EXPONENTIAL IDEALS AND A NULLSTELLENSATZ

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Abstract. We prove a version of a Nullstellensatz for partial exponential fields (F, E), even though the ring of exponential polynomials $F[\mathbf{X}]^E$ is not a Hilbert ring. We show that under certain natural conditions, one can embed an ideal of $F[\mathbf{X}]^E$ into an exponential ideal. In case the ideal consists of exponential polynomials with one iteration of the exponential function, we show that these conditions can be met. We apply our results to the case of ordered exponential fields.

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1. Introduction

There is an extensive literature on *Nullstellensätze* for expansions of fields (ordered, differential, p-valued) and several versions of abstract Nullstellensätze attempting to encompass those cases in a general framework (see for instance [11], [16]). When K is a (pure) algebraically closed field, Hilbert's Nullstellensatz establishes the equivalence between the following two properties: a system of polynomial equations (with coefficients in K) has a common solution (in K) and the ideal generated by these polynomials is nontrivial in the polynomial ring $K[\mathbf{X}]$.

It is well-known that this equivalence no longer hold in, for instance, the field of complex numbers endowed with the exponential function (see Remark 2.2). Nevertheless in this note we will give a version of a Nullstellensatz for exponential fields, namely fields (K, E) endowed with a partially defined exponential function E. We start by recalling the notion of E-algebraic closure, first defined by A. Macintyre, then used by A. Wilkie in an o-minimal setting and then investigated by J. Kirby in general. Then we show how to construct maximal (or prime) ideals which are also exponential ideals, using the fact that exponential polynomial rings are union of group rings, pointing out that there are prime ideals that are not an intersection of maximal ideals. Finally under a natural condition on the system of exponential polynomials we are dealing with, we prove a Nullstellensatz in this setting (see below for a precise statement).

In this line of research, let us point out two former works. In the eighties, K. Manders investigated the notion of exponential ideals [10], and pinpointed several obstructions to develop an analog of the classical Nullstenllensatz. Later in her thesis [14], G. Terzo pointed out that very few results are known about exponential ideals

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and that even seemingly basic questions are not yet answered (see Remark 3.1). Let us now describe our main result.

Let (F, E) be an exponential field, namely a field endowed with a morphism E from its additive group $(F, +, 0)$ to its multiplicative group $(F^*, ., 1)$ with $E(0) = 1$.

Using the construction of the free E-ring $F[\mathbf{X}]^E$ of exponential polynomials over F on $\mathbf{X}$ as an increasing union of group rings, namely letting $F_0 := F[\mathbf{X}]$ and $F\mathbf{X}]^E = \bigcup_{\ell \geq 0} F_\ell$ (see section 2.1 and [3]), we show the following. First let us state a corollary of our main theorem, where we restrict ourselves to elements of F_1, namely exponential polynomials with at most one iteration of the exponential function.

Corollary (Later Corollary 3.11) Let $f_1, \ldots, f_m, g \in F_1$ and let I be the ideal of F_1 generated by $f_1, \ldots, f_m$. Assume that g vanishes at each common zero of $f_1, \ldots, f_m$ in any partial exponential field containing (F, E). Then g belongs to $J(I)$, the Jacobson radical of I.

In order to consider the general case of exponential polynomials in $F[\mathbf{X}]^E$, we introduce on an ideal I of F_ℓ, the following (expected) E-compatibility condition at level ℓ :

$$(\mathrm{Ext})_\ell : \quad \forall \ h \in F_{\ell-1} \cap I, \ \ E(h) - 1 \in I.$$

Theorem (Later Theorem 3.10) Let $f_1, \ldots, f_m, g \in F[\mathbf{X}]^E$ and let ℓ be minimal such that $f_1, \ldots, f_m, g \in F_\ell$. Suppose that there is a maximal ideal M of F_ℓ containing $f_1, \ldots, f_m$ and satisfying the E-compatibility condition $(\mathrm{Ext})_\ell$.

Assume that g vanishes at each common zero of $f_1, \ldots, f_m$ in any partial exponential field containing (F, E). Then g belongs to M.

The paper is organized as follows. In section 2, we recall the construction of free exponential rings (and the complexity function they can be endowed with), the definition of exponential ideals and of the E-algebraic closure (ecl). We note that free exponential rings are not Hilbert rings, namely that there are prime ideals which are not an intersection of maximal ideals.

In section 3, we prove our main result (Theorem 3.10) by constructing step by step maximal (respectively prime) exponential ideals, containing a given proper ideal under the above E-compatibility condition $(\mathrm{Ext})_\ell$, at some appropriate finite level ℓ. Finally, we apply our techniques when the field of coefficients is ordered, considering real exponential ideals.

2. Preliminaries

Throughout, all our rings R will be commutative rings of characteristic 0 with identity 1. Let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $R^* := R \setminus \{0\}$. We will work in the language of rings $\mathcal{L}_{rings} := \{+, ., -, 0, 1\}$ augmented by a unary function E; let $\mathcal{L}_E := \mathcal{L}_{rings} \cup \{E\}$.

Definition 2.1. [3] An E-ring (R, E) is a ring R equipped with a map $E : (R, +, 0) \to (R^*, \cdot, 1)$ satisfying $E(0) = 1$ and $\forall x \forall y \ (E(x + y) = E(x).E(y))$.

An E-domain is an E-ring with no zero-divisors; an E-field is an E-domain which is a field.

We will need to work with partial E-rings, namely rings where the exponential function is only partially defined and so the corresponding language contains a unary predicate for the domain of the exponential function.

Definition 2.2. (See [6, Definition 2.2]). A partial E-ring is a two-sorted structure $(R, A(R), +_R, \cdot, +_A, E)$, where $(R, +_R, \cdot, 0, 1)$ is a domain, $(A(R), +_A, 0)$ is an abelian monoid and $E : (A(R), +_A, 0) \to (R, \cdot, 1)$ is a morphism of monoids. Further we have an injective homomorphism of abelian monoids from $(A(R), +_A)$ to $(R, +_R)$; we identify $A(R)$ its image in R and write just one operation $+$. We get the corresponding notions of partial E-domains and partial E-fields.

Let R be a partial E-ring and $A(R)$ a group, then $E(A(R))$ is a subset of the set of invertible elements of R.

The above definition of partial E-rings differs from [6, Definition 2.2]. Indeed, there one requires in addition that R is a $\mathbb{Q}$-algebra, $A(R)$ is a $\mathbb{Q}$-vector space and as such endowed with scalar multiplication $\cdot q$ for each $q \in \mathbb{Q}$.

To simplify notations and in order to treat the cases of partial E-rings and E-rings simultaneously, we will denote the partial E-ring $(R, A(R), E)$ simply by (R, E).

Examples 2.1. We recall below classical examples of exponential (partial) E-rings and E-fields.

(1) Let $\mathbb{R}$ be the field of real numbers (respectively $\mathbb{C}$ the field of complexes) endowed with the exponential function $exp(x) := \sum_{n\geq 0} \frac{x^n}{n!}$.
(2) Let $\mathbb{Z}_p$ be the ring of p-adic integers endowed with the exponential function $x \mapsto exp(px)$, $p > 2$.
(3) Let $\mathbb{Z}$ be the ring of integers endowed with the exponential function $x \mapsto 2^x$ only defined on the positive integers.
(4) Let $\mathbb{Q}_p$ be the field of p-adic numbers (respectively $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$) endowed with the exponential function $exp(px)$ restricted to $\mathbb{Z}_p$ (respectively to $\mathcal{O}_p$ the valuation ring of $\mathbb{C}_p$), $p > 2$.

Let (F, E) be a partial E-field. Consider the field of Laurent series $F((t))$. Write $r \in F[[t]]$ as $r_0 + r_1$ where $r_0 \in F$ and $r_1 \in tF[[t]]$. For $r_0 \in A(F)$, extend E on $F[[t]]$ as follows: $E(r_0 + r_1) := E(r_0)exp(r_1)$. By Neumann's Lemma, the series $exp(r_1) \in F((t))$ [4, chapter 8, section 5, Lemma]. So $F((t))$ can be endowed with the structure of a partial E-field.

More generally let G be a totally ordered abelian group and consider the Hahn field $F((G))$. Let $\mathcal{M}_v$ denote the maximal ideal of $F((G))$. Then $F((G))$ can be endowed with the structure of a partial E-field by defining E on the elements r of the valuation ring which can be decomposed as $r_0 + r_1$ with $r_0 \in A(F)$ and $r_1 \in \mathcal{M}_v$. Then $exp(r_1) \in F((G))$ by Neumann's lemma cited above and $E(r_0 + r_1) := E(r_0)exp(r_1)$, for $r_0 \in A(F)$.

2.1. **Free exponential rings.** For the convenience of the reader not familiar with the construction of the ring of exponential polynomials, we recall its construction below (see for instance [3], [9]).

We denote by $\mathbb{Z}[\mathbf{X}]^E$ the construction on $\mathbf{X} := X_1, \ldots, X_n$ and by $R[\mathbf{X}]^E$ the construction on the E-ring (R, E).

The elements of these rings are called E-*polynomials* in the indeterminates $\mathbf{X}$.

The ring $R[\mathbf{X}]^E$ is constructed by stages as follows: let $R_{-1} := R$, $R_0 := R[\mathbf{X}]$ and A_0 the ideal generated by $\mathbf{X}$ in $R[\mathbf{X}]$. We have $R_0 = R \oplus A_0$. For $k \geq 0$, let t^{A_k} be a multiplicative copy of the additive group A_k.

Then, set $R_{k+1} := R_k[t^{A_k}]$ and let A_{k+1} be the free R_k-submodule generated by t^a with $a \in A_k - \{0\}$. Then $R_{k+1} = R_k \oplus A_{k+1}$.

By induction on $k \geq 0$, one shows the following isomorphism: $R_{k+1} \cong R_0[t^{A_0 \oplus \cdots \oplus A_k}]$, using the fact that $R_0[t^{A_0 \oplus \cdots \oplus A_k}] \cong R_0[t^{A_0 \oplus \cdots \oplus A_{k-1}}][t^{A_k}]$ [9, Lemma 2].

We define the map $E_{-1} : R_{-1} \to R_0$ as the map E on R composed by the embedding of R_{-1} into R_0. Then for $k \geq 0$, we define the map $E_k : R_k \to R_{k+1}$ as follows: $E_k(r' + a) = E_{k-1}(r').t^a$, where $r' \in R_{k-1}$ and $a \in A_k$. Since R_{k-1} and A_k are in direct summand this is well-defined. Moreover for $k \leq \ell$, E_ℓ extends E_k.

Finally let $R[\mathbf{X}]^E := \bigcup_{k \geq 0} R_k$ and extend E on $R[\mathbf{X}]^E$ by defining for $f \in R_k$, $E(f) := E_k(f)$.

Using the construction of $R[\mathbf{X}]^E$ as an increasing union of group rings, one can define on the elements of $R[\mathbf{X}]^E$ an analogue of the degree function for ordinary polynomials which measures the complexity of the elements; it will take its values in the class On of ordinals and was described for instance in [3, 1.9] or in [9, 1.8].

For $p \in R[\mathbf{X}]$, let us denote by $totdeg_{\mathbf{X}}(p)$ the total degree of p, namely the maximum of $\{\sum_{j=1}^n i_j :$ for each monomial $X_1^{i_1} \ldots X_n^{i_n}$ occurring (nontrivially) in $p\}$.

Then one defines a height function h (with values in $\mathbb{N}$) which detects at which stage of the construction the (non-zero) element is introduced.

Let $p(\mathbf{X}) \in R[\mathbf{X}]^E$, then $h(p(\mathbf{X})) = k$, if $p \in R_k \setminus R_{k-1}$, $k > 0$ and $h(p(\mathbf{X})) = 0$ if $p \in R[\mathbf{X}]$. Using the freeness of the construction, one defines a function rk

$$rk : R[\mathbf{X}]^E \to \mathbb{N} :$$

If $p = 0$, set $rk(p) := 0$,

if $p \in R[\mathbf{X}] \setminus \{0\}$, set $rk(p) := totdeg_X(p) + 1$ and

if $p \in R_k \setminus R_{k-1}$, $k > 0$, let $p = \sum_{i=1}^d r_i.E(a_i)$, where $r_i \in R_{k-1}$, $a_i \in A_{k-1} \setminus \{0\}$. Set $rk(p) := d$.

Finally, one defines the complexity function ord

$$ord : R[\mathbf{X}]^E \to On$$

as follows. For $k \geq 0$, write $p \in R_k$ as $p = \sum_{i=0}^k p_i$ with $p_0 \in R_0$, $p_i \in A_i$, $1 \leq i \leq k$. Define $ord(p) := \sum_{i=0}^k \omega^i.rk(p_i)$.

Remark 2.1. [3, Lemma 1.10] Let $p \in R_k \setminus R_{k-1}$, $k \geq 1$ and assume that $p = \sum_{i=1}^k p_i$ with $p_i \in A_i$, $1 \leq i \leq k$. Then there is $q \in R[\mathbf{X}]^E$ such that $ord(E(q).p) < ord(p)$.

Note that if we had started with a partial E-ring (R, E), then $R[\mathbf{X}]^E$ can be endowed with the structure of a partial E-ring. Indeed, let $f \in R_k$, written as $f_0 + f_1$, with $f_0 \in A(R)$ and $f_1 \in A_0 \oplus A_1 \oplus \cdots \oplus A_k$, then define $E(f)$ as $E(f_0).t^{f_1} \in R_{k+1}$, $k \geq -1$.

Finally let us recall the definition of the closure operator ecl.

Notation 2.1. Given $f_1, \dots, f_n \in R[\mathbf{X}]^E$, $\bar{f} := (f_1, \dots, f_n)$, we will denote by $J_{\bar{f}}(\mathbf{X})$, the Jacobian matrix $\begin{pmatrix} \partial_{X_1} f_1 & \cdots & \partial_{X_n} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{X_1} f_n & \cdots & \partial_{X_n} f_n \end{pmatrix}$ and by $\det(J_{\bar{f}}(\mathbf{X}))$ its determinant; note that $\det(J_{\bar{f}}(\mathbf{X}))$ is an E-polynomial.

Definition 2.3. [6, Definition 3.1] Let $(B, E) \subseteq (R, E)$ be partial E-domains. A *Khovanskii system over* B is a quantifier-free $\mathcal{L}_E(B)$-formula in $\mathbf{x} := (x_1, \dots, x_n)$ of the form

$$H_{\bar{f}}(\mathbf{x}) := \bigwedge_{i=1}^{n} f_i(\mathbf{x}) = 0 \ \wedge \ \det(J_{\bar{f}}(\mathbf{x})) \neq 0,$$

for some $f_1, \dots, f_n \in B[\mathbf{X}]^E$.

Let $a \in R$. Then $a \in \mathrm{ecl}^R(B)$ if $H_{\bar{f}}(a_1, \dots, a_n)$ holds for some $a_2, \dots, a_n \in R$ with $a = a_1$, where $H_{\bar{f}}$ is a Khovanskii system, $f_1, \dots, f_n \in B[\mathbf{X}]^E$ (assuming that $a_i \in A(R)$, $1 \leq i \leq n$, if necessary for the $f_i's$ to be defined).

The operator ecl was introduced in [9] and then used by A. Wilkie in his proof of the model-completeness of the field of reals with the exponential function [15].

Then in a general algebraic framework, J. Kirby showed that ecl coincides with another closure operator defined using E-derivations [6, Propositions 4.7, 7.1] and used that property in order to prove that ecl satisfies the exchange property [6, Theorem 1.1]. So the operator ecl induces a pregeometry on subsets of R and we get a dimension, that we denote by dim.

2.2. **Hilbert rings.**

The Jacobson radical $J(R)$ of a ring R is by definition the intersection of all maximal ideals of R. It is a definable subset of R, namely $J(R) = \{u \in R : \ \forall z \exists y \ (1+ u.z).y = 1\}$. Given an ideal I of R, we denote by $J(I)$ the intersection of all maximal ideals that contain I, so this is equal to $\{u \in R : \ \forall z \exists y \ (1 + u.z).y - 1 \in I\}$. We denote by $rad(I)$ the intersection of all prime ideals of R that contain I and one shows that $rad(I) = \{u \in R : \ \exists n \in \mathbb{N} \ u^n \in I\}$.

A Hilbert ring (also called Jacobson ring) is a ring R where any prime ideal is the intersection of maximal ideals. Therefore for any ideal I, $rad(I) = J(I)$.

The terminology of *Hilbert ring* comes from the fact that in a Hilbert ring R, Hilbert Nullstellensatz holds, namely if a polynomial vanishes at every zero of an ideal I, then $f \in rad(I)$.

Using a result of J. Krempa and J. Okninski, one can observe that the ring of exponential polynomials is not an Hilbert ring (see Corollary 2.5 below). But let us quickly recall in the case of polynomial rings a way to show Hilbert Nullstellensatz [5].

Given a maximal ideal M of $K[X]$, one first shows that the field $K[X]/M$ is algebraic over K. Then using that $K[\mathbf{X}]$ is a Hilbert ring, one proves by induction on $n \geq 1$, that $K[\mathbf{X}]/M$ is algebraic over K, for M a maximal ideal of $K[\mathbf{X}]$ [5, Corollary to Theorem 5]. The following properties are used:

(0) If K is a field, then $K[\mathbf{X}]$ is a Hilbert ring [5, Corollary to Theorem 3].

(1) If R is a Hilbert ring, then for any ideal I of R, then R/I is also a Hilbert ring [5, Theorem 1].

(2) R is a Hilbert ring if and only if every maximal ideal of the polynomial ring $R[X]$ contracts to a maximal ideal of R [5, Theorem 5].

Let (K, E) be now an exponential field. We will show that $K[\mathbf{X}]^E$ is not an Hilbert ring, and in the next section that a weak form of the algebraicity property mentioned above, holds in the exponential setting (see Lemma 2.8 and Remark 2.3).

In the following, G denotes a torsion-free abelian group; the *torsion-free rank* of G is the dimension of the $\mathbb{Q}$-vector space $G \otimes \mathbb{Q}$.

Following a result of Krull for polynomial rings, one can show [7, Proposition 1], that if F is a field and G a group of torsion-free rank $\alpha \geq \omega$, then the group ring $F[G]$ is a Hilbert ring if and only if $|F| > \alpha$. When F is not a field, a necessary and sufficient condition was obtained by Krempa and Okninski [7, Theorem 4]. Let us state below part of their result.

Theorem 2.4. [7, Theorem 4] *Let G be an abelian group of infinite torsion-free rank α and let R be a ring. If $R[G]$ is a Hilbert ring, then all homomorphic images of R have cardinality exceeding α.*

Corollary 2.5. *The E-ring $K[\mathbf{X}]^E$ is not a Hilbert ring.*

Proof: Let K be an E-field and consider the group rings K_ℓ, $\ell \geq 1$. Recall that $K_\ell \cong K_0[t^{A_0 \oplus \cdots \oplus A_\ell}]$ (see section 2.1) and $K[\mathbf{X}]^E \cong K_0[t^{\bigoplus_{\ell \geq 0} A_\ell}]$. Let α_ℓ be the torsion-free rank of the (multiplicative) group $t^{A_0 \oplus \cdots \oplus A_\ell}$. Then all the homomorphic images of K_0 do not have cardinality $> \alpha_\ell$ and so the first condition of Theorem 4 in [7] fails and so K_ℓ is not a Hilbert ring. A similar reasoning applies for $K[\mathbf{X}]^E$. $\square$

To conclude this section let us recall the following observation of P. D'Aquino, A. Fornasiero, G. Terzo [2]. Let $(\mathbb{C}, exp)$ be the field of complex numbers endowed with the classical exponential function exp.

Remark 2.2. Let $c \in \mathbb{C} \setminus exp(2\pi\mathbb{Z})$. Consider $f(X) = exp(X) - c$ and $g(X) = exp(iX) - 1$. Let I be the E-ideal in $\mathbb{C}[X]^E$ generated by f and g. Then even though I is a nontrivial ideal of $\mathbb{C}[X]^E$, f and g have no common zero in $\mathbb{C}$ (or in any pseudo-exponential field (as defined by B. Zilber) containing $\mathbb{C}$) [2].

2.3. **Exponential ideals.**

Definition 2.6. Let (R, E) be a partial E-ring.

An E-ideal J of (R, E) is an ideal such that for any $h \in J \cap A(R)$, $E(h) - 1 \in J$.

An E-ideal J is prime if R/J is a domain.

Given a subset A of (R, E), we denote by $\langle A \rangle$ the ideal generated by A and by $\langle A \rangle^E$ the smallest E-ideal containing A.

An example of an E-ideal is the set of E-polynomials in $R[\mathbf{X}]^E$ which vanish on a subset of R^n. When R is a field, such ideal is a prime E-ideal.

Let J be an E-ideal of (R, E). Then the ring R/J can be endowed with an exponential function $E_J : R/J \to R/J$ sending $g + J$ to $E(g) + J$, $g \in A(R)$. This is well-defined since for $h,\ h' \in J \cap A(R)$, $E(h) - E(h') = (E(h - h') - 1).E(h') \in J$.

The following lemma is well-known. For convenience of the reader we give a proof below.

Lemma 2.7. *Let* $(F_0, E) \subseteq (F, E)$ *be two partial E-fields. Assume that* $c \in F$ *is such that there is an E-polynomial* $p(X) \in F_0[X]^E$ *such that* $p(c) = 0$ *(X a single variable). Then* $c \in \mathrm{ecl}(F_0)$.

Proof: Recall that in section 2.1, we have defined the ring of exponential polynomials $F_0[X]^E$ by induction setting $R_0 = F_0[X] = F_0 \oplus A_0$ and $R_i := R_{i-1}[t^{A_{i-1}}] = R_{i-1} \oplus A_i$, $i > 0$.

W.l.o.g. we may assume that $ord(p)$ is minimal such that $p(c) = 0$. Write p as: $p = p_0 + \sum_{i=1}^{k} p_i$, with $p_0 \in F_0[X]$ and $p_i \in A_i$, $i > 0$.

First let us note that p_0 is non trivial. Suppose otherwise that $p_0 = 0$, then by Lemma 1.10 in [3] (see also Remark 2.1), there exists $q \in F_0[X]^E$ such that $ord(E(q)p) < ord(p)$. Since if $p(c) = 0$, then $E(q(c))p(c) = 0$, we get a contradiction. We may assume further that p_0 is a monic polynomial.

Denote by $\partial_X p$ the partial derivative of p with respect to the variable X. Since $p_0 \neq 0$, $ord(\partial_X p) < ord(p)$ [3, Lemma 3.3]. So, by choice of p, $\partial_X p(c) \neq 0$. So $u \in \mathrm{ecl}^{F_0}(F)$. □

Lemma 2.8. *Let* $\mathcal{P}$ *be a prime E-ideal of* $K[X]^E$ *and let* F *be the fraction field of* $K[X]^E/\mathcal{P}$. *Then* F *is included in* $\mathrm{ecl}^F(K)$.

Proof: Let $u := X + \mathcal{P} \in K[X]^E/\mathcal{P}$. Then let $p(X) \in \mathcal{P}$ an element of minimal order, so we have $p(u) = 0$ and w.l.o.g. we may assume that p is monic. Consider the partial E-field extension F of K. Then by Lemma above, $u \in \mathrm{ecl}^F(K)$. Since, $\mathrm{ecl}^F(K)$ is a partial E-subfield containing u [6, Lemma 3.3], it contains F. □

Remark 2.3. More generally, letting $\mathcal{P}$ be a prime E-ideal of $K[\mathbf{X}]^E$, then $\dim(K[\mathbf{X}]^E/\mathcal{P}) < n$ [12, Corollary 3.8]. The proof of that last result uses a theorem of Macintyre [9, Theorem 15 and Corollary].

2.4. **Group rings and augmentation ideals.** Now let S_0 be any ring of characteristic 0 (not necessarily a polynomial ring) and let G be a torsion-free abelian group.

Definition 2.9. We consider the group ring $S_1 := S_0[G]$ and we define a map ϕ^a from $S_1 \to S_0 : \sum_i r_i.g_i \to \sum_i r_i$, with $g_i \in G$, $r_i \in S_0$. The kernel of the map ϕ^a is called the *augmentation ideal* of S_1.

Recall that the augmentation ideal is generated by elements of the form $g - 1$, $g \in G$ (write $\sum_i r_i.g_i$ as $\sum_i r_i.(g_i - 1) + \sum_i r_i$).

3. Exponential ideals and a Nullstellensatz

3.1. Embedding an E-ideal into a maximal ideal that is an E-ideal.

Notation 3.1. Let B_0 be a ring of characteristic 0, let G be a torsion-free abelian group and let B_1 be the group ring $B_0[G]$. Let I be an ideal of B_0. Then compose the augmentation map $\phi^a : B_1 \to B_0$ with the map sending B_0 to B_0/I. Denote the composition of these two maps: ϕ^a_I and denote by I_1 the kernel of ϕ^a_I in B_1.

Lemma 3.1. *Let $B_1 = B_0[G]$ be the group ring $B_0[G]$. Let I be an ideal of B_0 and let ϕ^a, ϕ^a_I and I_1 as in Notation 3.1. Then,*

(1) *$I_1 \cap B_0 = I$,*
(2) *if I prime, then I_1 prime,*
(3) *if I maximal, then I_1 maximal.*

$\square$

Now we will place ourselves in the group rings R_n, $n \geq 1$, defined in 2.1, assuming that (R, E) is an E-field and keeping the same notations as in subsection 2.1. (In particular all ideals of R_n are, as additive groups, $\mathbb{Q}$-vector spaces.) Given an ideal of R_n, we want to find a natural condition under which we can extend it to an exponential ideal of $R[\mathbf{X}]^E$. We do it by steps, using the above lemma. In order to extend a proper ideal I_n of R_n to a proper ideal of R_{n+1}, we will modify the augmentation ideal map "along I_n". We will require on I_n the following property:

$$(\mathrm{Ext})_n \quad u \in I_n \cap R_{n-1} \Rightarrow E(u) - 1 \in I_n.$$

Note that if I_n embeds in an E-ideal I with the property that $I \cap R_n = I_n$, then I_n has the property $(\mathrm{Ext})_n$.

Notation 3.2. Recall that $R_n = R_{n-1} \oplus A_n$ and denote by π_{A_n} (respectively by $\pi_{R_{n-1}}$) the projection on A_n (respectively on R_{n-1}). Recall also that $I_n \cap R_{n-1}$ is a divisible abelian subgroup of I_n. Therefore $I_{n-1} := I_n \cap R_{n-1}$ has a direct summand $\tilde{I}_n \subseteq I_n$ in I_n, namely

$$I_n = \tilde{I}_n \oplus I_{n-1}$$

Note that π_{A_n} is injective on $\tilde{I}_n$: let $u, v \in \tilde{I}_n$, and write $u = u_0 + u_1$, $v = v_0 + v_1$, with $u_0, v_0 \in R_{n-1}$ and $u_1, v_1 \in A_n$. Suppose $u_1 = v_1$, then $u - v = u_0 - v_0 \in R_{n-1} \cap I_n \cap \tilde{I}_n = I_{n-1} \cap \tilde{I}_n = \{0\}$, consequently $u = v$.

Let

$$A_n = \pi_{A_n}(\tilde{I}_n) \oplus \tilde{A}_n.$$

In the statement of the following lemma, we will use Notation 3.2.

Lemma 3.2. *Let I_n be a proper ideal of R_n and let $u \in R_n[t^{A_n}]$, then u can be rewritten in a unique way as*

$$\sum_i r_i E(u_i)$$

where $r_i \in R_n$, $u_i \in \tilde{I}_n \oplus \tilde{A}_n$, and for $i \neq j$, $u_i \neq u_j$. In other words, the group ring $R_n[t^{A_n}]$ is isomorphic to the group ring $R_n[E(\tilde{I}_n \oplus \tilde{A}_n)]$.

Proof. Let $u = \sum_i r_i.t^{a_i} \in R_n[t^{A_n}]$, where $r_i \in R_n$ and $a_i \in A_n$. Decompose a_i as

$$a_{i0} + a_{i1}$$

with $a_{i0} \in \pi_{A_n}(\tilde{I}_n)$ and $a_{i1} \in \tilde{A}_n$. Since π_{A_n} is injective on $\tilde{I}_n$, there exists a unique $f_i \in \tilde{I}_n$ such that $a_{i0} = \pi_{A_n}(f_i)$ and so $f_i = \pi_{R_{n-1}}(f_i) + a_{i0}$. Set $f_{i0} := \pi_{R_{n-1}}(f_i)$.

We have $E(f_i) = E(f_{i0}).t^{a_{i0}}$ (since $E(a_{i0})$ as been defined as $t^{a_{i0}}$ (see section 2.1)), and $t^{a_i} = t^{a_{i0}}.t^{a_{i1}} = E(-f_{i0}).E(f_i).t^{a_{i1}}$. Observe that both $E(-f_{i0}) \in R_n$, $E(f_{i0}) \in R_n$ and $E(f_i) \in R_n[t^{A_n}]$. Moreover, since $a_{i1} \in A_n$, $t^{a_{i1}} = E(a_{i1})$.

So we may re-write u as

$$\sum_i (r_i E(-f_{i0}))E(f_i)t^{a_{i1}} = \sum_i (r_i E(-f_{i0}))E(f_i + a_{i1})$$

with $r_i E(-f_{i0}) \in R_n$ and $f_i + a_{i1} \in \tilde{I}_n \oplus \tilde{A}_n$. Such expression is unique since the projection π_{A_n} on $\tilde{I}_n$ is injective: for $f \neq g \in \tilde{I}_n$, $\pi_{A_n}(f) \neq \pi_{A_n}(g) \in A_n$. So if $u_i \neq u_j \in \tilde{I}_n \oplus \tilde{A}_n$, then $\pi_{A_n}(u_i) \neq \pi_{A_n}(u_j)$. $\square$

Proposition 3.3. *Let $n \in \mathbb{N}$ and I_n be a proper ideal of R_n with the property* $(\mathrm{Ext})_n$. *Then, I_n embeds in a (proper) ideal I_{n+1} of R_{n+1} such that*

$$\begin{aligned} (\mathrm{Ext})_{n+1} \quad & E(f) - 1 \in I_{n+1} \text{ for any } f \in I_n, \quad \text{and} \\ (\mathrm{Tr})_{n+1} \quad & I_{n+1} \cap R_n = I_n. \end{aligned}$$

Moreover if I_n is prime (respectively maximal), then I_{n+1} is prime (respectively maximal).

Proof: Let I_n be a proper ideal of R_n, $n \in \mathbb{N}$ (in particular $I_n \cap R_{-1} = \{0\}$). By the preceding lemma, any $u \in R_{n+1}$ can be rewritten in a unique way as $\sum_{i=1}^{\ell} r_i E(u_i)$, where $r_i \in R_n$, $u_i \in \tilde{I}_n \oplus \tilde{A}_n$, and the u_i's are distinct. So the map ϕ sending $\sum_{i=1}^{\ell} r_i E(u_i)$ to $\sum_{i=1}^{\ell} r_i \in R_n$ is well-defined and it is a ring morphism from $R_n[E(\tilde{I}_n \oplus \tilde{A}_n)]$ to R_n.

The kernel $ker(\phi)$ contains $\{E(f) - 1 : \ f \in \tilde{I}_n\}$. Let ϕ_{I_n} be the map sending $\sum_{i=1}^{\ell} r_i E(u_i)$ to $\sum_{i=1}^{\ell} r_i + I_n \in R_n/I_n$.

Define I_{n+1} as $ker(\phi_{I_n})$. By Lemma 3.1, I_{n+1} is an ideal of R_{n+1} with the property that $ker(\phi_{I_n}) \cap R_n = I_n$ $(\mathrm{Tr})_{n+1}$.

It remains to show that I_{n+1} contains $E(f)-1$ for any $f \in I_n$ $(\mathrm{Ext})_{n+1}$. Let $f \in I_n$ and write f as $f_0 + f_1$ with $f_0 \in I_{n-1}$ and $f_1 \in \tilde{I}_n$.

Then $E(f) - 1 = (E(f_1) - 1).E(f_0) + (E(f_0) - 1)$. We already observe that $E(f_1) - 1 \in ker(\phi)$ and by assumption $(\mathrm{Ext})_n$, $E(f_0) - 1 \in I_n$ for $f_0 \in I_{n-1}$. So, $E(f) - 1 \in ker(\phi_{I_n})$.

Applying Lemma 3.1 with $S_0 = R_n$ and $G = E(\tilde{I}_n \oplus \tilde{A}_n)$, if I_n is prime (respectively maximal), then I_{n+1} is prime (respectively maximal). $\square$

Corollary 3.4. *Let I_{n_0} be a proper ideal of R_{n_0}, $n_0 \geq 0$, with the property* $(\mathrm{Ext})_{n_0}$. *Then, $\langle I_{n_0}\rangle^E$ is such that $\langle I_{n_0}\rangle^E \cap R_{n_0} = I_{n_0}$. Moreover if I_{n_0} is prime (respectively maximal), then $\langle I_{n_0}\rangle^E$ is prime (respectively maximal).*

Proof. Proposition 3.3 allows to construct a proper (respectively prime, maximal) ideal I_{n_0+1} of R_{n_0+1} containing $E(I_{n_0}) - 1$ and satisfying (Tr_{n_0+1}). Therefore we may reiterate the construction. Then let $I^E := \bigcup_{n \geq n_0} I_n$. It is an E-ideal by construction, and it is proper because for all $n \geq n_0$, $I^E \cap R_n = I_n$. If I_{n_0} is prime, then each I_n is prime for $n \geq n_0$ (by Proposition 3.3). So $\langle I_{n_0}\rangle^E$ is prime as the union of a chain of prime ideals. If I_{n_0} is maximal, then for $n \geq n_0$, each I_n is maximal by Proposition 3.3. So $\langle I_{n_0}\rangle^E$ is maximal as the union of a chain of maximal ideals. □

G. Terzo in her thesis asked several questions on E-ideals in $R_0[X]$ [14]. In particular whether the E-ideal generated by an irreducible polynomial is a prime ideal. (By irreducible E-polynomial we mean an E- polynomial which cannot be written as a product of two non invertible elements in $R[X]^E$.)

Remark 3.1. Let $p(X) \in R_0$ and suppose $p(X)$ is irreducible. Since the ideal generated by $p(X)$ in R_0 is a maximal ideal, by the above corollary, $\langle p(X)\rangle^E$ is a maximal ideal.

A natural question is when an ideal $I \subseteq R_n$ satisfies the condition $(\mathrm{Ext})_n$. Assume that (R, E) has an extension (S, E) where there is a tuple of elements $\bar{\alpha}$ with the property that for any $f \in I$, $f(\bar{\alpha}) = 0$. Then consider $I_{\bar{\alpha}} := \{g \in R_n : \ g(\bar{\alpha}) = 0\}$. By definition $I \subseteq I_{\bar{\alpha}}$, $I_{\bar{\alpha}}$ is a (prime) ideal and for any $f \in I \cap R_{n-1}$, $E(f) - 1 \in I_{\bar{\alpha}}$.

In the following proposition, we will examine the condition $(\mathrm{Ext})_1$ that we put on an ideal I of R_1 in order to embed it in an E-ideal. (This corresponds to the case of E-polynomials with only one iteration of E.) In this particular case, we can use the fact that R_0 is a Noetherian ring.

Proposition 3.5. *Let I be a proper ideal of R_1. Then we can embed I in a proper E-ideal of $R[\mathbf{X}]^E$.*

Proof: It suffices to show that we can embed I in an ideal J of R_1 such that for any $f \in J \cap R_0$, $E(f) - 1 \in J$ and use Corollary 3.4. Since I is a proper ideal of R_1, $I \cap R_{-1} = \{0\}$. Set $I_0 := R_0 \cap I$. Using Notation 3.2, we get that $\tilde{I}_0 = I_0$. Recall that $\tilde{A}_0$ is a direct summand of $\pi_{A_0}(I_0)$ in A_0. By Lemma 3.2, $R_1 \cong R_0[E(I_0 \oplus \tilde{A}_0)]$.
Set $R_0' := R_0[E(\tilde{A}_0)]$, so we get $R_1 \cong R_0'[E(I_0)]$. Consider the ideal $I_0' := I \cap R_0'$. (Note that $I_0 = I_0' \cap R_0$, since $R_0' \cap R_0 = R_0$.) Let $u \in R_1$ and write it as $\sum_j r_j.E(u_j)$, where $r_j \in R_0'$, $u_j \in I_0$, with the u_j's distinct. The map ϕ^+ sending u to $\sum_{j=1}^{\ell} r_j \in R_0'$ is well-defined. Define J as $ker(\phi^+_{I_0'})$, then J is an ideal of R_1 containing $E(f) - 1$ for any $f \in I_0$ with the property that $J \cap R_0' = I_0'$ by Lemma 3.1. (It implies that $J \cap R_0 = J \cap R_0' \cap R_0 = I_0' \cap R_0 = I_0$.
Let $u \in \langle J, I\rangle \cap R_0'$; then $u = \sum_i u_i a_i$ with $u_i \in J, a_i \in I$. Since $u \in R_0'$, we have that $\phi^+(u) = u$. So $u = \sum_i \phi^+(u_i).\phi^+(a_i)$. But $\phi^+(u_i) \in I_0'$ and so since $\phi^+(a_i) \in R_0'$ and I_0' is an ideal, we get that $u \in I_0'$. In particular $\langle J, I\rangle$ is proper.
So we repeat the same procedure replacing I_0 by $\langle J, I\rangle \cap R_0$. Since R_0 is noetherian, the process will stop. So we get a proper ideal $\tilde{J}$ containing I with the property that for any $f \in \tilde{J} \cap R_0$, $E(f) - 1 \in \tilde{J}$. So we may apply Corollary 3.4 and embed $\tilde{J}$ in an exponential ideal of $R[X]^E$. □

3.2. **Rabinowitsch's trick.** Recall that Rabinowitsch's trick corresponds to the introduction of an extra variable, allowing one to deduce the algebraic strong Nullstellensatz from the weak one. Given $f_1(\mathbf{X}), \ldots, f_m(\mathbf{X}) \in R[\mathbf{X}]$ and another polynomial

$g(\mathbf{X}) \in R[\mathbf{X}]$ vanishing on all common zeroes of $f_1, \ldots, f_m$, introducing the new variable Y, one gets: $f_1, \ldots, f_m, 1 - Yg$ do not have any common zeroes.

By the weak Nullstellensatz, this implies that the ideal generated by these polynomials is not proper. So one expresses (by an equality) that 1 belongs to the ideal generated by $f_1, \ldots, f_m$ and $1 - Y.g$. Then one substitutes g^{-1} to Y in the equality, and clears denominators. This entails that some power of g belongs to the ideal generated by $f_1, \ldots, f_m$ in $R[\mathbf{X}]$.

To mimick Rabinowitsch's trick in $R[\mathbf{X}]^E$, where (R, E) is as previously an E-field, we proceed as follows introducing a "non-E" variable, extending $R[\mathbf{X}]^E$ to $R[\mathbf{X}]^E \otimes_R R[Y]$. This partial E-ring is isomorphic to the following chain of partial E-rings. Recall that $R_{-1} := R$ and $R_0 := R[\mathbf{X}]$. Denote by $S_0 := R[\mathbf{X}, Y] = R_0[Y] \cong R_0 \otimes_R R[Y]$. Let $S_1 := S_0[t^{A_0}] \cong R_1 \otimes_R R[Y]$, and by induction on $n \geq 2$, let $S_n = S_{n-1}[t^{A_{n-1}}] \cong R_n \otimes_R R[Y]$. Let $\mathcal{S} := \bigcup_{n \geq 0} S_n$. Then $\mathcal{S} \cong R[\mathbf{X}]^E \otimes_R R[Y]$.

Now we consider an ideal J of $R[\mathbf{X}]^E \otimes_R R[Y]$ and we want to extend it into an E-ideal, namely an ideal containing $E(f) - 1$, for $f \in J \cap R[\mathbf{X}]^E$.

Proposition 3.6. *Let J_n be a proper ideal of S_n, let $I_n := J_n \cap R_n$. Suppose that I_n satisfies* $(\mathrm{Ext})_n$ *as an ideal of R_n. Then J_n embeds into a proper ideal J_{n+1} of S_{n+1} such that $I_{n+1} := J_{n+1} \cap R_{n+1}$ satisfies* $(\mathrm{Ext})_{n+1}$ *and* $(\mathrm{Tr})_{n+1}$. *Moreover if J_n is prime (respectively maximal), then J_{n+1} is prime (respectively maximal).*

Proof: By Lemma 3.2, $R_n[t^{A_n}] \cong R_n[E(\tilde{I}_n \oplus \tilde{A}_n)]$ (see Notation 3.2). Since $S_{n+1} = S_n[t^{A_n}] \cong R_n[t^{A_n}] \otimes_R R[Y]$, we get that $S_{n+1} \cong S_n[E(\tilde{I}_n \oplus \tilde{A}_n)]$. Rewrite $s \in S_{n+1}$ as $\sum_{i=1}^{\ell} s_i.E(u_i) : \ s_i \in S_n, \ u_i \in \tilde{I}_n \oplus \tilde{A}_n$, with u_i distinct. Let ϕ be the map sending $\sum_{i=1}^{\ell} s_i.E(u_i)$ to $\sum_{i=1}^{\ell} s_i \in S_n$ and let $\phi \restriction R_{n+1}$ be the restriction of the map ϕ to R_{n+1}. Let ϕ_{J_n} be the map sending $\sum_{i=1}^{\ell} s_i.E(u_i)$ to $\sum_{i=1}^{\ell} s_i + J_n \in S_n/J_n$. Define J_{n+1} as the kernel of ϕ_{J_n}; by Lemma 3.1, it is an ideal of S_{n+1} and $ker(\phi_{J_n}) \cap S_n = J_n$. Furthermore $ker(\phi_{J_n})$ contains $ker((\phi \restriction R_{n+1})_{I_n})$ and so by Proposition 3.3, it contains $E(f) - 1$ for any $f \in I_n$.

Applying Lemma 3.1 with $B_0 = S_n$ and $G = E(\tilde{I}_n \oplus \tilde{A}_n)$, if J_n is prime (respectively maximal), then J_{n+1} is prime (respectively maximal). □

Corollary 3.7. *Let J be a proper ideal of S_n with $n \geq 0$ chosen minimal such. Assume that $J \cap R_n$ satisfies the property* $(\mathrm{Ext})_n$ *(as an ideal of R_n). Then $\langle J \rangle^E$ is a proper E-ideal of $S[\mathbf{X}]^E$ with $\langle J \rangle^E \cap S_n = J$. Moreover whenever J is prime (respectively maximal) in S_n, then $\langle J \rangle^E$ is prime (respectively maximal) in $S[\mathbf{X}]^E$.*

Proof: We apply Proposition 3.6. □

3.3. **Nullstellensatz.**

Corollary 3.8. (Weak Nullstellensatz) *Let (R, E) be an E-field and $f_1, \ldots, f_m \in R[\mathbf{X}]^E$. Let $n \in \mathbb{N}$ be chosen minimal such that $f_1, \ldots, f_m \in R_n$. Assume the ideal I_n generated by $f_1, \ldots, f_m$ is proper and that there is a maximal ideal M_n of R_n containing I_n with the property* $(\mathrm{Ext})_n$ *(as an ideal of R_n). Then $f_1, \ldots, f_m$ have a common zero in an E-field extending (R, E).*

Proof. By Proposition 3.6, the E-ideal $\langle M_n\rangle^E$ is a proper maximal E-ideal of $R[\mathbf{X}]^E$. The quotient $R[\mathbf{X}]^E/\langle M_n\rangle^E$ is an E-field in which $(X_1+\langle M_n\rangle^E,\ldots,X_n+\langle M_n\rangle^E)$ is a common zero of $f_1,\ldots,f_m$. □

Remark 3.2.

(1) In the statement of the corollary above we may assume that the E-polynomials $f_1,\ldots,f_m$ have non-trivial polynomial parts, namely that for $1\leq i\leq m$, $ord(f_i)$ is of the form $\omega^n.m_n+\ldots+m_0$ with $m_n.m_0\neq 0$, $m_\ell\in\mathbb{N}$, $0\leq\ell\leq n$ (see Remark 2.1).

(2) If, in the above corollary we simply assume that the ideal I_n generated by $f_1,\ldots,f_m$ is proper and contained in a prime ideal P_n with the property $(\mathrm{Ext})_n$ (as an ideal of R_n), then $f_1,\ldots,f_m$ have a common zero in an E-domain extending (R,E) (and so in the field of fractions of this E-domain).

Definition 3.9. Let I be an ideal of R_ℓ, $\ell\geq 1$. We define $J_E(I)$ as the intersection of all maximal ideals M of R_ℓ containing I with the property$(\mathrm{Ext})_\ell$. If there are no such maximal ideals, then we set $J_E(I)=R_\ell$.

Theorem 3.10. *Let $f_1,\ldots,f_m,g\in R[\mathbf{X}]^E$ and let ℓ be minimal such that $f_1,\ldots,f_m,g\in R_\ell$. Let I be the ideal of R_ℓ generated by $f_1,\ldots,f_m$. Suppose that $J_E(I)$ is a proper ideal. Assume that g vanishes at each common zero of $f_1,\ldots,f_m$ in any (partial) E-field containing (R,E). Then g belongs to $J_E(I)$.*

Proof: Consider the ring $\mathcal{S}$ and the element $1-Y.g$, then $1-Y.g\in S_\ell$. Since $J_E(I)$ is proper, there is a maximal ideal M of R_ℓ containing I with property $(\mathrm{Ext})_\ell$. Let J be an ideal of S_ℓ containing M and $1-Y.g$. Assume that J is proper, so $J\cap R_\ell=M$ and it embeds into a maximal ideal of S_ℓ. Then, we embed it in a maximal E-ideal $\tilde{M}^E$ of $\mathcal{S}$, by Proposition 3.6. Since $\tilde{M}^E\cap R[\mathbf{X}]^E$ is a prime ideal, the quotient $R[\mathbf{X}]^E/\tilde{M}^E\cap R[\mathbf{X}]^E$ is an E-domain containing (R,E) which embeds in the partial E-field $\mathcal{S}/\tilde{M}^E$ where $f_1,\ldots,f_m,1-gY$ have a common zero, a contradiction.

Therefore we have

$$1=\sum_i t_i(X,Y).h_i(X)+(1-Y.g).r(X,Y),$$

with $h_i(X)\in M$, $t_i(X,Y),r(X,Y)\in S_\ell$. Note that the elements of S_ℓ, $\ell\geq 1$, are of the form $S_0[t^{A_0\oplus\ldots\oplus A_{\ell-1}}]\cong R_\ell\otimes_R R[Y]$. So we may substitute g^{-1} to Y and find a sufficiently big power g^d, $d>0$, of g such that by multiplying each $t_i(X,g^{-1})$, $r(X,Y)$, we obtain again an element in S_ℓ. Since $t_i(X,g^{-1}).g^d\in R_\ell$, we get

$$g^d=\sum_i(t_i(X,g^{-1}).g^d).h_i(X)\in R_\ell.$$

Therefore $g^d\in M$ and since M is maximal, $g\in M$. Since we can do that for any maximal ideal M containing I with property $(\mathrm{Ext})_\ell$, we get that $g\in J_E(I)$. □

Remark 3.3. The problem with replacing in the theorem above, maximal ideals M with prime ideals $\mathcal{P}$ is that $J\cap R_\ell$ might be bigger than $\mathcal{P}$ and so the condition $(\mathrm{Ext})_\ell$ might not hold anymore.

Using Proposition 3.5, we may deduce a stronger result in the case when the E-polynomials have only one iteration of the exponential function (namely the case $\ell = 1$).

Corollary 3.11. *Let $f_1, \dots, f_m, g \in R_1$ and let I be the ideal of R_1 generated by $f_1, \dots, f_m$. Assume that g vanishes at each common zero of $f_1, \dots, f_m$ in any partial E-field containing (R, E). Then g belongs to $J(I)$, the Jacobson radical of I.*

Proof: We apply Theorem 3.10 and we may replace $J_E(I)$ by the ordinary Jacobson radical of I since by Proposition 3.5, every maximal ideal of R_1 satisfies $(\mathrm{Ext})_1$. $\square$

Finally we ask ourselves the question of when does a system of E-polynomials with coeffcients in F have a common zero in an Ecl-closure of R.

Corollary 3.12. *Let $f_1, \dots, f_n \in R_k$ and assume that $k \in \mathbb{N}$ is minimal such. Assume that the ideal $I := (f_1, \dots, f_n)$ of R_k generated by $f_1, \dots, f_n$ is disjoint from $\{det(J_{\bar{f}})^m : \ m \in \mathbb{N}^*\}$. Let $\mathcal{P}$ be an ideal of R_k containing I and maximal for the property of being disjoint from $\{det(J_{\bar{f}})^m : \ m \in \mathbb{N}^*\}$. Suppose $\mathcal{P}$ satisfies $(\mathrm{Ext})_k$. Then $f_1, \dots, f_n$ have a zero $\bar{\alpha} \in \mathrm{ecl}^F(R)$ where (F, E) in a partial E-field extending (R, E).*

Proof: It is well-known that $\mathcal{P}$ is a prime ideal of R_k. Since we assumed that it satisfies the hypothesis that for any $f \in \mathcal{P} \cap R_{k-1}$, $E(f) - 1 \in \mathcal{P}$, we may apply Corollary 3.4 and get that $\langle \mathcal{P} \rangle^E$ is a prime E-ideal of $R[\mathbf{X}]^E$ with $\langle \mathcal{P} \rangle^E \cap R_k = \mathcal{P}$. Then the element $\bar{\alpha} := (X_1 + \langle \mathcal{P} \rangle^E, \dots, X_n + \langle \mathcal{P} \rangle^E)$ satisfies the formula $H_{\bar{f}}$ in the E-domain $R[\mathbf{X}]^E / \langle \mathcal{P} \rangle^E$. Let F be the fraction field of $R[\mathbf{X}]^E / \langle \mathcal{P} \rangle^E$. Then we have that $\bar{\alpha}$ belongs to $\mathrm{ecl}^F(R)$ (see Definition 2.3). $\square$

3.4. **Real Nullstellensatz.** In this section we adapt our previous results to the case of E-fields (or E-rings) which are ordered (or admits an ordering, i.e. are orderable). We will refer to such E-fields as ordered E-fields (or ordered E-rings), respectively orderable ones.

Definition 3.13. [8, Definition 2.1] Let (R, E) be an E-field. An ideal I of R is real if for any $u_1, \dots, u_n \in R$ such that $\sum_{i=1}^n u_i^2 \in I$, then $u_i \in I$ for all $i = 1, \dots, n$. A real E-ideal I of (R, E) is a real ideal which is also an E-ideal.

We denote by Σ the set of sums of squares in R.

Examples 3.1. Let R be an ordered E-domain, then the set of E-polynomials in $R[\mathbf{X}]^E$ which vanish on a subset of R^n is a real E-ideal.

Lemma 3.14. *Let B_0 be a ring of characteristic 0, G a torsion-free abelian group and B_1 be the group ring $B_0[G]$. Let I be a real ideal of B_0 and let ϕ^a, ϕ^a_I as in Notation 3.1. The kernel I_1 in B_1 of the map ϕ^a_I is a real ideal of B_1.*

Proof: Suppose that $\sum_{i=1}^n \ u_i^2 \in I_1$, so $\sum_{i=1}^n \ \phi^a_I(u_i)^2 = 0$. Therefore, $\sum_{i=1}^n \ \phi^a(u_i)^2 \in I$. Since I is real, it implies that $\phi^a(u_i) \in I$ for all $1 \leq i \leq n$, equivalently that $\phi^a_I(u_i) = 0$ for all $1 \leq i \leq n$. $\square$

Lemma 3.15. *Let $n \in \mathbb{N}$, $I_n \subseteq R_n$ be a prime real ideal with the property $(\mathrm{Ext})_n$. Then I_n embeds in a prime real ideal I_{n+1} of R_{n+1} with properties $(\mathrm{Ext})_{n+1}$ and $(\mathrm{Tr})_{n+1}$.*

Proof. By Lemma 3.14 and Proposition 3.3. □

Corollary 3.16. *Let $(R, E, <)$ be an ordered E-field. Let $f_1, \dots, f_m \in R[\mathbf{X}]^E$ and let $k \in \mathbb{N}$ be chosen minimal such that $f_1, \cdots, f_m \in R_k$. Assume the ideal I generated by $f_1, \cdots, f_m$ is disjoint from the set $1 + \Sigma$. Let $\mathcal{P}$ be an ideal of R_k maximal for the property of containing I and being disjoint from $1 + \Sigma$. Suppose that $\mathcal{P}$ satisfies* $(\mathrm{Ext})_k$ *Then $f_1, \cdots, f_m$ have a common zero in a partial ordered E-domain extending (R, E).*

Proof: It is well-known that such ideal $\mathcal{P}$ is a prime real ideal by [8, Lemma 1.2, Remark 1.3, Definition 2.1]. It remains to apply Lemma 3.15 and Corollary 3.4, checking that a union of a chain of real ideals is a real ideal. So $\langle \mathcal{P} \rangle^E$ is a prime, real E-ideal. The quotient $D := R[\mathbf{X}]^E / \langle \mathcal{P} \rangle^E$ is an orderable E-domain [8, Theorem 3.9]. Moreover, $f_1, \cdots, f_m$ have a common zero in D. □

Remark 3.4. Note that when the exponential function satisfies the growth condition: $E(f) \geq 1 + f$ for any $f \in R[\mathbf{X}]^E / \langle \mathcal{P} \rangle^E$, then by results of Dahn and Wolter [1, Theorem 24], one can embed this partial E-domain in a real-closed E-field where the exponential function is surjective on the positive elements.

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