_2c_3} - \tfrac{1}{2}\,\varepsilon^{i[a_1a_2(c_1}Y_{i;}{}^{c_2c_3)a_3]}\ , && (3.60)\\
\widehat{Z}^{a_1a_2a_3,b_1b_2,c} &:= -\tfrac{1}{15}\,\varepsilon^{ia_1a_2a_3}Z_{i;}{}^{c[b_1,b_2]} + \tfrac{1}{15}\,\varepsilon^{i[a_1a_2|[b_1}Z_{i;}{}^{b_2]c,|a_3]} - \tfrac{1}{15}\,\varepsilon^{i[b_1|[a_1a_2}Z_{i;}{}^{a_3]c,|b_2]} \\
&\quad -\tfrac{1}{15}\,\varepsilon^{ic[a_1a_2}Z_{i;}{}^{a_3][b_1,b_2]} - \tfrac{1}{15}\,\varepsilon^{ib_1b_2[a_1|}Z_{i;}{}^{c|a_2,a_3]} - \tfrac{1}{15}\,\varepsilon^{ic[a_1[b_1}Z_{i;}{}^{b_2]a_2,a_3]}\ , \\
& && (3.61)\\
\widehat{W}^{a_1a_2a_3a_4,c_1,c_2} &:= -\tfrac{1}{18}\,\varepsilon^{a_1a_2a_3a_4}W^{c_1c_2} - \tfrac{1}{9}\,\varepsilon^{[a_1a_2a_3(c_1}W^{c_2)a_4]}\ . && (3.62)
\end{aligned}
$$

At the second level of higher dualisation, decomposing $D_{a_1a_2;}{}^{c_1c_2c_3,d}$ recasts the action as

$$
S[D_{ab;}{}^{cde,f}, \widehat{Z}^{abc,d}] = S[X_{a_1a_2;}{}^{c_1c_2c_3,d}, Y_{a;}{}^{c_1c_2c_3}, Z_{a;}{}^{c_1c_2,d}, W^{c_1c_2}, \widehat{Z}^{abc,d}]\ , \tag{3.63}
$$

although (3.55)–(3.58) allows us to express this action as

$$
S[A^{a_1a_2,b_1b_2,c_1,c_2}, \widehat{Y}^{a_1a_2a_3,c_1,c_2,c_3}, \widehat{Z}^{a_1a_2a_3,b_1b_2,c}, \widehat{W}^{a_1a_2a_3a_4,c_1,c_2}, \widehat{Z}^{a_1a_2a_3,c}]\ . \tag{3.64}
$$

Off-shell dualisation from the dual graviton to the first higher dual graviton is given by

$$
\ydiagram{2} \quad \xrightarrow{\mathcal{D}_A} \quad 1\times \ydiagram{3,1} \quad \oplus \quad 1\times \ydiagram{2,1,1} \tag{3.65}
$$

where $\mathcal{D}_A$ denotes one round of off-shell dualisation applied only to the dual graviton at the previous level. At the next level, dualising only the first higher dual graviton $A^{a_1a_2,bc}$ produces a new action $\mathcal{D}_A^2(S[A_{ab}]) = \mathcal{D}_A(S[A^{ab,cd}, \widehat{Z}^{abc,d}])$ with the following set of fields

$$
\ydiagram{2} \quad \xrightarrow{\mathcal{D}_A^2} \quad 1\times \ydiagram{4,2} \oplus 1\times \ydiagram{4,1,1} \oplus 1\times \ydiagram{3,2,1} \oplus 1\times \ydiagram{3,1,1,1} \oplus 1\times \ydiagram{2,1,1} \tag{3.66}
$$

where we can see that $\widehat{Z}^{abc,d} \sim \mathbb{Y}[3,1]$ has been carried through to the new dual action in (3.64) with the same gauge symmetries as it had in $S[A^{ab,cd}, \widehat{Z}^{abc,d}] = \mathcal{D}_A(S[A_{ab}])$.

Taking $\mathcal{D}_A^2(S[A_{ab}])$ and dualising only the second higher dual graviton $A^{a_1a_2,b_1b_2,c,d}$ gives us

$$
\begin{aligned}
\ydiagram{2} \quad \xrightarrow{\mathcal{D}_A^3} \quad & 1\times \ydiagram{5,3} \oplus 1\times \ydiagram{5,2,1} \oplus 1\times \ydiagram{4,3,1} \oplus 1\times \ydiagram{4,2,1,1} \\
& \oplus 1\times \ydiagram{4,1,1} \oplus 1\times \ydiagram{3,2,1} \oplus 1\times \ydiagram{3,1,1,1} \oplus 1\times \ydiagram{2,1,1}
\end{aligned} \tag{3.67}
$$

The pattern is starting to become clear now. Label groups of k symmetric and k anti-symmetric indices by $a(k)$ and $a[k]$, respectively. For example, $A^{a_1a_2,b_1b_2} \equiv A^{a[2],b(2)}$ and

JHEP12(2022)152

$\widehat{Z}^{a_1a_2a_3,b} \equiv \widehat{Z}^{a[3],b}$. After dualising the $(n-1)^{\text{th}}$ higher dual graviton $A^{a^1[2],a^2[2],\,\ldots\,,a^{n-1}[2],c(2)}$, the set of independent fields will contain the n^{th} higher dual graviton

$$A^{(n)}_{[2,\ldots,2,1,1]} \equiv A^{(n)} := A^{a^1[2],a^2[2],\,\ldots\,,a^{n-1}[2],a^n[2],c(2)} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|}\hline a^1_1 & a^2_1 & \cdots & a^n_1 & c_1 & c_2 \\ \hline a^1_2 & a^2_2 & \cdots & a^n_2 \\ \cline{1-4}\end{array} \tag{3.68}$$

which is a GL(4)-irreducible field of type $\mathbb{Y}[2,\ldots,2,1,1] = \mathbb{Y}(n+2,2)$. The extra fields that are produced belong to the following families at the n^{th} level of higher dualisation:

$$\widehat{Y}^{(n)}_{[3,2,\ldots,2,1,1,1]} \equiv \widehat{Y}^{(n)} := \widehat{Y}^{a[3],b^1[2],\,\ldots\,,b^{n-2}[2],c(3)} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|c|}\hline a & b & \cdots & b & c & c & c \\ \hline a & b & \cdots & b \\ \cline{1-4} a \\ \cline{1-1}\end{array} \tag{3.69}$$

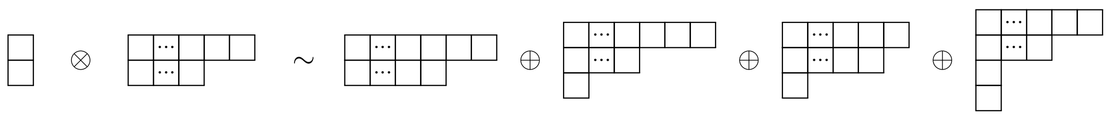
$$\widehat{Z}^{(n)}_{[3,2,\ldots,2,1]} \equiv \widehat{Z}^{(n)} := \widehat{Z}^{a[3],b^1[2],\,\ldots\,,b^{n-1}[2],c} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|}\hline a & b & \cdots & b & b & c \\ \hline a & b & \cdots & b & b \\ \cline{1-5} a \\ \cline{1-1}\end{array} \tag{3.70}$$

$$\widehat{W}^{(n)}_{[4,2,\ldots,2,1,1]} \equiv \widehat{W}^{(n)} := \widehat{W}^{a[4],b^1[2],\,\ldots\,,b^{n-2}[2],c(2)} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|}\hline a & b & \cdots & b & c & c \\ \hline a & b & \cdots & b \\ \cline{1-4} a \\ \cline{1-1} a \\ \cline{1-1}\end{array} \tag{3.71}$$

As with (A.19)–(A.22) in appendix A, these irreducible fields arise, respectively, from fields

$$\phi^{a(n+2),b(n)}\,, \qquad \psi_Y^{a(n+2),b(n-1),c}\,, \qquad \psi_Z^{a(n+1),b(n),c}\,, \qquad \psi_W^{a(n+1),b(n-1),c,d}\,, \tag{3.72}$$

in the symmetric convention. They themselves arise from the traceless components of the dual field $D_{a_1a_2;}{}^{c(n+2),d(n)}$ that must be introduced when moving from the $(n-1)^{\text{th}}$ to the n^{th} level of higher dualisation. This is a result of the Young diagram decomposition



(3.73)

which generalises (3.43). Ultimately, at the n^{th} level of higher dualisation, the action will be given in terms of the following set of independent fields:

$$\left\{A^{(n)},\widehat{Y}^{(n)},\widehat{Z}^{(n)},\widehat{W}^{(n)}\right\} \cup \left\{\widehat{Y}^{(n-1)},\widehat{Z}^{(n-1)},\widehat{W}^{(n-1)}\right\} \cup \cdots \cup \left\{\widehat{Y}^{(2)},\widehat{Z}^{(2)},\widehat{W}^{(2)}\right\} \cup \left\{\widehat{Z}^{(1)}\right\} \tag{3.74}$$

In parallel with the $A^{(n)}$ family of dual graviton fields, the $\widehat{Z}^{(n)}$ family of extra fields starts to appear at the first level of higher dualisation when $\widehat{Z}^{abc,d}$ enters the action, while the $\widehat{Y}^{(n)}$ and $\widehat{W}^{(n)}$ families both enter the action at the second level. With index structure explicit, we see that there is only $A_{ab} \sim A^{(0)}_{[1,1]}$ at level zero, i.e. at the level of the usual dual graviton.

JHEP12(2022)152

Dualisation at low levels. Up to this point, we have only dualised the $A^{(n)}$ family of fields and the extra fields have been completely untouched so that they are carried forward into every new dual action. This dualisation scheme may be extended by dualising some or all of the extra fields that we encounter at each stage. In this case, the second level of higher dualisation for the first higher dual graviton $A^{ab,cd}$ and the extra field $\widehat{Z}^{abc,d}$ is summarised as

$$\left\{A^{(1)}_{[2,1,1]}\right\} \xrightarrow{\mathcal{D}} \left\{A^{(2)}_{[2,2,1,1]}, \widehat{Y}^{(2)}_{[3,1,1,1]}, \widehat{Z}^{(2)}_{[3,2,1]}, \widehat{W}^{(2)}_{[4,1,1]}\right\} , \tag{3.75}$$

$$\left\{\widehat{Z}^{(1)}_{[3,1]}\right\} \xrightarrow{\mathcal{D}} \left\{\widehat{Z}^{(2)}_{[3,2,1]}, \widehat{W}^{(2)}_{[4,1,1]}, \widehat{P}^{(2)}_{[4,2]}, \widehat{Q}^{(2)}_{[3,3]}\right\} , \tag{3.76}$$

or equivalently as

$$\left\{A^{(0)}\right\} \xrightarrow{\mathcal{D}^2} \left\{1 \times A^{(2)}, 1 \times \widehat{Y}^{(2)}, 2 \times \widehat{Z}^{(2)}, 2 \times \widehat{W}^{(2)}, 1 \times \widehat{P}^{(2)}, 1 \times \widehat{Q}^{(2)}\right\} , \tag{3.77}$$

where $\mathcal{D}$ denotes one round of off-shell dualisation applied to *every* field at the previous level, not just the $A^{(n)}$ field. Moreover, $\widehat{P}^{(n)}$ and $\widehat{Q}^{(n)}$ denote two new families of fields with indices grouped as $\mathbb{Y}[4,2,\ldots,2]$ and $\mathbb{Y}[3,3,2,\ldots,2]$, respectively, which only appear after $\widehat{Z}^{abc,d}$ has been dualised. To further illustrate the ever growing number of families of fields, the third level of higher dualisation can be summarised as

$$\left\{A^{(2)}_{[2,2,1,1]}\right\} \xrightarrow{\mathcal{D}} \left\{A^{(3)}_{[2,2,2,1,1]}, \widehat{Y}^{(3)}_{[3,2,1,1,1]}, \widehat{Z}^{(3)}_{[3,2,2,1]}, \widehat{W}^{(3)}_{[4,2,1,1]}\right\} , \tag{3.78}$$

$$\left\{\widehat{Y}^{(2)}_{[3,1,1,1]}\right\} \xrightarrow{\mathcal{D}} \left\{\widehat{Y}^{(3)}_{[3,2,1,1,1]}, \widehat{W}^{(3)}_{[4,2,1,1]}, \widehat{R}^{(3)}_{[4,1,1,1,1]}, \widehat{S}^{(3)}_{[3,3,1,1]}\right\} , \tag{3.79}$$

$$\left\{\widehat{Z}^{(2)}_{[3,2,1]}\right\} \xrightarrow{\mathcal{D}} \left\{\widehat{Z}^{(3)}_{[3,2,2,1]}, \widehat{W}^{(3)}_{[4,2,1,1]}, \widehat{P}^{(3)}_{[4,2,2]}, \widehat{Q}^{(3)}_{[3,3,2]}, \widehat{S}^{(3)}_{[3,3,1,1]}, \widehat{T}^{(3)}_{[4,3,1]}\right\} , \tag{3.80}$$

$$\left\{\widehat{W}^{(2)}_{[4,1,1]}\right\} \xrightarrow{\mathcal{D}} \left\{\widehat{W}^{(3)}_{[4,2,1,1]}, \widehat{T}^{(3)}_{[4,3,1]}\right\} , \tag{3.81}$$

$$\left\{\widehat{P}^{(2)}_{[4,2]}\right\} \xrightarrow{\mathcal{D}} \left\{\widehat{P}^{(3)}_{[4,2,2]}, \widehat{T}^{(3)}_{[4,3,1]}, \widehat{O}^{(3)}_{[4,4]}\right\} , \tag{3.82}$$

$$\left\{\widehat{Q}^{(2)}_{[3,3]}\right\} \xrightarrow{\mathcal{D}} \left\{\widehat{Q}^{(3)}_{[3,3,2]}, \widehat{T}^{(3)}_{[4,3,1]}\right\} , \tag{3.83}$$

or equivalently as

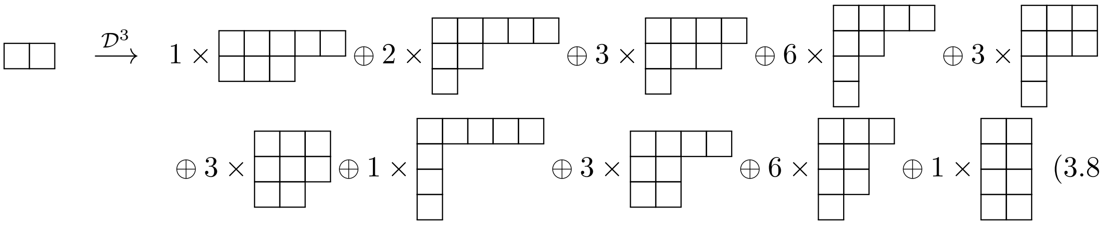
$$\begin{aligned}\left\{A^{(0)}\right\} \xrightarrow{\mathcal{D}^3} \Big\{&1 \times A^{(3)}, 2 \times \widehat{Y}^{(3)}, 3 \times \widehat{Z}^{(3)}, 6 \times \widehat{W}^{(3)}, 3 \times \widehat{P}^{(3)}, \\ &3 \times \widehat{Q}^{(3)}, 1 \times \widehat{R}^{(3)}, 3 \times \widehat{S}^{(3)}, 6 \times \widehat{T}^{(3)}, 1 \times \widehat{O}^{(3)}\Big\} .\end{aligned} \tag{3.84}$$

Let's take inventory. At levels one and two, dualising every field at every level, we have

$$\ydiagram{2} \quad \xrightarrow{\mathcal{D}} \quad 1 \times \ydiagram{3,1} \quad \oplus \quad 1 \times \ydiagram{2,1,1} \tag{3.85}$$

$$\ydiagram{2} \quad \xrightarrow{\mathcal{D}^2} \quad 1 \times \ydiagram{4,2} \oplus 1 \times \ydiagram{4,1,1} \oplus 2 \times \ydiagram{3,2,1} \oplus 2 \times \ydiagram{3,1,1,1} \oplus 1 \times \ydiagram{2,2,1,1} \oplus 1 \times \ydiagram{2,2,2} \tag{3.86}$$

whereas the third level is visualised as



(3.87)

The third higher dual graviton $A^{(3)} \equiv A^{a_1 a_2, b_1 b_2, c_1 c_2, (de)}$ dualises off-shell to produce

$$\mathbb{Y}[2,2,2,1,1] \longrightarrow \mathbb{Y}[2,2,2,2,1,1] \oplus \mathbb{Y}[3,2,2,2,1] \oplus \mathbb{Y}[3,2,2,1,1,1] \oplus \mathbb{Y}[4,2,2,1,1]$$

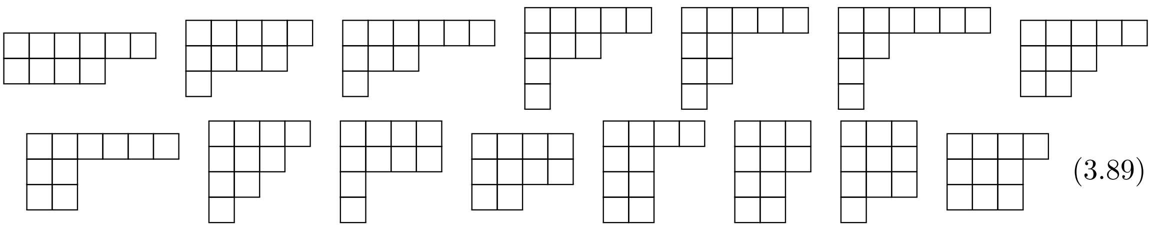
which consists of the fields $A^{(4)}$, $\widehat{Y}^{(4)}$, $\widehat{Z}^{(4)}$ and $\widehat{W}^{(4)}$. In addition to this, the extra fields at the third level of higher dualisation can also be dualised off-shell, and they produce the following fields at the fourth level of higher dualisation:

$$\begin{aligned}
\mathbb{Y}[3,2,1,1,1] &\longrightarrow \mathbb{Y}[4,3,1,1,1] \oplus \mathbb{Y}[4,2,2,1,1] \oplus \mathbb{Y}[4,2,1,1,1,1] \oplus \mathbb{Y}[3,3,2,1,1] \\
&\quad \oplus \mathbb{Y}[3,3,1,1,1,1] \oplus \mathbb{Y}[3,2,2,1,1,1] \\
\mathbb{Y}[3,2,2,1] &\longrightarrow \mathbb{Y}[4,3,2,1] \oplus \mathbb{Y}[4,2,2,2] \oplus \mathbb{Y}[4,2,2,1,1] \oplus \mathbb{Y}[3,3,2,2] \\
&\quad \oplus \mathbb{Y}[3,3,2,1,1] \oplus \mathbb{Y}[3,2,2,2,1] \\
\mathbb{Y}[4,2,1,1] &\longrightarrow \mathbb{Y}[4,2,2,1,1] \oplus \mathbb{Y}[4,3,2,1] \oplus \mathbb{Y}[4,3,1,1,1] \oplus \mathbb{Y}[4,4,1,1] \\
\mathbb{Y}[4,2,2] &\longrightarrow \mathbb{Y}[4,4,2] \oplus \mathbb{Y}[4,2,2,2] \oplus \mathbb{Y}[4,3,2,1] \\
\mathbb{Y}[3,3,2] &\longrightarrow \mathbb{Y}[4,3,3] \oplus \mathbb{Y}[4,3,2,1] \oplus \mathbb{Y}[3,3,3,1] \oplus \mathbb{Y}[3,3,2,2] \\
\mathbb{Y}[4,1,1,1,1] &\longrightarrow \mathbb{Y}[4,3,1,1,1] \oplus \mathbb{Y}[4,2,1,1,1,1] \\
\mathbb{Y}[3,3,1,1] &\longrightarrow \mathbb{Y}[4,3,2,1] \oplus \mathbb{Y}[4,3,1,1,1] \oplus \mathbb{Y}[3,3,3,1] \oplus \mathbb{Y}[3,3,2,1,1] \\
\mathbb{Y}[4,3,1] &\longrightarrow \mathbb{Y}[4,4,2] \oplus \mathbb{Y}[4,4,1,1] \oplus \mathbb{Y}[4,3,3] \oplus \mathbb{Y}[4,3,2,1] \\
\mathbb{Y}[4,4] &\longrightarrow \mathbb{Y}[4,4,2]
\end{aligned}$$

The irreducible fields at level four are given in column notation as

$$\begin{array}{lllll}
\mathbb{Y}[2,2,2,2,1,1]\,, & \mathbb{Y}[3,2,2,2,1]\,, & \mathbb{Y}[3,2,2,1,1,1]\,, & \mathbb{Y}[4,2,2,1,1]\,, & \mathbb{Y}[4,3,1,1,1], \\
\mathbb{Y}[4,2,1,1,1,1]\,, & \mathbb{Y}[3,3,2,1,1]\,, & \mathbb{Y}[3,3,1,1,1,1]\,, & \mathbb{Y}[4,3,2,1]\,, & \mathbb{Y}[4,2,2,2]\,, \\
\mathbb{Y}[3,3,2,2]\,, & \mathbb{Y}[4,4,1,1]\,, & \mathbb{Y}[4,4,2]\,, & \mathbb{Y}[4,3,3]\,, & \mathbb{Y}[3,3,3,1]
\end{array} \tag{3.88}$$

with corresponding Young diagrams



(3.89)

and respective multiplicities in the order presented above

$$(1\,,\ 4\,,\ 3\,,\ 12\,,\ 12\,,\ 3\,,\ 8\,,\ 2\,,\ 24\,,\ 6\,,\ 6\,,\ 12\,,\ 10\,,\ 9\,,\ 6)\,. \tag{3.90}$$

The number of families of fields will clearly continue to increase with further dualisation.

JHEP12(2022)152

Finally, note that $\widehat{W}^{(n)}$ is the same as $A^{(n-2)}$ with four extra antisymmetric indices, so they are Hodge dual. Dualising every field at every stage but only keeping track of the $A^{(n)}$ and $\widehat{W}^{(n)}$ families, we know that off-shell dualisation of $A^{(n)}$ produces both $A^{(n+1)}$ and $\widehat{W}^{(n+1)} = *A^{(n-1)}$. At the n^{th} level of higher dualisation, we have one copy of $A^{(n)}$ and at least one copy of the k^{th} Hodge dual $A^{(n-2k)}$ for every positive integer k such that $0 \leq n-2k \leq n$. When n is even, we find that our set of independent fields contains $\left\{A^{(0)}, A^{(2)}, A^{(4)}, \dots, A^{(n)}\right\}$ as a subset. Similarly, when n is odd, we find that it contains $\left\{A^{(1)}, A^{(3)}, \dots, A^{(n)}\right\}$ instead. However, if we only dualise the $A^{(n)}$ fields, and if we make the appropriate correspondence between the $W^{(n)}$ and $A^{(n)}$ families, then we find $\left\{A^{(n)}\right\} \cup \left\{A^{(n-2)}, A^{(n-3)}, \dots, A^{(1)}, A^{(0)}\right\}$ at level n.

Summary. Off-shell dualisation on empty columns in the Young tableaux of every field at the n^{th} level of higher dualisation produces independent fields in a one-to-one correspondance with the set of GL(4)-irreducible fields entering the decomposition of the tensor product

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \otimes \dots \otimes \ydiagram{1,1} \otimes \ydiagram{2} \tag{3.91}$$

with n factors of the antisymmetric Young diagram $\ydiagram{1,1}$.

3.4 Contact with A_1^{+++}

Before we explain the correspondence between the ℓ_2 representation of A_1^{+++} and the extra fields produced via off-shell dualisation, it will be useful to review an efficient method for computing fundamental representations [32]. First of all, add a new node labelled $*$ to the Dynkin diagram of A_1^{+++} and attach it to node i, say, by a single edge. A generic root in the corresponding enlarged algebra $A_1^{+++(i)}$ is given by

$$\alpha = m_* \alpha_* + m_4 \alpha_4 + \sum_{j=1}^{3} m_j \alpha_j \tag{3.92}$$

where α_* denotes the new simple root associated to the new node $*$ and where m_4 is the level in the usual decomposition of A_1^{+++} with respect to its A_3 subalgebra. The structure of A_1^{+++} is studied at each level by looking at the representation content (i.e. the weight space) of the A_3 subalgebra. Any generic A_3 weight can be expressed as

$$\lambda = \sum_{i=1}^{3} p_i \lambda_i \tag{3.93}$$

where λ_i is the i^{th} fundamental weight of A_3. This weight may also be written as $\lambda = [p_1, p_2, p_3]$ and we may depict this weight (and its corresponding representation) by a Young diagram with p_3 columns of height 1, p_2 columns of height 2, and p_1 columns of height 3. With this, we have

$$\lambda = [p_1, p_2, p_3] \sim \mathbb{Y}[\underbrace{3, \dots, 3}_{p_1}, \underbrace{2, \dots, 2}_{p_2}, \underbrace{1, \dots, 1}_{p_3}] =: \mathbb{Y}[3^{p_1}, 2^{p_2}, 1^{p_3}]\,. \tag{3.94}$$

JHEP12(2022)152

l	A_3 weight	A_1^{+++} root α	α^2	mult.	field	label
0	$[1,0,1]$	$(1,1,1,0)$	2	1	${h_a}^b$	h
1	$[0,0,2]$	$(0,0,0,1)$	2	1	$A^{(0)}_{[1,1]}$	a_0
2	$[0,1,2]$	$(0,0,1,2)$	2	1	$A^{(1)}_{[2,1,1]}$	a_1
3	$[0,2,2]$	$(0,0,2,3)$	2	1	$A^{(2)}_{[2,2,1,1]}$	a_2
3	$[1,1,1]$	$(0,1,3,3)$	-4	1	$\widehat{Z}^{(2)}_{[3,2,1]}$	c_2
4	$[0,3,2]$	$(0,0,3,4)$	2	1	$A^{(3)}_{[2,2,2,1,1]}$	a_3
4	$[1,1,3]$	$(0,1,3,4)$	-2	1	$\widehat{Y}^{(3)}_{[3,2,1,1,1]}$	d_1
4	$[1,2,1]$	$(0,1,4,4)$	-6	2	$\widehat{Z}^{(3)}_{[3,2,2,1]}$	d_2
4	$[0,1,2]$	$(1,2,4,4)$	-10	1	$\widehat{W}^{(3)}_{[4,2,1,1]}$	d_3
4	$[2,1,0]$	$(0,2,5,4)$	-10	1	$\widehat{Q}^{(3)}_{[3,3,2]}$	d_5
4	$[2,0,2]$	$(0,2,4,4)$	-8	1	$\widehat{S}^{(3)}_{[3,3,1,1]}$	d_7
4	$[1,0,1]$	$(1,3,5,4)$	-14	1	$\widehat{T}^{(3)}_{[4,3,1]}$	d_8

Table 1. The adjoint representation of A_1^{+++} up to level four.

The relationship between the permitted A_3 Dynkin labels p_i of λ and the Kac labels m_i of the A_1^{+++} root associated with λ can be found in equation (16.6.3) of reference [33].

The notion of level is preserved by commutators, so the set of roots with $m_* = 1$ forms a representation of A_1^{+++} which one can show is equivalent to the i^{th} fundamental representation, denoted ℓ_i. The ℓ_1 and ℓ_2 representations of A_1^{+++} were calculated this way in the tables found in this paper. Note that the A_3 weights in the tables for ℓ_1 and ℓ_2 have their corresponding A_1^{+++} roots written as $A_1^{+++(i)}$ roots so that the new simple root α_* is included.

In previous sections, we have found the extra fields appearing in the action principles and duality relations at low levels. Now we are finally ready to show, level-by-level, that off-shell dualisation produces a set of extra fields that is closely correlated with the ℓ_2 representation. In particular, at the n^{th} level of higher dualisation, we count fields that appear in the adjoint representation at level $n+1$ and in the ℓ_2 representation at level n. This will then be compared against the set of extra fields that we obtain by off-shell dualising every field at every level.

In table 3, the 'adj' and 'ℓ_2' columns contain the field multiplicities in the adjoint and ℓ_2 representations, respectively, and the 'total' column gives their sum. The 'maximal off-shell' column tells us how many of each field is found by off-shell dualising every field at every level. Lastly, the 'net' column is the 'maximal off-shell' column minus the 'total' column. It tells us if we have too many, too few, or the right amount of fields with maximal off-shell dualisation.

From gravity to dual gravity. Recall the dualisation of gravity in section 3.1 which gave us the dual graviton with GL(4)-irreducible symmetry type $\mathbb{Y}[D-3,1]$ and an extra

JHEP12(2022)152

l	A_3 weight	$A_1^{+++(2)}$ root α	α^2	mult.	field	label
0	$[0,1,0]$	$(0,0,0,0,1)$	2	1	$\widehat{U}^{(0)}_{[2]}$	u
1	$[1,0,1]$	$(0,1,1,1,1)$	0	1	$\widehat{Z}^{(1)}_{[3,1]}$	b
2	$[1,0,3]$	$(0,1,1,2,1)$	2	1	$\widehat{Y}^{(2)}_{[3,1,1,1]}$	c_1
2	$[1,1,1]$	$(0,1,2,2,1)$	-2	1	$\widehat{Z}^{(2)}_{[3,2,1]}$	c_2
2	$[0,0,2]$	$(1,2,2,2,1)$	-4	1	$\widehat{W}^{(2)}_{[4,1,1]}$	c_3
2	$[0,1,0]$	$(1,2,3,2,1)$	-6	1	$\widehat{P}^{(2)}_{[4,2]}$	c_4
3	$[1,1,3]$	$(0,1,2,3,1)$	0	1	$\widehat{Y}^{(3)}_{[3,2,1,1,1]}$	d_1
3	$[1,2,1]$	$(0,1,3,3,1)$	-4	2	$\widehat{Z}^{(3)}_{[3,2,2,1]}$	d_2
3	$[0,1,2]$	$(1,2,3,3,1)$	-8	4	$\widehat{W}^{(3)}_{[4,2,1,1]}$	d_3
3	$[0,2,0]$	$(1,2,4,3,1)$	-10	2	$\widehat{P}^{(3)}_{[4,2,2]}$	d_4
3	$[2,1,0]$	$(0,2,4,3,1)$	-8	1	$\widehat{Q}^{(3)}_{[3,3,2]}$	d_5
3	$[0,0,4]$	$(1,2,2,3,1)$	-2	1	$\widehat{R}^{(3)}_{[4,1,1,1,1]}$	d_6
3	$[2,0,2]$	$(0,2,3,3,1)$	-6	1	$\widehat{S}^{(3)}_{[3,3,1,1]}$	d_7
3	$[1,0,1]$	$(1,3,4,3,1)$	-12	2	$\widehat{T}^{(3)}_{[4,3,1]}$	d_8

Table 2. The ℓ_2 representation of A_1^{+++} up to level three.

label	A_3 weight	adj	ℓ_2	total	maximal off-shell	net
b	$[1,0,1]$	0	1	1	1	0
c_1	$[1,0,3]$	0	1	1	1	0
c_2	$[1,1,1]$	1	1	2	2	0
c_3	$[0,0,2]$	0	1	1	2	$+1$
c_4	$[0,1,0]$	0	1	1	1	0
c_5	$[2,0,0]$	0	0	0	1	$+1$
d_1	$[1,1,3]$	1	1	2	2	0
d_2	$[1,2,1]$	2	2	4	3	-1
d_3	$[0,1,2]$	1	4	5	6	$+1$
d_4	$[0,2,0]$	0	2	2	3	$+1$
d_5	$[2,1,0]$	1	1	2	3	$+1$
d_6	$[0,0,4]$	0	1	1	1	0
d_7	$[2,0,2]$	1	1	2	3	$+1$
d_8	$[1,0,1]$	1	2	3	6	$+3$
d_9	$[0,0,0]$	0	0	0	1	$+1$

Table 3. Extra fields from off-shell dualisation compared with A_1^{+++} representations.

l	A_3 weight	$A_1^{+++(1)}$ root α	α^2	mult.	field
0	$[1,0,0]$	$(0,0,0,0,1)$	2	1	P^a
1	$[0,0,1]$	$(1,1,1,1,1)$	0	1	$Z_{[1]}$
2	$[0,1,1]$	$(1,1,2,2,1)$	-2	1	$Z_{[2,1]}$
2	$[0,0,3]$	$(1,1,1,2,1)$	2	1	$Z_{[1,1,1]}$
3	$[1,1,0]$	$(1,2,4,3,1)$	-8	1	$Z_{[3,2]}$
3	$[1,0,2]$	$(1,2,3,3,1)$	-6	1	$Z_{[3,1,1]}$
3	$[0,2,1]$	$(1,1,3,3,1)$	-4	2	$Z_{[2,2,1]}$
3	$[0,1,3]$	$(1,1,2,3,1)$	0	1	$Z_{[2,1,1,1]}$

Table 4. The ℓ_1 representation of A_1^{+++} up to level three.

$(D-2)$-form which may be gauged away with a shift symmetry [1]. In four dimensions, the decomposition

$$\square \otimes \square \;=\; \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \tag{3.95}$$

gives us a symmetric rank-2 field A_{ab} and a 2-form field. The symmetric field is the familiar dual graviton $a_0 := A^{(0)}_{[1,1]}$ in the adjoint representation of A_1^{+++} at level 1 in table 1, and the 2-form $u := \widehat{U}^{(0)}_{[2]}$ is found in the ℓ_2 representation at level 0 in table 2. In contrast to what happens later, we do not dualise the extra 2-form field as it can be shifted away.

The first level of higher dualisation. Taking the dual graviton $A^{(0)}_{[1,1]}$ and dualising again, we obtain the first higher dual graviton $A^{(1)}_{[2,1,1]}$ and the extra field $\widehat{Z}^{(1)}_{[3,1]}$, with symmetry types $\mathbb{Y}[2,1,1]$ and $\mathbb{Y}[3,1]$. We will now find each of these fields in the representations of A_1^{+++}.

The weight $[0,1,2]$ in the adjoint representation at level 2 (see table 1) corresponds to the first higher dual graviton field $a_1 := A^{(1)}_{[2,1,1]}$, and the weight $[1,0,1]$ in the ℓ_2 representation at level 1 (see table 2) corresponds to the extra field $b := \widehat{Z}^{(1)}_{[3,1]}$ required to build a consistent dual action principle. So, at the first level of higher dualisation, we see the complete correspondence between the fields produced during off-shell dualisation and the generators of the adjoint and ℓ_2 representations of A_1^{+++}. There is a perfect match at this level. In table 3, we see that b appears zero times in the adjoint at level 2 and once at level 1 in ℓ_2. It is also required exactly once at this off-shell dualisation at this level, hence the zero in the 'net' column. Note that, although they both have the same A_3 weight $[1,0,1]$, the two fields b and d_8 in ℓ_2 at levels 1 and 3, respectively, are not compared or counted together since they appear at different levels. In fact, d_8 should be viewed as a $\mathbb{Y}[4,3,1]$ field.

The second level of higher dualisation. Taking the first higher dual graviton $A^{(1)}_{[2,1,1]}$ and dualising again, we obtain the second higher dual graviton $A^{(2)}_{[2,2,1,1]}$ and three extra fields that are required for a consistent action principle: $\widehat{Y}^{(2)}_{[3,1,1,1]}$, $\widehat{Z}^{(2)}_{[3,2,1]}$ and $\widehat{W}^{(2)}_{[4,1,1]}$. This

JHEP12(2022)152

is the *minimal off-shell dualisation* of linearised gravity. In addition, we may dualise the extra field from the previous level for *maximal off-shell dualisation.* Recall (3.75) and (3.76) for convenience:

$$A^{(1)}_{[2,1,1]} \quad \longmapsto \quad A^{(2)}_{[2,2,1,1]} \oplus \widehat{Y}^{(2)}_{[3,1,1,1]} \oplus \widehat{Z}^{(2)}_{[3,2,1]} \oplus \widehat{W}^{(2)}_{[4,1,1]} \tag{3.96}$$

$$\widehat{Z}^{(1)}_{[3,1]} \quad \longmapsto \quad \widehat{Z}^{(2)}_{[3,2,1]} \oplus \widehat{W}^{(2)}_{[4,1,1]} \oplus \widehat{P}^{(2)}_{[4,2]} \oplus \widehat{Q}^{(2)}_{[3,3]} \tag{3.97}$$

We are now going to see where these fields are in the representations of A_1^{+++} .

The weight $[0,2,2]$ in the adjoint representation at level 3 corresponds to the second higher dual graviton $a_2 := A^{(1)}_{[2,1,1]}$. It is useful to look at a couple of extra fields in detail. In the ℓ_2 representation at level 2, we find the A_3 weight $[1,0,3]$ which corresponds to $c_1 := \widehat{Y}^{(2)}_{[3,1,1,1]}$. Since it does not appear in the adjoint, c_1 appears only once in our tables. Moreover, c_1 appears precisely once in maximal off-shell dualisation at this level, so there is a perfect match for c_1 . Moving onto the next field, we see in table 3 that $c_2 := \widehat{Z}^{(2)}_{[3,2,1]}$ appears once in the adjoint and once in the ℓ_2 , and it also appears twice in maximal off-shell dualisation: once from the dualisation of a_1 and once more from the dualisation of b . Another perfect match. The reader might like to check that $c_4 := \widehat{P}^{(2)}_{[4,2]}$ gives yet another match.

Unfortunately, we do not find a match for every extra field. For example, $c_3 := \widehat{W}^{(2)}_{[4,1,1]}$ does not appear in the adjoint, and it appears once in the ℓ_2 . However, two of them are required for maximal off-shell dualisation. In other words, although ℓ_2 contains enough for the minimal off-shell description, we go slightly over when we dualise every field at every level. It is even more peculiar with $c_5 := \widehat{Q}^{(2)}_{[3,3]}$ since it is not contained in the tables at all, yet it is needed for maximal off-shell dualisation. We suggest that c_5 should be thought of as thrice Hodge dual to the extra field c_3 .

There is a perfect match for the $\mathrm{GL}(D)$ *types* of fields required but, as for multiplicities, we appear to be lacking a small number of fields in the tables. One possible solution could be to dualise some extra fields but not all of them. By carefully selecting which fields to dualise, this would provide an off-shell description of gravity that does not exceed the field content of the adjoint and ℓ_2 representations of A_1^{+++} but nonetheless, within this restriction, as many fields as possible are dualised. This lies somewhere between the minimal and maximal off-shell dualisations, and we call it the *optimal off-shell dualisation* of linearised gravity.

The third level of higher dualisation. Moving onto the next level, we find that maximal off-shell dualisation produces the fields in (3.84) with Young tableaux (3.87). This set of fields contains the third higher dual graviton $a_3 := A^{(3)}_{[2,2,2,1,1]}$ in the adjoint at level 4, and a number of extra fields that are obtained by dualising the set independent fields $\{a_2, c_1, \ldots, c_5\}$ from the previous level. As before, looking at table 3, we find that some fields are a perfect match and some are not. In fact, for almost all of the extra fields introduced at this level, we have a surplus of fields in the maximal off-shell description compared with the fields that are available from the adjoint and ℓ_2 representations. With the exception of the rogue scalar field $d_9 := \widehat{O}^{(3)}_{[4,4]}$, the Young tableaux for the extra fields perfectly match those of the spectrum of ℓ_2 at level 3. Rogue scalars like this one are found in the maximal off-shell description at all odd levels of higher dualisation greater than or equal to this level.

Optimal off-shell dualisation. In order to understand the differences between these higher dualisation schemes, it is useful to give examples at low levels. All three of them coincide at the first level of higher dualisation where the dual graviton A_{ab} is dualised to give $A^{ab,cd}$ and $\widehat{Z}^{abc,d}$. We then have the choice of whether to dualise only the first higher dual graviton or to dualise both fields. Unfortunately, we exceed the adjoint and ℓ_2 representations of A_1^{+++} if both of them are dualised. Optimal and minimal off-shell dualisations coincide at this level. In fact, we do not have an exact match with the fields coming from A_1^{+++} because there is an extra c_2 field in the A_1^{+++} tables that is not obtained in the optimal scheme.

At the next level of dualisation, we can choose any of the fields in $\{a_1, c_1, c_2, c_3\}$ to dualise. It turns out that all of them may be dualised to produce 16 new fields which are contained in the A_1^{+++} representations. However, we do not find a perfect match because there are six fields in the adjoint and ℓ_2 representations that cannot be obtained this way. In other words, the representations of A_1^{+++} contain slightly more fields than optimal off-shell dualisation.

The adjoint and ℓ_2 representations at the next level contain 96 fields. If we dualise all of the fields at the previous level in the optimal scheme, we obtain 69 fields. However, there are six $\mathbb{Y}[3,3,1,1]$ fields in the optimal scheme, whereas the adjoint and ℓ_2 representations of A_1^{+++} only contain five. The set of fields that produce $\mathbb{Y}[3,3,1,1]$ upon dualisation is $\{4\times d_3, 2\times d_8\}$, so optimal off-shell dualisation is attained by choosing any one of these six fields not to dualise at the previous level. This is important: optimal off-shell dualisation is, in general, *not* unique. Then again, we do not yet know what will happen at higher levels. It is possible that these various pathways to optimal off-shell dualisation may converge at higher levels. It would be interesting to draw the graph of optimal pathways at higher levels and to study its topology.

More general statements. We have just observed what happens at low levels, but there is more to say. It can easily be checked that the n^{th} higher dual graviton $A^{(n)} \sim \mathbb{Y}[2,\ldots,2,1,1]$ corresponds to the A_3 weight $[0,n,2]$ with associated A_1^{+++} root $(0,0,n,n+1)$ whose squared length is equal to 2. That is, the n^{th} higher dual graviton appears in the adjoint at level $n+1$. To calculate this, we have used equation (16.6.3) from [33] while requiring that the Kac labels are non-negative. It can also be shown that the ℓ_2 representation of A_1^{+++} contains the $\widehat{Y}^{(n)}$, $\widehat{Z}^{(n)}$ and $\widehat{W}^{(n)}$ families of extra fields. They correspond to the A_3 weights $[1,n-2,3]$, $[1,n-1,1]$ and $[0,n-2,2]$ with associated $A_1^{+++(2)}$ roots $(0,1,n-1,n,1)$, $(0,1,n,n,1)$ and $(1,2,n,n,1)$. However, even at low levels, this will turn out not to produce the entire spectrum of ℓ_2 and, indeed, less than half of the spectrum of ℓ_2 at level 3 is found if we only dualise the $A^{(n)}$ fields.

The A_1^{+++} algebra has been shown to contain the minimal off-shell dualisation of linearised gravity in four dimensions. Of course, extra fields may also be dualised off-shell, but dualising too many of them leads to field multiplicities that exceed those provided by the adjoint and ℓ_2 representations of A_1^{+++}. Maximal off-shell dualisation contains too many fields, but it is quite interesting nonetheless. Despite some discrepancies in the multiplicities of table 3, the correct Young tableaux shapes appear in this maximal scheme. More work is needed to fully understand the role of the rogue scalar $\widehat{O}^{(3)}_{[4,4]}$ at the third level of higher dualisation, and the other scalars at odd higher levels of dualisation.

The situation at the fourth level of maximal off-shell dualisation is more severe with fields that have a surplus as high as $+7$. However, ignoring multiplicities, the Young tableaux at this level in the maximal off-shell description perfectly matches the spectrum of ℓ_2 at level 4.

In this section, we have identified a possible solution to the tricky problem of mismatched multiplicities at each level: optimal off-shell dualisation. In this scheme, one carefully chooses which extra fields to dualise so that, at each level of higher dualisation, the set of extra fields is contained in the relevant representations of A_1^{+++} with multiplicities that do not exceed those in the 'total' column in table 3.

4 The graviton tower action at low levels

The off-shell dualisation procedure has the advantage that the extra fields contained in the ℓ_2 representation of A_1^{+++} are made explicit. See, for example, the dual action (3.28) where the GL(4)-irreducible field variables $A_{ab,cd}$ and $\widehat{Z}_{abc,d}$ are in direct contact with representations of A_1^{+++}, namely the adjoint and ℓ_2 representations.

In this section, at the first level of higher dualisation, we show that the fields $A^{ab,cd}$ and $\widehat{Z}_{abc,d}$ may be repackaged into new fields: $\widetilde{A}_{ab}$ and $\widetilde{A}^{ab,cd}$ with the respective symmetry types of $A^{(0)} = A_{ab}$ and $A^{(1)} = A^{ab,cd}$. We will show that the gauge transformation laws of these two fields are almost identical to those of the Fierz-Pauli field for $\widetilde{A}_{ab}$ and the Labastida gauge field [23] with symmetry type $\mathbb{Y}(3,1) = \mathbb{Y}[2,1,1]$ for $\widetilde{A}^{ab,cd}$, with additional terms that entangle the two gauge transformation laws. In order to make contact with the Labastida formalism where mixed-symmetry fields are given in the symmetric convention for Young tableaux, we will also use this convention for the first higher dual graviton in this section. However, it should be noted that this convention is not used in the context of E-theory.

The equivalent formulation we will present for the action of the first higher dual graviton in terms of $\widetilde{A}^{ab,cd}$ and $\widetilde{A}_{ab}$ has the advantage of showing more explicitly the number of degrees of freedom through an on-shell duality relation between the gauge invariant curvature tensors of the two fields, as is usual in this context [11, 19, 34].

Change of variables. Recall that the fields appearing in the action (3.24) were $X_{ab;}{}^{ij}$ and $Z_{a;}{}^{e}$, the latter being the Hodge dual of $\widehat{Z}^{bcd,e}$, see (3.25). The Hodge dual of the former field $X_{ab;}{}^{ef}$ is related to $A^{cd,ef}$ via $A^{ab,cd} \equiv \phi^{cd[a,b]}$ and $\phi^{abc,d} = \frac{3}{4}\,\varepsilon^{ijd(a}X_{ij;}{}^{bc)}$. This field, the first higher dual graviton, transforms in the GL(4)-irreducible representation $\mathbb{Y}[2,1,1] = \mathbb{Y}(3,1)$.

From the independent field variables $X_{ab;}{}^{ij}$ and $Z_{a;}{}^{e}$ we introduce the two-form field

$$U^{ab} := \tfrac{1}{4}\,\varepsilon^{abcd}(X_{cd;e}{}^{e} - 4\,Z_{c;d}) \equiv -\tfrac{1}{2}\,\eta_{cd}\,\phi^{cd[a,\,b]} - \epsilon^{abcd}\,Z_{c;d} \equiv -\tfrac{1}{2}\,A^{ab,c}{}_{c} + \widehat{Z}^{abc,}{}_{c} \qquad (4.1)$$

that transforms like

$$\delta U^{ab} = 2\,\partial^{[a}\tau^{b]} + \epsilon^{abcd}\,\partial_c\epsilon_d\;, \qquad \tau^a := \lambda^{ab}{}_b - \mu_b{}^{b,\,a}\;, \tag{4.2}$$

while we recall from section 3.2 that the field $\phi^{abc,\,d}$ transforms like

$$\delta\phi^{abc,\,d} = 3\,\partial^d\lambda^{abc} - 3\,\partial^{(a}\lambda^{bc)d} + 3\,\partial^{(a}\mu^{bc),\,d} - \tfrac{3}{2}\,\eta^{(ab}\,\varepsilon^{c)dij}\,\partial_i\epsilon_j\;. \tag{4.3}$$

We define the $\mathbb{Y}(3,1)$-type gauge field

$$\widetilde{\phi}^{abc,d} := \phi^{abc,d} + \tfrac{3}{4}\,\eta^{(ab}\,U^{c)d} \tag{4.4}$$

that transforms like

$$\delta\widetilde{\phi}^{abc,d} = 3\,\partial^d\widetilde{\lambda}^{abc} - 3\,\partial^{(a}\widetilde{\lambda}^{bc)d} + 3\,\partial^{(a}\widetilde{\mu}^{bc),d} - \tfrac{3}{4}\,\eta^{(ab}\,\varepsilon^{c)def}\,\partial_e\epsilon_f\;, \tag{4.5}$$

where

$$\widetilde{\lambda}^{abc} := \lambda^{abc} - \tfrac{1}{4}\,\eta^{(ab}\,\tau^{c)}\;, \qquad \widetilde{\mu}^{ab,c} := \mu^{ab,c} + \tfrac{1}{6}\left(\eta^{ab}\tau^c - \eta^{c(a}\tau^{b)}\right)\;. \tag{4.6}$$

The advantage of this change of variable is that the newly defined gauge parameters $\widetilde{\lambda}^{abc}$ and $\widetilde{\mu}^{ab,c}$ have the same trace:

$$\widetilde{\tau}^a := \widetilde{\lambda}^{ab}{}_b - \widetilde{\mu}_b{}^{b,a} = \tfrac{1}{2}\,\sigma^a - \tfrac{1}{2}\,\sigma^a \;\equiv\; 0\;, \qquad \sigma^a := \lambda^{ab}{}_b + \mu_b{}^{b,a}\;. \tag{4.7}$$

This also implies that, among the three linearly independent gauge fields $\{\widetilde{\phi}^{abc,d}, U^{ab}, Z_{(a;b)}\}$, only U^{ab} transforms with τ^a. As a result, the dependence of the action $S[\widetilde{\phi}^{abc,d}, U^{ab}, Z_{(a;b)}]$ on U^{ab} comes entirely through its field strength $H^{abc}(U) := 3\,\partial^{[a}U^{bc]}$.

From (A.3) together with (4.4) and

$$\widetilde{X}_{ab;}{}^{cd} = \tfrac{1}{2}\,\varepsilon_{abef}\,\widetilde{\phi}^{cde,f}\;, \qquad \widetilde{\phi}^{abc,d} = \tfrac{3}{4}\,\varepsilon^{ijd(a}\,\widetilde{X}_{ij;}{}^{bc)}\;, \tag{4.8}$$

we obtain

$$\widetilde{X}_{ab;}{}^{cd} = X_{ab;}{}^{cd} + \tfrac{3}{8}\,\varepsilon_{abef}\eta^{(cd}U^{e)f}\;. \tag{4.9}$$

We now express the action (3.24) in terms of the independent fields $\widetilde{X}_{ab;}{}^{cd}$, U^{ab} and $f_{ab} := Z_{(a;b)}$:

$$\begin{aligned}\mathcal{L}(\widetilde{X},U,f) = {}& \tfrac{7}{72}\,H_{abc}(U)\,H^{abc}(U) + \partial_a f_{bc}\,\partial^a f^{bc} - \tfrac{4}{3}\,\partial_a f^{ab}\,\partial_c f_b{}^c - \tfrac{5}{3}\,\partial_a f^{ab}\,\partial_c\widetilde{X}^c{}_{b;i}{}^i\\ &+ 2\,\partial^c f^{ab}\,\partial^d\widetilde{X}_{da;bc} - \tfrac{1}{18}\,\varepsilon_{abcd}\,H^{abc}(U)\left(\partial_i\widetilde{X}^{id;e}{}_e + 4\,\partial_i f^{id}\right) - \tfrac{1}{12}\,\partial_a\widetilde{X}^{ab;c}{}_c\,\partial^i\widetilde{X}_{ib;d}{}^d\\ &+ \tfrac{1}{2}\,\partial_a\widetilde{X}^{ab;cd}\,\partial_i\widetilde{X}^i{}_{b;cd} + \tfrac{1}{2}\,\partial^d\widetilde{X}_{da;bi}\,\partial^i\widetilde{X}^{ab;c}{}_c + \tfrac{1}{16}\,\partial_i\widetilde{X}_{ab;c}{}^c\,\partial^i\widetilde{X}^{ab;d}{}_d\;,\end{aligned} \tag{4.10}$$

where $H_{abc}(U) := 3\,\partial_{[a}U_{bc]}$.

Now we can dualise the field U^{ab} into a scalar field S by letting H_{abc} be an independent field and creating a new parent Lagrangian $\mathcal{L}(\widetilde{X},H,f)$ with the additional term $\frac{1}{18}\,\varepsilon^{abcd}\,\partial_a S\,H_{bcd}$. Solving the field equation for the auxiliary field H_{abc} yields

$$H_{abc} = \tfrac{2}{7}\,\varepsilon_{abcd}\left(\partial_i\widetilde{X}^{id;e}{}_e + 4\,\partial_i f^{id} + \partial^d S\right)\;. \tag{4.11}$$

JHEP12(2022)152

Substituting this into the parent Lagrangian, we obtain the following dual Lagrangian which is given, up to a total derivative, by

$$\begin{aligned}\mathcal{L}(\widetilde{X},S,f) = &\tfrac{1}{21}\,\partial_a S\,\partial^a S + \tfrac{8}{21}\,\partial_a S\,\partial_b f^{ab} - \tfrac{4}{7}\,\partial_a f^{ab}\,\partial^c f_{cb} + \partial_a f_{bc}\,\partial^a f^{bc}\\ &- \tfrac{1}{28}\,\partial_a \widetilde{X}^{ab;c}{}_c\,\partial^i \widetilde{X}_{ib;d}{}^d + \tfrac{1}{2}\,\partial_a \widetilde{X}^{ab;cd}\,\partial^i \widetilde{X}_{ib;cd} + \tfrac{1}{2}\,\partial^d \widetilde{X}_{da;be}\,\partial^e \widetilde{X}^{ab;c}{}_c\\ &+ \tfrac{1}{16}\,\partial_d \widetilde{X}_{ab;e}{}^e\,\partial^d \widetilde{X}^{ab;c}{}_c + 2\,\partial^c f^{ab}\,\partial^d \widetilde{X}_{da;bc} - \tfrac{9}{7}\,\partial_a f^{ab}\,\partial^c \widetilde{X}_{cb;i}{}^i\,. \end{aligned} \tag{4.12}$$

This action is invariant under

$$\delta\widetilde{\phi}_{abc,d} = 3\,\partial_d \widetilde{\lambda}_{abc} - 3\,\partial_{(a}\widetilde{\lambda}_{bc)d} + 3\,\partial_{(a}\widetilde{\mu}_{bc),d} - \tfrac{3}{4}\,\eta_{(ab}\,\varepsilon_{c)dij}\,\partial^i\epsilon^j\,, \tag{4.13}$$

$$\delta f_{ab} = \tfrac{1}{4}\left(\varepsilon_a{}^{cde}\,\partial_c\widetilde{\mu}_{bd,e} + \varepsilon_b{}^{cde}\,\partial_c\widetilde{\mu}_{ad,e}\right) + \partial_{(a}\epsilon_{b)} - \tfrac{1}{4}\,\eta_{ab}\,\partial_c\epsilon^c\,, \tag{4.14}$$

$$\delta S = -3\,\partial_a\epsilon^a\,. \tag{4.15}$$

Finally, we combine the scalar field S with the traceless symmetric field $f_{ab} := Z_{(a;b)}$ to get

$$\widetilde{A}_{ab} := 2\,f_{ab} - \tfrac{1}{6}\,\eta_{ab}\,S \quad\Leftrightarrow\quad S = -\tfrac{3}{2}\,\widetilde{A}^a{}_a\,,\quad f_{ab} = \tfrac{1}{2}\,(\widetilde{A}_{ab} - \tfrac{1}{4}\,\eta_{ab}\,\widetilde{A}^c{}_c)\,. \tag{4.16}$$

We therefore obtain a new action $S[\widetilde{A}_{ab},\widetilde{A}^{ab,cd}]$ which takes the form

$$\begin{aligned} &S[\widetilde{A}_{ab},\widetilde{A}^{ab,cd}]\\ &= -\tfrac{1}{2}\int \mathrm{d}^4x\,\Big[-\tfrac{1}{2}\,\partial_a\widetilde{A}_{bc}\,\partial^a\widetilde{A}^{bc} - \tfrac{3}{14}\,\partial_a\widetilde{A}_b{}^b\,\partial^a\widetilde{A}^c{}_c + \tfrac{2}{7}\,\partial_a\widetilde{A}^{ab}\,\partial^c\widetilde{A}_{bc} + \tfrac{3}{7}\,\partial_a\widetilde{A}^{ab}\,\partial_b\widetilde{A}_c{}^c\\ &\quad + \tfrac{9}{7}\,\partial_a\widetilde{A}^{ab}\,\partial^c\widetilde{X}_{cb;d}{}^d - 2\,\partial^c\widetilde{A}^{ab}\,\partial^d\widetilde{X}_{da;bc}\\ &\quad + \tfrac{1}{14}\,\partial_a\widetilde{X}^{ab;c}{}_c\partial^d\widetilde{X}_{db;e}{}^e - \tfrac{1}{8}\,\partial_d\widetilde{X}_{ab;e}{}^e\,\partial^d\widetilde{X}^{ab;c}{}_c - \partial^d\widetilde{X}^{ab;c}{}_c\,\partial^e\widetilde{X}_{ea;bd} - \partial_a\widetilde{X}^{ab;cd}\,\partial^e\widetilde{X}_{eb;cd}\Big], \end{aligned} \tag{4.17}$$

where

$$\widetilde{A}^{ab,cd} = \widetilde{\phi}^{cd[a,b]}\,,\qquad \widetilde{\phi}^{abc,d} = -\tfrac{3}{2}\widetilde{A}^{d(a,bc)} \tag{4.18}$$

and

$$\widetilde{X}_{a_1a_2;}{}^{c_1c_2} \equiv \tfrac{1}{2}\,\varepsilon_{a_1a_2b_1b_2}\,\widetilde{\phi}^{c_1c_2b_1,b_2}\,,\qquad \widetilde{\phi}^{c_1c_2c_3,d} \equiv \tfrac{3}{4}\,\varepsilon^{ijd(c_1}\,\widetilde{X}_{ij;}{}^{c_2c_3)} \tag{4.19}$$

are understood throughout. This allows us to write the repackaged first higher dual graviton $\widetilde{A}^{ab,cd}$ in a variety of useful ways. For example, it was convenient to write the above action in terms of $\widetilde{A}_{ab}$ and $\widetilde{X}_{ab;}{}^{cd}$. It is invariant under the following intertwined gauge transformations:

$$\delta\widetilde{A}_{ab} = 2\,\partial_{(a}\epsilon_{b)} - \varepsilon_{cde(a}\partial^c\,\widetilde{\mu}_{b)}{}^{d,e}\,,\qquad \widetilde{\lambda}^{ab}{}_b \equiv \widetilde{\mu}_b{}^{b,a}\,, \tag{4.20}$$

$$\delta\widetilde{\phi}^{abc,d} = 3\,\partial^d\widetilde{\lambda}^{abc} - 3\,\partial^{(a}\widetilde{\lambda}^{bc)d} + 3\,\partial^{(a}\widetilde{\mu}^{bc),d} - \tfrac{3}{4}\,\eta^{(ab}\,\varepsilon^{c)dij}\,\partial_i\epsilon_j\,. \tag{4.21}$$

Trivially, one would need to make use of (4.18) before checking gauge invariance under (4.21). The $\widetilde{\lambda}^{abc}$ and $\widetilde{\mu}^{ab,c}$ parts of the gauge transformations for $\widetilde{\phi}^{abc,d}$ coincide with the Labstida gauge transformations for a gauge field of type $\mathbb{Y}(3,1)$. In particular, the two gauge parameters are constrained to have equal trace. The ϵ_a part of the gauge transformations for the (traceful) symmetric rank-two tensor $\widetilde{A}_{ab}$ corresponds to linearised diffeomorphisms. However, notice that $\widetilde{A}_{ab}$ also transforms with the $\widetilde{\mu}^{ab,c}$ gauge parameter, and that $\widetilde{\phi}^{abc,d}$ transforms with the gauge parameter ϵ_a. As we have seen in [30] for higher dualisation of gauge fields, we find that the action contains fields that resemble the original dual graviton A_{ab} and the first higher dual graviton $\phi^{abc,d}$ with entangled gauge transformation laws. By the construction of our dual action using the parent action procedure, we know that the on-shell degrees of freedom are only those of a single massless spin-2 field around four-dimensional Minkowksi spacetime. Nevertheless, we will rederive this fact from the field equations. It is clear that the dual action is more than just the sum of the Fierz-Pauli and Labastida actions.

It is also important to remember that this repackaging approach seeks to drastically redefine our fields for reasons that will become clear towards the end of this section. As a result, gauge transformations (4.20) and (4.21) are not expected to resemble the gauge transformations for A_{ab} and $A^{ab,cd}$ that were found in section 2.2 and section 3.2.

Field equations. The equations of motion for the fields $\widetilde{A}_{ab}$ and $\widetilde{\phi}^{abc,d}$ are given by

$$\mathcal{E}[\widetilde{A}]_{ab} \approx 0 \qquad \text{and} \qquad \mathcal{E}[\widetilde{\phi}]^{abc,d} \approx 0 \,, \tag{4.22}$$

where $\mathcal{E}[\widetilde{A}]_{ab}$ and $\mathcal{E}[\widetilde{\phi}]^{abc,d}$ are given by

$$\begin{aligned}
\mathcal{E}[\widetilde{A}]_{ab} :=& -\tfrac{1}{2}\Box\widetilde{A}_{ab} + \tfrac{3}{14}\partial_a\partial_b\widetilde{A}_c{}^c + \tfrac{2}{7}\partial^i\partial_{(a}\widetilde{A}_{b)i} - \tfrac{3}{14}\eta_{ab}\left(\Box\widetilde{A}_c{}^c - \partial_i\partial_j\widetilde{A}^{ij}\right) \\
&- \tfrac{9}{14}\partial^i\partial_{(a}\widetilde{X}_{b)ij}{}^j - \partial^i\partial^j\widetilde{X}_{i(ab)j} \,, \qquad (4.23)\\
\mathcal{E}[\widetilde{\phi}]^{abc,d} :=& \tfrac{1}{4}\Box\widetilde{\phi}^{abc,d} - \tfrac{1}{4}\Box\widetilde{\phi}^{d(ab,c)} - \tfrac{1}{4}\partial_i\partial^d\widetilde{\phi}^{abc,i} + \tfrac{1}{4}\partial_i\partial^{(a}\widetilde{\phi}^{bc)d,i} + \tfrac{1}{4}\partial^d\partial_i\widetilde{\phi}^{i(ab,c)} \\
&- \tfrac{1}{4}\partial_i\partial^{(a}\widetilde{\phi}^{bc)i,d} - \tfrac{1}{8}\partial^{(a}\partial^b\widetilde{\phi}^{c)}{}_i{}^{i,d} + \tfrac{1}{8}\partial^{(a}\partial^{b|}\widetilde{\phi}^d{}_i{}^{i,|c)} + \tfrac{1}{8}\eta^{d(a}\partial_j\partial^b\widetilde{\phi}^{c)}{}_i{}^{i,j} - \tfrac{1}{8}\eta^{(ab}\partial^{c)}\partial_i\widetilde{\phi}^{dij,}{}_j \\
&- \tfrac{1}{8}\eta^{d(a}\partial^{b|}\partial_j\widetilde{\phi}_i{}^{ij,|c)} + \tfrac{1}{8}\eta^{(ab|}\partial_i\partial^d\widetilde{\phi}^{|c)ij,}{}_j + \tfrac{1}{8}\eta^{(ab|}\partial_i\partial_j\widetilde{\phi}^{ijd,|c)} - \tfrac{1}{8}\eta^{(ab}\partial_i\partial_j\widetilde{\phi}^{c)ij,d} \\
&- \tfrac{1}{7}\eta^{(ab|}\partial^d\partial_j\widetilde{\phi}_i{}^{ij,|c)} - \tfrac{1}{7}\eta^{(ab}\partial^{c)}\partial_j\widetilde{\phi}^d{}_i{}^{i,j} + \tfrac{15}{56}\eta^{(ab}\partial^{c)}\partial_j\widetilde{\phi}_i{}^{ij,d} + \tfrac{1}{56}\eta^{(ab|}\partial_j\partial^d\widetilde{\phi}^{|c)}{}_i{}^{i,j} \\
&+ \tfrac{5}{112}\eta^{(ab}\Box\widetilde{\phi}^{c)}{}_i{}^{i,d} - \tfrac{5}{112}\eta^{(ab|}\Box\widetilde{\phi}^d{}_i{}^{i,|c)} + \tfrac{1}{2}\varepsilon^{ijd(a}\partial_i\partial^b\widetilde{A}^{c)}{}_j + \tfrac{9}{28}\eta^{(ab}\varepsilon^{c)dij}\partial_i\partial^k\widetilde{A}_{jk} \,. \qquad (4.24)
\end{aligned}$$

Alternatively, we may vary with respect to $\widetilde{X}_{ab;}{}^{cd}$ to find $\mathcal{E}[\widetilde{X}]_{ab;}{}^{cd} \approx 0$, where

$$\begin{aligned}
\mathcal{E}[\widetilde{X}]_{ab;}{}^{cd} :=& -\partial_{[a}\partial^{(c}\widetilde{A}_{b]}{}^{d)} + \tfrac{1}{4}\delta_{[a}{}^{(c}\Box\widetilde{A}_{b]}{}^{d)} - \tfrac{1}{4}\delta_{[a}{}^{(c}\partial_{b]}\partial^{d)}\widetilde{A}_e{}^e + \tfrac{1}{14}\delta_{[a}{}^{(c}\partial_{b]}\partial_i\widetilde{A}^{d)i} - \tfrac{1}{14}\delta_{[a}{}^{(c}\partial^{d)}\partial^i\widetilde{A}_{b]i} \\
&+ \tfrac{9}{14}\eta^{cd}\partial^i\partial_{[a}\widetilde{A}_{b]i} + \partial^i\partial_{[a}\widetilde{X}_{b]i;}{}^{cd} - \tfrac{1}{2}\partial^{(c}\partial_{[a}\widetilde{X}_{b]}{}^{d);i}{}_i - \tfrac{1}{2}\eta^{cd}\partial^i\partial^j\widetilde{X}_{i[a;b]j} \\
&+ \tfrac{1}{4}\delta_{[a|}{}^{(c|}\partial_j\partial^i\widetilde{X}_{i|b];}{}^{|d)j} + \tfrac{1}{4}\delta_{[a|}{}^{(c|}\partial^j\partial_i\widetilde{X}^{i|d);}{}_{|b]j} - \tfrac{1}{8}\eta^{cd}\Box\widetilde{X}_{ab;i}{}^i - \tfrac{1}{14}\eta^{cd}\partial^i\partial_{[a}\widetilde{X}_{b]i;j}{}^j \\
&+ \tfrac{5}{56}\delta_{[a}{}^{(c}\partial_{b]}\partial_i\widetilde{X}^{d)i;j}{}_j + \tfrac{9}{56}\delta_{[a}{}^{(c}\partial^{d)}\partial^i\widetilde{X}^{b]i;j}{}_j \,. \qquad (4.25)
\end{aligned}$$

They obey Noether identities associated with the gauge parameters. For ϵ_a, we have

$$\partial^a\mathcal{E}[\widetilde{A}]_{ab} - \tfrac{3}{8}\,\eta_{ij}\,\varepsilon_{klab}\partial^a\mathcal{E}[\widetilde{\phi}]^{ijk,l} \equiv 0 \,. \tag{4.26}$$

In addition, associated with the traceless part of the $\widetilde{\mu}^{ab,c}$ gauge parameter, we find

$$\left(\varepsilon^{ijd(b}\partial_i\mathcal{E}[\widetilde{A}]_j{}^{c)} + 3\partial_a\mathcal{E}[\widetilde{\phi}]^{abc,d} - 3\partial_a\mathcal{E}[\widetilde{\phi}]^{a(bc,d)}\right) - \text{trace} \equiv 0 \,, \tag{4.27}$$

where "trace" indicates the terms needed to remove the trace of the expression in the brackets. There are also Noether identities related to the traceless part of the gauge parameter $\widetilde{\lambda}^{abc}$ and the shared trace of $\widetilde{\lambda}^{abc}$ and $\widetilde{\mu}^{ab,c}$ although we will not write them here.

Gauge-invariant tensors. Associated with the gauge transformations (4.20) and (4.21) for $\widetilde{A}_{ab}$ and $\widetilde{\phi}^{abc,d}$, respectively, we find the following gauge-invariant tensor with two derivatives:

$$\begin{aligned}K_{ma,nb} := 4\,\partial_{[m}\partial_{[n}\widetilde{A}_{b]a]} + \tfrac{10}{7}\,\eta_{[m[n}\partial_{b]}\partial^i\widetilde{A}_{a]i} + \tfrac{10}{7}\,\eta_{[n[m}\partial_{a]}\partial^i\widetilde{A}_{b]i} - \tfrac{20}{7}\,\eta_{[m[n}\partial_{b]}\partial_{a]}\widetilde{A}_e{}^e \\ - 4\,\partial_{[m|}\partial^i\widetilde{X}_{i[n;b]|a]} - 4\,\partial_{[n|}\partial^i\widetilde{X}_{i[m;a]|b]} - \tfrac{5}{7}\,\eta_{[n[m}\partial_{a]}\partial^i\widetilde{X}_{b]i;j}{}^j - \tfrac{5}{7}\,\eta_{[m[n}\partial_{b]}\partial^i\widetilde{X}_{a]i;j}{}^j\,.\end{aligned} \tag{4.28}$$

Notice that $I := \Box\widetilde{A}_e{}^e - \partial^a\partial^b\widetilde{A}_{ab} \equiv -\frac{7}{16}\,K^{ab,}{}_{ab}$ is a gauge invariant scalar. This can be seen as resulting from the gauge transformation of $\widetilde{V}_b := \partial^a[\widetilde{X}_{ab;c}{}^c + 2(\widetilde{A}_{ab} - \eta_{ab}\widetilde{A}_c{}^c)]$:

$$\delta\widetilde{V}_b = 7\,\partial^a\partial_{[a}\epsilon_{b]}\,. \tag{4.29}$$

We find that the left-hand side of the field equation for $\widetilde{A}_{ab}$ is related to the trace of $K_{ma,nb}$:

$$-2\,\mathcal{E}[\widetilde{A}]_{ab} \equiv K_{ab} - \tfrac{1}{2}\,\eta_{ab}K =: G_{ab}\,, \tag{4.30}$$

where $K_{ab} := \eta^{mn}K_{am,bn}$ and $K := \eta^{ab}K_{ab}$. Obviously, on-shell, we have $K_{ab} \approx 0$ which is to be compared with the field equation (2.28) in section 2.3. This is analogous to the Ricci-flat equation in linearised gravity.

We also have the following gauge-invariant quantity with three derivatives:

$$\begin{aligned}G_{mn,pq;}{}^d := 4\,\varepsilon^{abcd}\,\partial_a\partial_{[m}\partial_{[p}\widetilde{\phi}_{q]n]b,c} \\ - \tfrac{8}{7}\,\Big(\eta_{[m[p}\partial_{q]}\partial_{n]}\widetilde{V}^d + \tfrac{1}{4}\,(\delta^d{}_{[m}\partial_{n]}\partial_{[p}\widetilde{V}_{q]} + \delta^d{}_{[p}\partial_{q]}\partial_{[m}\widetilde{V}_{n]}) - \tfrac{1}{2}\,\eta_{m[p}\eta_{q]n}\partial^d\partial_a\widetilde{V}^a\Big)\end{aligned} \tag{4.31}$$

that possesses the algebraic symmetries of the Riemann tensor in its first four indices and also satisfies two additional tracelessness constraints: $G_{mn,pq;}{}^m \equiv G_{mn,pq;}{}^p \equiv 0$. These algebraic constraints on the tensor $G_{mn,pq;}{}^r$ imply that the dual tensor $\widetilde{G}_{abc,mn,pq} := \varepsilon_{abcd}\,G_{mn,pq;}{}^d$ is of GL(4)-irreducible type $\mathbb{Y}[3,2,2]$.

The Bianchi identity. We find that $\partial_{[a}K_{bc],de}$ is expressed in terms of $\widetilde{A}_{ab}$ and $\widetilde{\phi}^{abc,d}$ as

$$\partial_{[a}K_{bc],de} = 4\,\partial^i\partial_{[d}\partial_{[a|}\widetilde{X}_{i|b;c]e]} + \tfrac{5}{7}\eta_{[a[d}\partial^i\partial_{e]}\partial_b\widetilde{X}_{c]i;j}{}^j - \tfrac{10}{7}\eta_{[a[d}\partial^i\partial_{e]}\partial_b\widetilde{A}_{c]i}\,. \tag{4.32}$$

It is possible to write this in terms of the left-hand-side of the field equation for $\widetilde{X}_{ab;}{}^{cd}$:

$$\partial_{[a}K_{bc],de} \equiv 8\,\partial_{[a}\mathcal{E}[\widetilde{X}]_{b[d;e]c]} - \eta_{[a[d}\partial_{e]}\mathcal{E}[\widetilde{X}]_{bc];i}{}^i + \tfrac{4}{3}\,\eta_{[d[a}\partial^i\mathcal{E}\widetilde{X}_{bc];e]i} + \tfrac{8}{3}\,\eta_{[a[d}\partial^i\mathcal{E}[\widetilde{X}]_{e]b;c]i}\,. \tag{4.33}$$

Therefore, on-shell, we find the following relation that will be instrumental in showing that the degrees of freedom are those of a single graviton:

$$\partial_{[a}K_{bc],de} \approx 0\,. \tag{4.34}$$

On-shell duality relation. We find that the gauge-invariant tensor $G_{ab,cd;}{}^{e}$ is related to $K_{ab,cd}$ and the left-hand-sides of the equations of motion in the following way:

$$\begin{aligned} G_{ab,cd;}{}^{e} &\equiv -\partial^{e} K_{ab,cd} \\ &+ \eta_{[a[c}\partial_{d]}\mathcal{E}[\widetilde{A}]_{b]}{}^{e} + \eta_{[c[a}\partial_{b]}\mathcal{E}[\widetilde{A}]_{d]}{}^{e} - \delta_{[a}{}^{e}\partial_{[c}\mathcal{E}[\widetilde{A}]_{d]b]} - \delta_{[c}{}^{e}\partial_{[a}\mathcal{E}[\widetilde{A}]_{b]d]} + \eta_{a[c}\eta_{d]b}\partial^{e}\mathcal{E}[\widetilde{A}]_{i}{}^{i} \\ &+ 8\,\partial_{[a}\mathcal{E}[\widetilde{X}]^{e}{}_{[c;d]b]} + 8\,\partial_{[c}\mathcal{E}[\widetilde{X}]^{e}{}_{[a;b]d]} + 2\,\eta_{[a[c}\partial_{d]}\mathcal{E}[\widetilde{X}]_{b]}{}^{e;i}{}_{i} + 2\,\eta_{[c[a}\partial_{b]}\mathcal{E}[\widetilde{X}]_{d]}{}^{e;i}{}_{i}\,. \end{aligned} \tag{4.35}$$

Consequently, on-shell, we have the following duality relation:

$$\widetilde{G}_{a[3],b[2],c[2]} \approx -\,\varepsilon_{a[3]d}\,\partial^{d} K_{b[2],c[2]} \qquad \Leftrightarrow \qquad G_{mn,pq;}{}^{r} \approx -\,\partial^{r} K_{mn,pq}\,. \tag{4.36}$$

These equations are important in several respects. The tensors $K_{ab,cd}$ and $\widetilde{G}_{abc,de,fg}$ can be called the field strengths for the repackaged dual graviton and first higher dual graviton, resepctively. Indeed, they do not vanish on-shell and they are gauge invariant. The duality relation (4.36) sets equal the two curvature tensors, on-shell, thereby showing that the physical degrees of freedom carried by the field $\widetilde{A}_{ab}$ are also contained in $\widetilde{\phi}_{abc,d}$. There is no doubling of the degrees of freedom. Secondly, from (4.36) and the on-shell Bianchi identity (4.34), we find

$$\widetilde{G}^{ab}{}_{c,ab,de} \approx 0\,, \tag{4.37}$$

which is exactly the form of the field equation (2.34) that we derived for the first higher dual graviton in section 2.3. Finally, by taking the trace of the duality relation (4.36) on the indices b_2 and c_2 and using the Ricci-flat equation $K_{ab} \approx 0$ that we derived above, we find

$$\widetilde{G}_{abc,de,}{}^{d}{}_{f} \approx 0\,. \tag{4.38}$$

This field equation completes those found in section 2.3.

With these field equations, we have found a strong parallel with the analogous equations derived in section 2.3. However, since no action principle was considered in that section, each field strength was a function of a single field. Instead, in the off-shell formulation found in the present section that requires the extra field $\widehat{Z}^{abc,d}$ to be repackaged, equations necessarily entangle both fields due to the nature of the gauge transformation laws.

We conjecture that this dualisation and repackaging procedure creates an increasingly tall tower of new repackaged dual gravitons whose gauge transformation laws are intertwined. In particular, for a given tower with highest level N, the field $\widetilde{\phi}^{(n)}$ at level $n \leq N$ should transform as a Labastida gauge field of symmetry type $\mathbb{Y}[2,\ldots,2,1,1]$. Its gauge transformation law should contain terms that entangle it with the repackaged fields at every level lower than n. Moreover, if $n < N$, then its gauge transformation law will also be entangled with that of the repackaged field at level $n+1$. It may even be possible to redefine fields so that the repackaged dual graviton at level n is entangled only with those at level $n+1$ and $n-1$.

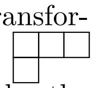
For the graviton tower action $S[\widetilde{A}_{ab}, \widetilde{A}^{ab,cd}]$ in (4.17), the extra field $\widetilde{Z}^{abc,d}$ was completely hidden by the specific field and gauge parameter redefinitions used to construct $\widetilde{A}_{ab}$. However, this may only be possible at low levels, so we cannot yet exclude the possibility that some extra fields may still be present in the graviton tower actions at higher levels.

5 Conclusion

In this paper, we started to make precise connections between the non-linear realisation based on A_1^{+++} [14–18] and the off-shell dualisation programme for pure gravity in four dimensions [9]. The non-linear realisation contains an infinite number of dualisations of gravity. It consists of an infinite set of duality relations, the first of which involves only the graviton and the dual graviton. This relation was worked out at the full non-linear level in [17, 18]. In section 2, we have used the non-linear realisation to work out the linearised equations of motion for the first higher dual graviton.

While on the other hand, in [9] it was shown that pure linearised gravity could be described by any member of an infinite family of action principles, each involving more and more fields. Some of these fields were shown in [9] to have a direct connection with the adjoint representation of the very-extended A_{D-3}^{+++} algebra, while other fields received no interpretation at that time.

In the present paper, where we focus on $D = 4$ for the sake of concreteness, we showed that the aforementioned fields are all associated with generators in the ℓ_2 representation of A_1^{+++} in the sense that there exist generators in ℓ_2 that have the same GL(4) types. We have carried out this match up to level four and, while there is a striking agreement at low levels, some of the multiplicities differ for the extra fields.

We also constructed, at the level of the first higher dual graviton, a new action principle featuring two fields $\widetilde{A}_{ab}$ and $\widetilde{A}_{ab,cd}$ with the GL(4) symmetry types $\mathbb{Y}[1,1]$ and $\mathbb{Y}[2,1,1]$ of the dual graviton and the first higher dual graviton, respectively. The gauge transformations of these two fields are those of the dual graviton and the corresponding  Labastida field, along with extra terms that entangle the two fields. Remarkably, the field equations can be obtained from a duality relation between the gauge invariant curvatures of these repackaged fields, which further demonstrates that our original action only propagates a single graviton. That the field equations can be encapsulated in a set of duality relations is in full agreement with the method of obtaining the field equations in the non-linear realisation of $A_1^{+++} \ltimes \ell_1$.

In a future work in preparation, we will extend our analysis to pure gravity in five dimensions where the relevant algebra is A_2^{+++}. We will also consider pure gravity and the bosonic sector of maximal supergravity in eleven dimensions. It is well-known that the relevant Kac-Moody algebras for these theories are A_8^{+++} and E_{11}, respectively. There, we will also show how their ℓ_2 representations are related to the set of off-shell fields entering higher dual action principles. It will be important to modify the coset space used to construct the non-linear realisation for these algebras in order to incorporate ℓ_2. Consequently, this will account for the extra fields that were thought to be missing from E-theory until now.

Finally, it would be interesting to make a contact with [35] where the importance of the ℓ_2 representation of E_{11} was noticed in a similar context. It is not yet clear to us that there is a connection since their equations of motion are obtained from the E_{11} pseudo-Lagrangian by a variational principle supplemented by extra duality relations that are not derived by variation. More specifically, variations with respect to constrained fields

JHEP12(2022)152

(which carry a section constraint index) vanish only when these extra duality relations are imposed. In contrast, our off-shell dualisation approach produces equations of motion and duality relations that are all obtained by varying dual actions. Nothing external needs to be imposed here. Another line of research is to investigate the possible non-linear extensions of the higher dual actions considered here.

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A Young tableaux in the symmetric convention

During the construction of the first higher dual action for gravity, we might want to express the irreducible field content $\{A^{ab,cd}\,,\widehat{Z}^{abc,d}\}$ of (3.28) in terms of fields with blocks of symmetric indices, corresponding to the manifestly symmetric convention for Young tableaux. Recall that $\{A^{ab,cd}\,,\widehat{Z}^{abc,d}\}$ and $\{X_{ab;}{}^{cd}\,,\widehat{Z}_{a;}{}^{b}\}$ are related by (3.25) and (3.26) as follows:

$$A^{ab,cd} := -\tfrac{1}{2}\,\varepsilon^{abij}\,X_{ij;}{}^{cd}\,, \qquad \widehat{Z}^{abc,d} := \varepsilon^{abce}\,Z_{e;}{}^{d}\,, \tag{A.1}$$

$$X_{ab;}{}^{cd} = \tfrac{1}{2}\,\varepsilon_{abij}\,A^{ij,cd}\,, \qquad Z_{a;}{}^{e} = \tfrac{1}{6}\,\varepsilon_{abcd}\,\widehat{Z}^{bcd,e}\,. \tag{A.2}$$

We can now introduce equivalent fields in the symmetric convention:

$$\phi^{c_1c_2c_3,d} := \tfrac{3}{4}\,\varepsilon^{ijd(c_1}\,X_{ij;}{}^{c_2c_3)}\,, \qquad \psi^{c_1c_2,d,e} := \tfrac{1}{2}\,\varepsilon^{aed(c_1}\,Z_{a;}{}^{c_2)}\,. \tag{A.3}$$

Inverse relations are given by

$$X_{a_1a_2;}{}^{c_1c_2} = \tfrac{1}{2}\,\varepsilon_{a_1a_2b_1b_2}\,\phi^{b_1c_1c_2,b_2}\,, \qquad Z_{a;}{}^{c} = \tfrac{1}{2}\,\varepsilon_{ab_1b_2b_3}\psi^{b_1c,b_2,b_3}\,. \tag{A.4}$$

These GL(4)-irreducible fields satisfy over-symmetrisation constraints:

$$\phi^{abc,d} = \phi^{(abc),d}\,, \qquad \phi^{(abc,d)} \equiv 0\,, \tag{A.5}$$

$$\psi^{ab,c,d} = \psi^{(ab),c,d}\,, \qquad \psi^{(ab,c),d} \equiv \psi^{(ab|,c,|d)} \equiv \psi^{ab,(c,d)} \equiv 0\,. \tag{A.6}$$

This is an opportunity to summarise and exemplify the two equivalent conventions for Young tableaux of GL(D). A finite-dimensional irreducible representation of GL(D) may be described by a tensor field with groups of manifestly symmetric indices, each group corresponding to a row on the Young tableau associated with it, such that the tensor satisfies over-symmetrisation identities. Alternatively, we may choose to describe the same finite-dimensional irreducible representation of $\mathrm{GL}(D)$ by a tensor with groups of manifestly antisymmetric indices, each group corresponding to a column on the Young tableau, such that the corresponding tensor satisfies over-antisymmetrisation identities. The $\mathrm{GL}(4)$ symmetries of the fields introduced so far are depicted by the following Young tableaux:

$$\phi^{abc,d} \sim \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d \\ \cline{1-1}\end{array} \sim A^{ad,bc} , \qquad \psi^{ab,c,d} \sim \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1} d \\ \cline{1-1}\end{array} \sim \widehat{Z}^{acd,b} . \tag{A.7}$$

The relation between the two conventions for Young tableaux is given by

$$A^{ab,cd} \equiv \phi^{cd[a,b]} , \qquad \widehat{Z}^{abc,d} \equiv 3\,\psi^{d[a,b,c]} , \tag{A.8}$$

with inverse relations

$$\phi^{abc,d} \equiv -\tfrac{3}{2}\,A^{d(a,bc)} , \qquad \psi^{ab,c,d} \equiv \tfrac{1}{2}\,\widehat{Z}^{cd(a,b)} . \tag{A.9}$$

These relations describe nothing more than a change of basis between irreducible tensor fields in the manifestly symmetric and antisymmetric conventions for Young tableaux.

The second level of higher dualisation involves a reducible field $D_{ab;}{}^{c_1c_2c_3,d}$ which must be decomposed into traceless components as in (3.50). These components then need to be Hodge dualised on their first blocks of indices in the same way that $X_{ab;}{}^{cd}$ and $Z_{a;}{}^{b}$ were. As before, this will create $\mathrm{GL}(4)$-irreducible fields with symmetric blocks of indices. Their symmetry types are $\mathbb{Y}(4,2)$, $\mathbb{Y}(4,1,1)$, $\mathbb{Y}(3,2,1)$ and $\mathbb{Y}(3,1,1,1)$:

$$\phi^{c_1c_2c_3c_4,d_1d_2} = \tfrac{4}{5}\,\varepsilon^{a_1a_2(d_1(c_1}X_{a_1a_2;}{}^{c_2c_3c_4),d_2)} , \tag{A.10}$$

$$\psi_{(Y)}^{c_1c_2c_3c_4,d,e} = \tfrac{2}{3}\,\varepsilon^{aed(c_1}Y_{a;}{}^{c_2c_3c_4)} , \tag{A.11}$$

$$\psi_{(Z)}^{c_1c_2c_3,d_1d_2,e} = \tfrac{4}{5}\,\varepsilon^{ae(d_1(c_1}Z_{a;}{}^{c_2c_3),d_2)} , \tag{A.12}$$

$$\psi_{(W)}^{c_1c_2c_3,d,e,f} = -\tfrac{1}{6}\,\varepsilon^{fed(c_1}W^{c_2c_3)} , \tag{A.13}$$

Inverse relations are given by

$$X_{a_1a_2;}{}^{c_1c_2c_3,d} = \tfrac{1}{2}\varepsilon_{a_1a_2b_1b_2}\phi^{b_1c_1c_2c_3,b_2d} , \tag{A.14}$$

$$Y_{a;}{}^{c_1c_2c_3} = \tfrac{1}{2}\varepsilon_{ab_1b_2b_3}\psi_{(Y)}^{b_1c_1c_2c_3,b_2,b_3} , \tag{A.15}$$

$$Z_{a;}{}^{c_1c_2,d} = \tfrac{1}{2}\varepsilon_{ab_1b_2b_3}\psi_{(Z)}^{b_1c_1c_2,b_2d,b_3} , \tag{A.16}$$

$$W^{c_1c_2} = \tfrac{1}{2}\varepsilon_{b_1b_2b_3b_4}\psi_{(W)}^{b_1c_1c_2,b_2,b_3,b_4} . \tag{A.17}$$

The symmetries of the fields on the right-hand-sides of (A.14)–(A.17) are depicted as

$$\begin{array}{|c|c|c|c|}\hline b_1 & c_1 & c_2 & c_3 \\ \hline b_2 & d \\ \cline{1-2}\end{array} , \qquad \begin{array}{|c|c|c|c|}\hline b_1 & c_1 & c_2 & c_3 \\ \hline b_2 \\ \cline{1-1} b_3 \\ \cline{1-1}\end{array} , \qquad \begin{array}{|c|c|c|}\hline b_1 & c_1 & c_2 \\ \hline b_2 & d \\ \cline{1-2} b_3 \\ \cline{1-1}\end{array} , \qquad \begin{array}{|c|c|c|}\hline b_1 & c_1 & c_2 \\ \hline b_2 \\ \cline{1-1} b_3 \\ \cline{1-1} b_4 \\ \cline{1-1}\end{array} . \tag{A.18}$$

Generators of A_1^{+++} are usually written with antisymmetric indices. To match that, we take the irreducible fields in the symmetric convention, namely $\phi^{c_1c_2c_3c_4,d_1d_2}$, $\psi_Y^{c_1c_2c_3c_4,d,e}$,

$\psi_Z^{c_1c_2c_3,d_1d_2,e}$ and $\psi_W^{c_1c_2c_3,d,e,f}$, and use them to construct irreducible fields with antisymmetric blocks of indices which obey over-antisymmetrisation identities. They are defined by

$$
\begin{aligned}
A^{a_1a_2,b_1b_2,cd} &:= \phi^{cd[a_1[b_1,b_2]a_1]} \, , && \text{(A.19)}\\
\widehat{Y}^{a_1a_2a_3,c_1,c_2,c_3} &:= \psi_{(Y)}^{c_1c_2c_3[a_1,a_2,a_3]} \, , && \text{(A.20)}\\
\widehat{Z}^{a_1a_2a_3,b_1b_2,c} &:= \psi_{(Z)}^{c[a_1[b_1,b_2]a_2,a_3]} \, , && \text{(A.21)}\\
\widehat{W}^{a_1a_2a_3a_4,c_1,c_2} &:= \psi_{(W)}^{c_1c_2[a_1,a_2,a_3,a_4]} \, , && \text{(A.22)}
\end{aligned}
$$

with inverse relations

$$
\begin{aligned}
\phi^{c_1c_2c_3c_4,d_1d_2} &= -\tfrac{12}{5}\, A^{(c_1|(d_1,d_2)|c_2,c_3,c_4)} \, , && \text{(A.23)}\\
\psi_{(Y)}^{c_1c_2c_3c_4,d,e} &= 2\, \widehat{Y}^{de(c_1,c_2,c_3,c_4)} \, , && \text{(A.24)}\\
\psi_{(Z)}^{c_1c_2c_3,d_1d_2,e} &= \tfrac{16}{5}\, \widehat{Z}^{(c_1|e(d_1,d_2)|c_2,c_3)} \, , && \text{(A.25)}\\
\psi_{(W)}^{c_1c_2c_3,d,e,f} &= 2\, \widehat{W}^{fed(c_1,c_2,c_3)} \, . && \text{(A.26)}
\end{aligned}
$$

B Representations of A_1^{+++} at the next level

l	A_3 weight	A_1^{+++} root α	α^2	mult.	field
5	$[0,1,0]$	$(2,4,7,5)$	-24	1	$A_{[4,4,2]}$
5	$[2,0,0]$	$(1,4,7,5)$	-22	1	$A_{[4,3,3]}$
5	$[0,0,2]$	$(2,4,6,5)$	-22	2	$A_{[4,4,1,1]}$
5	$[1,1,1]$	$(1,3,6,5)$	-20	5	$A_{[4,3,2,1]}$
5	$[3,0,1]$	$(0,3,6,5)$	-16	2	$A_{[3,3,3,1]}$
5	$[1,0,3]$	$(1,3,5,5)$	-16	3	$A_{[4,3,1,1,1]}$
5	$[0,3,0]$	$(1,2,6,5)$	-16	2	$A_{[4,2,2,2]}$
5	$[2,2,0]$	$(0,2,6,5)$	-14	3	$A_{[3,3,2,2]}$
5	$[0,2,2]$	$(1,2,5,5)$	-14	3	$A_{[4,2,2,1,1]}$
5	$[2,1,2]$	$(0,2,5,5)$	-12	4	$A_{[3,3,2,1,1]}$
5	$[0,1,4]$	$(1,2,4,5)$	-8	1	$A_{[4,2,1,1,1,1]}$
5	$[2,0,4]$	$(0,2,4,5)$	-6	2	$A_{[3,3,1,1,1,1]}$
5	$[1,3,1]$	$(0,1,5,5)$	-8	3	$A_{[3,2,2,2,1]}$
5	$[1,2,3]$	$(0,1,4,5)$	-4	2	$A_{[3,2,2,1,1,1]}$
5	$[0,4,2]$	$(0,0,4,5)$	2	1	$A_{[2,2,2,2,1,1]}$

Table 5. The adjoint representation of A_1^{+++} at level five.

l	A_3 weight	$A_1^{+++(2)}$ root α	α^2	mult.	field
4	$[0,1,0]$	$(2,4,6,4,1)$	-22	3	$A_{[4,4,2]}$
4	$[2,0,0]$	$(1,4,6,4,1)$	-20	3	$A_{[4,3,3]}$
4	$[0,0,2]$	$(2,4,5,4,1)$	-20	3	$A_{[4,4,1,1]}$
4	$[1,1,1]$	$(1,3,5,4,1)$	-18	12	$A_{[4,3,2,1]}$
4	$[3,0,1]$	$(0,3,5,4,1)$	-14	3	$A_{[3,3,3,1]}$
4	$[1,0,3]$	$(1,3,4,4,1)$	-14	7	$A_{[4,3,1,1,1]}$
4	$[0,3,0]$	$(1,2,5,4,1)$	-14	6	$A_{[4,2,2,2]}$
4	$[2,2,0]$	$(0,2,5,4,1)$	-12	3	$A_{[3,3,2,2]}$
4	$[0,2,2]$	$(1,2,4,4,1)$	-12	9	$A_{[4,2,2,1,1]}$
4	$[2,1,2]$	$(0,2,4,4,1)$	-10	5	$A_{[3,3,2,1,1]}$
4	$[0,1,4]$	$(1,2,3,4,1)$	-6	4	$A_{[4,2,1,1,1,1]}$
4	$[2,0,4]$	$(0,2,3,4,1)$	-4	1	$A_{[3,3,1,1,1,1]}$
4	$[1,3,1]$	$(0,1,4,4,1)$	-6	3	$A_{[3,2,2,2,1]}$
4	$[1,2,3]$	$(0,1,3,4,1)$	-2	2	$A_{[3,2,2,1,1,1]}$

Table 6. The ℓ_2 representation of A_1^{+++} at level four.

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