Appendix: To appear as supplementary material

A Binary relations

We use a standard vocabulary for binary relations. For the convenience of the reader and in order to avoid any misunderstanding, we detail our vocabulary here. A binary relation T on a set Z is a subset of $Z \times Z$. For $x, y \in Z$, as is usual, we will often write $x\ T\ y$ instead of $(x, y) \in T$.

Let T be a binary relation on Z. We define:

- the asymmetric part T^α of T as $x\ T^\alpha\ y \Leftrightarrow [x\ T\ y$ and $Not[y\ T\ x]]$,
- the symmetric part T^ι of T as $x\ T^\iota\ y \Leftrightarrow [x\ T\ y$ and $y\ T\ x]$,
- the symmetric complement T^σ of T as $x\ T^\sigma\ y \Leftrightarrow [Not[x\ T\ y]$ and $Not[y\ T\ x]]$,

for all $x, y \in Z$.

A binary relation T on Z is said to be:

(i) *reflexive* if $x\ T\ x$,

(ii) *irreflexive* if $Not[x\ T\ x]$,

(iii) *complete* if $x\ T\ y$ or $y\ T\ x$,

(iv) *symmetric* if $x\ T\ y$ implies $y\ T\ x$,

(v) *asymmetric* if $x\ T\ y$ implies $Not[y\ T\ x]$,

(vi) *antisymmetric* if $[x\ T\ y$ and $y\ T\ x] \Rightarrow x = y$,

(vii) *transitive* if $[x\ T\ y$ and $y\ T\ z] \Rightarrow x\ T\ z$,

(viii) *Ferrers* if $[x\ T\ y$ and $z\ T\ w] \Rightarrow [x\ T\ w$ or $z\ T\ y]$,

(ix) *semitransitive* if $[x\ T\ y$ and $y\ T\ z] \Rightarrow [x\ T\ w$ or $w\ T\ z]$,

for all $x, y, z, w \in Z$.

We list below a number of remarkable structures. A binary relation T on Z is said to be:

(i) a *weak order* (or *complete preorder*) if it is complete and transitive,

(ii) a *linear order* if it is an antisymmetric weak order,

(iii) a *semiorder* if it is reflexive, Ferrers and semitransitive,

(iv) a *strict semiorder* if it is irreflexive, Ferrers and semitransitive,

(v) an *equivalence* if it is reflexive, symmetric, and transitive,

(vi) a *partial order* if it is reflexive, antisymmetric and transitive.

Notice that a reflexive and Ferrers relation must be complete. Similarly an irreflexive and Ferrers relation must be asymmetric.

When T is an equivalence relation on Z, the set of equivalence classes of T on Z is denoted Z/T. A *partition* of Z is a collection of nonempty subsets of Z that are pairwise disjoint and such that the union of the elements in this collection is Z. It is clear that, when T is an equivalence relation on Z, Z/T is a partition of Z.

When T on Z is a semiorder, its asymmetric part T^α is irreflexive, Ferrers and semitransitive, i.e., a strict semiorder.

Any Ferrers and semitransitive T on Z (which includes semiorders and strict semiorders) induces a weak order T^{wo} on Z that is defined as follows:

$$a\ T^{wo}\ b \text{ if } \forall c \in Z, [b\ T\ c \Rightarrow a\ T\ c] \text{ and } [c\ T\ a \Rightarrow c\ T\ b]. \tag{7}$$

If T is a semiorder and V is its asymmetric part, it follows that $T^{wo} = V^{wo}$. The weak order induced by a semiorder is identical to the one induced by its asymmetric part.

Let T and V be two semiorders on Z such that $T \subseteq V$. We say that (T, V) is a *nested chain* of semiorders. Let T^{wo} (resp. V^{wo}) be the weak order on Z induced by T (resp. V). If a nested chain of semiorders $T \subseteq V$ is such that the relation $T^{wo} \cap V^{wo}$ is complete (and therefore is a weak order), we say that the nested chain of semiorders (T, V) is *homogeneous* (Doignon et al., 1988).

Finally, let us note that T is a semiorder on a finite set Z iff there are a real-valued function f on Z and a positive number $s > 0$ such that, for all $a, b \in Z$, $a\ T\ b \Leftrightarrow f(a) \geq f(b) - s$

B Sketch of the proof of Proposition 4

Suppose that $\text{Max}(T, B)$ is empty. Let $x \in B$. By hypothesis, x does not belong to $\text{Max}(T, B)$. This implies that there is $w_1 \in B$ such that $w_1\ T^\alpha\ x$. Clearly, this implies that w_1 is distinct from x, because T^α is irreflexive. But w_1 does not belong to $\text{Max}(T, B)$. This implies that there is $w_2 \in B$ such that $w_2\ T^\alpha\ w_1$. Clearly, this implies that w_2 is distinct from both w_1 and x. Continuing the reasoning leads to postulating the existence of a chain of elements w_i, $i \in \mathbb{N}^+$, that are all distinct (otherwise, the transitivity of T^α will lead to violate irreflexivity). This violates the finiteness of B. Hence, $\text{Max}(T, B)$ must be nonempty. The proof that

Min(T, B) must be nonempty is similar. The fact that, for all $x, y \in \text{Max}(T, B)$, we have $Not[x\ T^\alpha\ y]$ is clear from the definition of Max(T, B). The same is clearly true with Min(T, B).

Suppose now that $x \in B$ and there is no $y \in \text{Max}(T, B)$ such that $y\ T\ x$. If $x \in \text{Max}(T, B)$, the contradiction is established, because T is reflexive. Suppose, hence, that $x \notin \text{Max}(T, B)$. There is $w_1 \in B$ such that $w_1\ T^\alpha\ x$. But it is impossible that w_1 belongs to Max(T, B). This implies that there is $w_2 \in B$ such that $w_2\ T^\alpha\ w_1$. Because, T^α is transitive, it is impossible that $w_2 \in \text{Max}(T, B)$. Because T^α is asymmetric are transitive, it is impossible that w_2 is identical to w_1 or to x. Continuing the same reasoning, leads to postulating the existence of a chain of elements w_i, $i \in \mathbb{N}^+$, that are all distinct. This violates the finiteness of B. Hence, there exists $y \in \text{Max}(T, B)$ such that $y\ T\ x$. The proof that if $x \in B$, there is $z \in \text{Min}(T, B)$ such that $x\ T\ z$ is similar. □

C Example: Minimally acceptable alternatives for rational evaluations with one decimal digit

The ETri-nB model specified in Section 2.2 uses two profiles $p^1 = (8, 7, 5)$ and $p^2 = (5, 6, 8)$. The indifference, preference and veto thresholds are, respectively, $qt_i = 1$, $pt_i = 2$ and $vt_i = 4$, the same for all criteria $i = 1, 2, 3$. All criteria have the same weight $w_i = \frac{1}{3}$ and the cutting threshold $\lambda = .6$. We apply this model to the set X of alternatives whose evaluations are rational numbers with one decimal digit ranging in $[0, 10]$. The set of minimally acceptable alternatives is determined below.

For $x \in X$ to be acceptable, $c(x, p^1)$ or $c(x, p^2)$ has to be at least equal to $\lambda = .6$. We develop the consequences of this condition for p^1, the case of p^2 being similar. This condition entails that x_i must be strictly greater than $p^1_i - 2$ for at least two criteria. We distinguish two cases: Case 1: x_i is strictly greater than $p^1_i - 2$ on all three criteria; Case 2: x_i is strictly greater than $p^1_i - 2$ on exactly two criteria and less than this value on the third criterion.

C.1 Case 1

Let c_i be shorthand for $c_i(x, p^1)$, $i = 1, 2, 3$. If $x \in \mathcal{A}$, $c(x, p^1) = \sum_{i=1}^3 w_i c_i = 1/3 \sum_{i=1}^3 c_i \geq .6$. If x_i is strictly greater than $p^1_i - 2$ for all i, then we have $\sum_{i=1}^3 c_i \geq 1.8$ with $c_i > 0$ for all i. The alternative $y = (7, 6, 4)$ is the minimal one realizing $c(y, p^1) = 1$. With respect to $(7, 6, 4)$, we may decrease all coordinates by a total of 1.2 while remaining in $\mathcal{A}$. For instance, for $x = (6.5, 5.8, 3.5)$, we have $\sum_{i=1}^3 c_i = 1.8$ and x is minimally acceptable. There are actually $\binom{11}{2} = 55$ ordered partitions of 12 objects (12 tenths) in three nonempty subsets. Among them, 3 partitions have a class of cardinal 10, which we must exclude. Hence, there are 52 ways of decreasing each coordinate of $(7, 6, 4)$ by at least one tenth, for a total amount of 12/10 while yielding rational coordinates with one decimal digit, that are respectively strictly greater than $6, 5, 3$. To this we have to add the different ways of decreasing two of the coordinates of $(7, 6, 4)$ by a total amount of 12/10, while keeping unchanged the value of the third coordinate. There are $3 \times 7 = 21$ such alternatives. Hence there are $52 + 21 = 73$ minimally acceptable elements of this type for p^1 and 73 for p^2.

C.2 Case 2

The second type of minimally acceptable elements x satisfies $x_i > p_i^1 - 2$ for two values of i; the other coordinate does not satisfy this inequality. Assume that the latter coordinate is $i = 3$. The condition $c(x, p^1) \geq .6$ is only satisfied in the following 6 cases:

1. $(x_1, x_2) = (7, 6)$ and $x_3 < 3$; in such a case $c(x, p^1) = 2/3$;
2. $(x_1, x_2) = (6.9, 6)$ or $(7, 6.9)$ and $x_3 < 3$; in such a case $c(x, p^1) = 19/30$;
3. $(x_1, x_2) = (6.8, 6)$ or $(6.9, 5.9)$ or $(7, 5.8)$ and $x_3 < 3$; in such a case $c(x, p^1) = 6/10$;

Let us now compute in each case, the minimal value of x_3 such that $\sigma(x, p^1) \geq .6$.

1. If $x_3 = 1.6$, $d_3(1.6, 5) = 0.7 > 2/3 = c(x, p^1)$. We have $\frac{1-d_3(1.6,5)}{1-c(x,p^1} = \frac{3/10}{1/3} = 9/10$. Therefore $\sigma((7, 6, 1.6), p^1) = 2/3 \times 9/10 = .6$. Taking $x_3 < 1.6$ would lead to an unacceptable alternative. Hence $(7, 6, 1.6)$ is minimal in $\mathcal{A}$.
2. If $x_3 = 1.7$, $d_3(1.7, 5) = 0.65 > 19/30 = c(x, p^1)$. We have $\frac{1-d_3(1.7,5)}{1-c(x,p^1} = \frac{7/20}{11/30} = 21/22$. Therefore, for $(x_1, x_2) = (6.9, 6)$ or $(7, 6.9)$, $\sigma((x_1, x_2, 1.7), p^1) = 19/30 \times 21/22 \approx .6045 > .6$. Taking $x_3 < 1.7$ would lead to unacceptable alternatives. Hence $(6.9, 6, 1.7)$ and $(7, 5.9, 1.7)$ are minimal in $\mathcal{A}$.
3. If $x_3 = 1.8$, $d_3(1.8, 5) = 0.6 = c(x, p^1)$. We have $\sigma((x_1, x_2, 1.8), p^1) = c((x_1, x_2, 1.8), p^1) = .6$, for $(x_1, x_2) = (6.8, 6)$ or $(6.9, 5.9)$ or $(7, 5.8)$. Taking $x_3 < 1.8$ would lead to unacceptable alternatives. Hence $(6.8, 6, 1.8)$, $(6.9, 5.9, 1.8)$ and $(7, 5.8, 1.8)$ are minimal in $\mathcal{A}$.

There are thus 6 minimally acceptable alternatives with their third coordinate smaller than $p_3^1 - 3$. By symmetry, there are 6 minimally acceptable alternatives having a third coordinate smaller than $p_i^1 - 3$. Therefore, there are 18 minimally acceptable alternatives of the second type for p^1 and similarly for p^2.

Summing up, the total number of minimally acceptable alternatives is $2 \times (73 + 18) = 182$. None of these dominates another, as it can be easily verified.

D Example: A linear partition having only a unanimous representation in model (E)

The example has $n = 4$ and $X_1 = X_2 = X_3 = X_4 = \{0, 1, 2\}$. We let $\mathcal{A} = \{x \in X : \sum_{i=1}^{4} x_i \geq 6\}$. There are $3^4 = 81$ objects in X, 15 are in $\mathcal{A}$, while 66 are in $\mathcal{U}$.

Observe first that, on all attributes, we have $2 \succ_i 1 \succ_i 0$. Indeed, with $i = 1$, we have:

$$(2, 0, 2, 2) \in \mathcal{A}, \quad (1, 0, 2, 2) \in \mathcal{U},$$
$$(1, 1, 2, 2) \in \mathcal{A}, \quad (0, 1, 2, 2) \in \mathcal{U}.$$

The same relations clearly hold on all attributes since the problem is symmetric.

This partition clearly has a representation in Model (E) with $\mathcal{F} = \{N\}$ and a set of profiles consisting of all 10 objects in the class 6 (i.e., having a sum of components equal to 6). By construction, these 10 profiles are not linked by dominance (this is a representation in model (E^c)).

Our objective is to try obtaining a representation in Model (E) using a set $\mathcal{F}$ that is not reduced to $\{N\}$. Notice first that bringing the veto relations into play will not help us do so. Indeed, it is easy to check that if a representation exists in model (E), a representation exists in Model (E^c) (because whatever x_i, we can find a_{-i} such that $(x_i, a_{-i}) \in \mathcal{A}$). Hence, let us try to find a representation in Model (E^c).

This clearly excludes to take any object in the class 6 as a profile. Indeed, a family $\mathcal{F}$ that is not reduced to $\{N\}$ would then imply that some object in a class strictly lower than 6 belongs to $\mathcal{A}$, which is false. Hence, we must take as profiles objects belonging to the class 7 or 8.

Because profiles cannot dominate one another, if we take the object $(2, 2, 2, 2)$ as a profile, it must be the only one. We know that $(2, 2, 1, 1) \in \mathcal{A}$. Hence, we must have $\{1, 2\} \in \mathcal{F}$. This is contradictory. Indeed, since $\{1, 2\} \in \mathcal{F}$, we should have $(2, 2, 0, 0) \in \mathcal{A}$, a contradiction.

Hence the set of profiles must consist exclusively of objects belonging to the class 7.

Suppose that there is a unique profile, e.g., $(2, 2, 2, 1)$. It is clear that the set $\{1, 2, 3\}$ must be included in all elements of $\mathcal{F}$ (otherwise we would have an object in the class 5 belonging to $\mathcal{A}$). Because $(2, 2, 2, 0) \in \mathcal{A}$, it must be true that $\{1, 2, 3\}$ is an element of $\mathcal{F}$, which must therefore be equal to $\{\{1, 2, 3\}, \{1, 2, 3, 4\}\}$. This is contradictory since we know that $(0, 2, 2, 2) \in \mathcal{A}$. It is easy to see that, the problem being symmetric, it is therefore impossible to have a representation using a single profile from the class 7.

A similar reasoning can be made if we consider the cases of two or three profiles from the class 7 as profiles.

Suppose finally that we choose all four profiles from the class 7: $(1, 2, 2, 2)$, $(2, 1, 2, 2)$, $(2, 2, 1, 2)$, and $(2, 2, 2, 1)$. Using the same reasoning as above, the set $\mathcal{F}$ must contain the sets $\{2, 3, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 4\}$, and $\{1, 2, 3\}$, since $(0, 2, 2, 2)$, $(2, 0, 2, 2)$, $(2, 2, 0, 2)$ and $(2, 2, 2, 0)$ are all in $\mathcal{A}$. But this is contradictory since this would imply that $(0, 1, 2, 2) \in \mathcal{A}$ (since $(2, 1, 2, 2)$ is a profile and $\{2, 3, 4\} \in \mathcal{F}$).

Therefore, the only possible representation of this partition in Model (E) must use as profiles all 10 elements in the class 6 together with $\mathcal{F} = \{N\}$.

E An example of linear partition not representable in Model (Add)

Let $X = \prod_{i=1}^{4} X_i$, where $X_i = \{0, 1\}$. Let $\mathcal{A} = \{1100, 0011, 1110, 1101, 1011, 0111, 1111\}$ and $\mathcal{U}$ the complement of $\mathcal{A}$ in X. The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ respects the dominance relation determined by the natural order on X_i, for all i. This partition cannot be represented in Model (Add). Assuming the contrary would entail the following:

$$u_1(1) + u_2(1) + u_3(0) + u_4(0) > 0$$
$$u_1(0) + u_2(0) + u_3(1) + u_4(1) > 0.$$

This implies that $\sum_{i=1}^{4} u_i(1) + \sum_{i=1}^{4} u_i(0) > 0$. Since 1010 and 0101 belong yo $\mathcal{U}$, we should also a have:

$$u_1(1) + u_2(0) + u_3(1) + u_4(0) \leq 0$$
$$u_1(0) + u_2(1) + u_3(0) + u_4(1) \leq 0.$$

Therefore we must have $\sum_{i=1}^{4} u_i(1) + \sum_{i=1}^{4} u_i(0) \leq 0$, a contradiction.