Strong homotopy algebras in conformal field theory and higher spin gravity

Workshop on noncommutative and generalized geometry, Corfu

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September 21, 2022

European Research Council

Established by the European Commission

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Main Messages

Main message: higher spin symmetry is new Virasoro

- Is there any symmetry behind (Chern-Simons) vector models like Ising model? **Slightly-broken higher spin symmetry** (Maldacena, Zhiboedov) is clearly not a usual Lie-type symmetry ...
- The structure behind is L_∞ strong homotopy algebra, behind it is a simple associative algebra of fuzzy sphere. Invariants $=$ correlators and are uniquely fixed, which implies the **$3d$ bosonization duality**
- On a different note: there has to be a closed subsector of vector models (including the Ising), which has a local AdS_4 description as Chiral Higher Spin Gravity, yet unknown formality ...

Symmetry

99.99%: **Lie group** G/**Lie algebra** $\mathfrak{g}$ acting on some physical states. Group/Algebra = transformations without any info on what they act

Yangian: deformation of $U(\mathfrak{g}[z])$ as a Hopf algebra. Spin-chains, planar $\mathcal{N}=4$ SYM and scattering amplitudes therein

Strong homotopy algebras: multi-linear products on graded spaces (Lie and associative algebras are examples). Nice organizing tool for $\boldsymbol{QQ=0}$: BV-BRST, string field theory, higher spin gravities, ...

New: (Chern-Simons) vector models (e.g. $3d$ Ising, ...) have ∞-many almost conserved tensors $\partial^m J_{ma_2...a_s}\approx 0$ — **slightly-broken higher spin symmetry** (Maldacena, Zhiboedov). The right structure are certain L_∞-algebras. **Symmetry gets entangled with its representation.** Explains $3d$-bosonizationd duality

Plan

- Higher spin symmetry: from canonical QFT/CFT to algebraic viewpoint on free/very large-N CFT's
- Chern-Simons vector models and bosonization duality
- Slightly-broken higher spin symmetry as L_∞
- DQ of Poisson orbifolds, fuzzy sphere
- $3d$ bosonization duality via L_∞ glasses
- Chiral Higher Spin Gravity and yet unknown formality

Higher spin symmetry

Unbroken higher spin symmetry: free CFT's

Let's take any free CFT, e.g. free boson or free fermion

$$\Box\phi = 0 \qquad\qquad \partial\!\!\!/\psi = 0$$

We find the stress-tensor J_{ab} and infinitely many (even spin) *higher spin conserved tensors* $J_{a_1 \ldots a_s}$ (aka **higher spin currents**, old name — Zilch):

$$J_s = \phi\partial...\partial\phi + ... \qquad\qquad J_s = \bar{\psi}\gamma\partial...\partial\psi + ...$$

They are quasi-primary at the unitarity bound and have $\Delta = d + s - 2$.

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Stress-tensor is responsible for conformal symmetry $so(d, 2)$

$$Q_v = \int d^{d-1}x\, J_{0m}(x)v^m(x) \qquad\qquad \partial^n v^m + \partial^m v^n \sim \eta^{mn}$$

where conformal killing vector carries adjoint of $so(d, 2)$

$$v^a = \epsilon^a + x^a\epsilon + \epsilon^{a,b}x_b + \tilde{\epsilon}^a x^2 - 2x^a(\tilde{\epsilon}\cdot x)$$

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What are higher spin currents responsible for?

Unbroken higher spin symmetry

Any symmetry is certainly useful unless too much ... 

Imagine a CFT $d \geq 3$ with $J_2 \equiv T_{ab}$ and $J_s \equiv J_{a_1 \ldots a_s}$, all being traceless and conserved. Is it interesting?

One can show (Maldacena, Zhiboedov; Boulanger, Ponomarev, E.S, Taronna; Alba, Diab) that there are J_s with arbitrarily high spin (at least all even spins), and the correlators are

$$\langle \boldsymbol{J \ldots J} \rangle = \textbf{some free CFT}$$

In $3d$ there are two choices: free boson $\Box\phi = 0$ and free fermion $\not{\partial}\psi = 0$. Note: they have different correlators of J!

When something is completely fixed, usually it is thanks to some symmetry. What is the symmetry behind?

Unbroken higher spin symmetry

Conserved tensor $\rightarrow$ current $\rightarrow$ charge $\rightarrow$ symmetry

$$j_m(v) = J_{ma_2\dots a_s} v^{a_2\dots a_s} \qquad \partial^{(a_1} v^{a_2\dots a_s)} = \eta^{(a_1 a_2} u^{a_3\dots a_s)}$$

where $v^{a_1\dots a_{s-1}}$ is a conformal Killing tensor, $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \dots \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$ of $so(d,2)$

Unbroken higher spin symmetry

Conserved tensor $\rightarrow$ current $\rightarrow$ charge $\rightarrow$ higher-spin symmetry

$$j_m(v) = J_{ma_2\ldots a_s} v^{a_2\ldots a_s} \qquad\qquad \partial^{(a_1} v^{a_2\ldots a_s)} = \eta^{(a_1 a_2} u^{a_3\ldots a_s)}$$

where $v^{a_1\ldots a_{s-1}}$ is a conformal Killing tensor.

Miracle 1: the Lie algebra $Q_s = \int J_s$ originates from an associative one

Free CFT $=$ Associative algebra

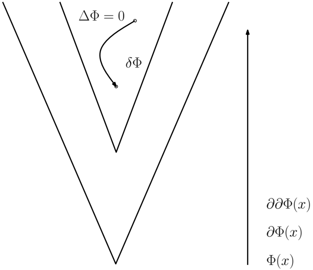
Indeed, $\Box\Phi = 0$ is $so(d,2)$-invariant:

$$\delta_v \Phi = v^m \partial_m \Phi + \frac{d-2}{2d} (\partial_m v^m) \Phi$$

We can multiply such symmetries due to linearity of $\Box\Phi = 0$

$$\delta\Phi = \delta_{v_1} \ldots \delta_{v_n} \Phi = \epsilon^{a_1\ldots a_k} \partial_{a_1} \ldots \partial_{a_k} \Phi + \ldots$$

Unbroken higher spin symmetry



Define $|V\rangle$ as the space of one-particle states $P_a ... P_c|\phi\rangle$, where $|\phi\rangle \equiv \phi(0)|0\rangle$

Higher spin algebra $\mathfrak{hs}$ is $\mathrm{gl}(V): V \to V$

$\mathfrak{hs}$ is DQ of the $so(d,2)$ coadjoint orbit ϕ

Higher spin currents are bilinear in ϕ or ψ

Miracle 2: $J \leftrightarrow \mathfrak{hs}$ upon identifying $|\phi\rangle|\phi\rangle$ with $|\phi\rangle\langle\phi|$ by inversion $\boldsymbol{R}$

$$J \sim |V\rangle \otimes |V\rangle \qquad \Longleftrightarrow \qquad \mathfrak{hs} \sim |V\rangle \otimes \langle V|$$

Unbroken higher spin symmetry: $3d$ specifics

In $3d$ the module of one-particle states of free boson/fermion CFT's is just the $2d$ harmonic oscillator (Dirac, 1963):

$$\begin{aligned} P_a ... P_a |\phi\rangle \quad &\sim \quad a^\dagger_\alpha a^\dagger_\beta ... a^\dagger_\alpha a^\dagger_\beta |0\rangle \\ P_a ... P_a |\psi\rangle \quad &\sim \quad a^\dagger_\alpha a^\dagger_\beta ... a^\dagger_\alpha a^\dagger_\beta \boldsymbol{a^\dagger_\gamma} |0\rangle \end{aligned}$$

This is thanks to $so(3,2) \sim sp(4,\mathbb{R})$ and thanks to the oscillator realization of $sp(2n)$, e.g. $P_m P^m \sim 0$, $P_m = \sigma^{\alpha\beta}_m a^\dagger_\alpha a^\dagger_\beta$, $\alpha, \beta, ... = 1, 2$

Now it is obvious that $\mathfrak{hs}$ is formed by even functions $f(a^\dagger, a)$. Formally, it is the even subalgebra of Weyl algebra A_2. Passing to p_i, q^j the product on $\mathfrak{hs}$ is the familiar Moyal-Weyl star-product:

$$(f \star g)(q,p) = f(q,p) \exp \frac{i\hbar}{2} (\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q) g(q,p)$$

Unbroken higher spin symmetry: correlators

Conserved tensor $\rightarrow$ current $\rightarrow$ symmetry $\rightarrow$ invariants=correlators

Is higher spin symmetry powerful enough to fix correlators?

All correlators are invariants (Sundell, Colombo; Didenko, E.S.; ...)

$$\langle J ... J \rangle = \mathrm{Tr}(\Psi \star ... \star \Psi) \qquad\qquad \Psi \leftrightarrow J$$

Ψ are coherent states representing J in the higher spin algebra. The correlators are invariant under full higher spin symmetry, $\delta\Psi = [\Psi, \xi]_\star$, which contains the conformal symmetry

Easy to say, but can we compute them?
yes, why not (Sundell, Colombo; Didenko, E.S.; ...)

$$\langle JJJJ \rangle_{F.B.} \sim \cos(Q^2_{13} + Q^3_{24} + Q^4_{31} + Q^1_{43}) \cos(P_{12}) \cos(P_{23}) \cos(P_{34}) \cos(P_{41}) + ...$$

Unbroken higher spin symmetry: Summary

In every free CFT one finds ∞-many higher spin currents $J_s \equiv J_{a_1\ldots a_s}$, which generate HS-charges $Q_s = \int J$. By construction, Q generate an ∞-dim Lie algebra, an extension of $so(3,2)$

$$[Q,Q] = Q \qquad\qquad [Q,J] = J \qquad\qquad [Q,\phi] = \phi$$

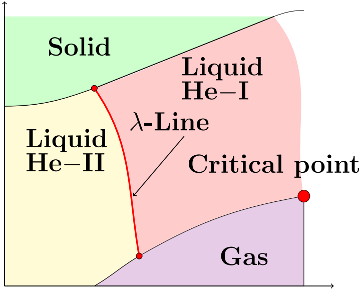
Miracle 1: the algebra originates from an associative HS-algebra $\mathfrak{hs}$ via $[a,b] = a \star b - b \star a$. **Free CFT = associative algebra**. **Miracle 2**: HS-currents J are isomorphic to $\mathfrak{hs}$, representation twisted by inversion.

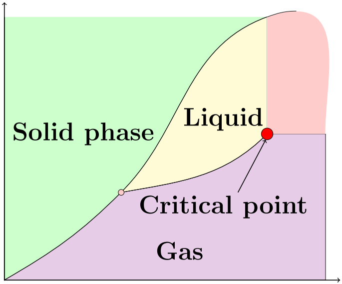
Correlators are invariants of this HS-algebra $\mathfrak{hs}$

$$\langle J...J\rangle = \mathrm{Tr}(\Psi \star ... \star \Psi) \qquad\qquad \Psi \leftrightarrow J$$

Important: in $3d$ $\mathfrak{hs}_{F.B.} \sim \mathfrak{hs}_{F.F.} \sim$ Weyl algebra of $f(a_i^\dagger, a^j)$ and the invariants are the unique invariants of HS-algebra (Sharapov, E.S.)

Chern-Simons Matter Theories and bosonization duality





Chern-Simons Matter theories and dualities

CFT_3: there exists a large class of models, vector models, Chern-Simons Matter theories (extends to ABJ(M))

$$\frac{k}{4\pi}S_{CS}(A) + \text{Matter}\begin{cases}(D\phi^i)^2 & \textbf{free boson}\\ (D\phi^i)^2 + g(\phi^i\phi^i)^2 & \textbf{Wilson-Fisher (Ising)}\\ \bar{\psi}\not{D}\psi & \textbf{free fermion}\\ \bar{\psi}\not{D}\psi + g(\bar{\psi}\psi)^2 & \textbf{Gross-Neveu}\end{cases}$$

- describe physics (Ising, quantum Hall, ...)
- break parity in general (Chern-Simons)
- two parameters $\lambda = N/k$, $1/N$ (λ continuous for N large)
- exhibit remarkable dualities, e.g. ***3d* bosonization duality** (Aharony, Alday, Bissi, Giombi, Karch, Maldacena, Minwalla, Prakash, Seiberg, Tong, Witten, Yacobi, Yin, Zhiboedov, ...)

Chern-Simons Matter theories and dualities

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\textbf{free boson} \arrow[r, shift left, "\tilde{\lambda}_b\text{-coupling}"] \arrow[d, dash, "\text{RG flow}"'] & \textbf{Gross-Neveu} \arrow[l, shift left, "\tilde{\lambda}_f\text{-coupling}"] \arrow[d, dash, "\text{RG flow}"] \\
\textbf{Wilson-Fisher} \arrow[r, shift left, "\tilde{\lambda}_b\text{-coupling}"] & \textbf{free fermion} \arrow[l, shift left, "\tilde{\lambda}_f\text{-coupling}"]
\end{tikzcd}
$$

The simplest gauge-invariant operators are higher spin currents

$$J_s = \phi D ... D\phi \qquad\qquad J_s = \bar{\psi}\gamma D ... D\psi$$

which are AdS/CFT dual to higher spin fields

Slightly-broken higher spin symmetry

What is going on in Chern-Simons-matter theories?

HS-currents are responsible for their own non-conservation:

$$\partial \cdot J_s = \sum_{s_1,s_2} C_{s,s_1,s_2}(\lambda)\frac{1}{N}[J_{s_1}J_{s_2}]$$

which is an exact non-perturbative quantum equation. In the large-N we can use classical (representation theory) formulas for $[JJ]$.

The worst case $\partial \cdot J$ = some other operator. The symmetry is gone, the charges are not conserved, do not form Lie algebra.

In our case the non-conservation operator $[JJ]$ is made out of J themselves, but charges are still not conserved.

Slightly-broken higher spin symmetry: what is it?

Initially we have well-defined charges and higher spin algebra $\mathfrak{hs}$

$$\partial \cdot J_s = 0 \quad \Longrightarrow \quad Q_s = \int J_s \quad \Longrightarrow \quad \begin{array}{ccc} [Q,Q] = Q & \& & [Q,J] = J \\ \boldsymbol{l(\xi_1,\xi_2)} & \& & \boldsymbol{l(\xi,J)} \end{array}$$

The higher spin symmetry does not disappear completely:

$$\partial \cdot J = \frac{1}{N}[JJ] \qquad\qquad [Q,J] = J + \frac{1}{N}[JJ]$$

What is the right math?

Slightly-broken higher spin symmetry: what is it?

Initially we have well-defined charges and higher spin algebra $\mathfrak{hs}$

$$\partial\cdot J_s=0 \quad\Longrightarrow\quad Q_s=\int J_s \quad\Longrightarrow\quad [Q,Q]=Q \quad \& \quad [Q,J]=J$$

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What is the right math?

We should deform the algebra together with its action on the module, so that the module (currents) can 'backreact':

$$\delta_\xi J=\boldsymbol{l(\xi,J)}+\boldsymbol{l(\xi,J,J)}+\ldots\,, \qquad\qquad [\delta_{\xi_1},\delta_{\xi_2}]=\delta_\xi\,,$$

where $\xi=\boldsymbol{l(\xi_1,\xi_2)}+\boldsymbol{l(\xi_1,\xi_2,J)}+\ldots$.

The consistency of such a structure leads to $\boldsymbol{L_\infty}$-algebras

Strong homotopy algebras

Strong homotopy algebra is a graded space, e.g. $V = V_{-1} \oplus V_0$ equipped with multilinear maps $l_k(x_1, ..., x_k)$ of degree-one. In our case

$$l_k(\xi, \xi, J, ..., J) \qquad\qquad l_k(\xi, J, ..., J)$$

that allow us to encode the deformed action

$$\delta_\xi J = l_2(\xi, J) + l_3(\xi, J, J) + ... , \qquad\qquad [\delta_{\xi_1}, \delta_{\xi_2}] = \delta_\xi ,$$

where $\xi = l_2(\xi_1, \xi_2) + l_3(\xi_1, \xi_2, J) + ...$ The maps obey 'Jacobi' relations

$$\sum_{i+j=n} (\pm) l_i(l_j(x_{\sigma_1}, ..., x_{\sigma_j}), x_{\sigma_{i+1}}, ..., x_{\sigma_n}) = 0$$

L_∞ **originates from** A_∞ **constructed from a certain deformation of** $\mathfrak{hs}$**, which is related to para-statistics/fuzzy sphere** (Sharapov, E.S.)

Deformations of Poisson Orbifold: Weyl Algebra

Everyone knows that the Weyl algebra A_1 is rigid

$$[q,p] = i\hbar \qquad \text{no deformation of} \qquad f(q,p) \star g(q,p)$$

Suppose that $\boldsymbol{R} f(q,p) = f(-q,-p)$, i.e. we can realize it as

$$\boldsymbol{R}^2 = 1 \qquad \boldsymbol{R} q \boldsymbol{R} = -q \qquad \boldsymbol{R} p \boldsymbol{R} = -p$$

The crossed-product algebra $A_1 \ltimes \mathbb{Z}_2$ is soft (Wigner; Yang; Mukunda; ...):

$$[q,p] = i\hbar + i\nu \boldsymbol{R}$$

Also known as para-bose oscillators. Even $\boldsymbol{R}(f) = f$ lead to $gl_\lambda = U(sp_2)/(C_2 - \lambda(\lambda-1))$ (Feigin), also (Madore; Bieliavsky et al) as fuzzy-sphere, NC hyperboloid, also (Plyushchay et al) as anyons.

Orbifold $\mathbb{R}^2/\mathbb{Z}_2$ admits 'second' quantization on top of the Moyal-Weyl $\star$-product, (Pope et al; Joung, Mrtchyan; Korybut; Basile et al; Sharapov, E.S., Sukhanov)

Slightly broken higher spin symmetry: summary

- necessary for bosonization: $\mathfrak{hs}$ (boson) $\sim$ $\mathfrak{hs}$ (fermion) (Dirac, 1963)
- there exist exactly one invariant, $\mathrm{Tr}(\Psi \star ... \star \Psi)$, to serve as n-point correlator $\langle J...J \rangle$ for free/large-N limit
- L_∞ depends on two pheno parameters, to be related to k, N
- invariants are unobstructed and have a quasi-free form

$$\mathrm{Tr}_\circ \log_\circ[1-\Psi] \mod \text{irr}\,, \qquad a \circ b = a \star b + \phi_1(a,b)\boldsymbol{R} + ...$$

- a simple consequence is that correlators are very special

$$\langle J...J \rangle = \sum \langle \text{fixed} \rangle_i \times \text{params}$$

This implies $3d$ bosonization since $\langle J...J \rangle$ know everything and it does not matter what matter J are made of, ϕ or ψ

Chiral higher spin gravity

Remarks on Higher Spin Gravity

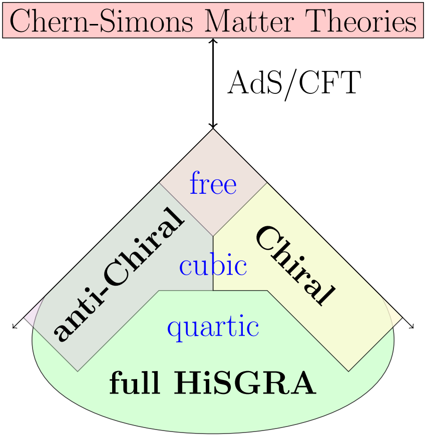
AdS/CFT duals of (Chern-Simons) vector models are HiSGRA since conserved tensor J_s is dual to (massless) gauge field in AdS_4 (Sundborg; Klebanov, Polyakov; Sezgin Sundell; Leight, Petkou; Giombi, Yin, ...)

$$\partial^m J_{ma_2\ldots a_s} = 0 \qquad \Longleftrightarrow \qquad \delta\Phi_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}$$

Instead of tedious quantum calculations in Chern-Simons matter one could do the standard holographic computation in the HiSGRA dual.

No free lunch: generic HiSGRA do not (want to) exist/too nonlocal (untamed); nice exceptions: $3d$, conformal, IKKT (Steinacker, ..., Tran), Chiral

Chiral HiSGRA (Metsaev; Ponomarev, E.S.): all spins $s = 0, (1), 2, (3), \ldots$; at least one-loop UV-finite (E.S., Tran, Tsulaia); any sign of the cosmological constant is allowed, including zero; the smallest and the only local HiSGRA with propagating massless fields; specific predictions for higher derivative modifications of gravity; some kind of self-dual theory (Ponomarev); trivial S-matrix in flat space, but not in AdS ...



Chiral HiSGRA is a local HiSGRA;

It has the right spectrum to be dual to Chern-Simons matter, but it cannot be;

Its very existence implies Chern-Simons matter have 2 closed subsectors;

Its very existence supports $3d$ bosonization duality;

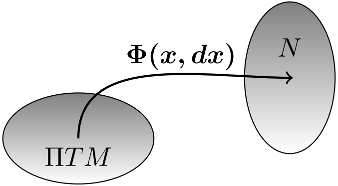
Bosonization is manifest!

(anti)-Chiral Theories provide a complete base for 3-pt amplitudes

$$\boldsymbol{V_3 = V_{chiral} \oplus \bar{V}_{chiral} \quad \leftrightarrow \quad \langle JJJ \rangle}$$

Let us be given a Q-manifold (better an $\boldsymbol{L_\infty}$-algebra)

$\Omega^\bullet(M)\,, d^2 = 0$



$\Phi\,, Q^2 = 0$

then we can always write a sigma-model:

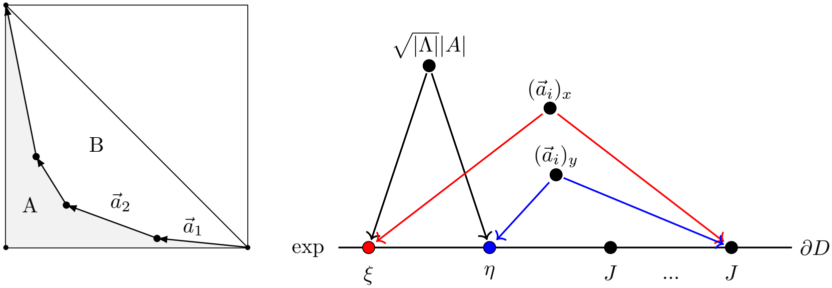
$$\boldsymbol{d\Phi = Q(\Phi)} \equiv l_2(\Phi, \Phi) + l_3(\Phi, \Phi, \Phi) + ...$$

Any PDE can be cast into such a form ... (Barnich, Grigoriev)

Other names: Free Differential Algebras (Sullivan), in physics: (van Nieuwenhuizen; Fre, D' Auria); FDA=unfolding (Vasiliev), AKSZ (AKSZ), gauged-PDE (Grigoriev, Kotov)

A piece of a yet unknown formality

Moyal-Weyl, $\mathfrak{hs}$, $2d$ Poisson/Symplectic $\pi^{AB} =$ const, only boundary graphs survive in (Shoikhet-Tsygan)-Kontsevich formality. The first maps: $\boldsymbol{m(\xi,\eta)} = \xi \star \eta$, $\boldsymbol{m(\xi,\eta,J)} \sim \phi_1(\xi,\eta) \star J\boldsymbol{R}$, but we need locality! Result for $\boldsymbol{m(\xi,\eta,J,...,J)}$:



(Sharapov, E.S., Van Dongen) the configuration space is of convex polygons B or swallowtails A, related to Grassmannian. The maps exponentiate like for Moyal-Weyl.

Our A_∞ is a pre-Calabi-Yau algebra (Kontsevich, ...). **Formality? TFT?**

Summary/Remarks/Comments/Speculations

- **Higher spin symmetry is new ($d = 3, \ldots$) Virasoro**
- Slightly-broken symmetry should be understood as $\boldsymbol{L_\infty}$
- Uniqueness of $\boldsymbol{L_\infty}$-invariants implies the $3d$ bosonization duality
- **New type of a physical symmetry** where transformations (algebra) and the object (module) deform together
- Anomalous dimensions of HS-currents are small even for Ising model, $N = 1$, e.g. $\Delta(J_4) = 5.02$ instead of 5
- Chiral Higher Spin Gravity is dual to a closed subsector of (Chern-Simons) vector models. How to find it? It should extend to small N due to integrability, implications for Ising?
- Locality implies specific A_∞-maps that have (Shoikhet-Tsygan)-Kontsevich as first two layers. Biger Formality? TFT?
- **More:** Snowmass paper, ArXiv: 2205.01567

Thank you for your attention!

May the higher spin symmetry
be with you

Consider functions $f(q, p, \boldsymbol{R})$ on the phase space

1st Poisson bracket:

$$\{f, g\} = m\,\pi_1(f \otimes g) \qquad\qquad \pi_1 = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$$

2nd Poisson bracket:

$$\{f, g\} = m\,\pi_1 \int f\big(q_{(1-2u_1)}, p_{(1-2u_1)}\big) \otimes g\big(q_{(1-2u_2)}, p_{(1-2u_2)}\big)\, \boldsymbol{R}$$

the integral over simplex $0 < u_1 < u_2 < 1$ is a leftover of the Kontsevich-Shoikhet graph. It can be obtained (Feigin, Felder, Shoikhet) from Kontsevich-Shoikhet-Tsygan Formality

In the general case of $A_n \rtimes \Gamma$ the complete deformation can be obtained (Sharapov, E.S., Sukhanov) via Homological Perturbation Theory. There is a relation to Dunkl derivative for $\hbar = 0$ (Tang)

L_∞ from A_∞; A_∞ from soft associative algebras

We need to construct L_∞ that 'deforms' our initial data = algebra + module, both originating from an associative algebra $A = \mathfrak{hs} \rtimes \mathbb{Z}_2$.

One can show (Sharapov, E.S.) that such L_∞ can be constructed as long as A is soft, i.e. can be deformed as an associative algebra:

$$a \circ (b \circ c) = (a \circ b) \circ c \qquad\qquad a \circ b = a \star b + \sum_{k=1} \phi_k(a,b)\, \hbar^k$$

The maps can be obtained from an auxiliary A_∞

$$\begin{aligned} m_3(a,b,u) &= \phi_1(a,b) \star u \quad \rightarrow \quad l_3 \\ m_4(a,b,u,v) &= \phi_2(a,b) \star u \star v + \phi_1(\phi_1(a,b),u) \star v \quad \rightarrow \quad l_4 \end{aligned}$$

Our algebra can be deformed thanks to para-statistics/anyons ... otherwise, any soft associative algebra leads to a certain A_∞

Empty Flat Space

Any PDE can be cast into such a form ...

Flat space is still a flat connection $dA = \frac{1}{2}[A, A]$

$$de^a = \omega^{a,}{}_b \wedge e^b \qquad \text{no torsion}$$
$$\boldsymbol{d\omega^{a,b} = \omega^{a,}{}_c \wedge \omega^{c,b}} \qquad \text{Riemann vanishes}$$

vielbein $e^a \equiv e^a_\mu \, dx^\mu$; spin-connection $\omega^{a,b} \equiv \omega^{a,b}_\mu \, dx^\mu$

Minkowski space = flat Poincare algebra connection*

How to add gravitatonal waves?

Einstein Equations

Any PDE can be cast into such a form ... but in practice cannot

Einstein equations are already painful enough

$$de^a = \omega^{a,}{}_b \wedge e^b \qquad \text{no torsion}$$

$$d\omega^{a,b} = \omega^{a,}{}_c \wedge \omega^{c,b} + e_m \wedge e_n C^{ab,mn} \qquad \text{Einstein is here!}$$

vielbein $e^a \equiv e^a_\mu \, dx^\mu$; spin-connection $\omega^{a,b} \equiv \omega^{a,b}_\mu \, dx^\mu$; Weyl tensor $C^{ab,cd}$

Any PDE can be cast into such a form ... but in practice cannot

Einstein equations are already painful enough

$$
\begin{aligned}
de^a &= \omega^{a,}{}_b \wedge e^b && \text{no torsion}\\
\boldsymbol{d\omega^{a,b}} &= \boldsymbol{\omega^{a,}{}_c} \wedge \boldsymbol{\omega^{c,b}} + e_m \wedge e_n \boldsymbol{C^{ab,mn}} && \text{Einstein is here!}\\
d\boldsymbol{C^{ab,mn}} &= \omega \boldsymbol{C^{ab,cd}} + e_k \boldsymbol{C^{ab,mn,k}} && \text{Bianchi for Weyl}
\end{aligned}
$$

vielbein $e^a \equiv e^a_\mu \, dx^\mu$; spin-connection $\omega^{a,b} \equiv \omega^{a,b}_\mu \, dx^\mu$; Weyl tensor $C^{ab,cd}$; **and more $C^{ab,cd,k}$, ... and nonlinear equations for them**

We have to introduce auxiliary fields to parameterize the on-shell jet:

$$\boldsymbol{R_{ab,mn}} = \boldsymbol{C_{ab,mn}} \qquad (\textbf{Riemann} = \textbf{its Weyl}) \sim \text{Einstein}$$

Closed-form for SDYM and SDGR (E.S., Van Dongen), l_2, l_3