# Multiple criteria sorting and maximal antichains

**Denis Bouyssou**[1] and **Thierry Marchant**[2] and **Marc Pirlot**[3]

**Abstract.** We analyze the expressivity difference between a very general multiple criteria sorting model and its dual version. The analysis amounts to assess the share of maximal antichains in the set of all antichains of a product of linear orders.

## 1 Introduction

Multiple Criteria Sorting (MCS) methods involve rules for assigning objects into ordered categories. A number of such methods, starting with ELECTRE TRI, assign objects by comparing them to lower limiting profiles of the categories. There are two main variants of such rules. One, referred to as pessimistic or pseudo-conjunctive, the other as optimistic or pseudo-disjunctive. These rules are dual of one another in the way they compare objects to category limits.

This work aims to analyze the duality between these rules in a general setting. In particular, we are interested in determining whether each partition determined by a type of rule can also be obtained by the other. If not, we want to tell which one is more general and by how much.

## 2 The framework

The set of objects to be sorted are represented by the elements of the Cartesian product $X = \Pi_{i=1}^n X_i$, where $X_i$ is the scale of criterion $i \in N = \{1, \ldots, n\}$. An element $x \in X$ is thus a $n$-tuple $(x_1, \ldots, x_n)$.

We assume that there is a preference order $\geq_i$ on the scale of each criterion $X_i$. We assume further that this relation is a linear order (reflexive, antisymmetric and transitive relation); its asymmetric part is a total order denoted $>_i$.

The set $X$ is thus endowed with a partial order $\geq$ (a reflexive, antisymmetric and transitive relation), which is the product of the linear orders $\geq_i$. In other words, for all $x, y \in X$, we have $x \geq y$ iff $x_i \geq_i y_i$ for all $i \in N$. We call this order the *dominance* relation in the sequel and we say $x$ dominates $y$ whenever $x \geq y$.

For simplicity we consider sorting the objects of $X$ in two categories, the subset $\mathcal{A}$ of "acceptable" objects and the subset $\mathcal{U}$ of "unacceptable" ones. These subsets form the two classes of a bipartition $\langle \mathcal{A}, \mathcal{U} \rangle$ of $X$. One of the categories may be empty.

### 2.1 MCS models

In multiple criteria sorting, it is generally assumed that the categories are ordered and respect the dominance relation $\geq$, i.e., if an object dominates another that is in the "acceptable" category, then this object is also acceptable.

[1] CNRS-LAMSADE, Université Paris-Dauphine, PSL Research University, email: denis.bouyssou@dauphine.fr
[2] Ghent University, Belgium, email: thierry.marchant@ugent.be
[3] Université de Mons, Belgium, email: marc.pirlot@umons.ac.be

**Definition 1** (Monotone bipartitions)**.** The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ of $X$ respects the dominance relation if, for all $x, y \in X$, $x \geq y$ and $y \in \mathcal{A} \Rightarrow x \in \mathcal{A}$. We call such bipartitions *monotone*

It is straightforward that this definition is equivalent to saying that $x \geq y$ and $x \in \mathcal{U} \Rightarrow y \in \mathcal{U}$.

Examples of models determining a monotone bipartition are:

- the Additive Value Function model [8]: $x \in \mathcal{A}$ iff $\sum_{i=1}^n u_i(x_i) \geq \lambda$, where the marginal value functions $u_i : X_i \to \mathbb{R}$ are nondecreasing w.r.t. $\geq_i$ and $\lambda$ is a threshold;
- the Non Compensatory Sorting (NCS) model [1, 2]: $x \in \mathcal{A}$ iff $\{i \in N : x_i \geq_i b_i\} \in \mathcal{F}$, where $b = (b_1, \ldots, b_n$ is a "lower limiting profile" of category $\mathcal{A}$ and $\mathcal{F}$ is the set of "sufficient coalitions" of criteria, which is an upset of $2^N$;
- the Majority Rule Sorting model (MR-Sort) [9, 12]: a particular case of the NCS model in which the sufficient coalitions $\mathcal{F}$ are determined by criteria weights $w_i$ and a threshold $\lambda$, i.e., $F \in \mathcal{F}$ iff $\sum_{i \in F} w_i \geq \lambda$.

The latter two models are idealizations of ELECTRE TRI [13, 11]. The expressivity of these models is limited: not all monotone bipartitions $\langle \mathcal{A}, \mathcal{U} \rangle$ can be represented in these models. In contrast, the ELECTRE TRI-nB model [6], in which an unbounded number of lower limiting profiles can be used to define $\mathcal{A}$, allows to represent any monotone bipartition [5].

### 2.2 Antichains

The objects in $X$ are partially ordered by the dominance relation $\mathcal{A}$. In case $\langle \mathcal{A}, \mathcal{U} \rangle$ is a monotone bipartition, the set $\mathcal{A}_*$ of minimal elements in $\mathcal{A}$ w.r.t. $\geq$ has the following properties:

- for all object $y \in \mathcal{A}$, there is an object $x \in \mathcal{A}_*$ such that $y \geq x$;
- if $x \in \mathcal{A}_*$, decreasing the evaluation $x_i$ of $x$ on any criterion $i$ leads to an unacceptable object belonging to $\mathcal{U}$;
- all elements in $\mathcal{A}_*$ are incomparable in terms of $\geq$, i.e., for all $x, y \in \mathcal{A}_*$, we have neither $x \geq y$ nor $y \geq x$.

The latter property means that $\mathcal{A}_*$ is an *antichain* in $(X, \geq)$.

For any antichain in $X$ we may thus define the following model.

**Definition 2** (Unanimous model U)**.** The monotone bipartition $\langle \mathcal{A}, \mathcal{U} \rangle$ of $(X, \geq)$ is representable in the *unanimous* model U if there is a set $\mathcal{P}$ of objects in $X$ which form an antichain and is such that $x \in \mathcal{A}$ iff there is $p \in \mathcal{P}$ with $x \geq p$.

In the unanimous model, to be acceptable, an object has to dominate one of the objects in $\mathcal{P}$. Clearly, any monotone bipartition is representable in model U. Therefore, model U is equivalent to ELECTRE TRI-nB, in the sense that any monotone bipartition – and only monotone bipartitions – can be represented in both models.

## 3 A dual of the unanimous model

Instead of defining the elements of $\mathcal{A}$ as these that dominate an element of the antichain $\mathcal{P}$, it is tempting to try also to define the elements of $\mathcal{U}$ as those dominated by an element of an antichain $\mathcal{P}$. Consider thus the following model that can be seen as a dual of model U.

**Definition 3** ("Dual" unanimous model V)**.** The monotone bipartition $\langle \mathcal{A}, \mathcal{U} \rangle$ of $(X, \geq)$ is representable in the model V if there is a set $\mathcal{P}$ of objects in $X$ which form an antichain and is such that $x \in \mathcal{U}$ iff there is $p \in \mathcal{P}$ with $p > x$.

Note that the elements of $\mathcal{P}$ in model V do not belong to $\mathcal{U}$; they belong to $\mathcal{A}$. Although any antichain $\mathcal{P}$ can be used to define a model V, the set of minimal elements $\mathcal{A}_*$ in $\mathcal{A}$ are not necessarily the elements of $\mathcal{P}$. We have the following result.

**Proposition 1.** *The monotone bipartition $\langle \mathcal{A}, \mathcal{U} \rangle$ is representable in model V iff $\mathcal{A}_*$ is a* maximal *antichain of $X$.*

A maximal antichain is an antichain that is not included in a larger antichain. Proposition 1 shows that model V is less general than model U. Not all monotone bipartitions $\langle \mathcal{A}, \mathcal{U} \rangle$ can be represented in model V. Only those for which $\mathcal{A}_*$ is a maximal antichain can. The apparently dual definitions of models U and V are not equivalent. This imperfect duality is related to the complex relationship between the pessimistic and optimistic assignment rules observed in ELECTRE TRI (see [3]).

## 4 Maximal antichains

In order to explore the gap between models U and V, we investigated two issues about maximal antichains. How is the number of maximal antichains as compared to the number of antichains ? How can we list all maximal antichains in a product of linearly ordered finite sets ?

### 4.1 Counting maximal antichains

We have only very partial results. We consider an homogenous product poset $X = [m]^n$ where $n$ is the number of criteria, $m$ is the number of distinct levels on the scales $X_i$ of all criteria $i$ and $[m] = \{1, \ldots, m\}$. The order $\geq_i$ on $X_i = [m]$ is the natural order on the integer interval $[m]$.

Let $D(m, n)$ (resp. $D_V(m, n)$) denote the number of antichains (resp. maximal antichains) in $X = [m]^n$.

**Case** $m = 2$**.** In this case, $X = \{0, 1\}^n$ and $D(2, n)$ is the sequence of Dedekind numbers (A000372 in [10]) and $D_V(2, n)$ is sequence A326358 in [10]. Table 1 shows these sequences for $n = 1$ to 7, as well as their ratio, which quickly decreases with $n$.

| $n$ | $D_V(2,n)$ | $D(2,n)$ | $D_V(2,n)/D(2,n)$ |
|---|---|---|---|
| 1 | 2 | 3 | 0.6666667 |
| 2 | 3 | 6 | 0.5 |
| 3 | 7 | 20 | 0.35 |
| 4 | 29 | 168 | 0.172619 |
| 5 | 376 | 7581 | 0.04959768 |
| 6 | 31746 | 7828354 | 0.004055259 |
| 7 | 123805914 | 2414682040998 | 0.00005127214 |

**Table 1.** Number of maximal antichains ($D_V(2, n)$), number of antichains ($D(2, n)$) and ratio of these numbers in $[2]^n$ for $n \in [7]$.

**Case** $n = 2$**.** In this case $X = [m] \times [m]$. $D(m, 2)$ (resp. $D_V(m, 2)$) is sequence A000984 (resp. A171155) in [10]. Their ratio decreases more slowly. For $m = 5$, it is about 32%; for $m = 10$, it is around 14.5%; it falls to around 6% for $m = 15$.

**Further results.** Computing $D(m, n)$ and $D_V(m, n)$ for $m$ and $n \neq 2$ is very difficult even for small values of $m, n$. Using formula (4) in [4] and the software system Macaulay 2 [7], we managed to compute the ratios $D_V(3, 3)/D(3, 3) \approx 14.7\%$ and $D_V(4, 3)/D(4, 3) \approx 4.6\%$. This seems to indicate that the share of maximal antichains in the set of antichains quickly decreases with the number of distinct levels in criteria scales.

### 4.2 Listing maximal antichains

We have elaborated an exact algorithm for listing the maximal antichains in $X = [m]^n$. We used it to list for instance the 144 maximal antichains in $[3]^3$. Having at disposal the list of maximal antichains for $[m]^n$ is useful for several purposes. It would allow to sample V models in view of simulation experiments aiming, for instance, to estimate the share of V models that can be represented in other sorting models (NCS, MR-Sort, additive value function, etc).

It cannot be expected however to list the maximal antichains of $[m]^n$ even for moderate values of $n$. It can be shown that the number $D_V(2, n)$ of maximal antichains in $[2]^n$ is at least the number of antichains in $[2]^{n-1}$. The latter is Dedekind number $D(2, n-1)$, growing so fast that they are known only up to $n = 10$ (see Table 1).

## 5 Conclusion

This paper investigates the share of *maximal* antichains in the set of antichains of a product of linear orders. It aims at shedding some light on the relationship between two simple sorting models, which in turn may help to better understand the pessimistic and optimistic assignment rules used in ELECTRE TRI. The latter is the focus of our current work which tries to analyze the version of ELECTRE TRI-nB that uses the optimistic assignment rule in a theoretical framework.

## REFERENCES


[1] D. Bouyssou and T. Marchant, 'An axiomatic approach to noncompensatory sorting methods in MCDM, I: The case of two categories', *European Journal of Operational Research*, **178**(1), 217–245, (2007).

[2] D. Bouyssou and T. Marchant, 'An axiomatic approach to noncompensatory sorting methods in MCDM, II: More than two categories', *European Journal of Operational Research*, **178**(1), 246–276, (2007).

[3] D. Bouyssou and T. Marchant, 'On the relations between ELECTRE TRI-B and ELECTRE TRI-C and on a new variant of ELECTRE TRI-B', *European Journal of Operational Research*, **242**(1), 201–211, (2015).

[4] D. Bouyssou, T. Marchant, and M. Pirlot, 'The size of the largest antichains in products of linear orders', *TOP*, 648–659, (2021).

[5] D. Bouyssou, T. Marchant, and M. Pirlot, 'A theoretical look at Electre Tri-nB and related sorting models', *4OR*, (February 2022).

[6] E. Fernández, J. R. Figueira, J. Navarro, and B. Roy, 'ELECTRE TRI-nB: A new multiple criteria ordinal classification method', *European Journal of Operational Research*, **263**(1), 214–224, (2017).

[7] M.E. Grayson, D.R. Stillman. Macaulay 2, a software for research in algebraic geometry, 2021. Available at `https://math.uiuc.edu/Macaulay2/`.

[8] E. Jacquet-Lagrèze and Y. Siskos, 'Assessing a set of additive utility functions for multicriteria decision making: the UTA method', *European Journal of Operational Research*, **10**, 151–164, (1982).

[9] A. Leroy, V. Mousseau, and M. Pirlot, 'Learning the parameters of a multiple criteria sorting method', in *Algorithmic Decision Theory*, eds., R.I. Brafman, F.S. Roberts, and A. Tsoukiàs, volume 6992 of *Lecture Notes in Artificial Intelligence*, 219–233, Springer, (2011).

[10] OEIS. The On-Line Enciclopedia of Integer Sequences, 2022. N.J.A. Sloane (ed.). url = `https://oeis.org`.

[11] B. Roy and D. Bouyssou, *Aide multicritère à la décision : méthodes et cas*, Economica, Paris, 1993.

[12] O. Sobrie, V. Mousseau, and M. Pirlot, 'Learning monotone preferences using a majority rule sorting model', *International Transactions in Operational Research*, **26**(5), 1786–1809, (2019).

[13] W. Yu, *Aide multicritère à la décision dans le cadre de la problématique du tri : concepts, méthodes et applications*, Thèse de doctorat, Université Paris Dauphine, Paris, France, 1992. (in French).