Higher-spins from higher dualisations

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Work in collaboration with Victor Lekeu , [2012.11356]

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1. Introduction

2. Higher dualisations of linearised gravity and Maxwell's

   2.1) Higher dual of vector field in dimensions 3 & 4

   2.2) Double dualisation of graviton in $n=5$

   2.3) Higher dualisation of the graviton in 3D

   2.4) Higher dualisations of Maxwell's in 3D

3. Higher dualisations of linearised gravity in 4D and A_1^{+++}

   3.1) The first higher dual of graviton in 4D

   3.2) A few words by a non-expert on NLR of A_1^{+++}

   3.3) Higher duals of graviton in 4D

4. Conclusions and outlook

1 Introduction

- Fierz-Pauli programme → all possible off-shell descriptions of spin-s massless field ?

  Interactions may not choose the most economical description

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$\boxed{\text{Higher-spin Gravity}}$

Hidden symmetries of Gravity

$\boxed{\text{Non-linear realisation of } e_{11} \ltimes \ell_1}$

$\boxed{\text{Higher, or "exotic" descriptions}}$

② Higher dualisations of linearised gravity and Maxwell's

An off-shell dualisation was initiated in 2001 by P. West completed by N.B., S. Cnockaert and M. Henneaux.

In [N.B., P. Cook, D. Ponomarev] $\Rightarrow$ Other off-shell dualisations schemes proposed.

First, review the dualisation of [West, N.B.-Cnockaert-Henneaux]

Parent action

$$S[Y^{abc|}{}_d, \omega_{ab|c}] = \int d^n x \left(\text{“}\omega\,\omega\text{''} + \partial_a \omega_{bc|}{}^d \, Y^{abc|}{}_d \right)$$

$$Z = \int \mathcal{D}\omega \, \mathcal{D}Y \exp \tfrac{i}{\hbar} S[\omega, Y]$$

enforces $\omega_{ab|c} = \partial_{[a} e_{b]c}$

semi-classical $S[e_{ab}] =$ Fierz-Pauli

with local Lorentz $\delta e_{ab} = \lambda_{ab}$

Field $\omega_{ab|c}$ auxiliary

$$\frac{\delta S}{\delta \omega} \approx 0 \Rightarrow \omega_{ab|c} \sim \partial^d Y_{abd|c}$$

$$S[Y^{abc|}{}_d] = \int d^n x \left(\partial^\circ Y_{ab\circ|c} \, \partial_\bullet Y^{\bullet ab|c} + \dots \right)$$

$$Y^{abc|}{}_e = \frac{1}{(n-3)!} \varepsilon^{abc\, d_1 \dots d_{n-3}} C_{d[n-3]|e}$$

Gauge inv. $\delta_\lambda Y_{abc|}{}^e = \delta^e_{[a} \lambda_{bc]} \Rightarrow \delta_\lambda C_{a[n-3]|b} = \varepsilon_{a[n-3]bcd} \lambda^{cd}$

$$C_{[n-3,1]} \rightsquigarrow C_{a_1 \dots a_{n-3},\, b} = C_{[a_1 \dots a_{n-3}],\, b} \quad \text{s.t.} \quad C_{[a_1 \dots a_{n-3},\, b]} \equiv 0$$

i.e. $C_{[n-3,1]} \sim$  of $GL(n)$ appears in Minkowski spacetime $\mathbb{R}^{1,n-1}$

$n-3$

that propagates the d.o.f. of Fierz_Pauli's graviton h_{ab} with $\eta^{bd} K_{ab,cd}(h) = 0$.

<u>Note</u> : Hull's [2001] twisted on_shell duality

relating

$$K_{a_1 \dots a_{n-2},}{}^{b_1 b_2}(C) := \partial_{[a_1} \partial^{[b_1} C_{a_2 \dots a_{n-2}],}{}^{b_2]}$$

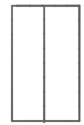
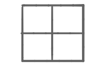
to

$$K_{ab,}{}^{cd}(h) := -\frac{1}{2} \partial_{[a} \partial^{c} h^{d]}{}_{b]}$$

via

$$\boxed{K_{[n-2,2]}(C) = *_1 K_{[2,2]}(h)}$$

In same work [Hull : 2001] a field

$D_{[n-3,\,n-3]} \sim$  of $Gl(n)$, $\xrightarrow[n=5]{}$ $D_{[2,2]} \sim$ 

$n-3 \quad n-3$

was introduced with field equations $\boxed{Tr^{n-3} K_{[n-2,n-2]}(D) = 0}$ $\xrightarrow[n=5]{}$ $Tr^2 K_{[3,3]} = 0$.

Twisted duality with h_{ab} : $\boxed{K_{[n-2,n-2]}(D) = *_1 *_2 K_{[2,2]}(h)}$ $\xrightarrow[n=5]{}$ $\boxed{K_{[3,3]} = *_1 *_2 K_{[2,2]}(h)}$.

EII $\quad d_1^{+} K_{[2,2]} \equiv 0 \iff d_1 K_{[3,3]} = 0 \Leftrightarrow K_{[3,3]} = d_1 d_2 D_{[2,2]} =: K_{[3,3]}(D)$ BII

⊡ [Dubois-Violette – Henneaux 2000]

But $\quad Tr\, K_{[2,2]} = 0$ (EI) $\longleftrightarrow Tr *_1 *_2 K_{[3,3]} = 0 \Leftrightarrow Tr^2 K_{[3,3]} = 0$ (EI) EoM for D .

Field equations mapped to Field equations .

[In presence of sources, Hull argued that $D_{[2,2]}$ couples to the usual T_{ab} stress tensor
confirming that $D_{[2,2]}$ is, on-shell, an avatar of h_{ab} .

Indeed, from twisted-duality $\boxed{K_{[3,3]}(D) \approx *_1 *_2 K_{[2,2]}(h).}$ $(*)$

$$K_{[3,3]}(D) = d_1 d_2 D_{[2,2]} = *_1 *_2 d_1 d_2 h_{[1,1]} \propto \eta_{[1,1]} d_1 d_2 h_{[1,1]} \qquad (\text{since } n=5)$$

$$\Leftrightarrow \quad d_1 d_2 \left(D_{[2,2]} - a\, \eta_{[1,1]} h_{[1,1]} \right) = 0 \qquad a \in \mathbb{R}_0$$

$$\Leftrightarrow \quad D_{[2,2]} = a\, \eta_{[1,1]} h_{[1,1]} + d^{\{2\}} \xi_{[2,1]} \qquad (**)$$

$\Rightarrow$ When the EoMs for h_{ab} Fierz-Pauli are satisfied (since twisted on-shell duality $(*)$ was used),

then the $D_{[2,2]}$-field is conformally flat up to a gauge transformation $(**)$

[Marc H., Victor Lekeu and Amaury Leonard 2019]

Therefore, inverting $(**)$ to express $h_{[1,1]} = H_{[1,1]}(\mathrm{Tr}\, D, \xi)$ and plugging in

$S^{FP}[H_{ab}(\mathrm{Tr}\, D, \xi)]$ gives field equations equivalent to $(**)$.

$\Rightarrow$ Obviously, the traceless part of $D_{[2,2]}$ does not enter this action $\Rightarrow$ not a genuine action principle for $D_{ab,cd}$!

However, before [Marc-Victor-Amaury 2019]'s note on the double-dual graviton,

an action principle had been proposed for a $D_{[2,2]}$ gauge field (plus other fields)

that propagates the degrees of freedom of a single graviton:

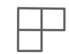
N.B, Paul Cook, Mitya Ponomarev [2012]

"Off-shell Hodge dualities in linearized gravity and E_{11}"

In the paper 2012.11356 (JHEP) with Victor Lekeu, we further analysed the action proposed earlier in 2012.

2.1) Higher dual of vector field in dimensions 3 & 4

Idea : A_b viewed as a $A_{[0,1]}$ bi-form

$$A_{[0,1]} \xrightarrow[\text{higher dualise}]{} C_{[n-0-2\,,\,1]} \underset{n=4}{=} C_{[2,1]} \sim$$ 

$$\underset{n=3}{=} h_{[1,1]} \sim$$ 

- Starts from Maxwell and IBP : $S[A_a] = -\frac{1}{2}\int d^n x \, (\partial_a A_b \, \partial^a A^b - \partial_a A^a \, \partial_b A^b)$
- Parent action $S[Y^{ab|}{}_c\,,\, P_{a|}{}^b] = \int d^n x \, (P_{a|b} \, \partial_c Y^{ca|b} - \frac{1}{2} P_{a|b} P^{a|b} + \frac{1}{2} P^{a|}{}_a \, P_{b|}{}^b)$

  . $\dfrac{\delta S[Y,P]}{\delta P_{a|b}} \approx 0 \iff P^{a|b} \approx \partial_c Y^{ca|b} - \eta^{ab} \dfrac{1}{n-1} \partial_c Y^{cd|}{}_d$

substitute to get

$$S[Y^{ab|}{}_c] = \int d^n x \left[\frac{1}{2} \partial_c Y^{ca|b} \, \partial_d Y^d{}_{a|b} - \frac{1}{2(n-1)} \partial_a Y^{ab|}{}_b \right]$$

. From $$\boxed{S[Y^{ab|}{}_c] = \int d^n x \left[\tfrac{1}{2}\, \partial_c Y^{ca|b}\, \partial_d Y^d{}_{a|b} - \frac{1}{2(n-1)}\, \partial_a Y^{ab|}{}_b \right]}$$

invariant under $$\delta Y^{ab|}{}_c = \delta_c^{[a}\, \partial^{b]} \lambda + \partial_d\, \tau^{abd|}{}_c \,,$$

one decomposes $$Y^{ab|}{}_c = X^{ab|}{}_c + \delta_c^{[a} Z^{b]} \,, \qquad X^{ab|}{}_b \equiv 0$$

irreducibly under $Gl(n)$.

A $\underline{\text{Hodge-dualise in 4D}}$: $X^{ab|}{}_c \overset{*_1}{\longleftrightarrow} T_{ab|c} \sim \begin{ytableau} a & c \\ b \end{ytableau}$ that gauge transforms as

$$\delta\, \ydiagram{2,1} = {}^{S}\begin{ytableau} \, & \, \\ \partial \end{ytableau} + {}^{A}\begin{ytableau} \, & \partial \\ \, \end{ytableau}$$

with $$\left\{ \begin{array}{l} \\ \delta Z_a = \partial_a \lambda + \partial^b A_{ab} \end{array} \right.$$ for the vector.

Perform some change of field variables and dualize $Z_a \leftrightarrow \tilde{A}_a$ to get

$$S[T_{ab|c}, \tilde{A}_a] = \int d^4x \left[\mathcal{L}^{\text{Curt.}}(T_{ab,c}) + \frac{1}{4} F^{ab}(\tilde{A}) F_{ab}(\tilde{A}) + \frac{1}{2} \tilde{A}^a K''_a(T) \right]$$

where $K^{a[3]}{}_{b[2]} := 6\, \partial^{[a} \partial_{[b} T^{aa],}{}_{b]}$ curvature, $K'' := \mathrm{Tr}^2 K$.

The gauge invariances are the ones expected for a Curtright field and a vector.

The field equations give $-\partial_a F^{ab}(\tilde{A}) + \frac{1}{2} K''^{b} = 0$ (1)

$$K'^{ab|}{}_c + \delta_c^{[a} K''^{b]} - \frac{1}{2} \partial_c F^{ab} - \delta_c^{[a} \partial_d F^{b]d} = 0 \qquad (2)$$

Take the trace of (2), combine with (1) to get the equations of motion and

duality relation $\boxed{d^\dagger F_{[2]} = 0 = \mathrm{Tr}^2 K_{[2,2]} \quad \& \quad \mathrm{Tr}\, K_{[3,2]} = d_2 F_{[2,0]}}$ $\Rightarrow$ no doubling of d.o.f. !

(B) Hodge-dualise in 3D

$S[Y^{ab|}{}_c]$ with $Y^{ab|}{}_c = \varepsilon^{abd} h_{cd} + 2\, \delta_c^{[a} Z^{b]}$, $\quad h_{ab} = h_{ba}$, giving an action $S[h_{ab}, Z_a]$

$$S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \tfrac{1}{2} \partial_a h_{bc} \partial^b h^{ac} + \tfrac{1}{2} \varepsilon^{bcd} \partial^a h_{ab} F_{cd}(Z) + \tfrac{1}{4} F^{ab}(Z) F_{ab}(Z) \right]$$

invariant under $\delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)}$, $\quad \delta Z_a = \partial_a \lambda + \varepsilon_{abc} \partial^b \epsilon^c$

mixing

- Dualise the vector Z_a in 3D to a scalar. After a field redefinition, one finds

$$S[h_{ab}, \phi] = \int d^3x \Big[-\tfrac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \tfrac{1}{2} \partial_a h \partial^a h - \partial_a h \partial_b h^{ab} + \partial_a h^{ab} \partial^c h_{bc}$$
$$+ \tfrac{1}{2} \partial_a \phi \partial^a \phi + \partial_a \phi \left(\partial_b h^{ab} - \partial^a h \right) \Big] \quad .$$

As consequence of field equations: $\boxed{\Box \phi \approx 0 \approx R(h) \quad \& \quad R_{ab}(h) \approx \partial_a \partial_b \phi}$

Field equations for propagation & duality relation = no doubling

2.2) Double dualisation of graviton in $n=5$

- The dualisation procedure for the double-dual graviton gives an action

$$S[D_{ab,cd}\,,\, Z^{ab|}{}_c] = \int d^5x \; \mathcal{L}(\partial D, \partial Z) \qquad , \quad D \sim \ydiagram{2,2}$$

$$Z^{ab|}{}_c = - Z^{ba|}{}_c \,.$$

where

$$\mathcal{L}(\partial D, \partial Z) = \mathcal{L}(\partial Z) - \mathcal{L}(\partial D) + \mathcal{L}^{cross}$$

features the complete $D_{ab,cd}$ field, including its traceless part.

- The action is gauge invariant under

$$\begin{cases} \delta\; \ydiagram{2,2} = \ytableaushort{{}{},{}{\partial}} \\[2ex] \delta Z^{mn|}{}_a = \lambda^{mn}{}_a + \partial^{[m}\xi^{n]}{}_a - \frac{1}{2}\,\delta^{[m}_a\,\partial_b\,\xi^{n]b} + \delta^{[m}_a\,\partial^{n]}\chi \end{cases}$$

where $\lambda_{abc} = \lambda_{[abc]}$.

- Perform the change of variable

$$Y^{ab|}{}_{c} := Z^{ab|}{}_{c} + \delta_c^{[a} Z^{b]} \qquad \text{where} \quad Z^b := Z^{bc|}{}_{c}$$

to get

$$S[Y^{ab|}{}_{c}, D_{ab,cd}] = \int d^5x \, \Big[\mathcal{L}^{curt.}(Y_{ab|c}) - \mathcal{L}(\partial D) + \tfrac{1}{2} \epsilon_{abcde} \, \partial^e D^{cd,}{}_{mn} \big(F^{ab|mn}(Y) - \tfrac{1}{2} F^{mn|ab} \big) \Big]$$

that propagates a single graviton.

- Dualize $Y^{ab|}{}_{c}$ in 5D to $f_{a|c} \sim h_{ab} + B_{[ab]}$ to get (B drops out)

$$S[D_{ab,cd}, f_{a|b}] = \int \big[\mathcal{L}^{FP}(h) - \mathcal{L}(\partial D) - \tfrac{3}{2} h_{ab} \tilde{K}^{,ab}(D) \big]$$

where $\tilde{K}_{[2,2]}(D) := *_1 *_2 \, d_1 d_2 \, D_{[2,2]}$. <u>Field equations</u>:

$$\boxed{\begin{aligned} &\mathrm{Tr}\, K_{[2,2]}(h) = 0 = \mathrm{Tr}^2 K_{[3,3]}(D) \\ \& \quad &K_{[2,2]}(h) \propto \mathrm{Tr}\, K_{[3,3]}(D). \end{aligned}}$$

$\hookrightarrow$ The on-shell duality relation of Hull is now derived from an <u>action principle</u>.

2.3) Higher dualisation of the graviton in 3D

Gravity in 3D is topological. Dual formulation thereof by higher dualisation $\rightarrow$ higher spin

- $S[G_{a|bc}, D_{ab|}{}^{cd}] = \int d^3x \left[-\frac{1}{2} G_{a|bc} G^{a|bc} + \frac{1}{2} G_{a|c}{}^{c} G^{a|b}{}_{b} - G_{a|}{}^{ab} G_{b|c}{}^{c} + G_{a|}{}^{ab} G^{c|}{}_{cb} + G^{d}{}_{|bc} \partial^a D_{ad|}{}^{bc} \right]$

$$G_{a|bc} = G_{a|cb} \quad , \quad D_{ab|}{}^{cd} = -D_{ba|}{}^{cd} = -D_{ba|}{}^{dc}$$

Gauge-invariant under

$$\delta G^{a}{}_{|bc} = 2\, \partial^a \partial_{(b} \epsilon_{c)} \quad , \quad \delta D_{ab|}{}^{cd} = \varepsilon_{abp}\, \partial^p v^{cd} + 2\, \eta^{cd} \partial_{[a} \epsilon_{b]} + 4\, \delta^{(c}_{[a} \partial_{b]} \epsilon^{d)} \; .$$

- $\dfrac{\delta S}{\delta D} \approx 0 \implies G_{a|bc} \approx \partial_a h_{bc}$ with $h_{ab} = h_{ba}$ symmetric

$\hookrightarrow S[\partial_a h_{bc}, D_{ab|}{}^{cd}] = \int d^3x \left[-\frac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \frac{1}{2} \partial_a h\, \partial^a h - \partial_a h^{ab} \partial_b h + \partial_a h^{ab} \partial^c h_{bc} \right]$

Fierz-Pauli

- $\dfrac{\delta S}{\delta G_{a|bc}} \approx 0 \implies G_{a|bc} \approx \partial^d D_{da|bc} + \partial D =: G_{a|bc}(\partial D)$

$\hookrightarrow S[G_{a|bc}(\partial D), D_{ab|}{}^{cd}] =: S[D_{ab|}{}^{cd}]$ Dual action

$$D_{ab|}{}^{pq} =: \varepsilon_{abm}\,\tilde{D}^{m|pq} \quad , \quad \tilde{D}^{a|cd} = \tilde{\varphi}^{acd} + 2\,\varepsilon^{ab(c}\,Z_{b|}{}^{d)} \quad , \quad Z_{a|}{}^{a} \equiv 0$$

Gl(3) decomposition

- Further decompose under $SO(3)$:

$$\tilde{D}^{a|bc} \sim \square \otimes \left(\square\square \oplus \bullet\right) \simeq \square\square\square \oplus \square\square \oplus \square \oplus \square \qquad \in SO(3)$$

- A linear combination of the two vectors can be dualised to a scalar, in 3D.

- Combining the scalar with $\square\square \longrightarrow$ traceful $h_{ab} \sim \square\square \in Gl(3)$

- Traceless rank-3 with remaing vector $\longrightarrow$ traceful $\varphi_{abc} \sim \square\square\square \in Gl(3)$

Final spectrum : $\{\varphi_{abc}, h_{ab}\}$.

Gauge transformations :

$$\begin{cases} \delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - \frac{2}{3}\, \varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q \,, \\ \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} + 2\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)} \,, \end{cases}$$

Preserve action

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2} \int d^3x \Big[- \partial_a \varphi_{bcd}\, \partial^a \varphi^{bcd} + \partial^a \varphi^b\, \partial^c \varphi_{abc} + \partial_a \varphi^{abc}\, \partial^d \varphi_{bcd} - \frac{1}{7}\, \partial_a \varphi_b\, \partial^a \varphi^b - \frac{31}{28}\, \partial_a \varphi^a\, \partial^b \varphi_b$$

$$+ \frac{1}{2}\, \partial_a h_{bc}\, \partial^a h^{bc} + \frac{1}{14}\, \partial_a h\, \partial^a h - \frac{3}{7}\, \partial^a h_{ab}\, \partial_c h^{bc} - \frac{1}{7}\, \partial^a h\, \partial_c h_a{}^c$$

$$+ \frac{10}{7}\, \varepsilon_{apq}\, \partial^b h_b{}^a\, \partial^p \varphi^q - 2\, \varepsilon_{apq}\, \partial^b h^{ac}\, \partial^p \varphi^q{}_{bc} \Big]$$

- Gauge transformations entangled
- "Wrong" relative kinetic term
- (Abelian) Chern - Simons reformulation

2.4) <u>Higher dualisations of Maxwell's in 3D</u>

- Maxwell in 3D ~ massless scalar in 3D : 1 propagating d.o.f.

- Start from the dual action obtained by the second higher dualisation of scalar theory,

  i.e. the first higher dualisation of Maxwell's vector reviewed above

$$S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2}\partial_a h_{bc}\partial^a h^{bc} + \tfrac{1}{2}\partial_a h_{bc}\partial^b h^{ac} + \tfrac{1}{2}\varepsilon^{bcd}\partial^a h_{ab} F_{cd}(Z) + \tfrac{1}{4}F^{ab}(Z)F_{ab}(Z)\right]$$

We perform the higher dualisation $\mathcal{D}$ of h_{ab} via parent action $S[G_{a|bc}, D_{ab|}{}^{cd}, A_a]$,

eliminate the auxiliary field $G_{a|bc}$ to obtain $S[D_{ab|}{}^{cd}, A_a]$. As before, $\tilde{D}^{a|ij} := -\tfrac{1}{2}\varepsilon^{abc}D_{bc|}{}^{ij}$.

Perform a field redefinition $\{\underset{\underline{18}}{\tilde{D}_{a|cd}}, \underset{\underline{3}}{A_a}\} \longleftrightarrow \{\underset{\underline{10}}{\phi_{abc}}, \underset{\underline{5}}{f_{ab}}, \underset{\underline{3}}{U_a}, \underset{\underline{3}}{A_a}\}$ where $f_{ab} \sim \square\square$ of $SO(3)$.

s.t. U_a enters the action only via $F_{ab}(U) = 2\partial_{[a}U_{b]}$. Dualise U_a in 3D to a scalar σ that one adds

to $f_{ab} \longrightarrow h_{ab}$ traceful.

The final action is invariant under

$$\begin{cases} \delta \phi_{abc} = 3\, \partial_{(a} \xi_{bc)} \\[2ex] \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} + 2\, \varepsilon_{pq(a} \partial^p \xi^q{}_{b)} \\[2ex] \delta A_a = \tfrac{3}{2}\, \partial_a \xi + \varepsilon_{abc}\, \partial^b \epsilon^c \end{cases}$$

Rem : $\delta\big(\underbrace{\phi_{abc} - \tfrac{2}{3}\eta_{(ab} A_{c)}}_{\varphi_{abc}}\big) = 3\, \partial_{(a} \hat{\xi}_{bc)} - \tfrac{2}{3}\eta_{(ab}\, \varepsilon_{c)pq}\, \partial^p \epsilon^q$.

Field equations : seemingly too many propagating d.o.f. since one finds that

- $\bar{\bar{K}}_{ab}(\phi) \approx 0$ where $\bar{\bar{K}}_{ab} := \eta^{cd}\eta^{ij} K_{ab|ci|dj}$ and $K_{ab|cd|ef} := 8\, \boxed{\begin{array}{c|c|c} a & c & e \\ \hline \partial_b & \partial_d & \partial_f \end{array}}$ curvature of ϕ_{abc} .

- $\tilde{R}^{ab|}{}_{ab} \approx 0$ where $\tilde{R}_{ab|}{}^{cd} = K_{ab|}{}^{cd}(h) - 2\left(\varepsilon_{abm}\partial^{[c}\Psi^{d]m} + \varepsilon^{cdm}\partial_{[a}\Psi_{b]m}\right)$, $\Psi^a{}_b := \partial_b \phi^a - \partial^c \phi^a{}_{bc}$

  $\hookrightarrow$ Riemann - like curvature for h_{ab} .

- $\partial_a \tilde{F}^{ab} \approx 0$ where $\tilde{F}_{ab} := F_{ab}(A) + \partial_{[a}\phi_{b]c}{}^c + \varepsilon_{abc}\left(\partial_d h^{cd} - \partial^c h\right)$ fieldstrength for A_a

However, combining the field equation, one finds the duality relations

$$\boxed{\begin{aligned} K_{ab|cd|ef}(\phi) &\approx -\frac{8}{21}\,\varepsilon_{efq}\,\partial^q\,\tilde{R}_{ab|cd} \\ \tilde{R}^{ab|}{}_{cd} &\approx \frac{7}{4}\,\varepsilon_{cdm}\,\partial^m\,\tilde{F}^{ab} \end{aligned}}$$

$$\Longrightarrow \quad K_{ab|cd|ef}(\phi) \approx -\frac{2}{3}\,\varepsilon_{cdp}\,\varepsilon_{efq}\,\partial^p\partial^q\,\tilde{F}_{ab} \quad ,$$

All the duality relations and E.o.M come out of the action.

③ Higher dualisations of linearised gravity in 4D and A_1^{+++}

3.1) The first higher dual of graviton in 4D

$$S[G_{a;bc}\,,\, D_{ab;}{}^{cd}] = \int d^4x \left(-\tfrac{1}{2}\, G_{a;bc}\, G^{a;bc} + \ldots + G_{a,bc}\, \partial_d D^{da;bc}\right)$$

Reproduces FP on one hand, and $S[D_{ab;}{}^{cd}]$ dual action on the other.

$$D_{ab;}{}^{cd} \sim \begin{array}{|c|}\hline a\\ \hline b\\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline c & d\\ \hline\end{array}$$

$$D_{ab;}{}^{cd} = X_{ab;}{}^{cd} + 4\, \delta^{(c}_{[a}\, Z_{b];}{}^{d)} \qquad GL(n)\text{-irreducible for } X_{ae;}{}^{eb} \equiv 0 \equiv Z_{a;}{}^{a}\,.$$

Hodge-dualise in 4D : $A^{ab,cd} := -\frac{1}{2}\, \varepsilon^{abij}\, X_{ab;}{}^{cd}\,, \quad \hat{Z}^{abc,d} := \varepsilon^{abce}\, Z_{e;}{}^{d}\,.$

$$A^{ab,cd} \sim \begin{array}{|c|c|c|}\hline a & c & d\\ \hline b \\ \cline{1-1}\end{array}\,, \qquad \hat{Z}^{abc,d} \sim \begin{array}{|c|c|}\hline a & d\\ \hline b \\ \cline{1-1} c \\ \cline{1-1}\end{array}$$

$h_{ab} \xrightarrow{\mathfrak{D}} \{ A^{ab,cd} , \hat{Z}^{abc,d} \}$ with action $S[A^{ab,cd}, \hat{Z}^{abc,d}]$ invariant under

$D_{ab;cd} \sim \ydiagram{1,1} \otimes \ydiagram{2} \sim \ydiagram{3,1} \oplus \ydiagram{2,1,1}$

- $\delta\ \ydiagram{3,1} = \ytableaushort{{}{}{},{\partial}} + \ytableaushort{{}{}{\partial},{}} + \ytableaushort{{\bullet}{\triangle}{\triangle},{\bullet}}$

  $\eta \sim \eta_{\triangle\triangle}$ and $\ytableaushort{{\bullet},{\bullet}} \rightsquigarrow \varepsilon^{ab\bullet\bullet}\, \partial_{[a} \epsilon_{b]}$.

  $\lambda_{abc} \sim \ydiagram{3}$, $m_{ab,c} \sim \ydiagram{2,1}$

- $A^{ab,cd} \rightsquigarrow$ generator $R_{ab,cd}$ at level 2 in the adjoint of A_1^{+++}

- $\hat{Z}^{abc,d} \rightsquigarrow$ in ℓ_2 representation of A_1^{+++} at level 1

- λ_{abc} & $m_{ab,c} \rightsquigarrow$ in ℓ_1 representation at level 2 .

- The field equations derived from $S[A_{ab,cd}, \tilde{Z}_{abc,d}]$ imply that

$$\mathrm{Tr}_{12}^{2}\, \hat{G}_{[3,2,2]} \approx 0 \approx \mathrm{Tr}_{23}\, \hat{G}_{[3,2,2]}$$

where $\hat{G}_{[3,2,2]} = d_1\, d_2\, d_3\, A_{[2,1,1]} + \cdots$ the curvature

- These are exactly of the form obtained in [A. Tumanov & P. West, M. Pettit & West, K. Glennon & P. West]

when studying the Nonlinear Realisation (NLR) of $A_1^{+++} \ltimes \ell_1$.

- By taking more and more dualisations $\mathcal{D}^N$,

one recognises many fields in adjoint & ℓ_2 rep. of A_1^{+++}

Second higher dualisation

explicitly here. The $GL(4)$-irreducible field content of the new field $D_{ab;}{}^{cd,ef}$ can be read off from its Hodge dual $\widetilde{D}^{ab;cd,ef} = \frac{1}{2}\,\varepsilon^{abij}\,D_{ij;}{}^{cd,ef}$ and the decomposition of its Young diagram:

$$\ydiagram{1,1} \otimes \ydiagram{3,1} \sim \ydiagram{4,2} \oplus \ydiagram{4,1,1} \oplus \ydiagram{3,2,1} \oplus \ydiagram{3,1,1,1}\,, \tag{3.51}$$

$$\Leftrightarrow \quad \mathbb{Y}[2] \otimes \mathbb{Y}[2,1,1] \sim \mathbb{Y}[2,2,1,1] \oplus \mathbb{Y}[3,1,1,1] \oplus \mathbb{Y}[3,2,1] \oplus \mathbb{Y}[4,1,1]\,, \tag{3.52}$$

$$\Leftrightarrow \quad \mathbb{Y}(1,1) \otimes \mathbb{Y}(3,1) \sim \mathbb{Y}(4,2) \oplus \mathbb{Y}(4,1,1) \oplus \mathbb{Y}(3,2,1) \oplus \mathbb{Y}(3,1,1,1)\,, \tag{3.53}$$

$$\Leftrightarrow \quad D \sim A \quad \widehat{Y} \quad \widehat{Z} \quad \widehat{W}\,. \tag{3.54}$$

3.2) A few words by a non-expert on NLR of A_1^{+++}

Dynkin diagram A_1^{+++}

- Computation of fundamental representations

↳ add a new node labelled $*$ to the Dynkin diagram and attach it to node i by a single link.

A generic root in the corresponding enlarged algebra $A_1^{+++(i)}$ given by

$$\alpha = m_* \alpha_* + \ell \, \alpha_4 + \sum_{j=1}^{3} m_j \alpha_j$$

Generators sorted by their Kac level ℓ.

It turns out that $\ell = \frac{1}{2}$(number of upper indices minus number of lower indices)

One presents the A_3 content of an A_1^{+++} rep. ℓ_i level by level w.r.t. ℓ

- The notion of level is preserved by commutators, so the set of roots with $m_* = 1$ forms a representation of A_1^{+++} which one can show is equivalent to the i^{th} fundamental representation denoted ℓ_i.

$$\alpha = m_* \alpha_* + \ell \alpha_4 + \sum_{j=1}^{3} m_j \alpha_j$$

- A generic $A_3 \simeq sl_4$ weight can be expressed as $\lambda = \sum_{i=1}^{3} p_i \lambda_i$ where λ_i is the i^{th} fundamental weight.

$$\lambda = [p_1, p_2, p_3] \sim \Upsilon[3, \dots, 3, 2, \dots, 2, 1, \dots, 1] = \Upsilon[3^{p_1}, 2^{p_1}, 1^{p_3}] .$$

- The relationship between the permitted A_3 Dynkin labels $(p_i)_{i=1,2,3}$ of λ and the Kac labels $(m_i)_{i=1,2,3}$ of the A_1^{+++} root α associated with λ is known.

- Generators at non-negative levels :

$$R^{\underline{\alpha}} = \{ K^a{}_b \; (0) \, , \; R^{(ab)} \; (1) \, , \; R^{a_1a_2,(b_1b_2)} \; (2) \, , \; R^{a_1a_2,b_1b_2,(c_1c_2)} \; (3) \, , \; R^{a_1a_2a_3,b_1b_2,(c_1c_2)} \; (3) \, , \dots \}$$

$$\boxed{[R^{\underline{\alpha}}, R^{\underline{\beta}}] = f^{\underline{\alpha}\underline{\beta}}{}_{\underline{\gamma}} \, R^{\underline{\gamma}}}$$

- Generators in the vector ℓ_1 representation :

$$L_A = \{ P_a \; (0) \, , \; Z^a \; (1) \, , \; Z^{a_1a_2a_3} \; (2) \, , \; Z^{a_1a_2,a_3} \; (2) \, , \dots \}$$

$$\boxed{[L_A, L_B] = 0} \, .$$

$$\boxed{\sigma(\xi) = g_L(z)\, g_A(\varphi(z))} \in A_1^{+++} \ltimes \ell_1$$

non-negative level $\quad \bullet\ g_A = \exp(A_{\underline{\alpha}}(z) R^{\underline{\alpha}}) = \cdots \exp(A_{ab,cd} R^{ab,cd}) \exp(A_{ab} R^{ab}) \exp(h_a{}^b K^a{}_b)$,

$\bullet\ g_L = \exp(z^A L_A) = \exp(x^a P_a) \exp(z_a Z^a) \exp(z_{abc} Z^{abc} + z_{ab,c} Z^{ab,c}) \ldots$

• The fields $A_{\underline{\alpha}}(z^A)$ depend on z^A coordinates.

G action $\quad \sigma \underset{G}{\longmapsto} g_0 \sigma \qquad g_0 \in G = A_1^{+++} \ltimes \ell_1 \quad$ rigid

G/H : $\quad \sigma \underset{H}{\longmapsto} \sigma h \qquad h(z^A)$ transformation used to set the coefficient of negative-level generators to zero in g_A

$h \in H = I_c(A_1^{+++})$

Cartan involution subalgebra $\subset A_1^{+++}$

$E_i \mapsto -F_i \ ,\ F_i \mapsto -E_i \ ,\ H_i \mapsto -H_i$

$$\boxed{\sigma(\xi) \underset{G}{\longmapsto} g_0\, \sigma(\xi) = \sigma(\xi')\, h(g_0, \xi)}$$

- Field equations invariant under $\sigma \mapsto g_0 \sigma$ and $\sigma \underset{H}{\mapsto} \sigma h$

Maurer-Cartan 1-form $\mathcal{V} := \sigma^{-1} d\sigma = \mathcal{V}_A + \mathcal{V}_L$,

$$\mathcal{V}_A = g_A^{-1} d g_A = dz^\pi G_{\pi,\underline{\alpha}} R^{\underline{\alpha}} \quad , \quad \mathcal{V}_L = g_A^{-1} (g_L^{-1} d g_L) g_A = g_A^{-1} (dz^B L_B) g_A = dz^\pi \underbrace{E_\pi{}^A}_{\text{"vierbein"}} L_A .$$

$e_\mu{}^a = (\exp(h))_\mu{}^a$.

- One derives a set of duality relations from which the E.o.M. are deduced.

- Gauge transformations $\delta A_{\underline{\alpha}} = C^{-1}_{\underline{\alpha},\beta} (D^{\beta})_E{}^F \partial_F \Lambda^E$ for the linearised theory

$C_{\underline{\alpha},\beta}$ Cartan-Killing metric of A_1^{+++} $\quad [R^{\underline{\alpha}}, L_A] = -(D^{\underline{\alpha}})_A{}^B L_B$

Preserve the linearised E.o.M.

- In [N.B., P.P. Cook, J. O'Connor and P. West],

a connection is made between the NLR and higher dualisation programmes.

3.3) Higher duals of graviton in 4D

- One shows, level by level n in $\mathcal{D}^n$, that off-shell dualisation produces a set of extra fields closely correlated with the ℓ_2 representation.

- At the n^{th} level of higher dualisation $\mathcal{D}^n$, counts fields that appear in the adjoint representation at level $n+1$ and in the ℓ_2 rep. at level n.

- In 4D, $\mathcal{D}^{n=0}$: $\quad C_{a|b} \sim \square \otimes \square \simeq \ydiagram{2} \oplus \ydiagram{1,1}$

dual graviton

adj. of A_1^{+++} level 1

2-form

in ℓ_2 rep. at level 0

In the standard dualisation of gravity, the dual graviton $\sim \mathbb{Y}[D-3,1]$ and an extra $(D-2)$-form that can be gauged away with the Hodge dual of Lorentz parameter λ_{ab}.

- At first $\mathcal{D}$ level in higher dualisation $\quad D_{ab;cd} \sim \ydiagram{1,1} \otimes \ydiagram{2} \sim \ydiagram{3,1} \oplus \ydiagram{2,1,1}$

$A_{ab,cd}$ in adjoint A_1^{+++}

level 2

in ℓ_2-rep. at level 1

Spectrum of fields in higher dualisation :

$$\yng(2) \xrightarrow{\mathscr{D}} 1 \times \yng(3,1) \oplus 1 \times \yng(2,1,1) \tag{3.110}$$

$$\yng(2) \xrightarrow{\mathscr{D}^2} 1 \times \yng(4,2) \oplus 1 \times \yng(4,1,1) \oplus 2 \times \yng(3,2,1) \oplus 2 \times \yng(3,1,1,1) \oplus 1 \times \yng(2,2,1,1) \oplus 1 \times \yng(2,2,2) \tag{3.111}$$

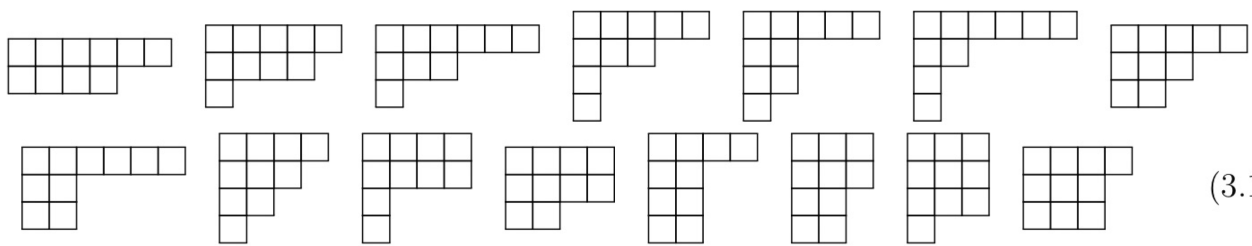
whereas the third level is visualised as

$$\begin{aligned} \yng(2) \xrightarrow{\mathscr{D}^3} \; & 1 \times \yng(5,3) \oplus 2 \times \yng(5,2,1) \oplus 3 \times \yng(4,3,1) \oplus 6 \times \yng(4,2,1,1) \oplus 3 \times \yng(3,3,1,1) \\ & \oplus 3 \times \yng(3,3,2) \oplus 1 \times \yng(5,1,1,1) \oplus 3 \times \yng(4,2,2) \oplus 6 \times \yng(3,2,2,1) \oplus 1 \times \yng(2,2,2,2) \end{aligned} \tag{3.112}$$

The irreducible fields at level four are given in column notation as

$$\begin{gathered}\mathbb{Y}[2,2,2,2,1,1]\,, \quad \mathbb{Y}[3,2,2,2,1]\,, \quad \mathbb{Y}[3,2,2,1,1,1]\,, \quad \mathbb{Y}[4,2,2,1,1]\,, \quad \mathbb{Y}[4,3,1,1,1], \\ \mathbb{Y}[4,2,1,1,1,1]\,, \quad \mathbb{Y}[3,3,2,1,1]\,, \quad \mathbb{Y}[3,3,1,1,1,1]\,, \quad \mathbb{Y}[4,3,2,1]\,, \quad \mathbb{Y}[4,2,2,2]\,, \\ \mathbb{Y}[3,3,2,2]\,, \quad \mathbb{Y}[4,4,1,1]\,, \quad \mathbb{Y}[4,4,2]\,, \quad \mathbb{Y}[4,3,3]\,, \quad \mathbb{Y}[3,3,3,1] \end{gathered} \tag{3.113}$$

with corresponding Young diagrams



(3.114)

and respective multiplicities in the order presented above

$$(1\,,\ 4\,,\ 3\,,\ 12\,,\ 12\,,\ 3\,,\ 8\,,\ 2\,,\ 24\,,\ 6\,,\ 6\,,\ 12\,,\ 10\,,\ 9\,,\ 6)\ . \tag{3.115}$$

Minimal off-shell dualisation

After dualising the $(n-1)^{\text{th}}$ higher dual graviton $A^{a^1[2],a^2[2],\dots,a^{n-1}[2],c_1,c_2}$, the set of independent fields will contain the n^{th} higher dual graviton

$$A^{(n)}_{[2,\dots,2,1,1]} \equiv A^{(n)} := A^{a^1[2],a^2[2],\dots,a^{n-1}[2],a^n[2],c_1,c_2} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|}\hline a_1^1 & a_1^2 & \cdots & a_1^n & c_1 & c_2 \\ \hline a_2^1 & a_2^2 & \cdots & a_2^n \\ \cline{1-4}\end{array} \tag{3.93}$$

which is a $GL(4)$-irreducible field of type $\mathbb{Y}[2,\dots,2,1,1] = \mathbb{Y}(n+2,2)$. The extra fields that are produced belong to one of the following families at the n^{th} level of higher dualisation:

$$\widehat{Y}^{(n)}_{[3,2,\dots,2,1,1,1]} \equiv \widehat{Y}^{(n)} := \widehat{Y}^{a[3],b^1[2],\dots,b^{n-2}[2],c_1,c_2,c_3} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|c|}\hline a & b & \cdots & b & c & c & c \\ \hline a & b & \cdots & b \\ \cline{1-4} a \\ \cline{1-1}\end{array} \tag{3.94}$$

$$\widehat{Z}^{(n)}_{[3,2,\dots,2,1]} \equiv \widehat{Z}^{(n)} := \widehat{Z}^{a[3],b^1[2],\dots,b^{n-1}[2],c} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|c|}\hline a & b & \cdots & b & b & c & c \\ \hline a & b & \cdots & b & b \\ \cline{1-5} a \\ \cline{1-1}\end{array} \tag{3.95}$$

$$\widehat{W}^{(n)}_{[4,2,\dots,2,1,1]} \equiv \widehat{W}^{(n)} := \widehat{W}^{a[4],b^1[2],\dots,b^{n-2}[2],c_1,c_2} \quad \sim \quad \begin{array}{|c|c|c|c|c|c|}\hline a & b & \cdots & b & c & c \\ \hline a & b & \cdots & b \\ \cline{1-4} a \\ \cline{1-1} a \\ \cline{1-1}\end{array} \tag{3.96}$$

Table 4: Extra fields from A_1^{+++} and off-shell dualisation.

label	A_3 weight	adj	ℓ_2	total	maximal off-shell	net
b	$[1,0,1]$	0	1	1	1	0
c_1	$[1,0,3]$	0	1	1	1	0
c_2	$[1,1,1]$	1	1	2	2	0
c_3	$[0,0,2]$	0	1	1	2	+1
c_4	$[0,1,0]$	0	1	1	1	0
c_5	$[2,0,0]$	0	0	0	1	+1
d_1	$[1,1,3]$	1	1	2	2	0
d_2	$[1,2,1]$	2	2	4	3	−1
d_3	$[0,1,2]$	1	4	5	6	+1
d_4	$[0,2,0]$	0	2	2	3	+1
d_5	$[2,1,0]$	1	1	2	3	+1
d_6	$[0,0,4]$	0	1	1	1	0
d_7	$[2,0,2]$	1	1	2	3	+1
d_8	$[1,0,1]$	1	2	3	6	+3
d_9	$[0,0,0]$	0	0	0	1	+1

$\hat{Z}^{abc,d}$

One can show that there is a dualisation scheme where the set of fields does not exceed the content of the adjoint and ℓ_2 rep. of A_1^{+++}.

At first level. Contact with Labastida formulation.

- $S[A_{ab,cd}\,,\,\hat{Z}^{abc,d}]$ obtained with its gauge transformations

$$A_{ab,cd} \longleftrightarrow \phi_{abc,d} \sim \ydiagram{3,1} \qquad\qquad \hat{Z}^{abc,d} \longleftrightarrow Z_{a;}{}^{d}$$

$$\delta\ \ydiagram{3,1} = \begin{ytableau} \ & \ & \ \\ \partial \end{ytableau} + \begin{ytableau} \ & \ & \partial \\ \ \end{ytableau} + \begin{ytableau} \bullet & \blacktriangle & \blacktriangle \\ \bullet \end{ytableau}$$

$\hookrightarrow$ " $\varepsilon^{\cdots\cdot}\ \eta^{\cdot\cdot}\ \partial_{\cdot}\epsilon_{\cdot}$ "

- $\delta\phi_{abc,d} = 3\,\partial_d\lambda_{abc} - 3\,\partial_{(a}\lambda_{bc)d} + 3\,\partial_{(a}\mu_{bc),d} - \frac{3}{2}\,\eta_{(ab}\,\varepsilon_{c)dij}\,\partial^i\epsilon^j$

$$\delta\ \begin{ytableau} a & d \\ b \\ c \end{ytableau} = \begin{ytableau} \ & \ \\ \ \\ \partial \end{ytableau} + \varepsilon^{abce}\,\partial^d\epsilon_e + \cdots$$

- Define $\quad U^{ab} := -\frac{1}{2}\,A^{ab,c}{}_{c} + \hat{Z}^{abc,}{}_{c}$

s.t. $\quad \delta U^{ab} = 2\,\partial^{[a}\tau^{b]} + \epsilon^{abcd}\,\partial_c\epsilon_d \quad , \quad \tau^a := \lambda^{ab}{}_b - \mu^b{}_{b,}{}^{a}\ .$

• Set $\tilde{\phi}^{abc,d} := \phi^{abc,d} + \frac{3}{4}\,\eta^{(ab}\,U^{c)d}$

$$\boxed{\delta\tilde{\Phi}_{abc,d} = 3\,\partial_d\tilde{\lambda}_{abc} - 3\,\partial_{(a}\tilde{\lambda}_{bc)d} + 3\,\partial_{(a}\tilde{\mu}_{bc),d} - \frac{3}{4}\,\eta_{(ab}\,\varepsilon_{c)dpq}\,\partial^p\epsilon^q}$$

• $\tilde{\lambda}_{abc} := \lambda_{abc} - \frac{1}{4}\,\eta_{(ab}\tau_{c)}$, $\tilde{\mu}_{ab,c} := \mu_{ab,c} + \frac{1}{6}\left(\eta_{ab}\tau_c - \eta_{c(a}\tau_{b)}\right)$

The new gauge parameters are s.t. $\tilde{\mu}^b{}_{b,a} \equiv \tilde{\lambda}_{ab}{}^b$, as in Labastida

• Independent fields $\{\tilde{\Phi}_{abc,d}\,,\,U_{ab}\,,\,Z_{(a;b)}\}$ $\longrightarrow$ only U_{ab} transforms with τ_a .

$\Longleftrightarrow$ U_{ab} enters $S[\tilde{\Phi}_{abc,d}\,,\,U_{ab}\,,\,Z_{(a;b)}]$ through $H_{abc} = 3\,\partial_{[a}U_{bc]}$.

$\Longrightarrow$ Dualise $U_{ab} \xleftrightarrow{4D}$ Scalar $\sigma(x)$. Combine $Z_{(a;b)}$ with $\sigma(x)$ $\longrightarrow$ $\tilde{A}_{ab}(x)$ traceful

$$\boxed{\delta\tilde{A}_{ab} = 2\,\partial_{(a}\epsilon_{b)} - \varepsilon_{cd(a}\,\partial^c\tilde{\mu}_{b)}{}^{d,e}}$$ entangled gauge transformations

Field equations of $S[\tilde{\Phi}_{abc,d}, \tilde{A}_{ab}]$ and duality

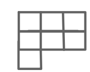
Gauge-invariant tensors

1) $K_{ma,nb} := 4\, \partial_{[m} \partial_{[n} \tilde{A}_{b]a]} + \text{``}\, \eta_{..}\, \partial.\partial \tilde{A}_{..} + \partial.\partial \tilde{\Phi}_{...,.}\, \varepsilon^{....} \,\text{''}$

where $-\frac{7}{16} K^{ab,}{}_{ab} = \Box \tilde{A}_a{}^a - \partial^a \partial^b \tilde{A}_{ab}$ Ricci scalar K.

↳ E.o.m. $-2 \frac{\delta S}{\delta \tilde{A}^{ab}} = K_{ab} - \frac{1}{2} \eta_{ab} K \approx 0$ $\Leftrightarrow$ Ricci flat

2) $G_{mn,pq,}{}^{d} := 4\, \varepsilon^{abcd}\, \partial_a \partial_{[m} \partial_{[p} \tilde{\Phi}_{q]n]b,c}$ + more

s.t. $\tilde{G}_{abc,mn,pq} := \varepsilon_{abcd}\, G_{mn,pq,}{}^{d}$ is 

Find Bianchi on-shell $\boxed{\partial_{[a} K_{bc],de} \approx 0}$

Field equations $\boxed{\begin{array}{l} K_{ab} \approx 0 \\ \\ \mathrm{Tr}^{2}_{12}\, \tilde{G}_{[3,2,2]} \approx 0 \end{array}}$

& consequence of $\dfrac{\delta S}{\delta \Phi^{abc,d}} \approx 0$

There is no doubling of d.o.f. because one has on-shell duality relation

$$\boxed{\tilde{G}_{a[3],b[2],c[2]} \approx -\varepsilon_{a[3]d}\, \partial^{d} K_{b[2],c[2]}} \iff \boxed{G_{mn,pq;}{}^{r} \approx -\partial^{r} K_{mn,pq}}$$

between the first higher dual graviton $\yng(3,1)$

and the (dual) graviton $\yng(2)$ in 4D

④ Conclusions and outlook

- Infinitely many off-shell covariant descriptions of linearised gravity
- Higher dualisation $\longleftrightarrow$ higher spins
- These are relevant to $A_1^{+++} \ltimes \ell_1$ Non-linear realisation of West et collaborators

Off-shell dualisation requires extra fields in ℓ_2 rep. not present in West et al.

- No multiplication of on-shell d.o.f. since (twisted) duality relation consequence of E.o.M

Questions:
- Interactions (given by NLR equations) ?
- Repackaging to Zabastida frame to higher levels.
- Relation to Bossard-Kleinschmidt-Sezgin quasi-Lagrangian