Appendix: To appear as supplementary material

A Binary relations

We use a standard vocabulary for binary relations. For the convenience of the reader and in order to avoid any misunderstanding, we detail our vocabulary here. A binary relation \( T \) on a set \( Z \) is a subset of \( Z \times Z \). For \( x, y \in Z \), as is usual, we will often write \( x T y \) instead of \((x, y) \in T\).

Let \( T \) be a binary relation on \( Z \). We define:

- the asymmetric part \( T^\alpha \) of \( T \) as \( x T^\alpha y \Leftrightarrow [x T y \text{ and } \neg (y T x)] \),
- the symmetric part \( T^\iota \) of \( T \) as \( x T^\iota y \Leftrightarrow [x T y \text{ and } y T x] \),
- the symmetric complement \( T^\sigma \) of \( T \) as \( x T^\sigma y \Leftrightarrow [\neg (x T y) \text{ and } \neg (y T x)] \),

for all \( x, y \in Z \).

A binary relation \( T \) on \( Z \) is said to be:

(i) reflexive if \( x T x \),
(ii) irreflexive if \( \neg (x T x) \),
(iii) complete if \( x T y \) or \( y T x \),
(iv) symmetric if \( x T y \) implies \( y T x \),
(v) asymmetric if \( x T y \) implies \( \neg (y T x) \),
(vi) antisymmetric if \( [x T y \text{ and } y T x] \Rightarrow x = y \),
(vii) transitive if \( [x T y \text{ and } y T z] \Rightarrow x T z \),
(viii) Ferrers if \( [x T y \text{ and } z T w] \Rightarrow [x T w \text{ or } z T y] \),
(ix) semitransitive if \( [x T y \text{ and } y T z] \Rightarrow [x T w \text{ or } w T z] \),

for all \( x, y, z, w \in Z \).

We list below a number of remarkable structures. A binary relation \( T \) on \( Z \) is said to be:

(i) a weak order (or complete preorder) if it is complete and transitive,
(ii) a linear order if it is an antisymmetric weak order,
(iii) a semiorder if it is reflexive, Ferrers and semitransitive,
(iv) a strict semiorder if it is irreflexive, Ferrers and semitransitive,

(v) an equivalence if it is reflexive, symmetric, and transitive,

(vi) a partial order if it is reflexive, antisymmetric and transitive.

Notice that a reflexive and Ferrers relation must be complete. Similarly an ir-
reflexive and Ferrers relation must be asymmetric.

When $T$ is an equivalence relation on $Z$, the set of equivalence classes of $T$ on
$Z$ is denoted $Z/T$. A partition of $Z$ is a collection of nonempty subsets of $Z$ that
are pairwise disjoint and such that the union of the elements in this collection is
$Z$. It is clear that, when $T$ is an equivalence relation on $Z$, $Z/T$ is a partition of
$Z$.

When $T$ on $Z$ is a semiorder, its asymmetric part $T^\alpha$ is irreflexive, Ferrers and
semitransitive, i.e., a strict semiorder.

Any Ferrers and semitransitive $T$ on $Z$ (which includes semiorders and strict
semiorders) induces a weak order $T^{wo}$ on $Z$ that is defined as follows:

\[ a \ T^{wo} \ b \text{ if } \forall c \in Z, [b \ T c \Rightarrow a \ T c] \text{ and } [c \ T a \Rightarrow c \ T b]. \]  

If $T$ is a semiorder and $V$ is its asymmetric part, it follows that $T^{wo} = V^{wo}$. The
weak order induced by a semiorder is identical to the one induced by its asymmetric
part.

Let $T$ and $V$ be two semiorders on $Z$ such that $T \subseteq V$. We say that $(T, V)$ is
a nested chain of semiorders. Let $T^{wo}$ (resp. $V^{wo}$) be the weak order on $Z$ induced
by $T$ (resp. $V$). If a nested chain of semiorders $T \subseteq V$ is such that the relation
$T^{wo} \cap V^{wo}$ is complete (and therefore is a weak order), we say that the nested chain
of semiorders $(T, V)$ is homogeneous (Doignon et al., 1988).

Finally, let us note that $T$ is a semiorder on a finite set $Z$ iff there are a real-
valued function $f$ on $Z$ and a positive number $s > 0$ such that, for all $a, b \in Z$,
\[ a \ T b \iff f(a) \geq f(b) - s. \]

**B Sketch of the proof of Proposition 4**

Suppose that Max($T, B$) is empty. Let $x \in B$. By hypothesis, $x$ does not belong
to Max($T, B$). This implies that there is $w_1 \in B$ such that $w_1 \ T^\alpha x$. Clearly, this
implies that $w_1$ is distinct from $x$, because $T^\alpha$ is irreflexive. But $w_1$ does not belong
to Max($T, B$). This implies that there is $w_2 \in B$ such that $w_2 \ T^\alpha w_1$. Clearly,
this implies that $w_2$ is distinct from both $w_1$ and $x$. Continuing the reasoning
leads to postulating the existence of a chain of elements $w_i, i \in \mathbb{N}^+$, that are all
distinct (otherwise, the transitivity of $T^\alpha$ will lead to violate irreflexivity). This
violates the finiteness of $B$. Hence, Max($T, B$) must be nonempty. The proof that
Min$(T, B)$ must be nonempty is similar. The fact that, for all $x, y \in \text{Max}(T, B)$, we have $\text{Not}[x T^\alpha y]$ is clear from the definition of $\text{Max}(T, B)$. The same is clearly true with $\text{Min}(T, B)$.

Suppose now that $x \in B$ and there is no $y \in \text{Max}(T, B)$ such that $y T x$. If $x \in \text{Max}(T, B)$, the contradiction is established, because $T$ is reflexive. Suppose, hence, that $x \notin \text{Max}(T, B)$. There is $w_1 \in B$ such that $w_1 T^\alpha x$. But it is impossible that $w_1$ belongs to $\text{Max}(T, B)$. This implies that there is $w_2 \in B$ such that $w_2 T^\alpha w_1$. Because, $T^\alpha$ is transitive, it is impossible that $w_2 \in \text{Max}(T, B)$. Because $T^\alpha$ is asymmetric are transitive, it is impossible that $w_2$ is identical to $w_1$ or to $x$. Continuing the same reasoning, leads to postulating the existence of a chain of elements $w_i, i \in \mathbb{N}^+$, that are all distinct. This violates the finiteness of $B$. Hence, there exists $y \in \text{Max}(T, B)$ such that $y T x$. The proof that if $x \in B$, there is $z \in \text{Min}(T, B)$ such that $x T z$ is similar.

\[ \square \]

C Example: Minimally acceptable alternatives for rational evaluations with one decimal digit

The ETr1-nB model specified in Section 2.2 uses two profiles $p^1 = (8, 7, 5)$ and $p^2 = (5, 6, 8)$. The indifference, preference and veto thresholds are, respectively, $q_{t_i} = 1$, $p_{t_i} = 2$ and $v_{t_i} = 4$, the same for all criteria $i = 1, 2, 3$. All criteria have the same weight $w_i = \frac{1}{3}$ and the cutting threshold $\lambda = .6$. We apply this model to the set $X$ of alternatives whose evaluations are rational numbers with one decimal digit ranging in $[0, 10]$. The set of minimally acceptable alternatives is determined below.

For $x \in X$ to be acceptable, $c(x, p^1)$ or $c(x, p^2)$ has to be at least equal to $\lambda = .6$. We develop the consequences of this condition for $p^1$, the case of $p^2$ being similar. This condition entails that $x_i$ must be strictly greater than $p^1_i - 2$ for at least two criteria. We distinguish two cases: Case 1: $x_i$ is strictly greater than $p^1_i - 2$ on all three criteria; Case 2: $x_i$ is strictly greater than $p^1_i - 2$ on exactly two criteria and less than this value on the third criterion.

C.1 Case 1

Let $c_i$ be shorthand for $c_i(x, p^1)$, $i = 1, 2, 3$. If $x \in \mathcal{A}$, $c(x, p^1) = \sum_{i=1}^{3} w_i c_i = 1/3 \sum_{i=1}^{3} c_i \geq .6$. If $x_i$ is strictly greater than $p^1_i - 2$ for all $i$, then we have $\sum_{i=1}^{3} c_i \geq 1.8$ with $c_i > 0$ for all $i$. The alternative $y = (7, 6, 4)$ is the minimal one realizing $c(y, p^1) = 1$. With respect to $(7, 6, 4)$, we may decrease all coordinates by a total of 1.2 while remaining in $\mathcal{A}$. For instance, for $x = (6.5, 5.8, 3.5)$, we have $\sum_{i=1}^{3} c_i = 1.8$ and $x$ is minimally acceptable. There are actually $\binom{11}{2} = 55$ ordered partitions of 12 objects (12 tenths) in three nonempty subsets. Among them, 3
partitions have a class of cardinal 10, which we must exclude. Hence, there are 52 ways of decreasing each coordinate of \((7, 6, 4)\) by at least one tenth, for a total amount of \(12/10\) while yielding rational coordinates with one decimal digit, that are respectively strictly greater than 6, 5, 3. To this we have to add the different ways of decreasing two of the coordinates of \((7, 6, 4)\) by a total amount of \(12/10\), while keeping unchanged the value of the third coordinate. There are \(3 \times 7 = 21\) such alternatives. Hence there are \(52 + 21 = 73\) minimally acceptable elements of this type for \(p^1\) and 73 for \(p^2\).

### C.2 Case 2

The second type of minimally acceptable elements \(x\) satisfies \(x_i > p_i^1 - 2\) for two values of \(i\); the other coordinate does not satisfy this inequality. Assume that the latter coordinate is \(i = 3\). The condition \(c(x, p^1) \geq .6\) is only satisfied in the following 6 cases:

1. \((x_1, x_2) = (7, 6)\) and \(x_3 < 3\); in such a case \(c(x, p^1) = 2/3\);
2. \((x_1, x_2) = (6.9, 6)\) or \((7, 6.9)\) and \(x_3 < 3\); in such a case \(c(x, p^1) = 19/30\);
3. \((x_1, x_2) = (6.8, 6)\) or \((6.9, 5.9)\) or \((7, 5.8)\) and \(x_3 < 3\); in such a case \(c(x, p^1) = 6/10\);

Let us now compute in each case, the minimal value of \(x_3\) such that \(\sigma(x, p^1) \geq .6\).

1. If \(x_3 = 1.6\), \(d_3(1.6, 5) = 0.7 > 2/3 = c(x, p^1)\). We have \(1 - d_3(1.6, 5) = \frac{3}{10}\). Therefore \(\sigma((7, 6, 1.6), p^1) = 2/3 \times 9/10 = .6\). Taking \(x_3 < 1.6\) would lead to an unacceptable alternative. Hence \((7, 6, 1.6)\) is minimal in \(\mathcal{A}\).

2. If \(x_3 = 1.7\), \(d_3(1.7, 5) = 0.65 > 19/30 = c(x, p^1)\). We have \(1 - d_3(1.7, 5) = \frac{7/20}{11/30} = 21/22\). Therefore, for \((x_1, x_2) = (6.9, 6)\) or \((7, 6.9)\), \(\sigma((x_1, x_2, 1.7), p^1) = 19/30 \times 21/22 \approx .6045 > .6\). Taking \(x_3 < 1.7\) would lead to unacceptable alternatives. Hence \((6.9, 6, 1.7)\) and \((7, 5.9, 1.7)\) are minimal in \(\mathcal{A}\).

3. If \(x_3 = 1.8\), \(d_3(1.8, 5) = 0.6 = c(x, p^1)\). We have \(\sigma((x_1, x_2, 1.8), p^1) = c((x_1, x_2, 1.8), p^1) = .6\), for \((x_1, x_2) = (6.8, 6)\) or \((6.9, 5.9)\) or \((7, 5.8)\). Taking \(x_3 < 1.8\) would lead to unacceptable alternatives. Hence \((6.8, 6, 1.8)\), \((6.9, 5.9, 1.8)\) and \((7, 5.8, 1.8)\) are minimal in \(\mathcal{A}\).

There are thus 6 minimally acceptable alternatives with their third coordinate smaller than \(p^1_3 - 3\). By symmetry, there are 6 minimally acceptable alternatives having a third coordinate smaller than \(p^1_3 - 3\). Therefore, there are 18 minimally acceptable alternatives of the second type for \(p^1\) and similarly for \(p^2\).

Summing up, the total number of minimally acceptable alternatives is \(2 \times (73 + 18) = 182\). None of these dominates another, as it can be easily verified.
D Example: A linear partition having only a unanimous representation in model \((E)\)

The example has \(n = 4\) and \(X_1 = X_2 = X_3 = X_4 = \{0, 1, 2\}\). We let \(\mathcal{A} = \{x \in X : \sum_{i=1}^4 x_i \geq 6\}\). There are \(3^4 = 81\) objects in \(X\), 15 are in \(\mathcal{A}\), while 66 are in \(U\).

Observe first that, on all attributes, we have \(2 \succ_i 1 \succ_i 0\). Indeed, with \(i = 1\), we have:

\[
(2, 0, 2, 2) \in \mathcal{A}, \quad (1, 0, 2, 2) \in U,
\]

\[
(1, 1, 2, 2) \in \mathcal{A}, \quad (0, 1, 2, 2) \in U.
\]

The same relations clearly hold on all attributes since the problem is symmetric.

This partition clearly has a representation in Model \((E)\) with \(F = \{N\}\) and a set of profiles consisting of all 10 objects in the class 6 (i.e., having a sum of components equal to 6). By construction, these 10 profiles are not linked by dominance (this is a representation in model \((E^c)\)).

Our objective is to try obtaining a representation in Model \((E)\) using a set \(F\) that is not reduced to \(\{N\}\). Notice first that bringing the veto relations into play will not help us do so. Indeed, it is easy to check that if a representation exists in model \((E)\), a representation exists in Model \((E^c)\) (because whatever \(x_i\), we can find \(a_{-i}\) such that \((x_i, a_{-i}) \in \mathcal{A}\)). Hence, let us try to find a representation in Model \((E^c)\).

This clearly excludes to take any object in the class 6 as a profile. Indeed, a family \(F\) that is not reduced to \(\{N\}\) would then imply that some object in a class strictly lower than 6 belongs to \(\mathcal{A}\), which is false. Hence, we must take as profiles objects belonging to the class 7 or 8.

Because profiles cannot dominate one another, if we take the object \((2, 2, 2, 2)\) as a profile, it must be the only one. We know that \((2, 2, 1, 1) \in \mathcal{A}\). Hence, we must have \(\{1, 2\} \in F\). This is contradictory. Indeed, since \(\{1, 2\} \in F\), we should have \((2, 2, 0, 0) \in \mathcal{A}\), a contradiction.

Hence the set of profiles must consist exclusively of objects belonging to the class 7.

Suppose that there is a unique profile, e.g., \((2, 2, 2, 1)\). It is clear that the set \(\{1, 2, 3\}\) must be included in all elements of \(F\) (otherwise we would have an object in the class 5 belonging to \(\mathcal{A}\)). Because \((2, 2, 2, 0) \in \mathcal{A}\), it must be true that \(\{1, 2, 3\}\) is an element of \(F\), which must therefore be equal to \(\{\{1, 2, 3\}, \{1, 2, 3, 4\}\}\). This is contradictory since we know that \((0, 2, 2, 2) \in \mathcal{A}\). It is easy to see that, the problem being symmetric, it is therefore impossible to have a representation using a single profile from the class 7.

A similar reasoning can be made if we consider the cases of two or three profiles from the class 7 as profiles.
Suppose finally that we choose all four profiles from the class 7: (1, 2, 2, 2), (2, 1, 2, 2), (2, 2, 1, 2), and (2, 2, 2, 1). Using the same reasoning as above, the set $F$ must contain the sets $\{2, 3, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 4\}$, and $\{1, 2, 3\}$, since $(0, 2, 2, 2)$, $(2, 0, 2, 2)$, $(2, 2, 0, 2)$ and $(2, 2, 2, 0)$ are all in $A$. But this is contradictory since this would imply that $(0, 1, 2, 2) \in A$ (since $(2, 1, 2, 2)$ is a profile and $\{2, 3, 4\} \in F$).

Therefore, the only possible representation of this partition in Model $(E)$ must use all 10 elements of the class 6 together with $F = \{N\}$.

### E  An example of linear partition not representable in Model (Add)

Let $X = \prod_{i=1}^{4} X_i$, where $X_i = \{0, 1\}$. Let $\mathcal{A} = \{1100, 0011, 1110, 1011, 0111, 1111\}$ and $\mathcal{U}$ the complement of $\mathcal{A}$ in $X$. The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ respects the dominance relation determined by the natural order on $X_i$, for all $i$. This partition cannot be represented in Model (Add). Assuming the contrary would entail the following:

\[
\begin{align*}
&u_1(1) + u_2(1) + u_3(0) + u_4(0) > 0 \\
&u_1(0) + u_2(0) + u_3(1) + u_4(1) > 0.
\end{align*}
\]

This implies that $\sum_{i=1}^{4} u_i(1) + \sum_{i=1}^{4} u_i(0) > 0$. Since 1010 and 0101 belong yo $\mathcal{U}$, we should also have:

\[
\begin{align*}
&u_1(1) + u_2(0) + u_3(1) + u_4(0) \leq 0 \\
&u_1(0) + u_2(1) + u_3(0) + u_4(1) \leq 0.
\end{align*}
\]

Therefore we must have $\sum_{i=1}^{4} u_i(1) + \sum_{i=1}^{4} u_i(0) \leq 0$, a contradiction.