

Common upper frequent hypercyclicity

Monia Mestiri *

Abstract

Considering a family of upper frequently hypercyclic operators we care about the existence of vectors which are upper frequently hypercyclic for any operator of this family. We establish sufficient conditions for an uncountable family of operators to admit such vectors called common upper frequently hypercyclic vectors. Using this result, we then give a construction of such vectors. Finally we derive some applications to families of weighted shifts.

1 Introduction

In the last decades the notion of hypercyclicity has been subject to many investigations. An operator T on a Fréchet space X is called *hypercyclic* on X if there exists some vector $x \in X$ such that the set $\{T^n(x) \mid n \geq 0\}$ is dense in X . In this case, x is called *hypercyclic* for T . Birkhoff has established an equivalent formulation of hypercyclicity in [13]. In addition he proved that the set of hypercyclic vectors for a hypercyclic operator is a dense G_δ -set. In 1969 Rolewicz has found one of the first example of hypercyclic operators: the multiples of the backward shift λB , with $\lambda > 1$ on l^p .

Later Salas raised in [21] the question of the existence of vectors which are hypercyclic for any multiple of the backward shift λB , $\lambda > 1$. This initiated a new notion: common hypercyclicity. For an arbitrary family of hypercyclic operators $(T(\lambda))_{\lambda \in \Lambda}$, a vector is then called a *common hypercyclic vector* for $(T(\lambda))_{\lambda \in \Lambda}$ if it is hypercyclic for each operator of the family. The Birkhoff theorem combined with the Baire category theorem directly implies that any countable family of hypercyclic operators has a common hypercyclic vector. Actually in this case the set of common hypercyclic vectors is a dense G_δ -set. For uncountable families of operators this argument fails. Indeed even if Abakumov and Gordon have shown in [1] the existence of common hypercyclic vectors for $(\lambda B)_{\lambda > 1}$, there also exist families of operators without common hypercyclic vectors. For example Bayart and Matheron have proved in [9] that the family of hypercyclic weighted shifts on l^2 does not possess any common hypercyclic vector. However imposing structures on the index set, positive results have been obtained (see [8], [15], [23]). Most theorems on common hypercyclicity are based on the following generalization of the Birkhoff theorem for families of hypercyclic operators. The first part of this result is due to Saint-Raymond (see [1], [3]). The equivalence has been found by Shkarin in [23].

*The author is supported by a grant of F.R.S.-FNRS.

Theorem (Shkarin [23]). *Let Λ be a σ -compact metric space, X a separable Fréchet space and $(T(\lambda))_{\lambda \in \Lambda}$ a continuous family of operators on X . Then the set of common hypercyclic vectors for $(T(\lambda))_{\lambda \in \Lambda}$ is a G_δ -set. Moreover the following assertions are equivalent:*

- (i) *the set of common hypercyclic vectors for $(T(\lambda))_{\lambda \in \Lambda}$ is a dense G_δ -set;*
- (ii) *for any compact subset K of Λ , for any non-empty open subsets U, V of X , there exists some $x \in U$ such that*

$$\forall \lambda \in K, \exists n \geq 0, T^n(\lambda)(x) \in V.$$

The question of the existence of common hypercyclic vectors extends naturally to frequent hypercyclicity, a notion introduced in [7]. Recall that the *lower density* of a subset A of \mathbb{N}_0 is given by

$$\underline{\text{dens}}(A) = \liminf_{n \rightarrow \infty} \frac{\#(A \cap [0, n])}{n+1}.$$

On the other hand the *upper density* of a subset A of \mathbb{N}_0 is given by

$$\overline{\text{dens}}(A) = \limsup_{n \rightarrow \infty} \frac{\#(A \cap [0, n])}{n+1}.$$

An operator T on X is then called *frequently hypercyclic* if there exists some vector $x \in X$ such that for any non-empty open subset U of X , we have

$$\underline{\text{dens}}(\{m \geq 0 \mid T^m(x) \in U\}) > 0.$$

In this case, the vector x is called *frequently hypercyclic* for T . In contrast to hypercyclicity, Moothathu in [20], Grivaux and Matheron in [16] and Bayart and Ruzsa in [10] have proved that the set of frequently hypercyclic vectors for an operator is of first category. Therefore the methods used for common hypercyclicity fail in this case. Nonetheless some positive results have been established by Bayart and Grivaux in [5, 7]. Keeping this idea of frequency we then care about a weaker notion introduced by Shkarin in [22], the *upper frequent hypercyclicity*. An operator T on X is called *upper frequently hypercyclic* if there exists some vector $x \in X$ such that for any non-empty open subset U of X ,

$$\overline{\text{dens}}(\{m \geq 0 \mid T^m(x) \in U\}) > 0.$$

In this case the vector x is called *upper frequently hypercyclic* for T . The set of upper frequently hypercyclic vectors is denoted by $\mathcal{UFHC}(T)$. The properties of this set are more similar to those of the set of hypercyclic vectors. Indeed Bayart and Ruzsa have established in [10] that, for an upper frequently hypercyclic operator, this set is residual. Furthermore, Bonilla and Grosse-Erdmann have obtained in [14] the following theorem analogue to the Birkhoff theorem for upper frequent hypercyclicity.

Theorem (Bonilla - Grosse-Erdmann [14]). *Let X be a separable Fréchet space and T an operator on X . Then the following assertions are equivalent:*

- (i) *the set of upper frequently hypercyclic vectors for T is residual in X ;*

- (ii) for any non-empty open subset V of X , there exists some $\delta > 0$ such that for any non-empty open subset U of X and any $N \in \mathbb{N}$, we have

$$\exists x \in U, \exists n \geq N, \frac{\#\{0 \leq m \leq n \mid T^m(x) \in V\}}{n+1} > \delta.$$

Using this result we will obtain sufficient conditions for common upper frequent hypercyclicity. This yields the following theorem which acts as the generalization of the Birkhoff theorem.

Theorem 1. *Let Λ be a σ -compact metric space, X a separable Fréchet space and $(T(\lambda))_{\lambda \in \Lambda}$ a continuous family of operators on X . Suppose that for any non-empty open subset V of X and for any compact subset K of Λ there exists some $\delta > 0$ such that, for any non-empty open subset U of X and any $M \in \mathbb{N}$, we have*

$$\exists x \in U, \forall \lambda \in K, \exists n \geq M, \frac{\#\{0 \leq m \leq n \mid T^m(\lambda)(x) \in V\}}{n+1} > \delta.$$

Then the set of common upper frequently hypercyclic vectors for $(T(\lambda))_{\lambda \in \Lambda}$ is residual in X .

Thanks to this result, in order to prove the existence of common upper frequently hypercyclic vectors it suffices for each non-empty open subsets U, V of X and for each compact subset K of Λ to create a vector of U that visits frequently enough V under $T(\lambda)$, for any $\lambda \in K$. Adapting methods used for common hypercyclicity we will give a construction of such a vector in Section 2. We will then study applications to families of weighted shifts.

2 A generalization of the Birkhoff theorem for common upper frequent hypercyclicity

Our results take place in the same context as common hypercyclicity criteria. In other terms we consider families of operators indexed by a σ -compact metric space. We then introduce an additional continuity for the family of operators. A family of operators $(T(\lambda))_{\lambda \in \Lambda}$ indexed by a metric space is called *continuous* if for any $x \in X$, the map $\Lambda \rightarrow X : \lambda \mapsto T(\lambda)(x)$ is continuous on Λ . In this context, we have the two following results from [17, Chapter 11].

Proposition 2. *Let Λ be a metric space, X a Fréchet space and $(T(\lambda))_{\lambda \in \Lambda}$ a continuous family of operators on X . The map $\Lambda \times X \rightarrow X : (\lambda, x) \mapsto T(\lambda)(x)$ is then continuous on $\Lambda \times X$ with respect to the product topology.*

Corollary 3. *Let Λ be a metric space, X a Fréchet space and $(T(\lambda))_{\lambda \in \Lambda}$ a continuous family of operators on X . Then for any $n \geq 0$, the family of operators $(T^n(\lambda))_{\lambda \in \Lambda}$ is continuous.*

As announced in the introduction we start with Theorem 1 which acts as the generalization of the Birkhoff theorem for common hypercyclicity. Before stating it we have to introduce the following notation.

Notation. Let X be a Fréchet space, $(T(\lambda))_{\lambda \in \Lambda}$ a family of operators on X and V an open subset of X . For each $\lambda \in \Lambda$ and each $x \in X$, we denote by $N_\lambda(x, V)$ the set given by

$$N_\lambda(x, V) := \{n \geq 0 \mid T^n(\lambda)(x) \in V\}.$$

This is the set of visiting times for x in V under the operator $T(\lambda)$.

Proof of Theorem 1. By separability of X there exists some countable base $(V_k)_{k \geq 1}$ of non-empty open subsets of X . Moreover the σ -compactness of Λ ensures the existence of a sequence $(K_m)_{m \geq 1}$ of compact subsets of Λ such that

$$\Lambda = \bigcup_{m \geq 1} K_m. \quad (1)$$

From the hypotheses we deduce that for any $k, m \geq 1$ there exists some $\delta_{k,m} > 0$ such that for any $M \geq 1$ and any non-empty open subset U of X ,

$$\exists x \in U, \forall \lambda \in K_m, \exists n \geq M, \frac{\#(N_\lambda(x, V_k) \cap [0, n])}{n+1} > \delta_{k,m}.$$

For any $k, m \geq 1$ and any $M \geq 1$, we then define the set $E(k, m, M)$ as

$$E(k, m, M) := \left\{ x \in X \mid \forall \lambda \in K_m, \exists n \geq M, \frac{\#(N_\lambda(x, V_k) \cap [0, n])}{n+1} > \delta_{k,m} \right\}.$$

By the above these sets are clearly dense in X . So in order to prove the claim it is sufficient by the Baire category theorem to show the following assertions:

- (a) $\bigcap_{k, m \geq 1} \bigcap_{M \geq 1} E(k, m, M) \subset \bigcap_{\lambda \in \Lambda} \mathcal{UFHC}(T(\lambda));$
- (b) for any $k, m \geq 1$ and any $M \geq 1$, the set $E(k, m, M)$ is open.

We begin by the proof of (a). Let $x \in X$ be such that

$$x \in \bigcap_{k, m \geq 1} \bigcap_{M \geq 1} E(k, m, M).$$

This then implies that

$$\forall k \geq 1, \forall m \geq 1, \forall \lambda \in K_m, \overline{\text{dens}}(N_\lambda(x, V_k)) > 0.$$

Since $(K_m)_{m \geq 1}$ covers Λ , we obtain that

$$\forall k \geq 1, \forall \lambda \in \Lambda, \overline{\text{dens}}(N_\lambda(x, V_k)) > 0.$$

Moreover $(V_k)_{k \geq 1}$ is a base of non-empty open subsets of X . Therefore the previous assertion entails that x is a common upper frequently hypercyclic vector for the family $(T(\lambda))_{\lambda \in \Lambda}$, which proves (a). So it remains to show (b). Let $k, m, M \geq 1$ and $x \in E(k, m, M)$. By definition we then obtain the existence of a family of positive integers $(n_\lambda)_{\lambda \in K_m}$ such that

$$\forall \lambda \in K_m, n_\lambda \geq M \text{ and } \frac{\#(N_\lambda(x, V_k) \cap [0, n_\lambda])}{n_\lambda + 1} > \delta_{k,m}. \quad (2)$$

We must verify that there exists some open neighbourhood O of x such that $O \subset E(k, m, M)$. First we fix $\lambda \in K_m$. For any $n \in N_\lambda(x, V_k)$, V_k is an open neighbourhood of $T^n(\lambda)(x)$. Moreover from Proposition 2 and Corollary 3, we deduce that for each $n \geq 0$, the map $\Lambda \times X \rightarrow X : (\mu, y) \mapsto T^n(\mu)(y)$ is continuous at (λ, x) . This then implies that for any $n \in N_\lambda(x, V_k)$, there exists some open neighbourhood $U_{n,\lambda}$ of λ and some open neighbourhood $O_{n,\lambda}$ of x such that

$$\forall \mu \in U_{n,\lambda}, \forall y \in O_{n,\lambda}, T^n(\mu)(y) \in V_k.$$

As the set $N_\lambda(x, V_k) \cap [0, n_\lambda]$ is finite, it follows the existence of an open neighbourhood U_λ of λ and an open neighbourhood O_λ of x such that

$$\forall n \in N_\lambda(x, V_k) \cap [0, n_\lambda], \forall \mu \in U_\lambda, \forall y \in O_\lambda, T^n(\mu)(y) \in V_k.$$

Altogether we have obtained that for any $\lambda \in K_m$, there exists some open neighbourhood U_λ of λ and some open neighbourhood O_λ of x such that

$$\forall \mu \in U_\lambda, \forall y \in O_\lambda, N_\lambda(x, V_k) \cap [0, n_\lambda] \subset N_\mu(y, V_k). \quad (3)$$

In particular $(U_\lambda)_{\lambda \in K_m}$ is a family of open subsets of Λ whose union contains K_m . By compactness of K_m , there then exists some $I \geq 1$ and some $\lambda_1, \dots, \lambda_I \in K_m$ such that

$$K_m \subset \bigcup_{1 \leq i \leq I} U_{\lambda_i}.$$

We finally take O to be the open neighbourhood of x given by

$$O := \bigcap_{1 \leq i \leq I} O_{\lambda_i}.$$

By (3) we then obtain that

$$\forall y \in O, \forall \mu \in K_m, \exists 1 \leq i \leq I, N_{\lambda_i}(x, V_k) \cap [0, n_{\lambda_i}] \subset N_\mu(y, V_k).$$

On the other hand we deduce from (2) that

$$\forall 1 \leq i \leq I, n_{\lambda_i} \geq M \text{ and } \frac{\#(N_{\lambda_i}(x, V_k) \cap [0, n_{\lambda_i}])}{n_{\lambda_i} + 1} > \delta_{k,m}.$$

Combining these assertions we conclude that

$$\forall y \in O, \forall \mu \in K_m, \exists n > M, \frac{\#(N_\mu(y, V_k) \cap [0, n])}{n + 1} > \delta_{k,m}.$$

Therefore the assertion (b) is satisfied. Applying the Baire category theorem we end the proof. \square

3 Common upper frequent hypercyclicity criterion

The previous theorem gives sufficient conditions to have common upper frequently hypercyclic vectors. In this section we will use this result to establish a theorem in the same vein as the common hypercyclicity theorem from [17, Chapter 11, Theorem 11.9]. Before stating this theorem we have to recall the definition of the unconditional and uniform convergence of a family of series. To this end we will work with an F -norm on X . This notion can be found in [18].

Definition 4. Let X be a vector space. An F -norm on X is a functional $\|_-\| : X \rightarrow [0, +\infty[$ such that

- (i) for any $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$;
- (ii) for any scalar λ and any $x \in X$, if $|\lambda| \leq 1$ then $\|\lambda x\| \leq \|x\|$;
- (iii) for any $x \in X$, $\lim_{\lambda \rightarrow 0} \|\lambda x\| = 0$;
- (iv) for any $x \in X$, if $\|x\| = 0$ then $x = 0$.

Considering a Fréchet space X endowed with a separating increasing sequence of seminorms $(p_k)_{k \geq 1}$, we denote by $\|_-\|$ the F -norm on X defined, for $x \in X$, by

$$\forall x \in X, \|x\| := \sum_{k \geq 1} \frac{1}{2^k} \min(1, p_k(x)).$$

This F -norm induces the topology of X and allows to work with a norm-like functional.

Definition 5. Let X be a Fréchet space and $(x_{\lambda,n})_{(\lambda,n) \in \Lambda \times \mathbb{N}}$ a family of vectors of X . We say that the series $\sum_{n \geq 1} x_{\lambda,n}$ converges *unconditionally and uniformly* for $\lambda \in \Lambda$ if for any $\varepsilon > 0$, there exists some $N_0 \in \mathbb{N}$ such that

$$\text{for any } \lambda \in \Lambda \text{ and any finite set } F \subset \{N_0, N_0 + 1, \dots\}, \left\| \sum_{n \in F} x_{\lambda,n} \right\| < \varepsilon.$$

For the application to families of weighted shifts we will use the following sufficient conditions for uniform unconditional convergence from [17, Chapter 11].

Remark 6. Let X be a Fréchet space, $(e_n)_{n \geq 1}$ a sequence of vectors of X and $(a_{\lambda,n})_{(\lambda,n) \in \Lambda \times \mathbb{N}}$ a family of scalars. Suppose that there exists some sequence of positive numbers $(c_n)_{n \in \mathbb{N}}$ such that

- for any $\lambda \in \Lambda$ and any $n \geq 1$, $|a_{\lambda,n}| \leq c_n$;
- the series $\sum_{n \geq 1} c_n e_n$ converges unconditionally.

Then the series $\sum_{n \geq 1} a_{\lambda,n} e_n$ converges unconditionally and uniformly for $\lambda \in \Lambda$.

Theorem 7 (Common upper frequent hypercyclicity criterion). *Let Λ be a real interval, X a separable Fréchet space and $(T(\lambda))_{\lambda \in \Lambda}$ a continuous family of operators on X . Suppose that for any compact subinterval K of Λ , there exists some dense subset X_0 of X and maps $S_n(\lambda) : X_0 \rightarrow X$, $n \geq 0$, $\lambda \in K$ such that for any $x \in X_0$,*

- (i) *the series $\sum_{n=0}^m T^n(\lambda)(S_{m-n}(\mu_n)(x))$ converges unconditionally and uniformly for $m \geq 0$, $\mu_0 \geq \dots \geq \mu_m \in K$ and $\lambda \in K$;*
- (ii) *the series $\sum_{n \geq 0} T^n(\lambda)(S_{m+n}(\mu_n)(x))$ converges unconditionally and uniformly for $m \geq 0$, $(\mu_n)_{n \geq 0}$ any non-decreasing sequence from K and $\lambda \leq \mu_0 \in K$;*

(iii) for any $\varepsilon > 0$, there exists some decreasing sequence $(d_n)_{n \geq 1}$ of positive numbers such that

(a) for any $n \geq 1$ and any $\lambda, \mu \in K$,

$$\text{if } 0 \leq \mu - \lambda \leq d_n \text{ then } \|T^n(\lambda)(S_n(\mu)(x)) - x\| \leq \varepsilon;$$

(b) for any $c \in \mathbb{N}$, the series $\sum_{t \geq 1} d_{c^t}$ diverges;

(iv) $(T^n(\lambda)(x))_{n \geq 0}$ converges uniformly to 0 for $\lambda \in K$.

Then the set of common upper frequently hypercyclic vectors for $(T(\lambda))_{\lambda \in \Lambda}$ is residual in X , and in particular, non-empty.

Remark that the finite sums of the hypothesis (i) are regarded as infinite series by adding 0 terms.

Let us explain the main idea of this result before proving it.

Main idea. In view of Theorem 1, in order to obtain common upper frequent hypercyclicity it is sufficient to construct a vector z from an open subset U which visits frequently enough V under $T(\lambda_t)$, for some $\lambda_0 < \dots < \lambda_\tau \in \Lambda$. We create this vector by blocks. The first block will ensure that $z \in U$. Then in order to visit V we will add perturbations of a vector y in V . More specifically we take z to be of the following form:

$$z := x + \sum_{t=1}^{\tau} \underbrace{(S_{l_t}(\lambda_t)(y) + S_{l_t+s_0}(\lambda_t)(y) + \dots + S_{l_t+L_t s_0}(\lambda_t)(y))}_{\text{to visit } V \text{ under } T(\lambda_t) \text{ } L_t+1 \text{ times}},$$

where $x \in U$ and for any $t \in \{1, \dots, \tau\}$, we denote by

- s_0 the *gap* between two successive approximations in each block;
- l_t the *beginning* of the block for λ_t ;
- $L_t + 1$ the *number of visits* in the block for λ_t .

Notice that we only consider the visiting times with a constant gap. This is a technical necessity. In the proof we will choose $(L_t)_{t \geq 1} := (l_t)_{t \geq 1}$.

As announced this result is similar to the common hypercyclicity theorem from [17, Chapter 11, Theorem 11.9]. The main difference between these theorems is the third hypothesis. For common hypercyclicity we only ask the divergence of the series $\sum_{n \geq 1} d_n$. Here we must have the divergence of the series

but in a specific way. We select a particular subsequence of $(d_n)_{n \geq 1}$. This allows to have enough times between approximations for two successive parameters λ .

Proof. In order to obtain the claim we will apply Theorem 1. To this end we have to show that the family $(T(\lambda))_{\lambda \in \Lambda}$ satisfies the hypotheses of this result. Let K be a compact subset of Λ . We can assume without loss of generality that $K := [a, b]$ is a subinterval of Λ . From the hypotheses it then follows the existence of a dense subset X_0 of X and maps $S_n(\lambda) : X_0 \rightarrow X$, $n \geq 0$, $\lambda \in K$ such that the assertions (i), (ii), (iii) and (iv) are satisfied. Let V be a non-empty open subset of X . Since X_0 is dense in X we can suppose that $V := B(y, \varepsilon)$ with some $y \in X_0$ and some $\varepsilon > 0$. By (iii) there exists some decreasing sequence $(d_n)_{n \geq 1}$ of positive numbers such that

(a) for any $n \geq 1$ and any $\lambda, \mu \in K$,

$$\text{if } 0 \leq \mu - \lambda \leq d_n \text{ then } \|T^n(\lambda)(S_n(\mu)(y)) - y\| \leq \frac{\varepsilon}{4};$$

(b) for any $c \in \mathbb{N}$, the series $\sum_{t \geq 1} d_{c^t}$ diverges.

On the other hand we obtain by (i) and (ii) the existence of a positive integer s_0 such that for any finite subset F of $\{s_0, s_0 + 1, \dots\}$ and any $m \geq 0$ we have that

(I) for any $\mu_0 \geq \dots \geq \mu_m \in [a, b]$ and any $\lambda \in [a, b]$,

$$\left\| \sum_{n \in F \cap [0, m]} T^m(\lambda)(S_{m-n}(\mu_n)(y)) \right\| < \frac{\varepsilon}{4};$$

(II) for any non-decreasing sequence $(\mu_n)_{n \geq 0}$ from $[a, b]$ and any $\lambda \leq \mu_0 \in [a, b]$,

$$\left\| \sum_{n \in F} T^m(\lambda)(S_{m+n}(\mu_n)(y)) \right\| < \frac{\varepsilon}{4}.$$

We then take $\delta := \frac{1}{2+s_0} > 0$. Let U be a non-empty open subset of X and $M \geq 1$. Once again, as X_0 is dense in X we can assume that $U := B(x, r)$ with some $x \in X_0$ and some $r > 0$. From (ii) with $m = 0$ and (iv) we deduce the existence of a positive integer N_0 such that for any finite subset F of $\{N_0, N_0 + 1, \dots\}$,

(III) for any non-decreasing sequence $(\mu_n)_{n \geq 0}$ from $[a, b]$,

$$\left\| \sum_{n \in F} S_n(\mu_n)(y) \right\| < r;$$

(IV) for any $n \geq N_0$ and any $\lambda \in [a, b]$, $\|T^n(\lambda)(x)\| < \frac{\varepsilon}{4}$.

Furthermore by taking $c := \max(N_0, 2 + s_0, M)$, the assertion (b) ensures that the series

$$\sum_{t \geq 1} d_{c^{t+1}} \text{ diverges.}$$

Therefore there exists some integer $\tau \geq 1$ such that

$$a + \sum_{t=1}^{\tau-1} d_{c^{t+1}} \leq b < a + \sum_{t=1}^{\tau} d_{c^{t+1}}.$$

So we obtain the subdivision $(\lambda_t)_{0 \leq t \leq \tau}$ of $[a, b]$ defined by

$$\lambda_\tau := b \text{ and for any } 0 \leq t \leq \tau - 1, \lambda_t := a + \sum_{s=1}^t d_{c^{s+1}}.$$

We finally consider the sequence $(l_t)_{t \geq 1} := (c^t)_{t \geq 1}$ and the vector z given by

$$z := x + \sum_{t=1}^{\tau} \sum_{l=0}^{l_t} S_{l_t+l_{s_0}}(\lambda_t)(y).$$

We observe that for any $1 \leq t \leq \tau$, $l_t + l_s s_0 < l_{t+1}$, which implies that the integers $l_t + l_s s_0$, with $0 \leq l \leq l_t$ and $t \geq 1$ are pairwise distinct. Combining this with (III) this entails that

$$\|z - x\| = \left\| \sum_{t=1}^{\tau} \sum_{l=0}^{l_t} S_{l_t + l_s s_0}(\lambda_t)(y) \right\| < r.$$

Moreover, considering $1 \leq t \leq \tau$, $\lambda \in [\lambda_{t-1}, \lambda_t]$ and $0 \leq l \leq l_t$, we have that

$$\begin{aligned} T^{l_t + l_s s_0}(\lambda)(z) &= T^{l_t + l_s s_0}(\lambda)(x) + \sum_{s=1}^{t-1} \sum_{k=0}^{l_s} T^{l_t + l_s s_0}(\lambda)(S_{l_s + k s_0}(\lambda_s)(y)) \\ &\quad + \sum_{k=0}^{l-1} T^{l_t + l_s s_0}(\lambda)(S_{l_t + k s_0}(\lambda_t)(y)) + T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0}(\lambda_t)(y)) \\ &\quad + \sum_{k=l+1}^{l_t} T^{l_t + l_s s_0}(\lambda)(S_{l_t + k s_0}(\lambda_t)(y)) + \sum_{s=t+1}^{\tau} \sum_{k=0}^{l_s} T^{l_t + l_s s_0}(\lambda)(S_{l_s + k s_0}(\lambda_s)(y)), \end{aligned}$$

which may be written equivalently as

$$\begin{aligned} T^{l_t + l_s s_0}(\lambda)(z) &= T^{l_t + l_s s_0}(\lambda)(x) + T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0}(\lambda_t)(y)) \\ &\quad + \left(\sum_{s=1}^{t-1} \sum_{k=0}^{l_s} T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0 - (l_t + l_s s_0 - l_s - k s_0)}(\lambda_s)(y)) \right. \\ &\quad \left. + \sum_{k=0}^{l-1} T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0 - (l_t + l_s s_0 - l_t - k s_0)}(\lambda_t)(y)) \right) \\ &\quad + \left(\sum_{k=l+1}^{l_t} T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0 + (l_t + k s_0 - l_t - l_s s_0)}(\lambda_t)(y)) \right. \\ &\quad \left. + \sum_{s=t+1}^{\tau} \sum_{k=0}^{l_s} T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0 + (l_s + k s_0 - l_t - l_s s_0)}(\lambda_s)(y)) \right). \end{aligned}$$

As observed previously, the integers $l_s + k s_0$, with $0 \leq k \leq l_s$ and $s \geq 1$ are pairwise distinct. Since $\lambda \in [\lambda_{t-1}, \lambda_t]$, it follows from (I) and (II) that

$$\|T^{l_t + l_s s_0}(\lambda)(z) - y\| < \|T^{l_t + l_s s_0}(\lambda)(x)\| + \|T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0}(\lambda_t)(y)) - y\| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}.$$

Moreover by definition, we have that $l_t + l_s s_0 \geq c = \max(N_0, 2 + s_0, M)$. Whence $l_t + l_s s_0 \geq N_0$. From (IV) we then deduce that

$$\|T^{l_t + l_s s_0}(\lambda)(z) - y\| < \frac{3}{4}\varepsilon + \|T^{l_t + l_s s_0}(\lambda)(S_{l_t + l_s s_0}(\lambda_t)(y)) - y\|. \quad (4)$$

On the other hand, since $\lambda \in [\lambda_{t-1}, \lambda_t]$, we have that

$$0 \leq \lambda_t - \lambda \leq \lambda_t - \lambda_{t-1} \leq d_{c^{t+1}} = d_{l_{t+1}}.$$

But the sequence $(d_n)_{n \geq 1}$ is decreasing and $l_t + l_s s_0 < l_{t+1}$. This then entails that

$$0 \leq \lambda_t - \lambda \leq d_{l_t + l_s s_0}.$$

So by (a) we have that

$$\|T^{l_t+ls_0}(\lambda)(S_{l_t+ls_0}(\lambda_t)(y)) - y\| \leq \frac{\varepsilon}{4}.$$

Together with (4) we conclude that

$$\|T^{l_t+ls_0}(\lambda)(z) - y\| < \varepsilon.$$

Altogether we have shown that $z \in U$ and

$$\forall 1 \leq t \leq \tau, \forall \lambda \in [\lambda_{t-1}, \lambda_t], \forall 0 \leq l \leq l_t, T^{l_t+ls_0}(\lambda)(z) \in V.$$

In particular this implies that

$$\forall 1 \leq t \leq \tau, \forall \lambda \in [\lambda_{t-1}, \lambda_t], \frac{\#(N_\lambda(z, V) \cap [0, l_t + l_t s_0])}{l_t + l_t s_0 + 1} > \frac{1}{2 + s_0}.$$

As $l_1 \geq M$, we have obtained that $z \in U$ and

$$\forall \lambda \in [a, b], \exists n \geq M, \frac{\#(N_\lambda(z, V) \cap [0, n])}{n + 1} > \frac{1}{2 + s_0},$$

which ends the proof. \square

4 Application to weighted shifts

In this section we give examples of families that have common upper frequently hypercyclic vectors. To this end we will consider families of weighted shifts. These operators form a rich source of examples in linear dynamics.

Definition 8. Let $p \geq 1$ and $w = (w_n)_{n \geq 1}$ be a sequence of nonzero scalars. The *weighted shift* B_w is the map on l^p defined, for $x = (x_n)_{n \geq 0} \in l^p$, by

$$B_w(x) := (w_{n+1}x_{n+1})_{n \geq 0},$$

where l^p is the Banach space of sequences $(x_n)_{n \geq 0}$ such that

$$\sum_{n \geq 0} |x_n|^p < +\infty.$$

Weighted shifts on c_0 , the Banach space of sequences converging to 0, are defined similarly.

We mention that a weighted shift B_w is an operator on l^p or on c_0 if and only if the sequence w is bounded. Furthermore a weighted shift is upper frequently hypercyclic on l^p if and only if

$$\text{the series } \sum_{\nu \geq 1} \frac{1}{|w_1 \dots w_\nu|^p} \text{ converges.}$$

A characterisation of upper frequent hypercyclicity for c_0 is also known but more complicated. These characterisations are due to Bayart and Ruzsa [10].

More generally we may define these operators on Fréchet spaces which possess an unconditional basis.

Definition 9. Let X be a Fréchet space, $(e_n)_{n \geq 0}$ an unconditional basis of X and $w = (w_n)_{n \geq 1}$ a sequence of nonzero scalars. The *weighted shift* B_w is the map on X defined, for $x = \sum_{n \geq 0} x_n e_n \in X$, by

$$B_w(x) := \sum_{n \geq 0} w_{n+1} x_{n+1} e_n.$$

Remark that this map is not necessarily well defined.

We derive from Theorem 7 the following result for families of weighted shifts. This result is based on a special case of a common hypercyclicity theorem for families of weighted shifts due to Bayart and Matheron [8].

Theorem 10. *Let Λ be a real interval, X a Fréchet space, $(e_n)_{n \geq 0}$ an unconditional basis of X and $(w_n(\lambda))_{(n, \lambda) \in \mathbb{N} \times \Lambda}$ a family of positive numbers such that for any $\lambda \in \Lambda$, $B_{w(\lambda)}$ is an operator on X . Suppose that*

- (i) *for any $n \geq 1$, the function $w_n : \Lambda \rightarrow \mathbb{R} : \lambda \mapsto w_n(\lambda)$ is increasing;*
- (ii) *for any $\lambda \in \Lambda$, the series*

$$\sum_{\nu \geq 1} \frac{1}{w_1(\lambda) \dots w_\nu(\lambda)} e_\nu \text{ converges in } X;$$

- (iii) *for any compact subinterval K of Λ and any $n \in \mathbb{N}$, there exists some $L_n(K) > 0$ such that*

$$\forall \lambda, \mu \in K, |\log(w_n(\lambda)) - \log(w_n(\mu))| \leq L_n(K) |\lambda - \mu|;$$

- (iv) *for any compact subinterval K of Λ and any $s \in \mathbb{N}$, the series*

$$\sum_{t \geq 1} \left(\sum_{i=1}^{s^t} L_i(K) \right)^{-1} \text{ diverges.}$$

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda \in \Lambda}$ is residual in X , and in particular, non-empty.

Proof. We will apply Theorem 7. To this end we must prove that the family $(B_{w(\lambda)})_{\lambda \in \Lambda}$ satisfies the hypotheses of this result. We begin by proving that $(B_{w(\lambda)})_{\lambda \in \Lambda}$ is a continuous family of operators on X . Let $x \in X$ and $K = [a, b]$ be a compact subinterval of Λ . As $B_{w(b)}$ is an operator on X and $(e_n)_{n \geq 0}$ is an unconditional basis of X , the series

$$\sum_{n \geq 0} w_{n+1}(b) x_{n+1} e_n$$

converges unconditionally. Together with (i) this entails by Remark 6 that the series

$$\sum_{n \geq 0} w_{n+1}(\lambda) x_{n+1} e_n$$

converges unconditionally and uniformly for $\lambda \in K$. Using the continuity of maps $\Lambda \rightarrow \mathbb{C} : \lambda \mapsto w_n(\lambda)$, $n \geq 0$, we then conclude easily the continuity of the map $\Lambda \rightarrow X : \lambda \mapsto B_{w(\lambda)}(x)$ on K . This yields that $(B_{w(\lambda)})_{\lambda \in \Lambda}$ is continuous.

Now we care about the main conditions of Theorem 7. We fix $K = [a, b]$ a compact subinterval of Λ . We take $X_0 := \text{span}\{e_\nu \mid \nu \geq 0\}$. We know that this space is dense in X . We also consider for any $\lambda \in K$, the map $F(\lambda)$ on X_0 defined by

$$\forall x = \sum_{\nu=0}^J x_\nu e_\nu \in X_0, \quad F(\lambda)(x) = \sum_{\nu=0}^J \frac{1}{w_{\nu+1}(\lambda)} x_\nu e_{\nu+1}.$$

We then take for any $\lambda \in K$ and any $n \geq 0$, $S_n(\lambda) := F^n(\lambda)$. So we have to show that for any $x \in X_0$, the assertions (I), (II), (III) and (IV) are satisfied. We observe that for the assertions (I), (II) and (IV) it is sufficient to prove them for e_ν , $\nu \geq 0$. Firstly we remark that for any nonnegative integer ν ,

$$\forall \lambda \in K, \forall n \geq 0, \quad B_{w(\lambda)}^n(e_\nu) = \begin{cases} w_{\nu-n+1}(\lambda) \dots w_\nu(\lambda) e_{\nu-n} & \text{if } \nu \geq n, \\ 0 & \text{else.} \end{cases} \quad (5)$$

By definition of $(F(\lambda))_{\lambda \in K}$, we also obtain that for any nonnegative integer ν ,

$$\forall \lambda \in K, \forall n \geq 0, \quad F^n(\lambda)(e_\nu) = \frac{1}{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)} e_{\nu+n}. \quad (6)$$

We immediately conclude from (5) that the assertion (IV) is satisfied for each e_ν , $\nu \geq 0$. Now we care about the assertions (I) and (II). Let $\nu \geq 0$. It follows from (5) and (6) that

$$\forall \lambda, \mu \in K, \forall m \geq n > \nu, \quad B_{w(\lambda)}^m(F^{m-n}(\mu)(e_\nu)) = 0,$$

which implies (I). Let $m \geq 0$, $(\mu_n)_{n \geq 0}$ be a non-decreasing sequence from K and $\lambda \leq \mu_0 \in K$. By (5) and (6) we have that

$$\forall n \geq 0, \quad B_{w(\lambda)}^m(F^{m+n}(\mu_n)(e_\nu)) = \frac{w_{\nu+n+1}(\lambda) \dots w_{\nu+m+n}(\lambda)}{w_{\nu+1}(\mu_n) \dots w_{\nu+m+n}(\mu_n)} e_{\nu+n}.$$

Thus we obtain that

$$\sum_{n \geq 0} B_{w(\lambda)}^m(F^{m+n}(\mu_n)(e_\nu)) = \sum_{n \geq 0} \frac{w_{\nu+n+1}(\lambda) \dots w_{\nu+m+n}(\lambda)}{w_{\nu+1}(\mu_n) \dots w_{\nu+m+n}(\mu_n)} e_{\nu+n}. \quad (7)$$

Furthermore since $K = [a, b]$, we have by the hypothesis (i) that

$$\forall n \geq 0, \quad \left| \frac{w_{\nu+n+1}(\lambda) \dots w_{\nu+m+n}(\lambda)}{w_{\nu+1}(\mu_n) \dots w_{\nu+m+n}(\mu_n)} \right| \leq \frac{1}{w_{\nu+1}(a) \dots w_{\nu+n}(a)}. \quad (8)$$

However by the hypothesis (ii) the series

$$\sum_{n \geq 0} \frac{1}{w_{\nu+1}(a) \dots w_{\nu+n}(a)} e_{\nu+n} \text{ converges in } X.$$

Combining this with (7) and (8) we deduce from Remark 6 the assertion (II). Finally it remains to show the assertion (III) for each $x \in X_0$. Let $x \in X_0$. So, by definition of X_0 , there exists some $J \geq 0$ and scalars x_0, \dots, x_J such that

$$x := \sum_{\nu=0}^J x_\nu e_\nu.$$

Let $\varepsilon > 0$. By the hypothesis (iii), there exists some sequence $(L_n)_{n \geq 1}$ of positive numbers such that

$$\forall n \geq 1, \forall \lambda, \mu \in K, |\log(w_n(\lambda)) - \log(w_n(\mu))| \leq L_n |\lambda - \mu|. \quad (9)$$

On the other hand, by properties of F -norms, there exists some $\eta > 0$ such that

$$\forall \xi \in \mathbb{C}, \left[|\xi| \leq \eta \Rightarrow \left(\forall 0 \leq \nu \leq J, \|\xi x_\nu e_\nu\| \leq \frac{\varepsilon}{J+1} \right) \right]. \quad (10)$$

We take $(d_n)_{n \geq 1}$ the sequence of positive numbers given by

$$\forall n \geq 1, d_n := \eta \left(\sum_{i=1}^{n+J} L_i \right)^{-1}.$$

Now we have to prove the assertions (a) and (b) of (III). By definition of $(d_n)_{n \geq 1}$, the hypothesis (iv) directly gives (b). Let $n \geq 1$ and $\lambda, \mu \in K$ be such that $0 \leq \mu - \lambda \leq d_n$. By (5) and (6) we obtain that

$$\|B_{w(\lambda)}^n(F^n(\mu)(x)) - x\| \leq \sum_{\nu=0}^J \left\| \left(\frac{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \dots w_{\nu+n}(\mu)} - 1 \right) x_\nu e_\nu \right\|. \quad (11)$$

Moreover as $\lambda \leq \mu$ the hypothesis (i) implies that

$$\forall 0 \leq \nu \leq J, \left| \frac{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \dots w_{\nu+n}(\mu)} - 1 \right| = 1 - \frac{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \dots w_{\nu+n}(\mu)},$$

which may be written equivalently as

$$\forall 0 \leq \nu \leq J, \left| \frac{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \dots w_{\nu+n}(\mu)} - 1 \right| = 1 - e^{-\sum_{i=1}^n \log(w_{\nu+i}(\mu)) - \log(w_{\nu+i}(\lambda))}.$$

In other terms we have that

$$\forall 0 \leq \nu \leq J, \left| \frac{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \dots w_{\nu+n}(\mu)} - 1 \right| = \int_{-\sum_{i=1}^n \log(w_{\nu+i}(\mu)) - \log(w_{\nu+i}(\lambda))}^0 e^\xi d\xi.$$

Whence we obtain that

$$\forall 0 \leq \nu \leq J, \left| \frac{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \dots w_{\nu+n}(\mu)} - 1 \right| \leq \sum_{i=1}^n (\log(w_{\nu+i}(\mu)) - \log(w_{\nu+i}(\lambda))).$$

Thus by (9) this entails that

$$\forall 0 \leq \nu \leq J, \left| \frac{w_{\nu+1}(\lambda) \dots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \dots w_{\nu+n}(\mu)} - 1 \right| \leq \left(\sum_{i=1}^n L_{\nu+i} \right) (\mu - \lambda).$$

Since $0 \leq \mu - \lambda \leq d_n$, it then follows from the definition of d_n that

$$\forall 0 \leq \nu \leq J, \left| \frac{w_{\nu+1}(\lambda) \cdots w_{\nu+n}(\lambda)}{w_{\nu+1}(\mu) \cdots w_{\nu+n}(\mu)} - 1 \right| \leq \eta.$$

By (10) and (11) we then conclude that

$$\|B_{w(\lambda)}^n(F^n(\mu)(x)) - x\| \leq \varepsilon.$$

Therefore the assertion (a) is satisfied. Altogether we have shown that Theorem 7 can be applied to the family $(B_{w(\lambda)})_{\lambda \in \Lambda}$. This implies the claim. \square

Corollary 11. *Let Λ be a real interval, X a Fréchet space, $(e_n)_{n \geq 0}$ an unconditional basis of X and $(w_n(\lambda))_{(n, \lambda) \in \mathbb{N} \times \Lambda}$ a family of positive numbers such that for any $\lambda \in \Lambda$, $B_{w(\lambda)}$ is an operator on X . Suppose that*

- (i) *for any $n \geq 1$, the function $w_n : \Lambda \rightarrow \mathbb{R} : \lambda \mapsto w_n(\lambda)$ is increasing;*
- (ii) *for any $\lambda \in \Lambda$, the series*

$$\sum_{\nu \geq 1} \frac{1}{w_1(\lambda) \cdots w_\nu(\lambda)} e_\nu \text{ converges in } X;$$

- (iii) *for any compact subinterval K of Λ , there exists some sequence of positive numbers $(L_n(K))_{n \geq 1}$ such that*

- (a) *the series $\sum_{n \geq 1} L_n(K)$ converges;*
- (b) *for any $n \geq 1$ and any $\lambda, \mu \in K$,*

$$|\log(w_n(\lambda)) - \log(w_n(\mu))| \leq L_n(K)|\lambda - \mu|.$$

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda \in \Lambda}$ is residual in X , and in particular, non-empty.

Proof. This follows directly from Theorem 10. \square

As an application of this corollary we obtain the propositions below which give us a general form of some families with common upper frequently hypercyclic vectors on l^p and on c_0 .

Proposition 12. *Let $p \geq 1$ and $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ two sequences of nonnegative numbers. We consider the family of sequences $(w(\lambda))_{\lambda \in \mathbb{R}}$ defined by*

$$\forall \lambda \in \mathbb{R}, w(\lambda) := (a_n e^{\lambda b_n})_{n \geq 1}.$$

Suppose that

- (I) *the sequence $(a_n)_{n \geq 1}$ is a bounded sequence of positive numbers;*
- (II) *the series $\sum_{n \geq 1} b_n$ and $\sum_{n \geq 1} \left(\prod_{i=1}^n a_i \right)^{-p}$ converge.*

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda \in \Lambda}$ is residual in l^p , and in particular, non-empty.

Proposition 13. Let $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ be two sequences of nonnegative numbers. We consider the family of sequences $(w(\lambda))_{\lambda \in \mathbb{R}}$ defined by

$$\forall \lambda \in \mathbb{R}, w(\lambda) := (a_n e^{\lambda b_n})_{n \geq 1}.$$

Suppose that

- (I) the sequence $(a_n)_{n \geq 1}$ is a bounded sequence of positive numbers;
- (II) the series $\sum_{n \geq 1} b_n$ converges;
- (III) the sequence $\left(\prod_{i=1}^n a_i\right)_{n \geq 1}$ converges to $+\infty$.

Then the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda \in \Lambda}$ is residual in c_0 , and in particular, non-empty.

We also give an example which cannot be obtained with Proposition 12. This actually illustrates the fact that the hypotheses of Corollary 11 are stronger than those of Theorem 10.

Example 14. Let $p \geq 1$. We consider the family of sequences $(w(\lambda))_{\lambda > \frac{1}{p}}$ defined by

$$\forall \lambda > \frac{1}{p}, w(\lambda) := \left(\left(\frac{n+1}{n}\right)^\lambda\right)_{n \geq 1}.$$

We observe that for any $n \geq 1$ and any $\lambda, \mu > \frac{1}{p}$,

$$|\log(w_n(\lambda)) - \log(w_n(\mu))| = \log\left(\frac{n+1}{n}\right)|\lambda - \mu|.$$

By taking $(L_n(K))_{n \geq 1} := (\log(\frac{n+1}{n}))_{n \geq 1}$, we deduce from Theorem 10 that the set of common upper frequently hypercyclic vectors for $(B_{w(\lambda)})_{\lambda > \frac{1}{p}}$ is residual in l^p .

Finally we care about the historical example of families with common hypercyclic vectors: the multiples of the backward shift $(\lambda B)_{\lambda > 1}$. We easily observe that this family does not satisfy the hypotheses of our results. Actually we have shown in [19] that the family of the multiples of an operator does not admit any common upper frequently hypercyclic vector. This will be the subject of another paper.

5 Common \mathcal{A} -hypercyclicity

We end this paper with a generalization to \mathcal{A} -hypercyclicity. This notion, created by Bès, Menet, Peris and Puig in [12], generalizes hypercyclicity and upper frequent hypercyclicity. To define this we consider Furstenberg families.

Definition 15. Let \mathcal{A} be a non-empty family of subsets of \mathbb{N}_0 . We say that \mathcal{A} is a *Furstenberg family* if it is *hereditary upward*, that is, if

$$\forall A \in \mathcal{A}, \forall B \subset \mathbb{N}_0, (A \subset B \Rightarrow B \in \mathcal{A}).$$

Moreover a Furstenberg family \mathcal{A} is called *upper* if \mathcal{A} does not contain the empty set and if there exists some arbitrary set D , some countable set M and some family $(\mathcal{A}_{\delta,\mu})_{(\delta,\mu) \in D \times M}$ of subsets of \mathbb{N}_0 such that \mathcal{A} can be written as

$$\mathcal{A} = \bigcup_{\delta \in D} \mathcal{A}_\delta \text{ with } \mathcal{A}_\delta := \bigcap_{\mu \in M} \mathcal{A}_{\delta,\mu}$$

and satisfying the following properties:

(i) \mathcal{A} is *uniformly left-invariant*, that is,

$$\forall A \in \mathcal{A}, \exists \delta \in D, \forall k \in \mathbb{N}_0, (A - k) \cap \mathbb{N}_0 \in \mathcal{A}_\delta;$$

(ii) for any $\delta \in D$ and any $\mu \in M$, the family $\mathcal{A}_{\delta,\mu}$ is *finitely hereditary upward*, that is,

$$\forall A \in \mathcal{A}_{\delta,\mu}, \exists F \subset \mathbb{N}_0 \text{ finite}, \forall B \subset \mathbb{N}_0, (A \cap F \subset B \Rightarrow B \in \mathcal{A}_{\delta,\mu}).$$

Definition 16. Let X be a Fréchet space, T an operator on X and \mathcal{A} a Furstenberg family. We say that T is \mathcal{A} -*hypercyclic* on X if there exists some vector $x \in X$ such that

$$\text{for any non-empty open subset } V \text{ of } X, \{n \geq 0 \mid T^n(x) \in V\} \in \mathcal{A}.$$

In this case, x is called \mathcal{A} -*hypercyclic* for T .

Actually Bonilla and Grosse-Erdmann have obtained in [14] an analogue to the Birkhoff theorem for \mathcal{A} -hypercyclicity with \mathcal{A} an upper Furstenberg family. Using similar arguments as in the proof of Theorem 1, we establish the following generalization of Theorem 1 for common \mathcal{A} -hypercyclicity.

Theorem 17. *Let \mathcal{A} be an upper Furstenberg family written as*

$$\mathcal{A} := \bigcup_{\delta \in D} \bigcap_{\mu \in M} \mathcal{A}_{\delta,\mu}.$$

Let Λ be a σ -compact metric space, X a separable Fréchet space and $(T(\lambda))_{\lambda \in \Lambda}$ a continuous family of operators on X . Suppose that for any non-empty open subset V of X and for any compact subset K of Λ there exists some $\delta \in D$ such that, for any non-empty open subset U of X and any $\mu \in M$, we have

$$\exists x \in U, \forall \lambda \in K, N_\lambda(x, V) \in \mathcal{A}_{\delta,\mu}.$$

Then the set of common \mathcal{A} -hypercyclic vectors for $(T(\lambda))_{\lambda \in \Lambda}$ is residual in X .

This theorem allows to adapt results of Sections 2 and 3 to common \mathcal{A} -hypercyclicity. So we can obtain common \mathcal{A} -hypercyclicity criteria for some families \mathcal{A} (see [19]).

In this paper we only study the existence of common upper frequently hypercyclic vectors adapting results for common hypercyclicity. It will be interesting to investigate the existence of common upper frequently hypercyclic subspaces. Indeed there exists some results for common hypercyclic subspaces (see [2, 4]). Moreover Bès and Menet have recently obtained results for upper frequently hypercyclic subspaces in [11].

Acknowledgments I thank the referee for useful comments and suggestions.

References

- [1] E. Abakumov and J. Gordon. Common hypercyclic vectors for multiples of backward shift. *J. Funct. Anal.*, 200:494–504, 2003.
- [2] R. Aron, J. Bès, F. León and A. Peris. Operators with common hypercyclic subspaces. *J. Operator Theory*, 54:251–260, 2005.
- [3] F. Bayart. Common hypercyclic vectors for composition operators. *J. Operator Theory*, 52:353–370, 2004.
- [4] F. Bayart. Common hypercyclic subspaces. *Integral Equations Operator Theory*, 53(4):467–476, 2005.
- [5] F. Bayart. Common hypercyclic vectors for high-dimensional families of operators. *International Mathematics Research Notices*, 2016(21):6512–6552, 2015.
- [6] F. Bayart and S. Grivaux. Hypercyclicité: le rôle du spectre ponctuel unimodulaire. *C. R. Math. Acad. Sci. Paris*, 338(9):703–708, 2004.
- [7] F. Bayart and S. Grivaux. Frequently hypercyclic operators. *Trans. Amer. Math. Soc.*, 358(11):5083–5117, 2006.
- [8] F. Bayart and E. Matheron. How to get common universal vectors. *Indiana Univ. Math. J.*, 56:553–580, 2007.
- [9] F. Bayart and E. Matheron. *Dynamics of linear operators*. Cambridge University Press, Cambridge, 2009.
- [10] F. Bayart and I. Z. Ruzsa. Difference sets and frequently hypercyclic weighted shifts. *Ergodic Theory Dynam. Systems*, 35(03):691–709, 2015.
- [11] J. Bès and Q. Menet. Existence of common and upper frequently hypercyclic subspaces. *J. Math. Anal. Appl.*, 432(1):10–37, 2015.
- [12] J. Bès, Q. Menet, A. Peris, and Y. Puig. Recurrence properties of hypercyclic operators. *Math. Ann.*, 366:545–572, 2014.
- [13] G.D. Birkhoff. Surface transformations and their dynamical applications. *Acta Math.*, 43(1):1–119, 1922.
- [14] A. Bonilla and K.-G. Grosse-Erdmann. Upper frequent hypercyclicity and related notions. *Rev. Mat. Complut.*, To appear.
- [15] G. Costakis and M. Sambarino. Genericity of wild holomorphic functions and common hypercyclic vectors. *Adv. Math.*, 182:278–306, 2004.
- [16] S. Grivaux and E. Matheron. Invariant measures for frequently hypercyclic operators. *Adv. Math.*, 265:371–427, 2014.
- [17] K.-G. Grosse-Erdmann and A. Peris. *Linear Chaos*. Springer, London, 2011.
- [18] N.J. Kalton, N.T. Peck, and J.W. Roberts. *An F space sampler*. Cambridge University Press, Cambridge, 1984.

- [19] M. Mestiri. *Common \mathcal{A} -hypercyclicity*. Thesis, University of Mons, Mons, 2018.
- [20] T. S. Moothathu. Two remarks on frequent hypercyclicity. *J. Math. Anal. Appl.*, 408(2):843–845, 2013.
- [21] H.N. Salas. Supercyclic and weighted shifts. *Studia Math.*, 135:55–74, 1999.
- [22] S. Shkarin. On the spectrum of frequently hypercyclic operators. *Proc. Amer. Math. Soc.*, 137(1):123–134, 2009.
- [23] S. Shkarin. Remarks on common hypercyclic vectors. *J. Funct. Anal.*, 258:132–160, 2010.