

TOPOLOGICAL FIELDS WITH A GENERIC DERIVATION

PABLO CUBIDES KOVACSICS AND FRANÇOISE POINT^(†)

ABSTRACT. We study a class of tame \mathcal{L} -theories T of topological fields and their \mathcal{L}_δ -extension T_δ^* by a generic derivation δ . The topological fields under consideration include henselian valued fields of characteristic 0 and real closed fields. We show that the associated expansion by a generic derivation has \mathcal{L} -open core (i.e., every \mathcal{L}_δ -definable open set is \mathcal{L} -definable) and derive both a cell decomposition theorem and a transfer result of elimination of imaginaries. Other tame properties of T such as relative elimination of field sort quantifiers, NIP and distality also transfer to T_δ^* . As an application, we derive consequences for the corresponding theories of dense pairs. In particular, we show that the theory of pairs of real closed fields (resp. of p -adically closed fields and real closed valued fields) admits a distal expansion. This gives a partial answer to a question of P. Simon.

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INTRODUCTION

The study of topological fields with a derivation has been traditionally divided in two main branches. The first branch, as studied (in chronological order) in [36, 37, 2, 34], treats the case where some compatibility between the derivation and the topology is assumed (*e.g.*, continuity). The second branch, as studied in [41, 44, 16, 30, 17, 14], deals with the case where no such compatibility is required but rather a *generic* behaviour of the derivation occurs. An example of such a generic behaviour arises in existentially closed ordered differential fields, a class studied and axiomatized by M. Singer in [41].

2010 *Mathematics Subject Classification.* Primary 12L12, 12J25, 12H05; Secondary 13N15, 03C60.

Key words and phrases. Topological fields, differential fields, generic derivations, elimination of imaginaries, open core.

(†) Research director at the Fonds National de la Recherche Scientifique (FNRS-FRS).

Each branch seems to tackle different aspects of differential fields and has its own applications.

The purpose of this article is to further develop the study of generic derivations and show that many tame properties of theories of topological fields transfer to their expansions by such derivations. Examples of the topological fields under consideration include real closed fields and henselian valued fields of characteristic 0. We adopt a uniform treatment and development of such topological fields in the spirit of L. Mathews [26] and A. Pillay [29], which we consider interesting on its own. As an application of generic derivations, we derive consequences for the corresponding theories of dense pairs of topological fields (as studied in [35, 24, 46, 3, 13], to mention a few), supporting the idea that this framework is a useful tool to study such pairs of structures.

The following section gathers a detailed overview of our main results.

MAIN RESULTS

In the first section of this article we introduce the theories of topological fields we will be concerned with, which we call *open theories of topological fields*. Informally, an open theory of topological fields is a first order topological theory of fields in the sense of A. Pillay [29] (i.e., the topology is uniformly definable) in which definable sets are finite boolean combinations of Zariski closed sets and open sets. However, in contrast to [29], our setting explicitly allows multi-sorted languages. Examples include complete theories of henselian valued fields of characteristic 0 and the theory of real closed fields. The formal definition will be given in Section 1.

The main focus of the present article is to study expansions of open theories of topological fields by *generic* derivations. Let T be an open \mathcal{L} -theory of topological fields and T_δ be the theory T together with axioms stating that δ is a derivation on the field sort (in the extension \mathcal{L}_δ of \mathcal{L} by δ). We define an \mathcal{L}_δ -extension T_δ^* of T_δ informally as follows. Models of T_δ^* satisfy that, for any unary differential polynomial P , if the ordinary polynomial associated with P has a regular solution a , then one can find differential solutions of P arbitrarily close to a . A derivation δ on a model K of T is *generic* if (K, δ) is a model of T_δ^* .

When T is the theory of real closed fields, the theory T_δ^* corresponds to the theory of closed ordered differential fields CODF as originally introduced and axiomatized by M. Singer in [41]. The idea behind CODF has been generalized to many different contexts including work by M. Tressl [44] and N. Solancki [42] in the framework of large fields, and by N. Guzy and the second author in [16, 17]. As in [16, 17], we will closely follow Singer's original axiomatization. The main difference in the present setting with respect to previous work is the explicit allowance of multi-sorted languages (which permits us to include theories in languages where they admit relative quantifier elimination).

How much complexity is introduced by a generic derivation? Two of our main contributions show that the derivation introduces no new complexity concerning both open definable sets and imaginaries.

Theorem (Later Theorem 3.1.11 and Corollary 3.1.12). *Let T be an open \mathcal{L} -theory of topological fields. Then, the theory T_δ^* has \mathcal{L} -open core (i.e., every open \mathcal{L}_δ -definable set is already \mathcal{L} -definable). In particular, when T is either $\text{ACVF}_{0,p}$ (with $p \geq 0$), RCVF ,*

pCF or, in general, the \mathcal{L}_{div} -theory of a henselian valued field of characteristic 0, then, the theory T_δ^ has \mathcal{L} -open core.*

Theorem (Later Theorem 3.3.6). *Let T be an open \mathcal{L} -theory of topological fields. Let \mathcal{G} be a collection of sorts of \mathcal{L}^{eq} and $\mathcal{L}^{\mathcal{G}}$ denote the restriction of \mathcal{L}^{eq} to the sorts in \mathcal{G} . Suppose that T admits elimination of imaginaries in $\mathcal{L}^{\mathcal{G}}$. Then the theory T_δ^* admits elimination of imaginaries in $\mathcal{L}_\delta^{\mathcal{G}}$.*

The previous theorem yields another proof of the fact that CODF has elimination of imaginaries (the first proof was given by the second author in [30], a second proof was given in [8]). As a corollary we also obtain the following.

Corollary (Later Corollary 3.3.8). *Let $\mathcal{L}^{\mathcal{G}}$ denote the geometric language of valued fields as defined in [18]. The theories $(\text{ACVF}_{0,p})_\delta^*$, RCVF_δ^* and $p\text{CF}_\delta^*$ have elimination of imaginaries in $\mathcal{L}_\delta^{\mathcal{G}}$.*

It is worth mentioning that the \mathcal{L} -open core plays a crucial role to show the transfer of elimination of imaginaries.

In return, the proof of the \mathcal{L} -open core has two main ingredients. The first ingredient is a cell decomposition theorem for definable subsets of models of T (see later Theorem 1.5.3) in the spirit of results by P. Simon and E. Walsberg [40] for dp-minimal topological structures. As in [40], cells are graphs of continuous correspondences (“multi-valued functions”) but, in addition, our definition of cell ensures that the projection of a cell (onto an initial subset of coordinates) is again a cell (see Definition 1.5.1). The second ingredient is a parametric version of the density of differential elements in open sets (see Lemmas 2.4.1, 3.1.9).

Building on the notion of cell given for definable sets of models of T , we introduce a notion of cells for definable sets of models of T_δ^* which we call δ -cells (see Definition 3.2.1). Informally, X is a δ -cell if there is an \mathcal{L} -definable cell Y such that X is the pullback under (iterations of) δ of Y and the differential prolongation of X is dense in Y . Using both the \mathcal{L} -open core and cell decomposition (in T), we prove a δ -cell decomposition theorem for T_δ^* . In the case of CODF, it improves results by T. Brihaye, C. Michaux and C. Rivière in [5] (see Remark 3.2.4).

Theorem (Later Theorem 3.2.2). *Let T be an open \mathcal{L} -theory of topological fields and K be a model of T_δ^* . Every \mathcal{L}_δ -definable subset $X \subseteq K^n$ can be partitioned into finitely many δ -cells.*

Last but not least, we illustrate how the theory T_δ^* provides a useful setting to study the theory T_P of dense pairs of models of a one-sorted open \mathcal{L} -theory of topological fields T . We show that T_P has \mathcal{L} -open core (Theorem 4.2.4) and deduce that T_P admits a distal expansion (namely T_δ^*) whenever T is a distal theory (Corollary 4.2.8). In particular, we show that T_P admits a distal expansion when T is RCF, $p\text{CF}$ and RCVF . Our result gives a positive answer to a particular case of a question of P. Simon who asked if the theory of dense pairs of an o-minimal structure (extending a group) has a distal expansion (see [28] for a discussion). T. Nell provided a positive answer in the case of ordered vector fields [28]. Our result extends to pairs of real closed fields.

Recently, A. Fornasiero and E. Kaplan [14] extended this result to other o-minimal expansions of real closed fields.

The paper is laid out as follows. Open theories of topological fields are studied in Section 1. Correspondences are studied in (1.4) and the cell decomposition theorem is presented in (1.5). Topological fields with a generic derivation are introduced in Section 2: consistency results are presented in (2.3); relative quantifier elimination is given in (2.4). Section 3 is devoted to the open core property (3.1), the δ -cell decomposition theorem (3.2), and the transfer of elimination of imaginaries (3.3). The applications to dense pairs are presented in Section 4. Some transfer proofs which were known in the one-sorted case (such as the transfer of NIP and distality) are gathered in Appendix A, together with the elimination of the quantifier \exists^∞ .

The reader eager of differential algebra may take the following fast track reading of Section 1, before starting with Section 2: read the definition of open \mathcal{L} -theories of topological fields in Section 1.2 and have a quick look at the cell decomposition theorem given for open \mathcal{L} -theories of topological fields (Theorem 1.5.3) in Section 1.5.

1. OPEN EXPANSIONS OF TOPOLOGICAL FIELDS

1.1. Preliminaries.

1.1.1. *Model theory.* We follow standard model theoretic notation and terminology. Lower-case letters like a, b, c and x, y, z usually denote finite tuples and we let $\ell(x)$ denote the length of x . We sometimes use \bar{x} to denote a tuple $\bar{x} = (x_1, \dots, x_n)$ where each x_i is a tuple. Let \mathcal{L} be a possibly multi-sorted language and \mathcal{M} be an \mathcal{L} -structure. For $A \subseteq \mathcal{M}$ (possibly ranging over different sorts), we let $\langle A \rangle_{\mathcal{L}}$ denote the \mathcal{L} -substructure generated by A in \mathcal{M} . Sorts are in general denoted by bold letters like \mathbf{S} , and $\mathbf{S}(\mathcal{M})$ denotes the \mathbf{S} -points in \mathcal{M} . For a tuple of variables x (again possibly ranging over different sorts), we let \mathcal{M}^x denote the corresponding product of sorts in \mathcal{M} . For an \mathcal{L} -formula $\varphi(x)$ we let $\varphi(\mathcal{M})$ denote the set

$$\{a \in \mathcal{M}^x : \mathcal{M} \models \varphi(a)\}.$$

We let $\mathcal{L}(A)$ denote the extension of \mathcal{L} by constants for all elements in A . By an \mathcal{L} -definable set of \mathcal{M} we mean definable with parameters, that is, of the form $\varphi(\mathcal{M})$ for an $\mathcal{L}(\mathcal{M})$ -formula φ .

For a subset $X \subseteq Q \times R$ and for $a \in Q$, the fiber of X over a is denoted by $X_a := \{b \in R : (a, b) \in X\}$.

We let acl denote the model-theoretic algebraic closure operator. Given a sort \mathbf{S} in \mathcal{L} , we let $\text{acl}_{\mathbf{S}}$ denote the model-theoretic algebraic closure restricted to \mathbf{S} . Note that $\text{acl}_{\mathbf{S}}$ is again a closure operator.

1.1.2. *Topological fields.* Throughout this article, every topological field is assumed to be non-discrete and Hausdorff.

Let K be a field and τ be a topology on K making it into a topological field. On K^n , $n \geq 1$, we put the product topology. The topological closure of a set $X \subseteq K^n$ is denoted by \overline{X} and its interior by $\text{Int}(X)$. The topological dimension of a non-empty

subset $X \subseteq K^n$, denoted $\dim(X)$, is defined as the maximal $\ell \leq n$ such that there is a projection $\pi: K^n \rightarrow K^\ell$ such that $\text{Int}(\pi(X)) \neq \emptyset$ (and equal to -1 if $X = \emptyset$).

We let $\mathcal{L}_{\text{ring}}$ denote the language of rings $\{\cdot, +, -, 0, 1\}$ and $\mathcal{L}_{\text{field}} := \mathcal{L}_{\text{ring}} \cup \{-^1\}$ denote the language of fields. We treat every field as an $\mathcal{L}_{\text{field}}$ -structure by extending the multiplicative inverse to 0 by $0^{-1} = 0$. We let $\mathcal{L}_{\text{field}}^\Omega$ be the extension of $\mathcal{L}_{\text{field}}$ by a set Ω of additional constant symbols (allowing the trivial case $\Omega = \emptyset$).

For the rest of the article, unless otherwise stated, we work in a (possibly multi-sorted) language \mathcal{L} extending $\mathcal{L}_{\text{field}}^\Omega$. We distinguish the field sort and denote it by \mathbf{F} . Any other sort is called an *auxiliary sort*.

{nota:sorts}

1.1.1. Notation. Let \mathcal{K} be an \mathcal{L} -structure. Unconventionally, and in order to simplify notation, we use non-calligraphic K to denote $\mathbf{F}(\mathcal{K})$ (so K is *not* the underlying universe of \mathcal{K} but only the universe of the field sort of \mathcal{K}).

{def:deftop}

1.1.2. Definition. We say τ is an \mathcal{L} -definable field topology if there is an \mathcal{L} -formula $\chi_\tau(x, z)$ with x a single \mathbf{F} -variable such that $\{\chi_\tau(\mathcal{K}, a) : a \in K^z\}$ is a basis of neighbourhoods of 0.

If K is an ordered field and the order is \mathcal{L} -definable, then the order topology on K is an \mathcal{L} -definable field topology. Similarly, if (K, v) is a valued field and the relation $\text{div} := \{(x, y) \in K^2 : v(x) \leq v(y)\}$ is \mathcal{L} -definable, then the valuation topology on K is an \mathcal{L} -definable field topology. When \mathcal{K} is a dp-minimal field, a result of W. Johnson [22, Theorem 1.3] guarantees the existence of a definable V -topology on K (equivalently a field topology induced either by a non-trivial valuation or an absolute value [33]).

{sec:opentheo}

1.2. Open theories of topological fields. We work in a first-order setting of topological fields which follows the spirit of [29], [45, Section 2], [26] and [16]. The main new ingredient of the present account is that we explicitly allow multi-sorted structures.

{def:T}

1.2.1. Definition. Fix an \mathcal{L} -structure \mathcal{K}_0 with K_0 a field of characteristic 0. Suppose that

- (1) the restriction of \mathcal{L} to the sort \mathbf{F} is a relational extension of $\mathcal{L}_{\text{field}}^\Omega$,
- (2) for every \mathbf{F} -valued \mathcal{L} -term $t(x, z)$ with x a tuple of \mathbf{F} -variables and z a tuple of auxiliary sort variables, there is an \mathbf{F} -valued \mathcal{L} -term $\tilde{t}(x)$ such that

$$\mathcal{K}_0 \models (\forall z)(\forall x)(t(x, z) = \tilde{t}(x)),$$

- (3) K_0 has an \mathcal{L} -definable field topology.

Let T be the \mathcal{L} -theory of \mathcal{K}_0 . Any such theory T is called an \mathcal{L} -theory of topological fields.

{rem:terms}

1.2.2. Remark. In a model \mathcal{K} of an \mathcal{L} -theory of topological fields, it follows by (1) and (2) of the previous definition, that for any $A \subseteq K$

$$\langle A \rangle_{\mathcal{L}} \cap K = \langle A \cap K \rangle_{\mathcal{L}_{\text{field}}^\Omega}.$$

1.2.3. Definition. An \mathcal{L} -theory of topological fields T is called an *open \mathcal{L} -theory of topological fields* if it satisfies in addition the following condition:

{condA}

- (A) for every \mathcal{L} -formula $\varphi(x, z)$ with x a tuple of \mathbf{F} -variables and z a tuple of auxiliary sort variables, there are \mathcal{L} -formulas $\{\xi_h(z)\}_{h \in H}$ with H a finite set, such that $\varphi(x, z)$ is equivalent modulo T to

$$\bigvee_{h \in H} \left(\xi_h(z) \rightarrow \left(\bigvee_{i \in I_h} \bigwedge_{j \in J_{ih}} P_{ijh}(x) = 0 \wedge \theta_{ih}(x, z) \right) \right)$$

where I_h and J_{ih} are finite sets, $P_{ijh} \in \mathbb{Q}(\Omega)[x] \setminus \{0\}$ and $\theta_{ih}(x, z)$ is an \mathcal{L} -formula such that for every model \mathcal{K} of T and every $a \in \mathcal{K}^z$, $\theta_{ih}(\mathcal{K}, a)$ defines an open set.¹

{rem:sorts}

1.2.4. **Remark.** Suppose T is an open \mathcal{L} -theory of topological fields.

- Let T' be an expansion by definitions of T in a relational language $\mathcal{L}' \supseteq \mathcal{L}$. Then T' is an open \mathcal{L}' -theory of topological fields. In particular, the Morleyization T_{Mor} of T is an open \mathcal{L}_{Mor} -theory of topological fields with quantifier elimination.
- Given a collection of sorts \mathcal{G} of \mathcal{L}^{eq} , let $\mathcal{L}^{\mathcal{G}}$ be the restriction of \mathcal{L}^{eq} to the sorts in \mathcal{G} . Then, the $\mathcal{L}^{\mathcal{G}}$ -theory of some (any) model of T is an open $\mathcal{L}^{\mathcal{G}}$ -theory of topological fields.

{examples}

1.2.5. **Examples.** The theory $T = \text{Th}(\mathcal{K}_0)$ is an open \mathcal{L} -theory of topological fields in the following cases.

- (1) When \mathcal{K}_0 is a real closed field and \mathcal{L} is \mathcal{L}_{of} the language of ordered fields $\mathcal{L}_{\text{field}} \cup \{<\}$. The definable topology is given by the order topology. We use RCF for T .
- (2) When \mathcal{K}_0 is a henselian valued field of characteristic 0 and \mathcal{L} is the one-sorted language of valued fields given by $\mathcal{L}_{\text{div}} = \mathcal{L}_{\text{field}} \cup \{\text{div}\}$. The definable topology corresponds to the valuation topology. Condition (A) follows by quantifier elimination in the multi-sorted \mathcal{L}_{RV} -language as defined in [12]². For convenience, we will hereafter assume that \mathcal{L}_{RV} contains \mathcal{L}_{div} in the field-sort. Examples include algebraically, p -adically and real closed valued fields, and theories of classical valued fields such as $\mathbb{C}((t))$, $\mathbb{R}((t))$.
- (3) Let \mathcal{L}'_{RV} be an RV-extension of \mathcal{L}_{RV} , i.e., it only extends the RV-sorts. Any complete \mathcal{L}'_{RV} -expansion T' of a complete theory T of henselian valued fields of characteristic 0 is an open \mathcal{L}'_{RV} -theory of topological fields. Indeed, in such expansions we still have relative quantifier elimination of field quantifiers and, for a model \mathcal{K} of T' , since the preimage in K of a non-zero point in an RV-sort under its canonical quotient map is an open subset of K , condition (A) is also satisfied.
- (4) The following open \mathcal{L} -theories T have, in addition, quantifier elimination in a one-sorted language:
 - when K_0 is an algebraically closed valued field of characteristic $(0, p)$ with $p \geq 0$ and $\mathcal{L} = \mathcal{L}_{\text{div}}$ (we write $\text{ACVF}_{0,p}$ for T);

¹In [16], only the case when \mathcal{L} is one-sorted was considered and K_0 was called a *topological \mathcal{L} -field* in case the additional relation symbols and their complement were interpreted in K_0 as the union of an open set and a Zariski closed set.

²Without loss of generality, we abuse of notation and use \mathcal{L}_{RV} both for the residue characteristic 0 variant and the residue characteristic p variant.

- when K_0 is a real closed valued field and $\mathcal{L} = \mathcal{L}_{\text{ovf}}$ the language of ordered valued fields $\mathcal{L}_{\text{of}} \cup \{\text{div}\}$ (we write RCVF for T).
- when K_0 is a p -adically closed field of p -rank $d = ef$ and \mathcal{L} is

$$\mathcal{L}_{p,d} := \mathcal{L}_{\text{field}} \cup \{\text{div}, c_1, \dots, c_d\} \cup \{P_n : n \geq 1\}$$

as defined in [32] (abusing of notation, we use in this case $p\text{CF}$ for T).

1.2.6. Remark. Observe that not all theories in Examples 1.2.5 are dp-minimal. Indeed, there are various henselian fields of equicharacteristic 0 which are not dp-minimal. By a result of F. Delon [11] combined with results of Y. Gurevich and P. H. Schmitt [15], the Hahn valued field $k((t^\Gamma))$ is NIP if and only if k is NIP (as a pure field). Even assuming NIP, by a result of A. Chernikov and P. Simon in [9], when k is algebraically closed, the field $k((t^\Gamma))$ is dp-minimal if and only if Γ is dp-minimal. However, there are ordered abelian groups which are not dp-minimal, as follows by a characterization of pure dp-minimal ordered abelian groups due to F. Jahnke, P. Simon and E. Walsberg in [21, Proposition 5.1].

1.2.7. Question. Is there an open \mathcal{L} -theory of topological fields whose topology does not come from an order or a valuation (equivalently, is not a V -topology)?

1.3. Basic consequences. For the rest of Section 1, unless otherwise stated, we let T be an open \mathcal{L} -theory of topological fields and \mathcal{K} be a model of T . By definable we mean \mathcal{L} -definable. We gather some standard consequences that will be used throughout the article. Proofs are either elementary or follow by classical arguments, so they will be omitted. Let us first introduce some notation.

1.3.1. Notation. Let $x = (x_0, \dots, x_{n-1})$ be a tuple of \mathbf{F} -variables and y be a single \mathbf{F} -variable. Let \mathcal{A} be a finite subset of $K[x, y]$ and $R \in K[x, y]$. We let the $\mathcal{L}_{\text{ring}}(K)$ -formula $Z_{\mathcal{A}}(x, y)$ be

$$\bigwedge_{P \in \mathcal{A}} P(x, y) = 0,$$

and $Z_{\mathcal{A}}^R(x, y)$ be

$$Z_{\mathcal{A}}(x, y) \wedge R(x, y) \neq 0.$$

Given $P \in K[x, y]$, we let $Z_P(x, y)$ denote $Z_{\{P\}}(x, y)$. Finally, we define $\mathcal{A}^y := \{P \in \mathcal{A} \mid \deg_y(P) > 0\}$.

Note that every locally Zariski-closed subset of K^{n+1} is of the form $Z_{\mathcal{A}}^R(\mathcal{K})$ for some \mathcal{A} and some R . The following lemma follows by induction on degrees and iterated applications of Euclid's algorithm.

1.3.2. Lemma. *Every locally Zariski closed subset of K^{n+1} can be written as a union of sets of the form $Z_{\mathcal{B}}^S(\mathcal{K})$ where $S \in K[x, y]$ and either $\mathcal{B} \subseteq K[x]$, or $\mathcal{B}^y = \{P\}$ and $\frac{\partial}{\partial y}P$ divides S .* \square

From condition (A) we directly obtain the following corollary on the form of definable sets in the field sort.

{cor:niceform}

1.3.3. **Corollary.** *Every definable subset of K^{n+1} is defined by*

$$\bigvee_{j \in J} Z_{\mathcal{A}_j}^{S_j}(x, y) \wedge \theta_j(x, y)$$

where $x = (x_0, \dots, x_{n-1})$ are \mathbf{F} -variables, y a single \mathbf{F} -variable, and for each $j \in J$, θ_j is an $\mathcal{L}(\mathcal{K})$ -formula that defines an open subset of K^{n+1} , $S_j \in K[x, y]$, and either

- (1) $\mathcal{A}_j \subseteq K[x] \setminus \{0\}$ or
- (2) $\mathcal{A}_j \subseteq K[x, y] \setminus \{0\}$, $\mathcal{A}_j^y = \{P_j\}$ and $\frac{\partial}{\partial y} P_j$ divides S_j . □

The following proposition follows from condition (A) by standard arguments and classical results from [45].

{prop:conseAA1}

1.3.4. **Proposition.** *The field sort of every model of T is algebraically bounded (in the sense of [45]). The algebraic dimension alldim (in the sense of [45, Lemma 2.3]) on the field sort coincides with $\text{dim}_{\text{acl}_{\mathbf{F}}}$ and defines a dimension function (as defined in [45]). In particular, T eliminates the field sort quantifier \exists^∞ and the restriction T' of T to the field sort is a geometric theory. □*

The notion of geometric structures was introduced in [?, Definition 2.1] as a the natural setting in which one could relate groups definable in real, p-adics and pseudo-finite fields to algebraic groups. Here it is more convenient to speak of geometric theories i.e. complete \mathcal{L} -theories that eliminates the quantifier \exists^∞ and in any model \mathcal{M} of T the model-theoretic algebraic closure acl satisfies the exchange property.

The following lemma follows by a simple induction on n .

{lem:int_zar}

1.3.5. **Lemma.** *Let $P(x) \in K[x] \setminus \{0\}$ with $x = (x_0, \dots, x_{n-1})$. Then $\dim(Z_P(\mathcal{K})) < n$. In particular, the set $K^n \setminus Z_P(\mathcal{K})$ is open and dense in K^n with respect to the ambient topology. □*

As a corollary of the previous results one can show the following consequence on the topological dimension.

{prop:conseAA2}

1.3.6. **Proposition.** *For every $n \geq 1$ and every definable subset $X \subseteq K^n$, $\dim(X) = \dim_{\text{acl}_K}(X)$. The topological dimension satisfies thus the following properties for definable sets $X, Y \subseteq K^n$:*

- (D1) $\dim(X) = 0$ if and only if X is finite and non-empty,
- (D2) $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
- (D3) $\dim(\overline{X} \setminus X) < \dim(X) = \dim(\overline{X})$,
- (D4) \dim is additive, that is: for a non-empty definable set $X \subseteq K^{m+n}$ and $d \in \{0, 1, \dots, n\}$,

$$\dim\left(\bigcup_{a \in X(d)} X_a\right) = \dim(X(d)) + d,$$

where $X(d) := \{a \in K^m : \dim X_a = d\}$. In addition, $X(d)$ is definable. □

As a convention, given a definable set X , we say that a property holds *almost everywhere* on X if there is a definable subset $Y \subseteq X$ such that $\dim(X \setminus Y) < \dim(X)$ and the property holds on Y .

ec:correspon}

1.4. Almost continuity of definable correspondences. We will show in Section 1.5 a cell decomposition theorem for open \mathcal{L} -theories of topological fields. An essential part of its proof is to show that definable correspondences (“multi-valued functions”) are almost everywhere continuous. We show such a result adapting to our setting the strategy employed by Simon-Walsberg for dp-minimal theories [40]. Note that correspondences cannot be avoided in the absence of finite Skolem functions. Let us recall their definition, adopting the same notation.

1.4.1. Definition ([40, Section 3.1]). A *correspondence* $f: E \rightrightarrows K^\ell$ consists of a definable set $E \subseteq K^n$ together with a definable subset $\text{graph}(f)$ of $K^n \times K^\ell$ such that

$$0 < |\{y \in K^\ell : (x, y) \in \text{graph}(f)\}| < \infty, \text{ for all } x \in E.$$

The set $\{y \in K^\ell : (x, y) \in \text{graph}(f)\}$ is also denoted by $f(x)$. For a positive integer m , we say f is an *m-correspondence* if $|f(x)| = m$ for all $x \in E$. The correspondence f is *continuous* at $x \in E$ if for every open set $V \subseteq K^\ell$ containing $f(x)$, there is an open neighbourhood U of x such that $f(U) \subseteq V$.

Let $f: E \subseteq K^n \rightrightarrows K^\ell$ be a correspondence. As a convention, when $n = 0$, we identify $\text{graph}(f)$ with a finite subset of K^ℓ . If E is an open subset and $\ell = 0$, then we identify $\text{graph}(f)$ with the set E . Note that a 1-correspondence can be identified with a function. The following lemma is a reformulation of [40, Lemma 3.1].

1.4.2. Lemma. *Let $U \subseteq K^n$ be open and let $f: U \rightrightarrows K^\ell$ be a continuous m-correspondence. Every $a \in U$ has an open neighbourhood V such that there are continuous functions $g_1, \dots, g_m: V \rightarrow K^\ell$ such that $\text{graph}(g_i) \cap \text{graph}(g_j) = \emptyset$ when $i \neq j$ and*

$$\text{graph}(f \upharpoonright V) = \text{graph}(g_1) \cup \dots \cup \text{graph}(g_m).$$

In addition, if f is definable, we can further choose V and the functions g_i to be definable (all possibly with additional parameters). \square

The following result is a reformulation of [40, Proposition 3.7] in which we isolate the components of its proof in an axiomatic way. Recall that a family of sets F is said to be *directed* if for every $A, B \in F$ there is $C \in F$ such that $A \cup B \subseteq C$.

1.4.3. Proposition ([40, Proposition 3.7]). *Let T be an \mathcal{L} -theory of topological fields (not necessarily satisfying condition (A)) and \mathcal{K} be a model of T . Suppose \mathcal{K} satisfies the following properties*

- (1) *if A is a non-empty definable open subset of K^n and B is a definable subset of A which is dense in A , then $\text{Int}(B)$ is dense in A ; in particular, $\dim(A \setminus B) < \dim(A)$.*
- (2) *if A, A_1, \dots, A_k are non-empty definable subsets of K^n , A is open and $A = \bigcup_{i=1}^k A_i$, then, $\text{Int}(A_i) \neq \emptyset$ for some $1 \leq i \leq k$.*
- (3) *if $C \subseteq K^{m+n}$ is a definable set such that the definable family $\{C_a : a \in K^m\}$ is a directed family and $\bigcup_{a \in K^m} C_a$ has non-empty interior, then, there is $a \in K^m$ such that C_a has non-empty interior.*
- (4) *There is no infinite definable discrete subset of K^n .*

{def:correspo

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Then, for $V \subseteq K^n$ a definable open set, every definable correspondence $f: V \rightrightarrows K^\ell$ is continuous on an open dense subset of V , and thus is continuous almost everywhere on V .

Proof. The proof is a word by word analogue of the proof of [40, Proposition 3.7] after replacing [40, Lemma 2.6] by condition (1), [40, Corollary 2.7] by condition (2), [40, Lemma 3.5] by condition (3) and [40, Lemma 3.6] by condition (4). \square

We now show that all four conditions in Proposition 1.4.3 hold for open \mathcal{L} -theories of topological fields.

1.4.4. Lemma. *If A is a definable open subset of K^n and B is a definable subset of A which is dense in A , then $\text{Int}(B)$ is dense in A . In particular, $\dim(A \setminus B) < \dim(A)$.*

Proof. It suffices to show that $\text{Int}(B) \neq \emptyset$. By condition (A), B is defined by a formula of the form $\bigvee_{i \in I} \bigwedge_{j \in J_i} P_{ij}(x) = 0 \wedge \theta_i(x)$, where $P_{ij} \in K[x] \setminus \{0\}$ and $\theta_i(x)$ defines an open set. Suppose that the set of indices J_i is non-empty. Therefore, the algebraic dimension of B is strictly less than n . Since the topological and the algebraic dimension coincide, $\dim(B) < n$. By Proposition 1.3.6, $\dim(A \setminus B) = n$ which implies there is an open subset $U \subseteq A$ disjoint from B . This contradicts the density of B in A . Then, there must be $i \in I$ such that $J_i = \emptyset$, hence $\text{Int}(B) \neq \emptyset$. \square

1.4.5. Lemma. *Suppose $C \subseteq K^{m+n}$ is a definable set inducing a definable family $\{C_a : a \in K^m\}$ which is directed. If $\bigcup_{a \in K^m} C_a$ has non-empty interior, then there is $a \in K^m$ such that C_a has non-empty interior.*

Proof. Let $\varphi(x, y)$ with $\ell(x) = m$ and $\ell(y) = n$ be an $\mathcal{L}(K)$ -formula defining C . Let $Y \subseteq K^n$ denote the definable set $\bigcup_{a \in K^m} C_a$. By hypothesis, $\text{Int}(Y) \neq \emptyset$. Since the family $\{C_a : a \in K^m\}$ is directed, we may assume there are infinitely many different C_a in the family (as otherwise the result follows directly from Proposition 1.3.6).

For $y = (y_1, \dots, y_n)$, we let \tilde{y} denote the tuple (y_1, \dots, y_{n-1}) . By Corollary 1.3.3 applied to the formula $\varphi(x, y)$ with respect to the variable y_n , $\varphi(x, y)$ is equivalent to a finite disjunction $\bigvee_{i \in I} \varphi_i$ where each φ_i is of the form $Z_{\mathcal{A}_i}^{S_i}(x, y) \wedge \theta_i(x, y)$ where $\theta_i(K)$ defines an open subset of K^{m+n} and either

- (1) $\mathcal{A}_i \subseteq K[x, \tilde{y}]$, or
- (2) $\mathcal{A}_i^{y_n} = \{P_i\}$ and $\frac{\partial}{\partial y_n} P_i$ divides S_i .

Collect all the subformulas of the disjunction of form (1) (resp. form (2)) and denote by $\varphi^1(x, y)$ (resp. $\varphi^2(x, y)$) their disjunction. We have that $\varphi(x, y) = \varphi^1(x, y) \vee \varphi^2(x, y)$.

Note that if $\mathcal{A}_i = \emptyset$ for some $i \in I$, then each fiber C_a contains the open set $\theta_i(a, K)$ and has therefore non-empty interior. Thus, we may assume that $\mathcal{A}_i \neq \emptyset$ for all $i \in I$. We proceed by induction on n . Let d be the maximum of the degrees (in y_n) of the polynomials occurring in all \mathcal{A}_i 's.

Assume $n = 1$. Suppose first that $\mathcal{A}_i \subseteq K[x]$ for some $i \in I$. Then, the fiber C_a contains an open set whenever $Z_{\mathcal{A}_i}^S(a, K) \neq \emptyset$. If $Z_{\mathcal{A}_i}(a, K) = \emptyset$ for every $a \in K^m$, then we remove the corresponding member from the disjunction. Therefore, we are left with the case where $\varphi(x, y) = \varphi^2(x, y)$. We show this case cannot happen. First, note that in this situation each fiber C_a has finite cardinality bounded by $d|I|$. Since the

family $\{C_a : a \in K^m\}$ is directed, there is $a_0 \in K^m$ such that $\varphi(a_0, \mathcal{K}) = Y$. But this contradicts that Y contains an open set (and is thus infinite). This concludes the case $n = 1$.

Now assume $n > 1$. Let $\pi : K^n \rightarrow K^{n-1}$ denote the projection onto the first $n - 1$ coordinates. For $(a, u) \in K^m \times K^{n-1}$ we denote by $C_{a,u}$ the fiber $(C_a)_u = \{b \in K : (a, u, b) \in C\}$. By the form of each formula φ_i , each fiber $C_{a,u}$ either contains a non-empty open subset or is finite (and bounded by $d|I|$). We uniformly partition the projection $\pi(C_a)$ of each fiber C_a into sets $\pi(C_a)_1$ and $\pi(C_a)_2$ where

$$\begin{aligned} \pi(C_a)_1 &:= \{u \in K^{n-1} : C_{a,u} \text{ contains an open set}\} \text{ and} \\ \pi(C_a)_2 &:= \{u \in K^{n-1} : |C_{a,u}| \leq d|I|\}. \end{aligned}$$

Since the definition is uniform, we have

$$\pi(Y) = \pi\left(\bigcup_{a \in K^m} C_a\right) = \bigcup_{a \in K^m} \pi(C_a) = \bigcup_{a \in K^m} \pi(C_a)_1 \cup \bigcup_{a \in K^m} \pi(C_a)_2.$$

As Y contains an open set, so does $\pi(Y)$. Therefore, by Proposition 1.3.6, either

$$\pi(Y)_2 := \bigcup_{a \in K^m} \pi(C_a)_2 \setminus \left(\bigcup_{a \in K^m} \pi(C_a)_1\right) \text{ or } \pi(Y)_1 := \bigcup_{a \in K^m} \pi(C_a)_1,$$

contains an open set.

1.4.6. Claim. *The set $\pi(Y)_1$ must contain an open set.*

Suppose for a contradiction $\dim(\pi(Y)_1) < n - 1$. Partition Y into $Y_1 \cup Y_2$ where

$$Y_i = \{(u, b) \in K^n : u \in \pi(Y)_i\} \text{ for } i = 1, 2.$$

By Proposition 1.3.6, Y_1 or Y_2 contains an open subset. By construction, we have that $\pi(Y_1) = \pi(Y)_1$ and, by assumption, $\dim(\pi(Y)_1) < n - 1$. Therefore $\dim(Y_1) < n$. Hence we must have that $\dim(Y_2) = n$. By the additivity of the dimension function (Proposition 1.3.6), there must be $u \in \pi(Y_2)$ such that the fiber $(Y_2)_u$ is infinite. In particular, there are $k \in \mathbb{N}$ and $a_1, \dots, a_k \in K^m$ such that $\bigcup_{j=1}^k C_{a_j, u}$ has cardinality bigger than $d|I|$. Since the family $\{C_{a, u} : a \in K^m\}$ is directed, there is $a \in K^m$ such that $\bigcup_{j=1}^k C_{a_j, u} \subseteq C_{a, u}$, which contradicts the fact that the fiber $C_{a, u}$ has cardinality smaller or equal than $d|I|$. This completes the claim.

Consider the directed family $\{\pi(C_a)_1 : a \in K^m\}$. By the claim, $\pi(Y)_1 = \bigcup_{a \in K^m} \pi(C_a)_1$ contains a non-empty open set. Therefore, by induction, there is $a \in K^m$ such that $\pi(C_a)_1$ contains an open subset, say $U \subseteq K^{n-1}$. For each $i \in I$, set

$$U_i := \{u \in U : \dim(Z_{\mathcal{A}_i}^{S_i}(a, u, \mathcal{K}) \cap \theta_i(a, u, \mathcal{K})) = 1\}.$$

Given that $U \subseteq \pi(Y)_1$, we have that $U = \bigcup_{i \in I} U_i$. Then, by Proposition 1.3.6, there is $i \in I$ such that U_i contains an open set, say $V \subseteq U_i$. By the definition of U_i , the set $Z_{\mathcal{A}_i}^{S_i}(a, u, \mathcal{K}) \cap \theta_i(a, u, \mathcal{K})$ is infinite for every $u \in V$. Thus, since $\mathcal{A}_i \neq \emptyset$, we must have $\mathcal{A}_i \subseteq K[x, \tilde{y}]$. But in this situation $Z_{\mathcal{A}_i}(a, \mathcal{K}) = K^n$. Indeed, consider $\mathcal{B} = \mathcal{A}_i$ as a set of polynomials in $K[x, \tilde{y}]$. Then $Z_{\mathcal{B}}(a, \mathcal{K}) \subseteq K^{n-1}$ contains an open set, namely,

V . By Lemma 1.3.5, $Z_{\mathcal{B}}(a, \mathcal{K}) = K^{n-1}$, and thus $Z_{\mathcal{A}_i}(a, \mathcal{K}) = K^n$ (as a subset of K^n). Therefore, the fiber C_a contains the open set

$$\{(u, b) \in V \times K : S_i(a, u, b) \neq 0 \wedge \theta_i(a, u, b)\}.$$

This is indeed open, as the set defined by the formula $S_i(x, \tilde{y}, y_n) \neq 0 \wedge \theta_i(x, \tilde{y}, y_n)$ defines a non-empty open subset of K^{m+n} by Lemma 1.3.5. \square

1.4.7. **Lemma.** *There is no infinite definable discrete subset of K^n .*

Proof. The proof goes by induction on n exactly as the proof of [40, Lemma 3.6]. The base case follows directly straightforward by condition (A). The inductive case follows word by word the proof in [40]. \square

1.4.8. **Proposition.** *For a definable open set $V \subseteq K^n$, every definable correspondence $f: V \rightrightarrows K^\ell$ is continuous almost everywhere.*

Proof. This follows from Proposition 1.4.3 where conditions (1)-(4) correspond respectively to Lemma 1.4.4, Proposition 1.3.6, Lemma 1.4.5 and Lemma 1.4.7. \square

1.5. **Cell decomposition.** We finish this section with a cell decomposition theorem for open \mathcal{L} -theories of topological fields. As in Simon-Walsberg's [40, Proposition 4.1], cells correspond to definable continuous m -correspondences (for some m). However, the main difference with their result is that we provide an inductive definition of cells which remains closer to classical definitions (e.g., o-minimal cells). In particular, our definition of cell ensures that if $X \subseteq K^n$ is a cell and $\pi: K^n \rightarrow K^d$ is a coordinate projection onto the first d -variables, then $\pi(X)$ is also a cell.

In order to define cells, we use the following conventions. Let $X \subseteq K^n$, $\rho: K^n \rightarrow K^d$ be a coordinate projection and $f: \rho(X) \rightrightarrows K^{n-d}$ be a correspondence. Let $\rho^\perp: K^n \rightarrow K^{n-d}$ be the complement projection of ρ (i.e. to the complement set of coordinates). While considering the graph of f , it is convenient to make explicit this projection ρ ; namely the *graph of f along ρ* is the set

$$\{x \in K^n : \rho(x) \in \rho(X) \text{ and } \rho^\perp(x) \in f(\rho(x))\}.$$

1.5.1. **Definition (Cells).** For $A \subseteq \mathcal{K}$, we define the collection of A -definable cells of K^n by induction on n as pairs (X, ρ_X) where $X \subseteq K^n$ is a non-empty A -definable set and $\rho_X: K^n \rightarrow K^{\dim(X)}$ is a coordinate projection such that $\rho_X(X)$ is an open set (with the convention that if $\dim(X) = 0$, then $\rho_X(X)$ is open).

So when $n = 1$, a pair (X, ρ_X) with $X \subseteq K$ is an A -definable cell if and only if X is an A -definable non-empty open set and $\rho_X = \text{id}$, or X is an A -definable non-empty finite set and ρ_X is the projection to K^0 .

Assume A -definable cells have been defined for all $k \leq n$ and let $\pi: K^{n+1} \rightarrow K^n$ be the projection onto the first n -coordinates. A pair (X, ρ_X) with $X \subseteq K^{n+1}$ is an A -definable cell if and only if X is the graph along ρ_X of a continuous A -definable m -correspondence $f_X: \rho_X(X) \rightrightarrows K^{n+1-\dim(X)}$ (for some $m > 0$), there is an A -definable cell (C, ρ_C) such that $C = \pi(X)$ and one of the following three cases holds

- (i) $\rho_X = \text{id}$ (so X is an A -definable open set);

- (ii) $\rho_X = \rho_C \circ \pi$;
- (iii) ρ_X corresponds to

$$\rho_X(x_1, \dots, x_{n+1}) = (\rho_C(x_1, \dots, x_n), x_{n+1}),$$

the fiber X_c is a non-empty open subset of K for every $c \in C$, and for every $x \in X$

$$f_X(\rho_X(x)) = f_C(\rho_C(\pi(x))).$$

By a cell, we mean an \mathcal{K} -definable cell. We often omit the associated projection ρ_X of a cell, and simply write X for a cell.

1.5.2. Lemma. *Let (X, ρ_X) be an A -definable cell with $X \subseteq K^n$. If $Y \subseteq \rho_X(X)$ is a non-empty open A -definable subset, then the set $X' = \rho_X^{-1}(Y) \cap X$ together with the associated projection $\rho_{X'} = \rho_X$, is an A -definable cell.*

{lem:subcell}

Proof. We proceed by induction on n . If $X \subseteq K$, the result is obvious. So suppose the results holds for all $k \leq n$ and let $X \subseteq K^{n+1}$. We proceed by induction on $\dim(X)$. If X is finite there is nothing to show. So suppose the result for all $i < \dim(X)$. Since X is a cell, it is the graph along ρ_X of a continuous m -correspondence $f_X: \rho_X(X) \rightrightarrows K^{n+1-\dim(X)}$. Let $\pi: K^{n+1} \rightarrow K^n$ be the projection onto the first n coordinates and (C, ρ_C) be the corresponding cell such that $C = \pi(X)$. If X is of type (i) ($\dim(X) = n + 1$), then X is open and the result is again obvious. Hence we may suppose that $\dim(X) < n + 1$.

Suppose first X is of type (ii) as given in Definition 1.5.1. Then $\rho_X = \rho_C \circ \pi$ and $f_X: \rho_C(C) \rightrightarrows K^{n+1-\dim(X)}$. In particular, $Y \subseteq \rho_X(X) = \rho_C(C)$, so by induction $C' = \rho_C^{-1}(Y) \cap C$ is a cell with $\rho_{C'} = \rho_C$. So X' is the graph of $f_X \upharpoonright \rho_{C'}(C')$ along $\rho_{X'}$, hence a cell of type (ii).

It remains to show the result when X is of type (iii) as given in Definition 1.5.1. In this case, $\rho_X(x_1, \dots, x_{n+1}) = (\rho_C(x_1, \dots, x_n), x_{n+1})$, for every $c \in C$, the fiber X_c is open and $f_X(\rho_X(x)) = f_C(\rho_C(\pi(x)))$ for all $x \in X$. Let $\pi': \rho_X(X) \rightarrow K^{\dim(C)}$ be the coordinate projection such that $\rho_C \circ \pi = \pi' \circ \rho_X$ (i.e., dropping the last coordinate). Since $\pi'(Y) \subseteq \rho_C(C)$ is open, by induction, $C' = \rho_C^{-1}(\pi'(Y)) \cap C$ is a cell with $\rho_{C'} = \rho_C$. Note that $\pi(X') = C'$ and it holds that $X'_c = X_c$ for every $c \in C'$ (so in particular X'_c is open). Since X' is the graph of $f_X \upharpoonright \rho_{X'}(X')$ and $\rho_{X'}(X) = Y$, X' is a cell of type (iii). \square

{thm:newCD}

1.5.3. Theorem (Cell decomposition). *Let T be an open \mathcal{L} -theory of topological fields, \mathcal{K} be a model of T and $A \subseteq \mathcal{K}$. Let X be an A -definable subset of K^{n+1} , then X can be expressed as a finite disjoint union of A -definable cells.*

Proof. Let $X \subseteq K^{n+1}$ be an A -definable set. We proceed by induction on n .

For $n = 0$, note that $X = \text{Int}(X) \cup (X \setminus \text{Int}(X))$. Both sets are A -definable, and since $\dim(X \setminus \text{Int}(X)) < \dim(X)$, this already constitutes a cell decomposition.

Suppose now the result for all $k \leq n$ and proceed by induction on $\dim(X)$. Let $\pi: K^{n+1} \rightarrow K^n$ be the projection onto the first n -coordinates.

If $\dim(X) = 0$, then X is already a cell, so suppose the result holds for all $i < \dim(X)$. If $\dim(X) = n + 1$, then after removing $\text{Int}(X)$ from X (which is an open definable subset of K^{n+1} and so a cell), we may already suppose $\dim(X) < n + 1$. Writing an

element in X as a pair (a, b) with $a \in \pi(X)$ and $b \in K$, we partition X into the following two A -definable subsets of X :

$$(1.5.4) \quad W = \bigcup_{a \in H} \{a\} \times \text{Int}(X_a) \text{ and } Z = X \setminus W,$$

where $H = \{a \in \pi(X) : \dim(X_a) = 1\}$. Since W and Z form an A -definable partition of X , it suffices to show the result for W and Z .

Assume that $\dim(X) = \dim(Z)$ (if $\dim(Z) < \dim(X)$ the result, follows by induction). By the definition of Z and the fact that for every $a \in H$, $\dim(X_a \setminus \text{Int}(X_a)) = 0$, the fiber Z_a is finite for every $a \in \pi(Z)$. Hence, by elimination of \exists^∞ , there is a uniform bound on the size of Z_a . Possibly further partitioning Z , we may suppose that every fiber Z_a is of size m for some $m > 0$. By induction, since $\pi(Z) \subseteq K^n$, we may express $\pi(Z)$ as a finite disjoint union of cells (C, ρ_C) . Let us consider each cell in turn and assume that $\pi(Z) = C$. Since C is the graph along ρ_C of a continuous correspondence $f_C: \rho_C(C) \rightrightarrows K^{n-\dim(C)}$ (note C cannot be finite by assumption on $\dim(Z)$), Z is the graph along $\rho_C \circ \pi$ of an m' -correspondence $f: \rho_C(C) \rightrightarrows K^{n+1-\dim(C)}$ (for some $m' > 0$). By Proposition 1.4.8, f is continuous on an open dense definable subset U of $\rho_C(C)$. Then we partition Z as a disjoint union of

$$Z' := \text{graph}(f \upharpoonright U) \text{ and } Z'' := \text{graph}(f \upharpoonright (\rho_C(C) \setminus U)).$$

Setting $\rho_{Z'} = \rho_C \circ \pi$, since f is continuous on U , Z' is the graph along $\rho_{Z'}$ of an m' -continuous correspondence $f_{Z'} = f \upharpoonright U$ and therefore a cell of type (ii) by Lemma 1.5.2. For Z'' the result follows by induction since $\dim(Z'') < \dim(Z)$.

It remains to show the result for W . Assume that $\dim(X) = \dim(W)$ (if $\dim(W) < \dim(X)$, the result follows by induction). By the definition of W , $W_a = \text{Int}(X_a)$ is a non-empty open subset of K , for each $a \in \pi(W) = H$. By induction, we may suppose that $\pi(W)$ is a finite disjoint union of cells (C, ρ_C) . As in the previous case we consider each cell in turn and further assume that $\pi(W) = C$. Since C is a cell, C is the graph along ρ_C of a continuous m -correspondence $f_C: \rho_C(C) \rightrightarrows K^{n-\dim(C)}$, for some m . Note that C cannot be an open set since $\dim(X) < n + 1$. Set $\rho_W: K^{n+1} \rightarrow K^{\dim(W)}$ to be the coordinate projection given by

$$\rho_W(x_1, \dots, x_{n+1}) := (\rho_C(x_1, \dots, x_n), x_{n+1}),$$

and $\pi': \rho_W(W) \rightarrow K^{\dim(C)}$ be the coordinate projection such that $\rho_C \circ \pi = \pi' \circ \rho_W$. In particular, W is the graph along ρ_W of the m -correspondence

$$f: \rho_W(W) \rightrightarrows K^{n+1-\dim(X)}: x \mapsto f_C(\pi'(x)).$$

Set $Y' = \text{Int}(\rho_W(W))$. Note that $f \upharpoonright Y'$ is continuous. Set $W' \subseteq W$ to be the graph of $f \upharpoonright Y'$ along ρ_W and $W'' := W \setminus W'$. The set W' is already a cell of type (iii) with $\rho_{W'} = \rho_W$. Indeed, by Lemma 1.5.2, $C' = \rho_C^{-1}(\pi'(Y')) \cap C$ is a cell with $\rho_{C'} = \rho_C$ and $\pi(W') = C'$; for every $a \in C'$ we have that $W_a = W'_a$, so the fiber is open; and by construction W' is the graph along $\rho_{W'}$ of a continuous m -correspondence $f_{W'} = f \upharpoonright Y'$ satisfying $f_{W'}(\rho_{W'}(x)) = f_C(\rho_C(\pi(x)))$ for all $x \in W'$. For W'' the result follows by induction since

$$\dim(W'') = \dim(\rho_W(W) \setminus \text{Int}(\rho_W(W))) < \dim(\rho_W(W)) = \dim(W). \quad \square$$

The reader may have noticed that Theorem 1.5.3 is not really about open \mathcal{L} -theories of topological fields. Indeed, such a result holds in any structure with a uniform definable topology such that (i) the topological dimension satisfies the properties of Proposition 1.3.6 and (ii) definable correspondences are almost everywhere continuous. In particular, it applies to all dp-minimal structures with a uniform definable topology as in [40] (i.e. satisfying properties (1) and (2) in Corollary 1.5.5 below) which satisfy the exchange property. Indeed, (i) follows from Proposition [40, Proposition 2.2] and (ii) from [40, Proposition 3.7], together with the fact that the exchange property implies that the dimension is additive [39, Exercise 4.38].

{cor:DC-dp-mi}

1.5.5. Corollary. *Let \mathcal{M} be a dp-minimal structure with a uniform definable topology on a distinguished home sort M satisfying*

- (1) *M does not have any isolated points;*
- (2) *every infinite definable subset of M has nonempty interior;*
- (3) *acl_M has the exchange property.*

Then, every definable subset of M is a finite disjoint union of definable cells (as defined in Definition 1.5.1). \square

We finished this section with a technical lemma that will be later needed in Section 3.1.

{lem:Ldens}

1.5.6. Lemma. *Let $X \subseteq K^n$ be a cell. Let $1 \leq d < n$, $\pi: K^n \rightarrow K^{n-d}$ be the projection onto the first $n - d$ coordinates and $a = (a', a'') \in X$ with $a' = \pi(a)$. Then, given an open neighbourhood V of a'' , there exists an open neighbourhood U of a' such that for all $b' \in U \cap \pi(X)$, there is $b'' \in V$ such that $b = (b', b'') \in X$.*

Proof. Let us first show the proposition for $d = 1$. Let $C \subseteq K^{n-1}$ be the cell such that $\pi(X) = C$ and $f_X: \rho_X(X) \rightrightarrows K^{n-\dim(X)}$ be a continuous m -correspondence such that X is the graph of f_X along ρ_X . We split in cases depending on the form of X .

- If X is open (cell of type (i)), the result is straightforward.
- Assume that X is a cell of type (ii), so $\rho_X = \rho_C \circ \pi$. Let $\pi^\perp: K^{n-1-\dim(C)} \rightarrow K$ be the projection onto the last coordinate. By Lemma 1.4.2, there is an open neighbourhood W of $\rho_X(a) = \rho_C(a')$ and continuous definable functions $g_1, \dots, g_m: W \rightarrow K^{n-\dim(X)}$ such that $\text{graph}(g_i) \cap \text{graph}(g_j) = \emptyset$ when $i \neq j$ and

$$\text{graph}(f_X \upharpoonright W) = \text{graph}(g_1) \cup \dots \cup \text{graph}(g_m).$$

Without loss of generality, suppose g_1 is such that $a'' = \pi^\perp(g_1(\rho_X(a)))$. Let D be an open neighbourhood of $g_1(\rho_X(a))$ such that $\pi^\perp(D) \subseteq V$. Since g_1 is continuous, we can find an open neighbourhood O of $\rho_X(a)$ such that $g_1(O) \subseteq D$. Set $U := \rho_C^{-1}(O)$ and let $b' \in U \cap \pi(X) = U \cap C$. Then, $\rho_C(b') \in O$, and hence $g_1(\rho_C(b')) \in D$. By the choice of D , $b'' = \pi^\perp(g_1(\rho_C(b')))) \in V$, which shows that $b = (b', b'') \in X$, by the definition of g_1 .

- Assume now that X is a cell of type (iii), so that ρ_X corresponds to

$$\rho_X(x_1, \dots, x_n) = (\rho_C(x_1, \dots, x_{n-1}), x_n),$$

the fiber X_c is a non-empty open subset of K for every $c \in C$ and, $f_X(\rho_X(x)) = f_C(\rho_C(\pi(x)))$ for all $x \in X$. Since $\rho_X(a) = (\rho_C(a'), a'')$ and $\rho_X(X)$ is open, we can

find open neighbourhoods O of $\rho_C(a')$ and V' of a'' such that $O \times V' \subseteq \rho_X(X)$ and $V' \subseteq V$. Set $U = \rho_C^{-1}(O)$ and take $b' \in U \cap \pi(X) = U \cap C$. Let b'' be any element in V' . Then, $\rho_X((b', b'')) = (\rho_C(b'), b'') \in O \times V' \subseteq \rho_X(X)$, and by the definition of X , since $b' \in C$,

$$f_C(\rho_C(b')) = f_X(\rho_X(b', b'')),$$

this shows that $(b', b'') \in X$.

Now we consider the general case and we proceed by induction on d . So assume the property has been proven for all n and all $1 \leq e < d$. Let $\pi_{n-1}: K^n \rightarrow K^{n-1}$ be the projection onto the first $n-1$ coordinates and recall that π denotes the projection onto the first $n-d$ coordinates. Write $a = (a_1, \dots, a_n)$ as a triple (a'_1, a'_2, a_n) where

$$a'_1 = \pi(a) \text{ and } a'_2 = (a_{n-d+1}, \dots, a_{n-1}).$$

Without loss of generality suppose $V = V_1 \times V'$ with V' an open neighbourhood of a_n and V_1 an open neighbourhood of a'_2 . By the case $d = 1$, there is an open neighbourhood U' of $\pi_{n-1}(a)$ such that for any $b' \in U \cap \pi_{n-1}(X)$ there is $b'' \in V'$ such that $(b', b'') \in X$. By induction on $\pi_{n-1}(X)$, for any open neighbourhood V_2 of a'_2 there is an open neighbourhood U_1 of a'_1 such that for all $b_1 \in U_1 \cap \pi(X)$, there is $b_2 \in V_2$ such that $(b_1, b_2) \in \pi_{n-1}(X)$. Possibly shrinking U_1 and V_2 , we may suppose $U_1 \times V_2 \subseteq U'$. Let us show that $U = U_1$ has the desired property. Indeed, if $b_1 \in U \cap \pi(X)$, then there is $b_2 \in V_2$ such that $(b_1, b_2) \in \pi_{n-1}(X)$. Setting $b' = (b_1, b_2)$, since $b' \in U_1 \times V_2 \subseteq U'$, there is $b'' \in V'$ such that $(b', b'') = (b_1, b_2, b'') \in X$. By construction, $(b_2, b'') \in V_2 \times V' = V$, which completes the proof. \square

2. THEORIES OF TOPOLOGICAL FIELDS WITH A GENERIC DERIVATION

Let T be an \mathcal{L} -theory of topological fields and \mathcal{L}_δ be the language \mathcal{L} extended by a symbol for a derivation (in the field sort). From now on we will focus on the study of an \mathcal{L}_δ -extension T_δ^* of T . The derivation δ of any model of T_δ^* is called a *generic derivation*. Every such a derivation is highly non-continuous. The theory T_δ^* will be defined in Section 2.2. In Section 2.3 we show various examples of theories T for which the theory T_δ^* is consistent. Then, in Section 2.4 we show that if T is an open \mathcal{L} -theory of topological fields and T eliminates \mathbf{F} -quantifiers, T_δ^* also eliminates \mathbf{F} -quantifiers.

Before defining T_δ^* , let us fix some notation and recall the needed background on differential algebra. We work in a model \mathcal{K} of T .

2.1. Differential algebra background. Equip K with a derivation δ , that is, an additive morphism $\delta: K \rightarrow K$ which satisfies Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$. We let C_K denote the field of constants of K , namely, $C_K := \{a \in K : \delta(a) = 0\}$. It is a subfield of K .

For $m \geq 0$ and $a \in K$, we define

$$\delta^m(a) := \underbrace{\delta \circ \dots \circ \delta}_{m \text{ times}}(a), \text{ with } \delta^0(a) := a,$$

and $\bar{\delta}^m(a)$ as the finite sequence $(\delta^0(a), \delta(a), \dots, \delta^m(a)) \in K^{m+1}$. Similarly, given an element $a = (a_1, \dots, a_n) \in K^n$ and a tuple of non-negative integers $\bar{m} := (m_1, \dots, m_n)$, we write $\delta^{\bar{m}}(a)$ to denote the element $(\delta^{m_1}(a_1), \dots, \delta^{m_n}(a_n)) \in K^n$ and we define $\bar{\delta}^{\bar{m}}(a)$

accordingly. For notational clarity, we sometimes use $\nabla_{\bar{m}}$ instead of $\bar{\delta}^{\bar{m}}$, especially concerning the image of subsets of K^n . For example, when $A \subseteq K$, we use the notation $\nabla_m(A)$ for $\{\bar{\delta}^m(a) : a \in A\}$ instead of $\bar{\delta}^m(A)$. Likewise for $A \subseteq K^n$ and $\bar{m} := (m_1, \dots, m_n)$, $\nabla_{\bar{m}}(A) := \{\bar{\delta}^{\bar{m}}(a) : a \in A\} \subseteq K^{n(\bar{m}+1)}$, where $n(\bar{m}+1) := \sum_{i=1}^n (m_i + 1)$.

We assume that our tuples of variables are ordered with the convention that, given variables $x = (x_0, \dots, x_n)$ and a variable y , the tuple (x, y) is ordered such that y is bigger than x_n (and similarly when y is an ordered tuple of variables). We endow \mathbb{N}^n with the partial order: for $\bar{m}, \bar{k} \in \mathbb{N}^n$, we have $\bar{m} \leq \bar{k}$ if for all $1 \leq i \leq n$, $m_i \leq k_i$.

For x as above, we let $K\{x\}$ be the ring of differential polynomials in $n+1$ differential indeterminates x_0, \dots, x_n over K , namely it is the ordinary polynomial ring in formal indeterminates $\delta^j(x_i)$, $0 \leq i \leq n$, $j \in \mathbb{N}$, with the convention $\delta^0(x_i) := x_i$. We extend the derivation δ to $K\{x\}$ by setting $\delta(\delta^i(x_j)) = \delta^{i+1}(x_j)$. By a rational differential function we simply mean a quotient of differential polynomials.

2.1.1. Order and separant of a differential polynomial. For $P(x) \in K\{x\}$ and $0 \leq i \leq n$, we let $\text{ord}_{x_i}(P)$ denote the *order of P with respect to the variable x_i* , that is, the maximal integer k such that $\delta^k(x_i)$ occurs in a non-trivial monomial of P and -1 if no such k exists. We let *the order of P* be

$$\text{ord}(P) := \max\{\text{ord}_{x_i}(P) : 0 \leq i \leq n\}.$$

Similarly, for a finite subset \mathcal{A} of $K\{x\}$, we let

$$\text{ord}_{x_i}(\mathcal{A}) := \max\{\text{ord}_{x_i}(P) : P \in \mathcal{A}\} \text{ and } \text{ord}(\mathcal{A}) := \max\{\text{ord}(P) : P \in \mathcal{A}\}.$$

For $R \in K\{x\}$, we write $\text{ord}_{x_i}(\mathcal{A}, R)$ for $\text{ord}_{x_i}(\mathcal{A} \cup \{R\})$.

Suppose $\text{ord}(P) = m$. For $\bar{x} = (\bar{x}_0, \dots, \bar{x}_n)$ a tuple of variables with $\ell(\bar{x}_i) = m + 1$, we let $P^* \in K[\bar{x}]$ denote the corresponding ordinary polynomial such that $P(x) = P^*(\bar{\delta}^m(x))$. Suppose $\text{ord}_{x_n}(P) = m \geq 0$. Then, there are (unique) differential polynomials $c_i \in K\{x\}$ such that $\text{ord}_{x_n}(c_i) < m$ and

$$(2.1.1) \quad \{\text{eq:diffpol}\} \quad P(x) = \sum_{i=0}^d c_i(x) (\delta^m(x_n))^i.$$

The separant s_P of P is defined as $s_P := \frac{\partial}{\partial \delta^m(x_n)} P \in K\{x\}$. We extend the notion of separant to arbitrary polynomials with an ordering on their variables in the natural way, namely, if $P \in K[x]$, the separant of P corresponds to $s_P := \frac{\partial}{\partial x_n} P \in K[x]$. By convention, we induce a total order on the variables $\delta^j(x_i)$ by declaring that

$$\delta^k(x_i) < \delta^{k'}(x_j) \Leftrightarrow \begin{cases} i < j \\ i = j \text{ and } k < k'. \end{cases}$$

This order makes the notion of separant for differential polynomials compatible with the extended version for ordinary polynomials, i.e., $s_{P^*} = s_P^*$.

2.1.2. Minimal differential polynomials. Let $F \subseteq K$ be an extension of differential fields. Recall that an ideal I of $F\{x\}$ is a *differential ideal* if for every $P \in I$, $\delta(P) \in I$. For $a \in K$, let $I(a, F)$ denote the set of differential polynomials in $F\{x\}$ vanishing on a . The set $I(a, F)$ is a prime differential ideal of $F\{x\}$. Let $P \in I(a, F)$ be a differential

polynomial of minimal degree among the elements of $I(a, F)$ having minimal order. Any such differential polynomial is called a *minimal differential polynomial of a over F* . Let $\langle P \rangle$ denote the differential ideal generated by P and $I(P) := \{Q(x) \in F\{x\} : s_P^\ell Q \in \langle P \rangle \text{ for some } \ell \in \mathbb{N}\}$.

2.1.2. Lemma ([25, Section 1]). *If P is a minimal differential polynomial for a over F , then $I(a, F) = I(P)$.* \square

2.1.3. Rational prolongations. We define an operation on $K\{x\}$ sending $P \mapsto P^\delta$ as follows: for P written as in (2.1.1)

$$P(x) \mapsto P^\delta(x) = \sum_{i=0}^d \delta(c_i(x))(\delta^m(x_n))^i.$$

A simple calculation shows that

$$(2.1.3) \quad \delta(P(x)) = P^\delta(x) + s_P(x)\delta^{m+1}(x_n).$$

2.1.4. Lemma-Definition. Let $x = (x_0, \dots, x_n)$ be a tuple of variables and y be a single variable. Let $P \in K\{x, y\}$ be a differential polynomial such that $m = \text{ord}_y(P) \geq 0$. There is a sequence of rational differential functions $(f_i^P)_{i \geq 1}$ such that for every $a \in K^{n+1}$ and $b \in K$

$$K \models [P(a, b) = 0 \wedge s_P(a, b) \neq 0] \rightarrow \delta^{m+i}(b) = f_i^P(a, b).$$

In addition, each f_i^P is of the form

$$f_i^P(x, y) = \frac{Q_i(x, y)}{s_P(x, y)^{\ell_i}},$$

where $\ell_i \in \mathbb{N}$, $\text{ord}_y(Q_i) = \text{ord}_y(P)$ and

$$\text{ord}_{x_j}(Q_i) = \begin{cases} \text{ord}_{x_j}(P) + i & \text{if } \text{ord}_{x_j}(P) \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

We call the sequence $(f_i^P)_{i \geq 1}$ the *rational prolongation along P* .

Proof. It suffices to inductively define the polynomials Q_i . By (2.1.3), if $\delta(P(x, y)) = 0$ we obtain that

$$\delta^{m+1}(y) = \frac{-P^\delta(x, y)}{s_P(x, y)},$$

Setting $Q_1 = -P^\delta$, the rational differential function $f_1^P = \frac{Q_1}{s_P}$ satisfies the required property. Now suppose Q_i has been defined and that $f_i^P = \frac{Q_i}{(s_P)^{\ell_i}}$ satisfies $\delta^{m+i}(y) = f_i^P(x, y)$. By applying δ on both sides we obtain

$$\delta^{m+i+1}(y) = \frac{\delta(Q_i(x, y))s_P(x, y) - Q_i(x, y)\delta(s_P(x, y))}{s_P(x, y)^{2\ell_i}}.$$

By replacing instances of $\delta^{m+i}(y)$ in $\delta(Q_i(x, y))$ and $\delta(s_P(x, y))$ by $f_i^P(x, y)$, we obtain in the numerator a differential polynomial of order m with respect to y . Setting Q_{i+1} as such numerator shows the result. The last assertion is a straightforward calculation. \square

{not:prolong}

2.1.5. Notation. Let $x = (x_0, \dots, x_m)$ be a tuple of variables. For an integer $d \geq 0$, we define a new tuple of variables $x(d)$ which extends x by d new variables, that is,

$$x(d) := (x_0, \dots, x_m, x_{m+1}, \dots, x_{m+d}).$$

When $\bar{x} = (\bar{x}_0, \dots, \bar{x}_\ell)$ is a tuple of tuples of variables and $\bar{d} \in \mathbb{N}^{\ell+1}$, we let $\bar{x}[\bar{d}] := (\bar{x}_0(d_0), \dots, \bar{x}_\ell(d_\ell))$. To ease notation, if all $d_i = d$, $0 \leq i \leq \ell$, we denote $\bar{x}[\bar{d}]$ by $\bar{x}[d]$. Note that if $d \neq 0$ and \bar{x} is not a singleton, then $\bar{x}[d]$ and $\bar{x}(d)$ are different.

2.1.6. Notation. Let $x = (x_0, \dots, x_n)$ and y be a single variable. Let $P \in K\{x, y\}$ be a differential polynomial of order m and let $(f_i^P)_{i \geq 1}$ be its rational prolongation along P . Let $\bar{x} = (\bar{x}_0, \dots, \bar{x}_n)$ where $\bar{x}_i = (x_{i0}, \dots, x_{im_i})$ with $m_i \geq \text{ord}_{x_i}(P)$ and $\bar{y} = (y_0, \dots, y_m)$. For every $d \geq 0$, we let $\lambda_P^d(\bar{x}[d], \bar{y}(d))$ be the $\mathcal{L}(K)$ -formula:

$$P^*(\bar{x}, \bar{y}) = 0 \wedge s_P^*(\bar{x}, \bar{y}) \neq 0 \wedge \bigwedge_{i \geq 1}^d y_{m+i} = (f_i^P)^*(\bar{x}[i], \bar{y}).$$

We will allow ourselves to consider the tuple of functions $(f_i^P)^*$, $1 \leq i \leq d$, as formally depending on $\bar{x}[d]$.

2.1.4. Kolchin closed sets. Similarly as in Notation 1.3.1, we introduce the following notation:

2.1.7. Notation. Let x be a tuple of variables with $\ell(x) = n$. for a finite subset \mathcal{A} of $K\{x\}$ and $R \in K\{x\}$, we let $\mathcal{Z}_{\mathcal{A}}^R(x)$ denote the $\mathcal{L}_\delta(K)$ -formula

$$\bigwedge_{P \in \mathcal{A}} P(x) = 0 \wedge R(x) \neq 0.$$

We let $\mathcal{Z}_{\mathcal{A}}(x)$ be $\mathcal{Z}_{\mathcal{A}}^1(x)$.

Recall that a subset $X \subseteq K^n$ is called *Kolchin closed* if there is a finite subset $\mathcal{A} \subseteq K\{x\}$ such that $X = \mathcal{Z}_{\mathcal{A}}(K)$. It is called *locally Kolchin closed* if $X = \mathcal{Z}_{\mathcal{A}}^R(K)$ for some $R \in K\{x\}$. The following lemma is the differential analogue of Lemma 1.3.2. Its proof is completely algebraic and follows by induction on orders and degrees using Lemma 1.3.2. We leave it to the reader.

2.1.8. Lemma. *Let $x = (x_1, \dots, x_n)$, y be a single variable, \mathcal{A} be a finite subset of $K\{x, y\}$ and $R \in K\{x, y\}$. The set $\mathcal{Z}_{\mathcal{A}}^R(K)$ is the union of finitely many sets $\mathcal{Z}_{\mathcal{B}}^{S_{\mathcal{B}}}(K)$, with \mathcal{B} a finite subset of $K\{x, y\}$ and $S_{\mathcal{B}} \in K\{x, y\}$, such that $\text{ord}_y(S_{\mathcal{B}}) \leq \text{ord}_y(\mathcal{A}, R)$, $\text{ord}_{x_i}(\mathcal{B}, S_{\mathcal{B}}) \leq \text{ord}_{x_i}(\mathcal{A}, R) + \text{ord}_y(\mathcal{A}, R)$ for each $1 \leq i \leq n$, and either*

- $\mathcal{B} \subseteq K\{x\}$ or
- there is a unique $P_{\mathcal{B}} \in \mathcal{B}$ of non-negative order in y , $\text{ord}_y(P_{\mathcal{B}}) \leq \text{ord}_y(\mathcal{A}, R)$ and $s_{P_{\mathcal{B}}}$ divides $S_{\mathcal{B}}$. \square

2.2. The theory T_δ^* . Denote by T_δ the \mathcal{L}_δ -theory T together with the usual axiom of a derivation, namely,

$$\mu := \forall x \forall y (\delta(x + y) = \delta(x) + \delta(y) \wedge \delta(xy) = \delta(x)y + x\delta(y))$$

{not:prolong2}

{sec:kolchin}

{nota:kolchin}

{lem:minord3}

{sec:Tdelta}

and an axiom $\delta(c) = 0$ for each constant symbol $c \in \Omega$. Note that this is only one possible choice of how to define the derivation on $\mathbb{Q}(\Omega)$. For simplicity of presentation we focus on the case where $\mathbb{Q}(\Omega)$ is a subfield of the field of constants C_K for every model K of T_δ . This choice will only play an important role in Section 4.

2.2.1. Lemma-Definition. Let x be a tuple of \mathbf{F} -variables and z be a tuple of auxiliary sort variables. Let $t(x, z)$ be an \mathcal{L}_δ -term and $\varphi(x, z)$ be an \mathbf{F} -quantifier free \mathcal{L}_δ -formula.

- (1) Then, there is a tuple $\bar{m} := (m_1, \dots, m_{\ell(x)})$, $m_i \in \mathbb{N}$, a tuple \bar{x} of \mathbf{F} -variables with $\ell(\bar{x}_i) = m_i$ and an \mathcal{L} -term $t^*(\bar{x}, z)$, following the convention that if x_i does not occur in $t(x, z)$, then \bar{x}_i does not occur in t^* and $m_i = 0$, $1 \leq i \leq \ell(x)$, such that

$$T_\delta \models (\forall x)(\forall z)(t(x, z) = t^*(\bar{\delta}^{\bar{m}}(x), z)), \text{ and}$$

- (2) there is a tuple $\bar{m} := (m_1, \dots, m_{\ell(x)})$, $m_i \in \mathbb{N}$, a tuple \bar{x} of \mathbf{F} -variables with $\ell(\bar{x}_i) = m_i$ and an \mathbf{F} -quantifier free \mathcal{L} -formula $\varphi^*(\bar{x}, z)$, following the convention that if x_i , $1 \leq i \leq \ell(x)$, does not occur in $\varphi(x, z)$, then \bar{x}_i does not occur in φ^* and $m_i = 0$, $1 \leq i \leq \ell(x)$, such that

$$T_\delta \models (\forall x)(\forall z)(\varphi(x, z) \leftrightarrow \varphi^*(\bar{\delta}^{\bar{m}}(x), z)),$$

Proof. Point (1) follows by induction on terms. The main step consists in showing the result for an \mathcal{L}_δ -term of the form $\delta(t_0(x, z))$ where $t_0(x, z)$ is an \mathbf{F} -valued \mathcal{L}_δ -term. By induction, $t_0(x, z) = t_0^*(\bar{\delta}^{\bar{m}}(x), z)$ modulo T_δ for some \mathcal{L} -term $t_0^*(\bar{x}, z)$. Then, by part (2) of Definition 1.2.1, there is an \mathcal{L} -term $\tilde{t}_0^*(\bar{x})$ such that $t_0^*(\bar{x}, z) = \tilde{t}_0^*(\bar{x})$ modulo T . Thus, it remains to show the result for $\delta(\tilde{t}_0^*(\bar{x}))$. This follows from the usual axiom of a derivation and the fact that the restriction of \mathcal{L} to \mathbf{F} is a relational extension of $\mathcal{L}_{\text{field}}^\Omega$ (part (1) of Definition 1.2.1).

Part (2) follows by induction on formulas using part (1). \square

We now describe a scheme of \mathcal{L}_δ -axioms generalizing the axiomatization of CODF given by M. Singer in [41]. Let $\chi_\tau(x, z)$ be the \mathcal{L} -formula providing a basis of neighbourhoods of 0. Abusing of notation, when x is a tuple of \mathbf{F} -variables $x = (x_1, \dots, x_n)$ we let $\chi_\tau(x, z)$ denote the formula

$$\bigwedge_{i=1}^n \chi_\tau(x_i, z).$$

2.2.2. Definition. The \mathcal{L}_δ -theory T_δ^* is the union of T_δ and the following scheme of axioms (DL): given a model \mathcal{K} of T_δ , \mathcal{K} satisfies (DL) if for every differential polynomial $P(x) \in K\{x\}$ with $\ell(x) = 1$ and $\text{ord}_x(P) = m \geq 1$, for field sort variables $y = (y_0, \dots, y_m)$ it holds in \mathcal{K} that

$$\forall z((\exists y(P^*(y) = 0 \wedge s_P^*(y) \neq 0) \rightarrow \exists x(P(x) = 0 \wedge s_P(x) \neq 0 \wedge \chi_\tau(\bar{\delta}^m(x) - y, z))).$$

As usual, by quantifying over coefficients, the axiom scheme (DL) can be expressed in the language \mathcal{L}_δ . The scheme (DL) essentially expresses the following: given a differential polynomial $P(x)$ of order m , if its associated ordinary polynomial P^* has a regular zero a , then for every open neighbourhood U of a we may find a zero b of P such that $\bar{\delta}^m(b)$ is in U .

Letting $T = \text{RCF}$, we get back CODF as the theory RCF_δ^* .

2.3. Consistency. The main result of this section is Theorem 2.3.4 which shows that if T is a complete theory of topological fields for which the topology is induced by a henselian valuation, then T_δ^* is consistent. As a consequence we obtain the consistency of T_δ^* for all theories T described in Examples 1.2.5. For some of such theories, the consistency of T_δ^* has already been proved. Indeed, the consistency of CODF was proved in [41] and was later generalized in [44] for large fields and then in [16] for topological fields, using a notion of topological largeness. Although we will follow a very similar strategy to the known proofs, our argument is based on henselizations rather than using explicitly a notion of largeness (or topological largeness) for the fields under consideration. However, the fields for which consistency will be proven, all being henselian, are large fields.

Let us start by a general criterion to show that T_δ^* is consistent.

2.3.1. Proposition. *Let T be a complete \mathcal{L} -theory of topological fields and $\chi_\tau(x, z)$ be the \mathcal{L} -formula defining a basis of neighbourhoods of 0. Suppose that for every model \mathcal{K} of T and every derivation δ on K the following holds*

- (*) *for every $P \in K\{x\}$ ($\ell(x) = 1$) of order $m \geq 1$ for which there is $a \in K^{m+1}$ such that $P^*(a) = 0$ and $s_P^*(a) \neq 0$, there is an \mathcal{L} -elementary extension \mathcal{F} of \mathcal{K} , a derivation on F extending δ and $b \in F$ such that $P(b) = 0$, $s_P(b) \neq 0$ and for every $c \in K^z$*

$$\mathcal{F} \models \chi_\tau(\bar{\delta}^m(b) - a, c).$$

Then, for every model \mathcal{K} of T and every derivation δ on K , there is an \mathcal{L} -elementary extension \mathcal{M} of \mathcal{K} and an extension of δ to M making (\mathcal{M}, δ) into a model of T_δ^ . In particular, T_δ^* is consistent.*

Proof. Fix some model \mathcal{K} of T and some derivation δ on K . We use the following two step construction to build (\mathcal{M}, δ) .

Step 1: We construct an \mathcal{L} -elementary extension $\mathcal{K} \preceq_{\mathcal{L}} \mathcal{F}_{\mathcal{K}}$ and a derivation on $F_{\mathcal{K}}$ extending δ as follows. Let $(P_i)_{i < \lambda}$ be an enumeration of all differential polynomials $P_i \in K\{x\}$ with $\text{ord}(P_i) = m_i \geq 1$ together with tuples $a_i \in K^{m_i+1}$ such that $P_i^*(a_i) = 0$ and $s_{P_i}^*(a_i) \neq 0$. Consider the following chain $(\mathcal{F}_i, \delta_i)_{i < \lambda}$ defined by

- (i) $(\mathcal{F}_0, \delta_0) := (\mathcal{K}, \delta)$,
- (ii) $(\mathcal{F}_{i+1}, \delta_{i+1})$ is given by condition (*) with respect to $(\mathcal{F}_i, \delta_i)$ and P_i , that is, $\mathcal{F}_i \preceq_{\mathcal{L}} \mathcal{F}_{i+1}$, δ_{i+1} extends δ_i and there is $b_i \in \mathcal{F}_{i+1}$ such that $P(b_i) = 0$, $s_P(b_i) \neq 0$ and for every $c \in K^z$

$$\mathcal{F}_{i+1} \models \chi_\tau(\bar{\delta}^m(b_i) - a_i, c).$$

- (iii) At limit stages, we take unions.

Abusing notation, let $(\mathcal{F}_{\mathcal{K}}, \delta) := \bigcup_{i < \lambda} \mathcal{F}_i$, where δ denotes the union of the derivations δ_i . Observe that indeed $\mathcal{K} \preceq_{\mathcal{L}} \mathcal{F}_{\mathcal{K}}$ and $(K, \delta) \subseteq (F_{\mathcal{K}}, \delta)$ is an extension of differential fields.

Step 2: Define a chain $(\mathcal{M}_i, \delta_i)_{i < \omega}$ where $(\mathcal{M}_0, \delta_0) := (\mathcal{K}, \delta)$, and $(\mathcal{M}_{i+1}, \delta_{i+1})$ corresponds to $(\mathcal{F}_{\mathcal{M}_i}, \delta_{i+1})$ obtained in Step (1) with respect to $(\mathcal{M}_i, \delta_i)$, so that $\mathcal{M}_i \preceq_{\mathcal{L}}$

\mathcal{M}_{i+1} . Let $(\mathcal{M}, \delta) := \bigcup_{i < \omega} (\mathcal{M}_i, \delta_i)$. By construction, $\mathcal{K} \preceq_{\mathcal{L}} \mathcal{M}$. It is an easy exercise to show that (\mathcal{M}, δ) satisfies the axiom scheme (DL). \square

:consistency}

2.3.2. Proposition. *Let \mathcal{L} be an extension of \mathcal{L}_{div} and T be a complete \mathcal{L} -theory of topological fields where $\chi_{\tau}(x, z)$ corresponds to the \mathcal{L}_{div} -formula $v(x) > v(z) \wedge z \neq 0$. Suppose that for every model \mathcal{K} of T , (K, v) is henselian. Let δ be a derivation on K . Then, there is an \mathcal{L} -elementary extension \mathcal{M} of \mathcal{K} and an extension of δ to M making (\mathcal{M}, δ) into a model of T_{δ}^* . In particular, T_{δ}^* is consistent.*

Proof. It suffices to show condition $(*)$ in the statement of Proposition 2.3.1. Let \mathcal{K} be a model of T such that K is equipped with a derivation δ . Let v denote the henselian valuation on K and Γ^v denote the value group of (K, v) . Suppose $P \in K\{x\}$ is a differential polynomial of order $m \geq 1$ and $a = (a_0, \dots, a_m) \in K^{m+1}$ such that $P^*(a) = 0$ and $s_P^*(a) \neq 0$. Let \mathcal{K}^* be an \mathcal{L} -elementary extension of \mathcal{K} containing a tuple of elements $t = (t_0, \dots, t_m)$ such that $\Gamma^v < v(t_0) \ll v(t_1) \ll \dots \ll v(t_m)$. Consider the (ordinary) polynomial

$$Q(x) = P^*(a_0 - t_0, \dots, a_{m-1} - t_{m-1}, x)$$

in $K(t_0, \dots, t_{m-1})[x]$. Let $w: K(t) \rightarrow \mathbb{Z}_{\infty}^n$ denote the coarsening of v such that $w(K) = 0$ and $0 < w(t_0) \ll w(t_1) \ll \dots \ll w(t_m)$. Let $F = (K(t), w)^h$ and $L = (K(t), v)^h$ be their corresponding henselizations. Without loss, we may suppose that $F \subseteq L \subseteq K^*$ as sets (the first inclusion might be assumed since w is a coarsening of v) and that for all $y \in F$

$$w(y) > 0 \Leftrightarrow v(y) > \Gamma^v.$$

Let us show that there is $c \in F$ such that $Q(c) = 0$ and $w(c - a_m) > 0$. The reduction \tilde{Q} of Q modulo the maximal ideal of (F, w) corresponds to $P^*(a_0, \dots, a_{m-1}, x) \in K[x]$. By assumption, $\tilde{Q}(a_m) = 0$ and $\frac{\partial}{\partial x} \tilde{Q}(a_m) \neq 0$. Then, by Hensel's lemma, there is $c \in F$ such that $Q(c) = 0$ and $w(c - a_m) > 0$ (equivalently $v(c - a_m) > \Gamma^v$). This implies both that $c \notin K$ and that $\frac{\partial}{\partial x} Q(c) \neq 0$. We extend δ to the subfield $K(t_0, \dots, t_{m-1}, c) \subseteq F$ by inductively setting

- (1) $\delta(t_i) = \delta(a_i) + t_{i+1} - a_{i+1}$ for $0 \leq i < m-1$,
- (2) $\delta(t_{m-1}) = c$,

noting that, since $Q(c) = 0$ and $\frac{\partial}{\partial x} Q(c) \neq 0$, the derivative of c is already determined by the rational prolongation $f_1^Q(c)$. Setting $b := a_0 - t_0$, we have that

$$P(b) = P^*(\bar{\delta}^m(b)) = P^*(a_0 - t_0, \dots, a_{m-1} - t_{m-1}, c) = Q(c) = 0.$$

Similarly, $s_P(b) \neq 0$. In addition, for every $e \in K^{\times}$, $K^* \models \chi_{\tau}(\bar{\delta}^m(b) - a, e)$, since

$$(2.3.3) \quad v(\bar{\delta}^m(b) - a) = \min\{v(t_0), \dots, v(t_{m-1}), v(c - a_m)\} > v(e).$$

Extending the derivation from $K(t_0, \dots, t_{m-1}, b)$ to K^* (such an extension always exists by [23, Theorem 5.1]) completes the result. \square

The following consistency result slightly improves Proposition 2.3.2 by allowing cases in which the definable topology is induced by a henselian valuation, but such a valuation is not necessarily definable.

:consistency}

2.3.4. Theorem. *Let T be a complete \mathcal{L} -theory of topological fields. Assume every model \mathcal{K} of T has the following property: letting τ be the definable topology, there is an \mathcal{L} -elementary extension \mathcal{F} of \mathcal{K} with F equipped with a henselian valuation v such that the valuation topology coincides with τ . Then, for every derivation δ on K , there is an \mathcal{L} -elementary extension \mathcal{M} of \mathcal{K} and an extension of δ to M making (\mathcal{M}, δ) into a model of T_δ^* . In particular, T_δ^* is consistent.*

Proof. Consider the language $\mathcal{L}' = \mathcal{L} \cup \{\text{div}\}$ and let $T' = \text{Th}_{\mathcal{L}'}(\mathcal{F})$. In particular, T' is an \mathcal{L}' -theory of topological fields with henselian valued fields in the field sort. Extend the derivation δ to some (any) a derivation on F . By Proposition 2.3.2, there is an \mathcal{L}' -elementary extension \mathcal{M} of \mathcal{F} and an extension of δ to M such that $(\mathcal{M}, \delta) \models (T')_\delta^*$. Note that $\mathcal{K} \preceq_{\mathcal{L}} \mathcal{F} \preceq_{\mathcal{L}} \mathcal{M}$. Since the valuation topology coincides with τ on M , the reduct of \mathcal{M} to \mathcal{L}_δ is a model of T_δ^* . \square

{cor:cons_exa}

2.3.5. Corollary. *Let T be any theory from Examples 1.2.5. Then T_δ^* is consistent.*

Proof. Except for CODF, all examples in Examples 1.2.5 correspond to expansions of theories of henselian valued fields of characteristic 0 and the result follows already from Proposition 2.3.2. For CODF, the consistency follows by Theorem 2.3.4, since any real closed field can be elementarily embedded into a real closed valued field (which is henselian) for which the order topology and the valuation topology coincide. Alternatively (and essentially the same), the consistency of RCVF_δ^* implies the consistency of CODF, as every model of RCVF_δ^* is a model of CODF. \square

{rem:closedne}

2.3.6. Remark. Let (K, v) be a valued field of characteristic 0 endowed with a derivation δ . Let (K^h, v) be the henselization of (K, v) . Note that the derivation extends (uniquely) to K^h . Let T be the theory of (K^h, v) . Proposition 2.3.2 implies that (K, v, δ) embeds as an $\mathcal{L}_{\text{div}, \delta}$ -structure into a model of T_δ^* .

{rem:consiste}

2.3.7. Remark. Note that if T_δ^* is consistent, then every model \mathcal{K} of T embeds as an \mathcal{L} -structure into a model of T_δ^* . Indeed, take a model \mathcal{K}' of T_δ^* . Then the reduct of \mathcal{K}' to \mathcal{L} is a model of T , and since T is complete $\mathcal{K} \equiv_{\mathcal{L}} \mathcal{K}'$. The result follows by Keisler-Shelah's theorem.

{sec:relQE}

2.4. Relative quantifier elimination. For the rest of Section 2 we let T be an open \mathcal{L} -theory of topological fields and assume T_δ^* is a consistent theory.

We will need the following classical consequence of the axiom scheme (DL).

{fact:density}

2.4.1. Lemma ([16, Lemma 3.17]). *Let \mathcal{K} be a model of T_δ^* . Let O be an open subset of K^n . Then there is $a \in K$ such that $\delta^{n-1}(a) \in O$.* \square

{thm:QE}

2.4.2. Theorem. *Suppose T eliminates field sort quantifiers. Then, T_δ^* eliminates field sort quantifiers.*

Proof. Let Σ denote the set of \mathbf{F} -quantifier free \mathcal{L}_δ -formulas. Let x be a tuple of \mathbf{F} -variables, z a tuple of auxiliary sort variables, y a single variable and $\varphi(x, y, z)$ be a formula in Σ . Let $\mathcal{K}_1, \mathcal{K}_2$ be two models of T_δ^* , $b_i \in \mathcal{K}_i^x$ and $c_i \in \mathcal{K}_i^z$ such that

$$(b_1, c_1) \equiv_\Sigma (b_2, c_2).$$

Suppose there is $a \in \mathcal{K}_1^y$ such that $\mathcal{K}_1 \models \varphi(b_1, a, c_1)$. It suffices to show there is $a' \in \mathcal{K}_2^y$ such that $\mathcal{K}_1 \models \varphi(b_2, a', c_2)$. If y is an auxiliary sort variable, the result follows trivially, so suppose y is an \mathbf{F} -variable.

{cla:terms}

2.4.3. Claim. $\langle b_i, c_i \rangle_{\mathcal{L}_\delta} \cap K_i = \langle (\delta^n(b_i))_{n \in \mathbb{N}} \rangle_{\mathcal{L}_{\text{field}}^\Omega}$.

By Part (1) of Lemma-Definition 2.2.1, we have that $\langle b_i, c_i \rangle_{\mathcal{L}_\delta} = \langle (\delta^n(b_i))_{n \in \mathbb{N}}, c_i \rangle_{\mathcal{L}}$. By Remark 1.2.2, $\langle (\delta^n(b_i))_{n \in \mathbb{N}}, c_i \rangle_{\mathcal{L}} \cap K_i = \langle (\delta^n(b_i))_{n \in \mathbb{N}} \rangle_{\mathcal{L}_{\text{field}}^\Omega}$, which completes the claim.

Let \mathcal{F}_i denote $\langle (\delta^n(b_i))_{n \in \mathbb{N}} \rangle_{\mathcal{L}_{\text{field}}^\Omega}$ and $\sigma: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be an \mathcal{L}_δ -isomorphism. By elimination of \mathbf{F} -quantifiers in \mathcal{L} modulo T , we may extend σ to $\mathcal{F}_1 \cup (F_1^{\text{alg}} \cap K_1)$. Indeed, it follows from Lemma-Definition 2.1.4, that if $a_0 \in F_1^{\text{alg}} \cap K_1$ and P is its minimal polynomial over F_1 , then for each $n \in \mathbb{N}$, $\delta^n(a_0) = f_n^P(a_0)$ where f_n^P is a rational function with coefficients in F_1 .

By part (2) of Lemma-Definition 2.2.1 let $\varphi^*(\bar{x}, \bar{y}, z)$ be the \mathbf{F} -sort quantifier free \mathcal{L} -formula such that

$$T_\delta^* \models (\forall x)(\forall y)(\forall z)(\varphi(x, y, z) \leftrightarrow \varphi^*(\bar{\delta}^m(x), \bar{\delta}^m(y), z))$$

for some $m \geq 0$. By condition (A), the formula $\varphi^*(\bar{x}, \bar{y}, z)$ is equivalent to

$$\bigvee_{h \in H} \left(\xi_h(z) \rightarrow \left(\bigvee_{i \in I_h} \bigwedge_{j \in J_{ih}} P_{ijh}(\bar{x}, \bar{y}) = 0 \wedge \theta_{ih}(\bar{x}, \bar{y}, z) \right) \right)$$

Let $h \in H$ such that $\xi_h(c_1)$ holds and $i \in I_h$ such that

$$(2.4.4) \quad \bigwedge_{j \in J_{ih}} P_{ijh}(\bar{\delta}^m(b_1), \bar{\delta}^m(a)) = 0 \wedge \theta_{ih}(\bar{\delta}^m(b_1), \bar{\delta}^m(a), c_1),$$

holds.

We split in two cases depending on whether J_{ih} is empty or not.

Case 1: Suppose $J_{ih} \neq \emptyset$, so a is differentially algebraic over F_1 . Let $P \in F_1\{y\}$ be a minimal differential polynomial for a over F_1 of order k . We may assume a is not algebraic over F_1 , since σ was already extended to $F_1^{\text{alg}} \cap K_1$, and hence that $k \geq 1$. Since P is minimal, we must have both $k \leq m$ and that $s_P(a) \neq 0$. For $d = m - k$, we have then (see Notation 2.1.6)

$$(2.4.5) \quad \mathcal{K}_1 \models \lambda_P^d(\bar{\delta}^m(a)).$$

Let P^σ denote the corresponding polynomial over F_2 in which every coefficient of P is replaced by its image under σ . Since $I(a, F_1) = I(P)$ (by Lemma 2.1.2), for each $j \in J_{ih}$, it holds that

$$P_{ijh}(\bar{\delta}^m(b_1), \bar{\delta}^m(y)) \in I(P).$$

Our assumption on b_1 and b_2 implies that $P_{ijh}(\bar{\delta}^m(b_2), \bar{\delta}^m(y)) \in I(P^\sigma)$. Therefore, it suffices to show that there is $a' \in K_2$ such that

$$\mathcal{K}_2 \models P^\sigma(a') = 0 \wedge s_{P^\sigma}(a') \neq 0 \wedge \theta(\bar{\delta}^m(b_2), \bar{\delta}^m(a'), c_2),$$

as this will also imply that $\mathcal{K}_2 \models \bigwedge_{j \in J_{ih}} P_{ijh}(\bar{\delta}^m(b_2), \bar{\delta}^m(a')) = 0$. By (2.4.5), $\mathcal{K}_1 \models (\exists \bar{x}) \lambda_P^d(\bar{x})$ (which is an $\mathcal{L}(F_1)$ -formula), so by \mathbf{F} -quantifier elimination in T , there is $\bar{e} = (e_0, \dots, e_m) \in K_2^{m+1}$ such that $\mathcal{K}_2 \models \lambda_{P^\sigma}^d(\bar{e})$. Letting $\hat{e} = (e_0, \dots, e_k)$, the previous

formula yields that $e_{k+i} = (f_i^{P^\sigma})^*(\hat{e})$ for all $1 \leq i \leq d$, where $(f_i^{P^\sigma})_{i \geq 1}$ is the rational prolongation of P^σ . Since $(P^\sigma)^*(\hat{e}) = 0$ and $s_{P^\sigma}^*(\hat{e}) \neq 0$, the axiom scheme (DL) implies there is $a' \in K_2$ such that $P^\sigma(a') = 0$ and $s_{P^\sigma}(a') \neq 0$. Moreover, by the continuity of the functions $(f_i^{P^\sigma})^*$, we may further suppose that $\theta(\bar{\delta}^m(b_2), \bar{\delta}^m(a'), c_2)$ holds. This completes Case 1.

Case 2: Suppose $J_{ih} = \emptyset$. Since the set $\theta_{ih}^*(\bar{\delta}^m(b_2), \mathcal{K}_2, c_2)$ is an open subset of K_2^{m+1} , the result follows directly from Lemma 2.4.1. \square

{cor:QE-theor}

2.4.6. Corollary. *For T either RCF, ACVF_{0,p}, RCVF or pCF, T_δ^* has quantifier elimination. For T the \mathcal{L}_{RV} -theory of a henselian valued field of characteristic 0, the theory T_δ^* eliminates \mathbf{F} -quantifiers.* \square

By Remark 1.2.4, the theory T_{Mor} is an open \mathcal{L}_{Mor} -theory of topological fields, and therefore, by Theorem 2.4.2, the theory $(T_{\text{Mor}})_\delta^*$ eliminates \mathbf{F} -quantifiers. We exploit this fact in what follows.

{cor:complete}

2.4.7. Corollary. *The theory T_δ^* is complete.*

Proof. Note that T_δ^* is complete if and only if $(T_{\text{Mor}})_\delta^*$ is complete. Therefore, by Theorem 2.4.2 and possibly working in $(T_{\text{Mor}})_\delta^*$, we may suppose T_δ^* eliminates \mathbf{F} -quantifiers. Let φ be an \mathcal{L}_δ -sentence. We may suppose φ has no variable of field sort. Therefore, by Lemma-Definition 2.2.1, φ is equivalent, modulo T_δ^* , to an \mathcal{L} -sentence φ^* . The result follows from the completeness of T . \square

{cor:eq}

2.4.8. Corollary. *For every \mathcal{L}_δ -formula $\varphi(x, z)$, with x a tuple of \mathbf{F} -variables and z a tuple of auxiliary sort variables, there are an \mathcal{L} -formula $\tilde{\varphi}(\bar{x}, z)$ and $\bar{m} \in \mathbb{N}^{\ell(x)}$ such that*

$$T_\delta^* \models (\forall x)(\forall z)(\varphi(x, z) \leftrightarrow \tilde{\varphi}(\bar{\delta}^{\bar{m}}(x), z)).$$

Proof. Since $\varphi(x, z)$ is also an $(\mathcal{L}_{\text{Mor}})_\delta$ -formula, by Theorem 2.4.2, φ is equivalent modulo $(T_{\text{Mor}})_\delta^*$ to an \mathbf{F} -quantifier-free $(\mathcal{L}_{\text{Mor}})_\delta$ -formula $\psi(x, z)$. By Lemma-Definition 2.2.1, there are an \mathcal{L}_{Mor} -formula $\psi^*(\bar{x}, z)$ and a tuple $\bar{m} \in \mathbb{N}^{\ell(x)}$ such that $\psi(x, z)$ is equivalent modulo $(T_{\text{Mor}})_\delta^*$ to $\psi^*(\bar{\delta}^{\bar{m}}(x), z)$. Letting $\tilde{\varphi}(\bar{x}, z)$ be an \mathcal{L} -formula equivalent modulo T_{Mor} to $\psi^*(\bar{x}, z)$ completes the result. \square

2.4.9. Corollary. *Every \mathcal{L}_δ -definable set $X \subseteq K^n$ is of the form $\nabla_{\bar{m}}^{-1}(Y)$ for an \mathcal{L} -definable set $Y \subseteq K^{n(\bar{m}+1)}$.* \square

By Corollary 2.4.8, the following notion of order is well-defined.

{def:order}

2.4.10. Definition (Order). Let $\varphi(x, z)$ be an \mathcal{L}_δ -formula with x a tuple of \mathbf{F} -variables and z a tuple of auxiliary sort variables. Let X be any \mathcal{L}_δ -definable set.

- The *order* of φ (as an element of $\mathbb{N}^{\ell(x)}$) is defined as

$$\text{ord}(\varphi) := \min \left\{ \bar{m} : \begin{array}{l} \varphi \text{ is equivalent to } \psi(\bar{\delta}^{\bar{m}}(x), z) \text{ modulo } T_\delta^* \\ \text{with } \psi \text{ an } \mathcal{L}\text{-formula} \end{array} \right\}.$$

- The *order* of X , is defined as

$$\text{ord}(X) := \min \{ \text{ord}(\varphi) : \varphi \text{ is an } \mathcal{L}_\delta\text{-formula and } \varphi(\mathcal{K}) = X \}.$$

{rem:order_up}

2.4.11. Remark. Let $X \subseteq \mathcal{K}^{(x,z)}$ be an \mathcal{L}_δ -definable set where x a tuple of \mathbf{F} -variables and z a tuple of auxiliary sort variables. It holds that $\text{ord}(X) = \bar{0}$ if and only if X is \mathcal{L} -definable. In addition, for every tuple $(m, \dots, m) \geq \text{ord}(X)$ of length $\ell(x)$, there is an $\mathcal{L}(\mathcal{K})$ -formula $\psi(\bar{x}, z)$ such that $\psi(\bar{\delta}^m(x), z)$ defines X . This follows by the definition of $\text{ord}(X)$ and possibly adding formulas of the form $\delta^m(x_i) = \delta^m(x_i)$, $1 \leq i \leq \ell(x)$. In particular, when $X \subseteq \mathcal{K}^x$ we may always assume that X is of the form $X = \nabla_m^{-1}(Y)$ for some \mathcal{L} -definable set Y .

Let us end this section by pointing out some final consequences of relative quantifier elimination on definable sets. The corollary below shows that the derivation introduces no new structure on auxiliary sorts.

{cor:induced}

2.4.12. Corollary. *Let \mathcal{K} be a model of T_δ^* and z be tuple of auxiliary sort variables. Then every \mathcal{L}_δ -definable subset X of \mathcal{K}^z is \mathcal{L} -definable.*

Proof. Let $\varphi(x, w, z)$ be an \mathcal{L}_δ -formula with x a tuple of \mathbf{F} -variables and w be a tuple of auxiliary variables, such that $X = \varphi(a, b, \mathcal{K})$ for some $a \in \mathcal{K}^x$ and $b \in \mathcal{K}^w$. By Corollary 2.4.8, there are $m \geq 0$ and an \mathcal{L} -formula $\tilde{\varphi}(\bar{x}, z)$ such that $\varphi(x, z)$ is equivalent to $\tilde{\varphi}(\bar{\delta}^m(x), z)$. This shows that $X = \tilde{\varphi}(\bar{\delta}^m(a), b, \mathcal{K})$, and hence is \mathcal{L} -definable. \square

Somewhat orthogonal to the previous corollary, one can show that an \mathcal{L}_δ -definable family of definable subsets of the field sort with finite fibers and parametrized by auxiliary sorts is already \mathcal{L} -definable. We will delay the proof of that result (Lemma 3.3.2) until section 3.3 since we will need it there.

By classical arguments from [45], possibly working in $(T_{\text{Mor}})_\delta^*$, elimination of \mathbf{F} -quantifiers also yields the following result (known in the one-sorted case in [17]).

hm:delta-dim}

2.4.13. Corollary ([17, Corollary 3.10]). *The \mathcal{L}_δ -definable subsets of the field sort can be endowed with a dimension function (as defined by van den Dries in [45]).* \square

From Corollary 2.4.8, one may deduce some transfer results, like NIP, distality or elimination of \exists^∞ that will be stated and proven in the Appendix.

It is worthy to mention that other model-theoretic properties such as the existence of prime models or dp-minimality do not transfer from T to T_δ^* . Indeed, M. Singer showed in [41] (see also [31]) that CODF has no prime models (while RCF has) and Q. Brouette showed in his thesis [6] that CODF is not dp-minimal (whereas RCF is).

se{sepefPofE}

3. OPEN CORE AND TRANSFER OF ELIMINATION OF IMAGINARIES

3.1. Open core. Throughout this section we let T be an open \mathcal{L} -theory of topological fields and assume T_δ^* is consistent. We let \mathcal{K} be a model of T . Let us start by recalling the definition of open core.

In order to make notation lighter on indices, for the remaining of this section, given an \mathcal{L}_δ -definable set $X \subseteq K^n$, we note $\text{ord}(X) \in \mathbb{N}^n$ by $o(X)$.

ef:open_core}

3.1.1. Definition. Let $\tilde{\mathcal{L}}$ be an extension of \mathcal{L} and $\tilde{\mathcal{K}}$ be an $\tilde{\mathcal{L}}$ -expansion of \mathcal{K} . We say $\tilde{\mathcal{K}}$ has \mathcal{L} -open core if, for every $n \geq 1$, every $\tilde{\mathcal{L}}$ -definable open subset of K^n is \mathcal{L} -definable. An $\tilde{\mathcal{L}}$ -theory \tilde{T} extending T has \mathcal{L} -open core if every model of \tilde{T} has \mathcal{L} -open core.

Before proving the main results of this section, we give a useful characterization of the \mathcal{L} -open core for T_δ^* (which will also be used in our transfer result of elimination of imaginaries).

3.1.2. Definition. Let $X \subseteq K^n$ be a non-empty \mathcal{L}_δ -definable set. Given a tuple $\bar{m} \in \mathbb{N}^n$ and an \mathcal{L} -definable set $Z \subseteq K^{n(\bar{m}+1)}$, we call the triple (X, Z, \bar{m}) a *linked triple* if

- (1) $X = \nabla_{\bar{m}}^{-1}(Z)$ and
- (2) $\bar{Z} = \overline{\nabla_{\bar{m}}(X)}$.

Note that the tuple \bar{m} occurring in a linked triple might be bigger than $o(X)$.

3.1.3. Notation. For $0 \leq \bar{m} \leq \bar{d}$ we let $\pi_{\bar{m}}^{\bar{d}}: K^{n(\bar{d}+1)} \rightarrow K^{n(\bar{m}+1)}$ be the projection sending each block of d_i+1 coordinates to its block of first m_i+1 coordinates, $1 \leq i \leq n$, that is,

$$\pi_{\bar{m}}^{\bar{d}}(x_{1,0}, \dots, x_{1,d_1}, \dots, x_{n,0}, \dots, x_{n,d_n}) = (x_{1,0}, \dots, x_{1,m_1}, \dots, x_{n,0}, \dots, x_{n,m_n}).$$

In the next three lemmas, we relate the existence of linked triples for an \mathcal{L}_δ -definable set X when \bar{m} varies. We will use them in the proof of the \mathcal{L} -open core.

3.1.4. Lemma. Let $X \subseteq K^n$ be a non-empty \mathcal{L}_δ -definable set. Let $\bar{m}, \bar{d} \in \mathbb{N}^n$ be such that $o(X) \leq \bar{m} \leq \bar{d}$ and suppose (X, Z, \bar{d}) is a linked triple. Then, $\overline{\nabla_{\bar{m}}(X)} = \pi_{\bar{m}}^{\bar{d}}(\bar{Z})$. In particular, $\overline{\nabla_{\bar{m}}(X)}$ is \mathcal{L} -definable.

Proof. From left-to-right, let $a \in \nabla_{\bar{m}}(X)$. So $a = \delta^{\bar{m}}(x)$ with $x \in X$. Then by assumption $\delta^{\bar{d}}(x) \in Z$ and $a \in \pi_{\bar{m}}^{\bar{d}}(Z)$.

Conversely, let $b \in \pi_{\bar{m}}^{\bar{d}}(\bar{Z})$ and O an open neighbourhood of b . Then for some $z \in \bar{Z}$, $b = \pi_{\bar{m}}^{\bar{d}}(z)$. Let \tilde{O} an open neighbourhood of z such that $\pi_{\bar{m}}^{\bar{d}}(\tilde{O}) = O$. Since $z \in \bar{Z}$ and $\bar{Z} = \overline{\nabla_{\bar{d}}(X)}$, there is $x \in X$ such that $\delta^{\bar{d}}(x) \in \tilde{O}$ and so $\pi_{\bar{m}}^{\bar{d}}(\delta^{\bar{d}}(x)) \in O$, namely $\delta^{\bar{m}}(x) \in O$. So $b \in \overline{\nabla_{\bar{m}}(X)}$. \square

3.1.5. Lemma. Let $X \subseteq K^n$ be a non-empty \mathcal{L}_δ -definable set such that for some $\bar{m} \geq o(X)$, $\overline{\nabla_{\bar{m}}(X)}$ is \mathcal{L} -definable. Then there is a linked triple of the form (X, Z, \bar{m}) .

Proof. Let $Y \subseteq K^{n(o(X)+1)}$ be an \mathcal{L} -definable set such that $X = \nabla_{o(X)}^{-1}(Y)$. Set

$$Z := \{x \in \overline{\nabla_{\bar{m}}(X)} : \pi_{o(X)}^{\bar{m}}(x) \in Y\}.$$

The hypothesis yields that Z is \mathcal{L} -definable. Since $\nabla_{\bar{m}}(X) \subseteq Z \subseteq \overline{\nabla_{\bar{m}}(X)}$, both properties (1) and (2) are easily shown. \square

3.1.6. Lemma. Let $X \subseteq K^n$ be a non-empty \mathcal{L}_δ -definable set. Let $\bar{m}, \bar{d} \in \mathbb{N}^n$ be such that $o(X) \leq \bar{m} < \bar{d}$ and suppose (X, Z, \bar{d}) is a linked triple. Then, there is a linked triple of the form (X, Z', \bar{m}) .

Proof. By Lemma 3.1.4, $\overline{\nabla_{\bar{m}}(X)}$ is \mathcal{L} -definable. Therefore by Lemma 3.1.5, there is a linked triple of the form (X, Z', \bar{m}) . \square

3.1.7. Proposition. The theory T_δ^* has \mathcal{L} -open core if and only if for every $n \geq 1$ and every \mathcal{L}_δ -definable set $X \subseteq K^n$, there is a linked triple (X, Z, \bar{m}) for some $\bar{m} \geq o(X)$.

Proof. (\Rightarrow) Let $Y \subseteq K^{n(o(X)+1)}$ be an \mathcal{L} -definable set such that $X = \nabla_{o(X)}^{-1}(Y)$. The subset $\overline{\nabla_{o(X)}(X)}$ is both closed and \mathcal{L}_δ -definable, so it is \mathcal{L} -definable since T_δ^* as \mathcal{L} -open core. Then apply Lemma 3.1.5.

(\Leftarrow) It suffices to show that \overline{X} is \mathcal{L} -definable. By assumption there is a tuple \bar{m} and an \mathcal{L} -definable set Z such that (X, Z, \bar{m}) is a linked triple. By Lemma 3.1.4, we have $\overline{X} = \overline{\pi_0^{\bar{m}}(Z)}$. The result follows since, as Z is \mathcal{L} -definable, so is $\overline{\pi_0^{\bar{m}}(Z)}$. \square

We now prove a rather technical lemma, which is a parametric version of the density of differential points (Lemma 2.4.1). But before doing that we need to put the \mathbf{F} -quantifier free \mathcal{L}_δ -definable sets in an amenable form (in order to use the scheme (DL)).

3.1.8. Lemma. *Let $\varphi(x, y)$ be an $\mathcal{L}_\delta(\mathcal{K})$ -formula where $x = (x_1, \dots, x_n)$ are \mathbf{F} -variables and y is a single \mathbf{F} -variable. Then, given $m \geq m_i$, $1 \leq i \leq n+1$, with $(\bar{m}, m_{n+1}) = \text{ord}(\varphi)$ and $\bar{m} := (m_1, \dots, m_n)$, φ is equivalent to a finite disjunction of $\mathcal{L}_\delta(\mathcal{K})$ -formulas of the form*

$$\mathcal{Z}_{\mathcal{A}}^S(x, y) \wedge \theta(\bar{\delta}^m(x), \bar{\delta}^m(y)),$$

where θ is an $\mathcal{L}(\mathcal{K})$ -formula which defines an open subset of $K^{(n+1)(m+1)}$ and either $\mathcal{A} \subseteq K\{x\}$ or \mathcal{A} contains only one differential polynomial P of non-negative order in y and s_P divides S . In addition, $\text{ord}_{x_i}(\mathcal{A}, S) \leq 2m$ for $1 \leq i \leq n$ and $\text{ord}_y(\mathcal{A}, S) \leq m$.

Proof. By Remark 2.4.11 and condition (A), φ is equivalent to a disjunction of the form

$$Z_{\mathcal{A}}(\bar{\delta}^m(x), \bar{\delta}^m(y)) \wedge \theta(\bar{\delta}^m(x), \bar{\delta}^m(y)),$$

where θ is an $\mathcal{L}(\mathcal{K})$ -formula which defines an open subset of $K^{(n+1)(m+1)}$ and $\mathcal{A} \subseteq K[\bar{x}, \bar{y}]$. Define

$$\mathcal{A}' := \{Q(\bar{\delta}^m(x), \bar{\delta}^m(y)) : Q \in \mathcal{A}\}.$$

By definition, we have that φ is equivalent to the corresponding disjunction of $\mathcal{L}_\delta(\mathcal{K})$ -formulas of the form

$$\mathcal{Z}_{\mathcal{A}'}(x, y) \wedge \theta(\bar{\delta}^m(x), \bar{\delta}^m(y)).$$

By Lemma 2.1.8, the formula $\mathcal{Z}_{\mathcal{A}'}(x, y)$ is equivalent (modulo T_δ) to a finite disjunction of formulas of the form $\mathcal{Z}_{\mathcal{B}}^{S_{\mathcal{B}}}(x, y)$ such that $\text{ord}_y(S_{\mathcal{B}}) \leq m$, $\text{ord}_{x_i}(\mathcal{B}, S_{\mathcal{B}}) \leq 2m$ for each $1 \leq i \leq n$, and either

- $\mathcal{B} \subseteq K\{x\}$ or
- there is a unique $P \in \mathcal{B}$ of non-negative order in y , $\text{ord}_y(P_{\mathcal{B}}) \leq m$ and $s_{P_{\mathcal{B}}}$ divides $S_{\mathcal{B}}$.

Then $\varphi(x, y)$ is equivalent to the disjunction of the corresponding disjunction of $\mathcal{L}_\delta(\mathcal{K})$ -formulas

$$\mathcal{Z}_{\mathcal{B}}^{S_{\mathcal{B}}}(x, y) \wedge \theta(\bar{\delta}^m(x), \bar{\delta}^m(y)). \quad \square$$

3.1.9. Lemma. *Let X be an \mathcal{L}_δ -definable subset of K^{n+1} and $(\bar{m}, m_{n+1}) = \text{ord}(X)$ with $\bar{m} := (m_1, \dots, m_n)$. Let $m \geq m_i$, $1 \leq i \leq n+1$. Then, for $\bar{d} = (3m, \dots, 3m, m)$ with $\ell(\bar{d}) = n+1$, there is an \mathcal{L} -definable subset $Y \subseteq K^{(n+1)\bar{d}+1}$ such that*

$$(1) \ X = \nabla_{\bar{d}}^{-1}(Y) \text{ and}$$

(2) for every $a \in K^n$ and $c \in K^{m+1}$ such that $(\bar{\delta}^{3m}(a), c) \in Y$ it holds that for every open neighbourhood W of c there is $b \in K$ such that $\bar{\delta}^m(b) \in W$ and $(\bar{\delta}^{3m}(a), \bar{\delta}^m(b)) \in Y$.

In particular, $|X_a| = |Y_{\bar{\delta}^{3m}(a)}|$ for every $a \in K^n$ such that X_a is finite.

Proof. Let $\varphi(x, y)$ be an $\mathcal{L}_\delta(\mathcal{K})$ -formula where $x = (x_1, \dots, x_n)$, y is a single variable and $\varphi(\mathcal{K}) = X$. By Lemma 3.1.8, $\varphi(x, y)$ is equivalent, modulo T_δ^* , to a finite disjunction of the form

$$\bigvee_{j \in J} \mathcal{Z}_{\mathcal{A}_j}^{S_j}(x, y) \wedge \theta_j(\bar{\delta}^m(x), \bar{\delta}^m(y)),$$

where for each $j \in J$, θ_j is an $\mathcal{L}(\mathcal{K})$ -formula which defines an open subset of $K^{(n+1)(m+1)}$ and either $\mathcal{A}_j \subseteq K\{x\}$ or $\mathcal{A}_j \subseteq K\{x, y\}$, it only contains one differential polynomial P_j of non-negative order in y and s_{P_j} divides S_j . In addition, $\text{ord}_{x_i}(\mathcal{A}_j, S_j) \leq 2m$ for $1 \leq i \leq n$ and $\text{ord}_y(\mathcal{A}_j, S_j) \leq m$. For each $j \in J$, let $\tilde{\theta}_j(\bar{x}[m], \bar{y})$ be the $\mathcal{L}(\mathcal{K})$ -formula $\theta_j(\bar{x}, \bar{y}) \wedge S_j^*(\bar{x}[m], \bar{y}) \neq 0$, where $\bar{x} := (\bar{x}_1, \dots, \bar{x}_n)$ and each \bar{x}_i , $1 \leq i \leq n$, and \bar{y} have length $m+1$. Note that $\tilde{\theta}_j$ defines an open subset of $K^{n(2m+1)+m+1}$. For each $j \in J$, we define by cases an $\mathcal{L}(\mathcal{K})$ -formula $\psi_j(\bar{x}[m], \bar{y})$ depending on whether $\mathcal{A} \subseteq K\{x\}$ or not:

(i) if $\mathcal{A}_j \subseteq K\{x\}$ then $\psi_j(\bar{x}[m], \bar{y})$ is

$$Z_{\mathcal{A}_j}(\bar{x}[m], \bar{y}) \wedge \tilde{\theta}_j(\bar{x}[2m], \bar{y}).$$

(ii) otherwise, letting $k_j = \text{ord}_y(P_j)$ we define $\psi_j(\bar{x}[2m], \bar{y})$ as

$$Z_{\mathcal{A}_j}(\bar{x}[m], \bar{y}) \wedge \tilde{\theta}_j(\bar{x}[m], \bar{y}) \wedge \lambda_{P_j}^{m-k_j}(\bar{x}[2m], \bar{y}).$$

Let $\psi(\bar{x}[2m], \bar{y}(2m))$ be the disjunction $\bigvee_{j \in J} \psi_j(\bar{x}[2m], \bar{y})$ and Y be the subset of $K^{(n+1)(d+1)}$ defined by ψ . Let us show (1). The inclusion $\nabla_d^{-1}(Y) \subseteq X$ is clear. The converse follows by noting that for each $j \in J$ for which ψ_j is as in (ii)

$$T_\delta^* \models \forall x \forall y (\mathcal{Z}_{\mathcal{A}_j}^{S_j}(x, y) \rightarrow \lambda_{P_j}^{m-k_j}(\bar{\delta}^d(x), \bar{\delta}^m(y))).$$

It remains to show (2).

Fix $a \in K^n$ and $c = (c_0, \dots, c_d) \in K^{d+1}$ such that $(\bar{\delta}^d(a), c) \in Y$. Let $j \in J$ be such that $\psi_j(\bar{\delta}^d(a), c|m)$ holds, where $c|m := (c_0, \dots, c_m)$. We split in cases. If ψ_j is as in (i), then the result follows from Lemma 2.4.1. So suppose ψ_j is as in (ii). Let W be an open neighbourhood of (c_0, \dots, c_m) . Without loss of generality, we may suppose there is V an open neighbourhood of $\bar{\delta}^{2m}(a)$ such that $V \times W \subseteq \tilde{\theta}_j(\mathcal{K})$. Let $V \times V_1$ be an open neighbourhood of $\bar{\delta}^d(a)$. By the continuity of the functions $f_i^{P_\ell}$ (see Lemma-Definition 2.1.4), we may shrink $V \times V_1$ to a smaller open neighbourhood of $\bar{\delta}^d(a)$ and find an open neighbourhood W_1 of (c_0, \dots, c_{k_j}) such that, letting $U := V \times W_1$

$$W_1 \times f_1^{P_j}(U) \times \dots \times f_{m-k_j}^{P_j}(U) \subseteq W,$$

where $f_i^{P_j}$, $1 \leq i \leq m - k_j$ is a function from \cdot .

By the scheme (DL), we can find a differential tuple $\bar{\delta}^{k_j}(b) \in W_1$ such that

$$K \models P_j^*(\bar{\delta}^{2m}(a), \bar{\delta}^{k_j}(b)) = 0 \wedge \frac{\partial}{\partial y_{k_j}} P_j^*(\bar{\delta}^{2m}(a), \bar{\delta}^{k_j}(b)) \neq 0.$$

Since $\delta^{k_j+i}(b) = f_i^{P_j}(\bar{\delta}^{2m+i}(a), \bar{\delta}^{k_j}(b))$ for each $i \in \{1, \dots, m - k_j\}$, we have that $(\bar{\delta}^{2m}(a), \bar{\delta}^m(b)) \in V \times W$, and hence in $\tilde{\theta}_\ell(\mathcal{K})$. This shows that $\psi_j(\bar{\delta}^{2m}(a), \bar{\delta}^m(b))$ holds, so $(\bar{\delta}^d(a), \bar{\delta}^d(b)) \in Y$.

The last statement follows directly from part (2) and the fact the topology is Hausdorff. \square

3.1.10. Corollary. *Let $X \subseteq K$ be \mathcal{L}_δ -definable. Then for every $m \geq o(X)$, X has a linked triple of the form (X, Z, m) .*

Proof. Setting $d = 2m$, let $Y \subseteq K^{d+1}$ be the \mathcal{L} -definable subset given by Lemma 3.1.9 with $n = 0$. Conditions (1) and (2) of Lemma 3.1.9 imply that (X, Y, d) is a linked triple. By Lemma 3.1.6, X has a linked triple of the form (X, Z, m) for any $o(X) \leq m \leq d$. \square

3.1.11. Theorem. *Let T be an open \mathcal{L} -theory of topological fields. Then the theory T_δ^* has \mathcal{L} -open core.*

Proof. Let X be an \mathcal{L}_δ -definable subset of K^{n+1} . By Proposition 3.1.7, it suffices to show that X has a linked triple, for some $m \geq o(X)$. However, we will show by induction on n the following stronger statement:

(*) for every $m \geq o(X)$, X has a linked triple (X, Z, m) .

For $n = 0$, (*) follows from Corollary 3.1.10. Suppose $n > 0$. By Lemma 3.1.6, it suffices to show that X has a linked triple (X, Z, m) for arbitrarily large $m \geq o(X)$.

Consider $\pi(X) \subseteq K^n$ where $\pi: K^{n+1} \rightarrow K^n$ is the projection onto the first n -coordinates. Let $m \geq \max\{o(X), o(\pi(X))\}$. We will show that X has a linked triple of the form (X, Z, d) , with $d = 2m$.

By Lemma 3.1.9, there is an \mathcal{L} -definable subset $Y \subseteq K^{(n+1)(d+1)}$ such that $X = \nabla_d^{-1}(Y)$ and, if $x = \delta^d(a)$ for $a \in K^n$ and $(\delta^d(a), c) \in Y$ for some $c \in K^{d+1}$, given any open neighbourhood W of c , there is $b \in K$ such that $\delta^d(b) \in W$ and $(\delta^d(a), \delta^d(b)) \in Y$.

By induction hypothesis, $\pi(X)$ has a linked triple of the form $(\pi(X), Z, d)$ (note that $o(\pi(X)) \leq m \leq d$). Set $Y' = \{(x, y) \in Y : x \in Z\}$. We claim that (X, Y', d) is a linked triple for X . First note that $\nabla_d^{-1}(Y') = X$, and therefore, $\overline{\nabla_d(X)} \subseteq \overline{Y'}$.

To show $\overline{Y'} \subseteq \overline{\nabla_d(X)}$, let $(a, b) \in Y'$ and $U \times V$ be an open neighbourhood of (a, b) . By cell decomposition (Theorem 1.5.3), $Y' = \bigcup_{i \in I} Y_i$, where Y_i is a cell and I is a finite set. Let $i \in I$ be such that $(a, b) \in Y_i$. By Lemma 1.5.6, there is an open neighbourhood $U_1 \subseteq U$ of a such that for every $a' \in U_1$, there is $b' \in V$ such that $(a', b') \in Y_i$. Since $a' \in Z$ and $(\pi(X), Z, d)$ is a linked triple, there is $c \in \pi(X)$ such that $\bar{\delta}^d(c) \in U_1$. Let $b' \in V$ be such that $(\bar{\delta}^d(c), b') \in Y_i$. Since $Y_i \subseteq Y$, there is $c' \in K$ such that $\bar{\delta}^d(c') \in V$ and $(\bar{\delta}^d(c), \bar{\delta}^d(c')) \in Y$. Therefore $(c, c') \in X$ and $(\bar{\delta}^d(c), \bar{\delta}^d(c')) \in U \times V$, which shows that $\overline{Y'} \subseteq \overline{\nabla_d(X)}$. \square

3.1.12. Corollary. *Let T be one of the following theories: RCF, ACVF $_{0,p}$, RCVF, pCF or, in general the \mathcal{L}_{div} -theory of a henselian valued field of characteristic 0. Then, T_δ^* has \mathcal{L} -open core.*

Proof. All these theories are open \mathcal{L} -theories of topological fields (see Examples 1.2.5) and by Theorem 2.3.4 T_δ^* is consistent. \square

-cell-decomp}
f:delta-cell}

3.2. δ -Cell decomposition. Let us start by defining δ -cells.

3.2.1. Definition (δ -cells). A subset $X \subseteq K^n$ is a δ -cell if there are $d \geq 0$ and an \mathcal{L} -definable cell Y (as defined in Definition 1.5.1) such that $X = \nabla_d^{-1}(Y)$ and $\overline{\nabla_d(X)} = \overline{Y}$.

{delta-cell-d

3.2.2. Theorem (δ -Cell decomposition). *Let \mathcal{K} be a model of T_δ^* . Then, for every $n \geq 1$, every \mathcal{L}_δ -definable set $X \subseteq K^n$ is a disjoint union of finitely many δ -cells.*

Proof. By Theorem 3.1.11, T_δ^* has \mathcal{L} -open core. By Proposition 3.1.7 and Lemma 3.1.6, we may associate to X a linked triple (X, Y, d) with $d = o(X)$. We proceed by induction on $\dim(Y)$. If $\dim(Y) = 0$, then Y is a cell and hence X is already a δ -cell. Suppose $\dim(Y) > 0$.

By cell decomposition (Theorem 1.5.3), Y can be expressed as a finite disjoint union of cells Z_i , $i \in I$. Let $I_0 := \{i \in I : \dim(Z_i) = \dim(Y)\}$ and let $I_1 := I \setminus I_0$. Let $X_1 := \nabla_d^{-1}(\bigcup_{i \in I_1} Z_i)$ and for $i \in I_0$, let $X_i := \nabla_d^{-1}(Z_i)$. We show the result for X_1 and X_i for $i \in I_0$.

For X_1 , we apply the induction hypothesis. Indeed, by the \mathcal{L} -open core, $(X_1, \overline{\nabla_d(X_1)}, d)$ is a linked triple and $\dim(\overline{\nabla_d(X_1)}) \leq \dim(\bigcup_{i \in I_1} \overline{Z_i}) < \dim(Y)$.

For $j \in I_0$, let us show that X_j is already a δ -cell. It amounts to show that $\overline{\nabla_d(X_j)} = \overline{Z_j}$. One inclusion is clear, so let us show that $\overline{Z_j} \subseteq \overline{\nabla_d(X_j)}$. It suffices to show that for $z \in Z_j$, every open neighbourhood of z intersects $\nabla_d(X_j)$. First let us show the following claim.

{claim:densit

3.2.3. Claim. *Let $z \in Z_j$, then for every open neighbourhood U of z , there is $x \in X$ such that $\bar{\delta}^d(x) \in \overline{Z_j}$.*

Suppose for a contradiction that there is an open neighbourhood U of z such that $U \cap \overline{Z_j} \cap \nabla_d(X) = \emptyset$. Note that since (X, Y, d) is a linked triple, for every open neighbourhood U of z there is $x \in X$ such that $\bar{\delta}^d(x) \in \overline{Y} = \bigcup_{i \in I} \overline{Z_i}$. We split in two cases:

Case 1: Suppose that $U \cap Z_j \subseteq \bigcup_{i \in I \setminus \{j\}} \overline{Z_i}$. Since the $(Z_i)_{i \in I}$ are disjoint, we have $U \cap Z_j \subseteq \bigcup_{i \in I \setminus \{j\}} \overline{Z_i} \setminus Z_i$. This contradicts that $\dim(U \cap Z_j) = \dim(Z_j) = \dim(Y)$ and $\dim(\bigcup_{i \in I} \overline{Z_i} \setminus Z_i) < \dim(Y)$.

Case 2: Suppose there is $z' \in U \cap Z_j \setminus (\bigcup_{i \in I \setminus \{j\}} \overline{Z_i})$. Let $U' \subseteq U$ be an open neighbourhood of z' such that $U' \cap \overline{Z_i} = \emptyset$ for every $i \in I \setminus \{j\}$. Since $z' \in Z_j$ and (X, Y, d) is a linked triple, there is $x \in X$ such that $\bar{\delta}^d(x) \in U' \cap \overline{Y}$. By the choice of U' , $\bar{\delta}^d(x) \in \overline{Z_j}$, but $U' \subseteq U$, so we contradict the assumption on U . This completes the claim.

Let ρ_{Z_j} be the projection associated to the cell Z_j such that Z_j is the graph along ρ_{Z_j} of a continuous correspondence $f_{Z_j} : \rho_{Z_j}(Z_j) \rightarrow K^{n-\dim(Z_j)}$ (see Definition 1.5.1). To conclude, let U be an open neighbourhood of z . Without loss, we may assume that $\rho_{Z_j}(U) \subseteq \rho_{Z_j}(Z_j)$. By Claim 3.2.3, U contains an element $\bar{\delta}^d(x) \in \overline{Z_j}$ with $x \in X$. Let us show that $\bar{\delta}^d(x) \in Z_j$. Note that if ρ_{Z_j} is the identity, the result follows directly. For notational simplicity, suppose ρ_{Z_j} is onto the first m -coordinates for some $m < n(d+1)$.

Recall that $\rho_{Z_j}^\perp$ denotes the complement projection to ρ_{Z_j} . Let $w = \rho_{Z_j}(\bar{\delta}^d(x))$. Since $\rho_{Z_j}(U) \subseteq \rho_{Z_j}(Z_j)$, it suffices to show that $\rho_{Z_j}^\perp(\bar{\delta}^d(x)) \in f_{Z_j}(w)$. For suppose not. Then, since the topology is Hausdorff, there are disjoint open sets V_1 and V_2 such that V_1 contains $\{(w, y) : y \in f_{Z_j}(w)\}$ and V_2 contains $\bar{\delta}^d(x)$. Since $w \in \rho_{Z_j}(V_1) \cap \rho_{Z_j}(V_2)$, we may suppose without loss that $\rho_{Z_j}(V_1) = \rho_{Z_j}(V_2)$. This contradicts the continuity of f_{Z_j} and the fact that $\bar{\delta}^d(x) \in \overline{Z_j}$. \square

rem:delta-CD}

3.2.4. Remark. In [5], the authors proved a cell decomposition theorem for CODF (see [5, Theorem 4.9]). According to their definition, δ -cells are \mathcal{L}_δ -definable sets of the form $X = \nabla_d^{-1}(Y)$ where Y an o-minimal cell (of RCF). They do not require however the density condition $\overline{\nabla_d(X)} = \overline{Y}$ present in Definition 3.2.1. Although our definition of \mathcal{L} -definable cells does not coincide with o-minimal cells, o-minimal cells are cells in the sense of Definition 1.5.1, so the proof of Theorem 3.2.2 applies word for word replacing our cells by o-minimal cells.

{sec:EI}

3.3. Transfer of elimination of imaginaries. In this section, we show how to transfer elimination of imaginaries from T to T_δ^* . Our proof relies on properties of the topological dimension in models of T . The strategy we employ has been previously used by M. Tressl in the case of CODF. For background facts on the elimination of imaginaries, we refer to [43, Section 8.4].

lem:order_EI}

3.3.1. Lemma. *Let $X \subseteq K^n$ be an \mathcal{L}_δ -definable set. If $m_1 \leq m_2$ then*

$$\dim(\overline{\nabla_{m_1}(X)}) \leq \dim(\overline{\nabla_{m_2}(X)}).$$

Proof. Let $\pi: \overline{\nabla_{m_1}(X)} \rightarrow K^\ell$ be a projection such that $\pi(\overline{\nabla_{m_1}(X)})$ has non-empty interior. Then, letting ρ denote the projection from $\overline{\nabla_{m_2}(X)}$ onto $\overline{\nabla_{m_1}(X)}$, we have that $\pi \circ \rho(\overline{\nabla_{m_2}(X)})$ has non-empty interior. \square

{prop:fini}

3.3.2. Lemma. *Let $X \subseteq \mathcal{K}^{(x,z)}$ be a definable set where x is a tuple of \mathbf{F} -variables and z is a tuple of auxiliary sort variables. Suppose that for every $b \in \mathcal{K}^z$, the fiber X_b is finite. Then X is \mathcal{L} -definable.*

Proof. Let $\varphi(x, z)$ be an $\mathcal{L}_\delta(\mathcal{K})$ -formula defining X and $x = (x_1, \dots, x_{n+1})$ with $n \geq 0$. By Lemma 3.1.8, for $m := \text{ord}(\varphi)$, $\varphi(x, z)$ is equivalent to a finite disjunction of $\mathcal{L}_\delta(\mathcal{K})$ -formulas of the form

$$(3.3.3) \quad \mathcal{Z}_{\mathcal{A}}^S(x) \wedge \theta(\bar{\delta}^m(x), z),$$

where θ is an $\mathcal{L}(\mathcal{K})$ -formula such that for every each $b \in \mathcal{K}^z$, $\theta(\mathcal{K}, b)$ is an open set and either

- (1) $\mathcal{A} \subseteq K\{x_1, \dots, x_n\}$ or
- (2) \mathcal{A} contains only one differential polynomial P with $\text{ord}_{x_{n+1}}(P) = m_P \leq m$, $\text{ord}_{x_j}(P) \leq 2m$, $1 \leq j \leq n$ and s_P divides S .

Without loss of generality we may assume $\varphi(x, z)$ is already the formula given in (3.3.3). In addition, since each X_b is finite, case (1) above cannot hold, so we must be in case (2).

By induction on n , we show there are differential polynomials $P_j \in K\{X_1, \dots, X_j\}$ for $j \in \{1, \dots, n+1\}$ such that

$$(3.3.4) \quad T_\delta^* \models (\forall z)(\forall x)(\varphi(x, z) \rightarrow \bigwedge_{j=1}^{n+1} P_j(x_1, \dots, x_j) = 0 \wedge s_{P_j}(x_1, \dots, x_j) \neq 0),$$

and an $\mathcal{L}(\mathcal{K})$ -formula $\tilde{\varphi}(\bar{x}, z)$ where $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n+1})$, $\bar{x}_j = (x_j, x_{j,1}, \dots, x_{j,k_j})$ with $k_j := \text{ord}_{x_j}(P_j)$, $1 \leq j \leq n+1$, such that

$$(3.3.5) \quad \{\text{eq: formula 2}\} \models (\forall z)(\forall \bar{x})(\tilde{\varphi}(\bar{x}, z) \leftrightarrow (\varphi(x, z) \wedge \bigwedge_{j=1}^{n+1} \bar{\delta}^{k_j}(x_j) = \bar{x}_j)).$$

This implies the statement, since X is defined by the $\mathcal{L}(\mathcal{K})$ -formula

$$(\exists x_{1,1}) \cdots (\exists x_{1,k_1}) \cdots (\exists x_{n+1,1}) \cdots (\exists x_{n+1,k_{n+1}}) \tilde{\varphi}(\bar{x}, z).$$

Suppose $n = 0$. Set $P_1 \in K\{X_1\}$ to be the differential polynomial P from (2) above. The choice of P_1 ensures already (3.3.4). Define $\tilde{\varphi}(\bar{x}_1, z)$ to be the $\mathcal{L}(\mathcal{K})$ -formula given by

$$P_1^*(\bar{x}_1) = 0 \wedge S^*(g_1^{\text{ord}(S)}(\bar{x}_1)) \neq 0 \wedge \theta(g_1^m(\bar{x}_1), z),$$

where for any $d \geq 0$, $g_1^d(\bar{x}_1)$ is the definable function given by the prolongation of length d (possibly truncation depending on d) of \bar{x}_1 with respect to P_1 , namely,

$$g_1^d(\bar{x}_1) = \begin{cases} (x_1, x_{1,1}, \dots, x_{1,d}) & \text{if } d \leq k_1 \\ (\bar{x}_1, f_1^{P_1}(\bar{x}_1), \dots, f_{d-k_1}^{P_1}(\bar{x}_1)) & \text{if } d > k_1, \end{cases}$$

where $f_i^{P_1}$, $i \geq 1$, is defined as in Lemma-Definition 2.1.4 and by convention $g_1^0(\bar{x}_1) = x_1$. In particular, if $\bar{x}_1 = \bar{\delta}^{k_1}(x_1)$, then $\bar{\delta}^d(x_1) = g_1^d(\bar{x}_1)$ for every $d \geq 0$. Let us show $\tilde{\varphi}$ satisfies (3.3.5). The implication from right to left follows from (3.3.4), Lemma-Definition 2.1.4 and the fact that $\bar{x}_1 = \bar{\delta}^{k_1}(x_1)$. For the converse, let $\bar{a}_1 = (a_1, a_{1,1}, \dots, a_{1,k_1})$ and suppose that $\tilde{\varphi}(\bar{a}_1, b)$ holds for $b \in \mathcal{K}^z$. By the axiom scheme (DL) and similarly to the argument given in Lemma 3.1.9, for any neighbourhood of \bar{a}_1 there is an element $u \in K$ such that

$$P(u) = 0 \wedge S(u) \neq 0 \wedge \theta(\bar{\delta}^m(u), b)$$

holds. Since the topology is Hausdorff and X_b is finite, we must have that \bar{a}_1 is of the form $\bar{\delta}^{k_1}(a_1)$ and therefore, by Lemma-Definition 2.1.4, $\varphi(a, b)$ holds.

Suppose $n > 0$. Consider the $\mathcal{L}_\delta(\mathcal{K})$ -formula $\xi(x_1, \dots, x_n, z)$ given by $(\exists x_{n+1})\varphi(x, z)$. Note that for each $b \in \mathcal{K}^z$, the fiber $\xi(\mathcal{K}, b)$ is finite. Therefore, by induction, there are polynomials $P_j \in K\{X_1, \dots, X_j\}$ for $1 \leq j \leq n$ satisfying the corresponding (3.3.4) with respect to ξ , and an $\mathcal{L}(\mathcal{K})$ -formula $\tilde{\xi}(\bar{x}_1, \dots, \bar{x}_n, z)$ satisfying the corresponding (3.3.5). We set P_{n+1} as the differential polynomial P from (2) above. Since $\varphi(x, z)$ implies $\xi(x_1, \dots, x_n, z)$, the choice of P_{n+1} and the induction hypothesis ensure (3.3.4)

holds. Define the formula $\tilde{\varphi}(x, z)$ as

$$\begin{aligned} & P_{n+1}^*(g_1^{\text{ord}_{x_1}(P)}(\bar{x}_1), \dots, g_n^{\text{ord}_{x_n}(P)}(\bar{x}_1, \dots, \bar{x}_n), \bar{x}_{n+1})) = 0 \wedge \\ & S^*(g_1^{\text{ord}_{x_1}(S)}(\bar{x}_1), \dots, g_{n+1}^{\text{ord}_{x_{n+1}}(S)}(\bar{x}_1, \dots, \bar{x}_{n+1})) \neq 0 \wedge \\ & \theta(g_1^m(\bar{x}_1), \dots, g_{n+1}^m(\bar{x}_1, \dots, \bar{x}_{n+1}), z) \wedge \tilde{\xi}(\bar{x}_1, \dots, \bar{x}_n, z) \end{aligned}$$

where for each $d \geq 0$ and each $1 \leq j \leq n+1$, the function $g_j^d(\bar{x}_1, \dots, \bar{x}_j)$ is inductively defined and corresponds to the prolongation of length d (possibly truncation depending on d) of \bar{x}_j with respect to P_j and $g_1^{d'}(\bar{x}_1), \dots, g_{j-1}^{d'}(\bar{x}_1, \dots, \bar{x}_{j-1})$ for all $d' \geq 0$ (we omit the formulas, for simplicity). As before, for each $1 \leq j \leq n+1$, if $\bigwedge_{i=1}^j \bar{\delta}^{k_j}(x_j) = \bar{x}_j$, then $\bar{\delta}^d(x_j) = g_j^d(\bar{x}_1, \dots, \bar{x}_j)$ for all $d \geq 0$. It remains to show (3.3.5). The implication from right to left follows by induction using in addition (3.3.4), Lemma-Definition 2.1.4 and the fact that $\bar{x}_j = \bar{\delta}^{k_j}(x_j)$ for all $1 \leq j \leq n+1$ (note that the $\mathcal{L}_\delta(\mathcal{K})$ -formula $\mathcal{Z}_{\mathcal{A} \setminus \{P\}}(x_1, \dots, x_n)$ is implied by ξ). For the converse, suppose that $\tilde{\varphi}(\bar{a}, b)$ holds for some $b \in \mathcal{K}^z$. Then $\tilde{\xi}(\bar{a}_1, \dots, \bar{a}_n, b)$ holds and by induction we have that $\bar{a}_j = \bar{\delta}^{k_j}(a_j)$ for all $1 \leq j \leq n$. In particular, $\bar{\delta}^d(a_j) = g_j^d(\bar{a}_1, \dots, \bar{a}_j)$ for every $d \geq 0$ and $1 \leq j \leq n$. As in the case $n = 0$, by the axiom scheme (DL) and as in Lemma 3.1.9, for any neighbourhood of \bar{a}_{n+1} there is an element $u \in K$ such that

$$P(a_1, \dots, a_n, u) = 0 \wedge S(a_1, \dots, a_n, u) \neq 0 \wedge \theta(\bar{\delta}^m(a_1), \dots, \bar{\delta}^m(a_n), \bar{\delta}^m(u), b).$$

Since the topology is Hausdorff and X_b is finite, we must have that \bar{a}_{n+1} is of the form $\bar{\delta}^{k_{n+1}}(a_{n+1})$. This shows $\varphi(a, b)$ holds. \square

Let \mathcal{G} be a collection of sorts of \mathcal{L}^{eq} . We let $\mathcal{L}^{\mathcal{G}}$ denote the restriction of \mathcal{L}^{eq} to the field sort together with the new sorts in \mathcal{G} . The following proof will make use of properties (D1)-(D3) of the topological dimension listed in Proposition 1.3.6).

{thm:EI}

3.3.6. Theorem. *Suppose that T admits elimination of imaginaries in $\mathcal{L}^{\mathcal{G}}$. Then the theory T_δ^* admits elimination of imaginaries in $\mathcal{L}_\delta^{\mathcal{G}}$.*

Proof. Fix a sufficiently saturated model \mathcal{K} of T_δ^* . For the sake of exposition, we first assume \mathcal{L} is one-sorted and later explain how to modify the proof to obtain the general case.

Let $X \subseteq K^n$ be a non-empty \mathcal{L}_δ -definable set. It suffices to show that X has a code in $\mathcal{L}_\delta^{\mathcal{G}}$ (that is, an element $e \in \mathcal{G}$ such that $\sigma(e) = e$ if and only if $\sigma(X) = X$ for every \mathcal{L}_δ -automorphism σ of \mathcal{K}). Observe that every \mathcal{L} -definable set has a code in $\mathcal{L}^{\mathcal{G}}$, and therefore a code in $\mathcal{L}_\delta^{\mathcal{G}}$, as the \mathcal{L}_δ -automorphism group of \mathcal{K} is a subgroup of the \mathcal{L} -automorphism group of \mathcal{K} . Consider the set $\widetilde{X} \supseteq X$ defined by

$$\widetilde{X} := \nabla_{o(X)}^{-1}(\overline{\nabla_{o(X)}(X)}).$$

By Theorem 3.1.11, T_δ^* has \mathcal{L} -open core, so the set $\overline{\nabla_{o(X)}(X)}$ is \mathcal{L} -definable. We proceed by induction on $\dim(\overline{\nabla_{o(X)}(X)})$. If $\dim(\overline{\nabla_{o(X)}(X)}) = 0$, then X is finite (by (D1)) and in particular \mathcal{L} -definable, so it has a code in $\mathcal{L}_\delta^{\mathcal{G}}$. To show the inductive step we need the following claim:

{cla:dim}

3.3.7. Claim. $\dim(\overline{\nabla_{o(X)}(\widetilde{X} \setminus X)}) < \dim(\overline{\nabla_{o(X)}(X)})$.

Suppose the claim holds. Since $o(\widetilde{X} \setminus X) \leq o(X)$, by Lemma 3.3.1, we have that

$$\dim(\nabla_{o(\widetilde{X} \setminus X)}(\widetilde{X} \setminus X)) \leq \dim(\nabla_{o(X)}(\widetilde{X} \setminus X)).$$

Therefore, by Claim 3.3.7 and the induction hypothesis, let e_1 be a code for $\widetilde{X} \setminus X$. By the previous observation, let e_2 be a code for $\overline{\nabla_{o(X)}(X)}$ (which is \mathcal{L} -definable by the \mathcal{L} -open core hypothesis). It is an easy exercise to show that $e = (e_1, e_2)$ is a code for X .

It remains to prove the claim. By the \mathcal{L} -open core property and Proposition 3.1.7, let $(X, Z, o(X))$ be a linked triple. Applying (D3), we have

$$\begin{aligned} \dim(\nabla_{o(X)}(\widetilde{X} \setminus X)) &= \dim(\nabla_{o(X)}(\nabla_{o(X)}^{-1}(\overline{\nabla_{o(X)}(X)}) \setminus X)) \\ &= \dim(\nabla_{o(X)}(\{x \in K^n : \nabla_{o(X)}(x) \in \overline{Z}\} \setminus X)) \\ &= \dim(\nabla_{o(X)}(\{x \in K^n : \nabla_{o(X)}(x) \in \overline{Z} \setminus Z\})) \\ &\leq \dim(\overline{Z} \setminus Z) < \dim(\overline{Z}) = \dim(\overline{\nabla_{o(X)}(X)}). \end{aligned}$$

This completes the proof in the one-sorted case. For the general case, we need to code, more generally, a non-empty definable subset $X \subseteq K^n \times \mathcal{K}^z$ where z is a tuple of auxiliary sort variables. Note that if $n = 0$ then, by Corollary 2.4.12, X is already \mathcal{L} -definable and has therefore a code. Thus we may suppose $n \geq 1$. We proceed analogously, using the following notion of dimension on general \mathcal{L} -definable sets: for X as above and Z the projection of X onto the z -variables, we set

$$\dim(X) := \max\{\dim(X_b) : b \in Z\}.$$

If $\dim(X) = 0$, this means that for all $z \in Z$, the fiber X_z is finite. Then, by Lemma 3.3.2, X is \mathcal{L} -definable and has a code. For the inductive step, define for any $d \geq 0$, the operation $\nabla_d(X)$ (resp. for ∇_d^{-1}) and \overline{X} fiber-wise, that is,

$$\nabla_d(X) := \bigcup_{b \in Z} \nabla_d(X_b) \times \{b\} \quad \overline{X} := \bigcup_{b \in Z} \overline{X_b} \times \{b\}.$$

The proof now follows exactly the same strategy, after showing the corresponding version of Claim 3.3.7 and property (D3) for this new notion of dimension. However, note that both properties follow easily since they hold fiber-wise. \square

3.3.8. Corollary. *Let \mathcal{G} denote the geometric language of valued fields. The theories $(\text{ACVF}_{0,p})_\delta^*$, RCVF_δ^* and $p\text{CF}_\delta^*$ have elimination of imaginaries in $\mathcal{L}_\delta^\mathcal{G}$.*

Proof. Let T be either $\text{ACVF}_{0,p}$, RCVF or $p\text{CF}$. The theory T has elimination of imaginaries in $\mathcal{L}^\mathcal{G}$ by results of Haskell, Hrushovski and Macpherson for ACVF [18], of Mellor for RCVF [27], and of Hrushovski, Martin and Rideau for $p\text{CF}$ [20]. By Corollary 3.1.12, T_δ^* has \mathcal{L} -open core. The result follows by Theorem 3.3.6. \square

4. APPLICATIONS TO DENSE PAIRS

The study of pairs of models of a given complete theory is a classical topic in model theory (see [35], [24] and [46] to mention just a few). New developments have enclosed

{cor:EI_exam

{sec:app}

many classical results in different abstract frameworks. Two such frameworks are the theory of lovely pairs of geometric structures developed by A. Berenstein and E. Vassiliev [3], and the theory of dense pairs of theories with existential matroids developed by A. Fornasiero [13]. In this section we will study the theory T_P of dense pairs of models associated to a one-sorted open \mathcal{L} -theory of topological fields T . Our goal is to show that such theory is closely related with the theory T_δ^* . In Section 4.1, we will recall the definition of the theory T_P and show how it fits into the two above mentioned frameworks. In Section 4.2, we will show how to use T_δ^* to deduce properties of T_P . Although most of the results gather in this section concerning T_P are known, the proofs and methods will put in evidence the interesting connexion between the model theory of dense pairs and generic derivations.

{sec:DenseP1}

4.1. Dense pairs of models of T . Given that the literature of the model theory of pairs is quite extensive, we will unify references and cite [13] even if particular cases of cited results were proven before by many different authors.

Let T be a complete one-sorted geometric \mathcal{L} -theory T extending the theory of fields (not necessarily an open \mathcal{L} -theory of topological fields).³ Given a model \mathcal{M} of T and a definable subset $X \subseteq M$, we say that X is *dense* if $X \cap U \neq \emptyset$ for every M -definable subset U of M of acl-dimension 1 (see [13, Definition 7.1]). Let \mathcal{L}_P be the language of pairs, that is, defined as $\mathcal{L}_P := \mathcal{L} \cup \{P\}$ for P a new unary predicate. The theory T_P of dense pairs of models of T is defined as the \mathcal{L}_P -theory of pairs $(\mathcal{K}, P(K))$ such that $\mathcal{K} \models T$, $P(K)$ is acl-closed and dense in K (in the above sense). Equivalently (by [13, Lemma 7.4]), it corresponds to the \mathcal{L}_P -theory of pairs $(\mathcal{K}, P(K))$ such that $\mathcal{K} \models T$, $\langle P(K) \rangle_{\mathcal{L}} \preceq_{\mathcal{L}} \mathcal{K}$ and $P(K)$ is dense in K . Among various model-theoretic results which are proven in [13] about the theory T_P , what plays a crucial role in this section is the fact that T_P is a complete theory [13, Theorem 8.3].

Observe that when T is a one-sorted open \mathcal{L} -theory of topological fields, by Proposition 1.3.4, T is geometric and therefore, T_P is complete. Note also that in view of Proposition 1.3.6, the notion of density above defined coincides with the topological notion of density. It is not difficult to see that in this case the theory T_P also coincides with the theory of lovely pairs of geometric theories introduced by A. Berenstein and E. Vassiliev in [3].

{sec:DenseP2}

4.2. Dense pairs and generic derivations. Throughout this section we suppose T is a one-sorted \mathcal{L} -open theory of topological fields and \mathcal{K} be a model of T_δ^* . The connection of T_P with the theory T_δ^* arises via the field of constants C_K of a model \mathcal{K} of T_δ^* . Recall that by our definition of T_δ , $\mathbb{Q}(\Omega)$ is a subfield of C_K . One can readily observe that when $\mathcal{K} \models \text{CODF}$, the pair $(\mathcal{K}_{|\mathcal{L}}, C_K)$ is a dense pair of real-closed fields. The following lemma shows this holds in general for T_δ^* . For the reader's convenience, we give a proof here, following the references [13, Lemma 7.4], [3, Lemma 2.5] (see also [7, Corollary 1.7]).

{fact:model}

4.2.1. Lemma. *The pair $(\mathcal{K}_{|\mathcal{L}}, C_K)$ is a model of T_P and if \mathcal{K} is $|T|^+$ -saturated, then $(\mathcal{K}_{|\mathcal{L}}, C_K)$ is a lovely pair of models of T .*

³Fornasiero considers more generally the case where T admits an *existential matroid* (see [13, Definition 3.25]), but we will not need this level of generality in the present paper.

Proof. A direct consequence of the scheme (DL) is that $C_K \neq K$. Since C_K is topologically dense in K [16, Lemma 3.12], C_K is dense. So it remains to show that $C_K \preceq_{\mathcal{L}} \mathcal{K}_{|\mathcal{L}}$. We apply Tarski-Vaught test. Let $\varphi(x, \bar{y})$ be an \mathcal{L} -formula and let $\bar{b} \in C_K$. By condition (A) on T , $\varphi(\mathcal{K}, \bar{b})$ is a finite union of finite subsets and open sets. Since C_K is algebraically closed in K , either $\varphi(\mathcal{K}, \bar{b}) \subseteq C_K$ or $\varphi(\mathcal{K}, \bar{b})$ contains an open subset. Since C_K is topologically dense in K , we get the result. \square

{lem:elext}

4.2.2. Lemma. *For every model (\mathcal{K}, F) of T_P there is an model \mathcal{K}^* of T_δ^* such that $(\mathcal{K}, F) \preceq (K_{|\mathcal{L}}^*, C_{K^*})$.*

Proof. Since T_P is complete, by Lemma 4.2.1, there is a model \mathcal{K}' of T_δ^* with constant field F' such that $(\mathcal{K}'_{|\mathcal{L}}, F') \equiv_{\mathcal{L}_P} (\mathcal{K}, F)$. The result follows by Keisler-Shelah's theorem. \square

Let us now show how the previous results allows us to transfer properties of T_δ^* to T_P using the fact that, for models of T_δ^* , every \mathcal{L}_P -formula defines a set which is \mathcal{L}_δ -definable by replacing $P(t)$ by the formula $\delta(t) = 0$.

{cor:delta_ex}

4.2.3. Corollary. *The theory T_P eliminates \exists^∞ .*

Proof. This follows directly from the elimination of \exists^∞ in T_δ^* (Theorem A.0.6), and the fact that every \mathcal{L}_P -formula defines a set which is \mathcal{L}_δ -definable. \square

{thm:opencore}

4.2.4. Proposition. *The theory T_P has \mathcal{L} -open core.*

Proof. By Lemma 4.2.2, let \mathcal{K}^* be a model of T_δ^* such that $(\mathcal{K}_{|\mathcal{L}}^*, C_{K^*})$ is an \mathcal{L}_P -elementary extension of (\mathcal{K}, F) . Let $\varphi(x, y)$ be an \mathcal{L}_P -formula such that for $a \in \mathcal{K}^y$, $\varphi(\mathcal{K}, a)$ is open. Then, $\varphi(\mathcal{K}^*, a)$ is open too. Since \mathcal{K}^* is a model of T_δ^* and every \mathcal{L}_P -formula defines a set which is \mathcal{L}_δ -definable, $\varphi(\mathcal{K}^*, a)$ is \mathcal{L}_δ -definable. Now by Theorem 3.1.11, we have that $\varphi(\mathcal{K}^*, a)$ is definable by $\psi(x, c)$ where $\psi(x, z)$ is an \mathcal{L} -formula and $c \in (\mathcal{K}^*)^z$. Then we have that

$$(\mathcal{K}_{|\mathcal{L}}^*, C_{K^*}) \models (\forall x)(\varphi(x, a) \leftrightarrow \psi(x, c)),$$

and quantifying over c , we have that

$$(\mathcal{K}, F) \models (\exists c)(\forall x)(\varphi(x, a) \leftrightarrow \psi(x, c)),$$

which shows that $\varphi(\mathcal{K}, a)$ is \mathcal{L} -definable. \square

As a corollary of the above proposition and Corollary 3.1.12, we recover the following result shown by P. Hieronymi and G. Boxall [4, Corollary 3.4] and A. Fornasiero [13, Theorem 13.11].

{cor:opencore}

4.2.5. Corollary. *Let T be either RCF, p CF, RCVF, $\text{ACVF}_{0,p}$ or the \mathcal{L}_{div} -theory of a henselian valued field of characteristic 0. Then T_P has \mathcal{L} -open core.* \square

We finish this section with some remarks on distality. By a result of P. Hieronymi and T. Nell in [19], the theory of dense pairs of an o-minimal expansion of an ordered group is not distal. A natural question posed by P. Simon asks whether the theory of such pairs always admits a distal expansion [28, Question 1]. In [28], T. Nell provided a positive answer to the question for the theory of dense pairs of ordered vector spaces.

We show in Corollary 4.2.8 that the theory of dense pairs of real closed fields admits a distal expansion, namely, CODF. We derive analogous results for some theories of dense pairs of henselian valued fields using the following characterization of distality given in [1].

4.2.6. Fact ([1, Main Theorem]). *Let K be a henselian valued field of characteristic 0 in \mathcal{L}_{div} . Then K is distal if and only if it is finitely ramified and both its residue field and value group are distal.* \square

4.2.7. Remark. In particular, the following theories are distal: p CF, RCVF and the \mathcal{L}_{div} -theory of $k((t^\Gamma))$ where k is a field of characteristic 0 with a distal theory and where Γ is an ordered abelian group with a distal theory (for p CF one uses additionally that adding a small set of constants preserves distality, see [38]). It is worth mentioning that for theories of regular abelian groups, the following properties are equivalent by [1, Theorem 3.2]: being distal, being dp-minimal or satisfying the algebraic condition that $\Gamma/p\Gamma$ is finite for every prime p . Alternatively, for RCVF and p CF, distality also follows from dp-minimality.

4.2.8. Corollary. *If the theory T is distal, then T_δ^* is a distal expansion of T_P . In particular, T_P admits a distal expansion when T is either RCF, or p CF, or RCVF or the \mathcal{L}_{div} -theory of a finitely ramified henselian valued field with distal residue field and value group (e.g., $\mathbb{R}((t))$, or $\mathbb{Q}_p((t))$).*

Proof. Since distality transfers from T to T_δ^* (Theorem A.0.5), Lemma 4.2.2 implies that T_δ^* is a distal expansion of T_P whenever T is distal. Thus, for RCF the result follows from the well-known fact that any o-minimal theory is distal. The remaining cases follow from Remark 4.2.7. \square

It is worthy to note that A. Fornasiero and E. Kaplan [14] generalized the previous result for some o-minimal expansions of RCF.

As a consequence of results of A. Chernikov and S. Starchenko in [10], definable relations in models of T_P satisfied the so called strong Erdős-Hajnal property (see [10, Definition 1.6, Theorem 6.10 (3)]).

4.2.9. Corollary. *If T is distal, then definable relations in models of T_P satisfy the strong Erdős-Hajnal property. This holds in particular for models of T_P when T is RCF, p CF or RCVF, or when T is the \mathcal{L}_{div} -theory of a finitely ramified henselian valued field with distal residue field and value group.*

Proof. This follows from Corollary 4.2.8 and [10, Corollary 4.8]. \square

APPENDIX A. CLASSICAL TRANSFERS

Through this section we let T be an open \mathcal{L} -theory of topological fields. Let \mathbb{U} be a monster model of T_δ^* and A be some small subset.

A.0.1. Lemma. *Let x be a tuple of \mathbf{F} -variables and let z be a tuple of auxiliary sort variables. Then for $a \in \mathbb{U}^x$ and $b \in \mathbb{U}^z$, the \mathcal{L}_δ -type $tp_\delta(a, e/A)$ is determined by the infinite sequence of \mathcal{L} -types $\{tp(\delta^m(a), e/\langle A \rangle_{\mathcal{L}_\delta}) : m \in \mathbb{N}\}$.*

Proof. This Follows from Corollary 2.4.8. \square

A.0.2. Corollary. *Let x and z be as in the previous lemma. Let $(a_i, e_i)_{i \in I}$ be a sequence where $a_i \in \mathbb{U}^x$ and $e_i \in \mathbb{U}^z$. Then the sequence $(a_i, e_i)_{i \in I}$ is \mathcal{L}_δ -indiscernible over A if and only if for each $m \in \mathbb{N}$, the sequence $(\bar{\delta}^m(a_i), e_i)_{i \in I}$ is \mathcal{L} -indiscernible over $\langle A \rangle_{\mathcal{L}_\delta}$.* \square

A.0.3. Theorem. *T is NIP if and only if T_δ^* is NIP.*

Proof. Suppose T_δ^* is not NIP. Let $\varphi(x, z)$ be an \mathcal{L}_δ -formula with x a tuple of \mathbf{F} -variables and z a tuple of auxiliary sort variables and suppose φ has IP. By Corollary 2.4.8, φ is equivalent to $\tilde{\varphi}(\bar{\delta}^m(x), z)$, where $\tilde{\varphi}(\bar{x}, z)$ is an \mathcal{L} -formula. Then, since φ has IP so does the formula $\tilde{\varphi}$. The converse is clear, since being NIP is preserved by reducts (see Remark 2.3.7). \square

To show the transfer of distality, we use the following definition which appears in [19], which is slightly different from the original definition given by P. Simon in [38]. In the following definition we let \mathcal{L} be any first order language, T be a complete \mathcal{L} -theory and \mathbb{U} be a monster model of T .

A.0.4. Definition ([19, Definition 1.3]). Let $\varphi(x_1, \dots, x_n; y)$ be a partitioned \mathcal{L} -formula, where x_i , $1 \leq i \leq n$ is a p -tuple of variables and y is a q -tuple of variables, $p, q > 0$. Then φ is distal (in T) if for every $b \in \mathbb{U}^q$, and every indiscernible sequence $(a_i)_{i \in I}$ in \mathbb{U}^p such that

- (1) $I = I_1 + c + I_2$, where both I_1, I_2 are (countable) infinite dense linear orders without end points and c is a single element with $I_1 < c < I_2$,
- (2) the sequence $(a_i)_{i \in I_1 + I_2}$ in \mathbb{U}^p is indiscernible over b ,

then $\mathbb{U} \models \varphi(a_{i_1}, \dots, a_{i_n}; b) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_n}; b)$ with $i_1 < \dots < i_n, j_1 < \dots < j_n$ in I .

A theory T is distal if every formula is distal in T .

The transfer of distality from T to T_δ^* was already noted for some theories T by A. Chernikov, which by now have been included in [1] (where they consider more generally the transfer of distality for fields endowed with several operators). The converse has not been, to our knowledge, observed before. Note that since distality is not preserved under reducts, the converse implication is not straightforward as in Theorem A.0.3.

A.0.5. Theorem. *T is distal if and only if T_δ^* is distal.*

Proof. Let us check that in T_δ^* every formula is distal. Let $\varphi(x_1, z_1, \dots, x_n, z_n; y, w)$ be a partitioned \mathcal{L}_δ -formula where each x_i is a p -tuple of \mathbf{F} -variables, each z_i is a q -tuple of variables of fixed auxiliary sorts $\mathbf{S}_1, \dots, \mathbf{S}_q$, y is a tuple of \mathbf{F} -variables and w is a tuple of auxiliary sorts. Let \mathbb{U} be a monster model of T_δ^* . By Corollary 2.4.8, there is an \mathcal{L} -formula $\tilde{\varphi}(\bar{x}_1, z_1, \dots, \bar{x}_n, z_n; \bar{y}, w)$ such that $\varphi(x_1, z_1, \dots, x_n, z_n; y, w)$ is equivalent to $\tilde{\varphi}(\bar{\delta}^m(x_1), z_1, \dots, \bar{\delta}^m(x_n), z_n; \bar{\delta}^m(y), w)$ for some $m \geq 0$. Take an \mathcal{L}_δ -indiscernible sequence $(a_i, e_i)_{i \in I}$ in \mathbb{U} where $(a_i, e_i) \in \mathbb{U}^{(x_1, z_1)}$ and $I = I_1 + c + I_2$ with I_1, I_2 infinite dense linear orders without end points. Let (b, d) be a tuple in $\mathbb{U}^{(y, w)}$, and assume that $(a_i, e_i)_{i \in I_1 + I_2}$ is \mathcal{L}_δ -indiscernible over (b, d) . Then, by Corollary A.0.2, the sequence $(\bar{\delta}^m(a_i), e_i)_{i \in I}$ (resp. $(\bar{\delta}^m(a_i), e_i)_{i \in I_1 + I_2}$) is \mathcal{L} -indiscernible (resp. \mathcal{L} -indiscernible over B

{thm:ANIP}

{thm:distalit

where $B = \{\delta^m(b) : m \in \mathbb{N}\} \cup \{d\})$ for every $m \in \mathbb{N}$. Since T is distal, the \mathcal{L} -formula $\tilde{\varphi}$ is distal, which implies the distality of φ .

For the converse, suppose $\varphi(x_1, z_1, \dots, x_n, z_n; y, w)$ is an \mathcal{L} -formula which is not distal in T . Consider the \mathcal{L}_δ -formula ψ

$$\varphi(x_1, z_1, \dots, x_n, z_n; y, w) \wedge \bigwedge_{j=1}^n \delta(x_j) = 0 \wedge \delta(y) = 0.$$

Let $A \subseteq \mathbb{U}$ be such that all elements of the field sort are in the constant field $C_{\mathbb{U}}$ of \mathbb{U} . Let $(a_i, e_i)_{i \in I}$ be an \mathcal{L} -indiscernible sequence over A , where $a_i \in \mathbb{U}^{x_1}$ and $e_i \in \mathbb{U}^{z_1}$. If $a_i \in C_{\mathbb{U}}$ for each $i \in I$, then by Corollary A.0.2, we have that $(a_i, e_i)_{i \in I}$ is also \mathcal{L}_δ -indiscernible over A . Then if $(a_i, e_i)_{i \in I}$ and $(b, d) \in \mathbb{U}^{(y, w)}$ are a counterexample for the distality of φ in T , the same witnesses show that ψ is not distal in T_δ^* . \square

We finish by showing that T_δ^* eliminates the field quantifier \exists^∞ . It was proven for CODF by the second author in [30] and played a crucial role in the (first) proof that CODF has open core. The result is to our knowledge new in the general setting.

A.0.6. Theorem. *The theory T_δ^* eliminates the field sort quantifier \exists^∞ .*

Proof. Let \mathcal{K} be a model of T_δ^* . Let $X \subseteq K^{n+1}$ be an \mathcal{L}_δ -definable set of order m . Let $d = 2m$ and $Y \subseteq K^{(n+1)(d+1)}$ be the \mathcal{L} -definable set given by Lemma 3.1.9. Since T eliminates \exists^∞ , there is a finite bound n_Y such that for any $n(d+1)$ -tuple of elements \bar{e} of K , either $Y_{\bar{e}}$ is infinite or has cardinality $\leq n_Y$. By Lemma 3.1.9, if X_a is finite for $a \in K^n$, then $|X_a| = |Y_{\delta^d(a)}| \leq n_Y$, so the same bound shows the result for X . \square

Acknowledgements. We would like to thank: Marcus Tressl for encouraging discussions and for sharing with us his proof strategy to show elimination of imaginaries in CODF; Arno Fehm and Philip Dittmann for interesting discussions around henselian valued fields; the Institute of Algebra of the Technische Universität Dresden for its hospitality during a visit of the second author in May 2019; and l'Équipe de logique mathématique de l'IMJ-PRG for its hospitality during a visit of the first author in June 2021. The first author was partially funded by the ERC project TOSSIBERG (Grant Agreement 637027) and the Deutsche Forschungsgemeinschaft (DFG) - 404427454. Finally, we would like to point out that a previous version of the present work followed a slightly different approach to show the the open core property which can also be of interest (see arXiv:1912.0791v1 [math.LO]).

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MATHEMATISCHES INSTITUT DER HEINRICH-HEINE-UNIVERSITÄT DÜSSELDORF, UNIVERSITÄTSSTR. 1, 40225 DÜSSELDORF, GERMANY.

Email address: cubidesk@hhu.de

DEPARTMENT OF MATHEMATICS (DE VINCI), UMONS, 20, PLACE DU PARC 7000 MONS, BELGIUM

Email address: point@math.univ-paris-diderot.fr