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Chiral approach to partially-massless fields

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Abstract: We propose a new (chiral) description of partially-massless fields in $4d$, including the partially-massless graviton, that is similar to the pure connection formulation for gravity and massless higher spin fields, the latter having a clear twistor origin. The new approach allows us to construct complete examples of higher spin gravities with (partially-)massless fields that feature Yang-Mills and current interactions.

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Contents

1 Introduction

Partially-massless fields constitute a novel class of fields that appears in the presence of a non-vanishing cosmological constant [1–3] (see also [4–9]). They appear as fields whose mass take special values for which the corresponding action acquires a gauge symmetry of higher-derivative type, and hence propagate an intermediate number of degrees of freedom between those of a genuine massless field (subject to single-derivative gauge invariance), and a genuine massive field (without any gauge symmetry).

Partially-massless fields are unitary in de Sitter space and may have phenomenological applications (see e.g. [10, 11] and references therein). Despite being non-unitary around anti-de Sitter spacetime, partially-massless fields are nevertheless of interest,[1] if only because they are dual to partially-conserved currents, that is, currents which are annihilated after taking several divergences [12]. These kinds of currents naturally appear in free conformal field theories of higher-derivative scalar fields, i.e. scalar fields subject to polywave equations of the type $\Box^\ell \phi = 0$, with $\ell > 1$ [13], which are known to describe special RG fixed points called 'multi-critical isotropic Lifshitz points' [14]. The holographic dual of this theory would be a theory of both massless and partially-massless fields of arbitrary spin in anti-de Sitter space, which has been studied in [13, 15, 16] (see also [17–20] for works on the corresponding higher spin algebras), but not worked out in full details yet. One reason is that holographic duals of vector models feature severe nonlocalities that invalidate the usual field theory methods to construct them [21–24].

Nevertheless, cubic interactions for partially-massless fields of any spins have been studied [25–27], but complete interacting theories featuring partially-fields in the spectrum are still lacking. Particular attention has been given to the problem of finding gravitational interactions and constructing what one might want to call a theory of partially-massless

[1] Note that partially-massless fields have also been of interest recently in the context of inflation [10, 11].

gravity, i.e. an interacting theory of a massless and a partially-massless spin-2 field. Unfortunately, the search for such a non-linear theory led to several no-go theorems, whether it is in relation with massive and/or bimetric gravity [28–32], with conformal gravity [33, 34], or on general grounds [35, 36]. A notable exception is the recent work [37], wherein an interacting theory of a multiplet of spin-2 partially-massless fields has been proposed.

All of the aforementioned results were obtained by working with symmetric rank-s tensors to describe partially-massless fields of spin-s. In this paper, we introduce a new description of partially-massless fields in $4d$, inspired by twistor theory and the description of massless fields given in [38, 39], based on a pair of a 1-form and a 0-form which are also $\mathrm{SL}(2,\mathbb{C})$ spin-tensors (see also [40–42] for a pure connection formulation of gravity, which is closely related). In terms of these new field variables, the free action for partially-massless takes a fairly simple form, and more importantly, one can construct complete interacting theories featuring partially-massless fields. We will illustrate this last fact by spelling out a partially-massless higher spin extension of self-dual Yang-Mills, which is a generalisation of the higher spin extension discussed in [39], and a theory featuring current interactions between a couple of massless fields with a partially-massless one, which is complete at the cubic order.

The organisation of this paper is as follows: in section 2, we briefly recall the metric- and frame- like description of free partially-massless fields before introducing a new description based on two-component spin-tensors, in section 3 we present two simple examples of fully interacting theories featuring partially-massless fields, and we end up by some concluding remarks in section 4.

2 Free partially-massless fields, old and new

Metric-like approach. Free fields are known to be in one-to-one with irreducible representations of the spacetime isometry group. For de Sitter (dS) space in $(d+1)$-dimensions, the isometry algebra is $\mathfrak{so}(1,d+1)$, whereas for anti-de Sitter (AdS) space in $(d+1)$-dimensions, it is $\mathfrak{so}(2,d)$. We will hereafter denote these algebras collectively by $\mathfrak{g}_\Lambda$. One new feature of the representation theory of (anti-)de Sitter algebras, as compared to that of the Poincaré algebra, is that they admit irreducible representations which are realized as fields propagating an intermediate number of degrees of freedom between that of a massless field and that of a massive one, for a fixed value of the spin [1–3]. Consequently, these fields are called partially-massless (PM). A spin-s partially-massless field of depth-t, with $1 \leq t \leq s$, can be represented by a rank-s symmetric tensor $\Phi^{a_1\dots a_s} \equiv \Phi^{a(s)}$ that is subject to[2]

$$\delta_\xi \Phi^{a(s)} = \underbrace{\nabla^a \dots \nabla^a}_{t\ \text{times}} \xi^{a(s-t)} + \dots\,, \tag{2.1}$$

where the dots denote lower order derivatives terms. In other words, the depth of a partially-massless field is nothing but the number of derivatives in its gauge transformation, and

[2]In trying to save letters we abbreviate a group of symmetric indices $a_1 \dots a_s$ as $a(s)$ and, more generally, denote all indices to be symmetrized by the same letter.

the massless case corresponds to $t = 1$ in our convention. Omitting the transversality and tracelessness constraints for Φ and ξ, the equations of motion reduce to

$$(\Box - m^2)\,\Phi^{a(s)} = 0\,, \qquad\qquad m^2 = -\Lambda\left((d+s-t-1)(s-t-1)-s\right), \tag{2.2}$$

where, as for the massless case, the mass-like term is proportional to the cosmological constant and depends on the spin-s, depth-t and spacetime dimension $d+1$. The mass-like term is fixed by the gauge symmetry. While equations of motion are simple, the action requires an intricate pattern of auxiliary fields[3] [8].

Frame-like approach. The frame-like description of partially-massless fields was developed in [44], see also [45] for the specialization to $4d$ and [46–49] for purely massless higher spin fields.[4] The key idea is to consider a (generalized) connection of the (anti-)de Sitter algebra $W^{\mathbb{Y}}$, i.e. a one-form that takes values in a finite-dimensional representation $\mathbb{Y}$ of the algebra that is not necessarily the adjoint one. The simplest case is the adjoint itself, $\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}$ [56], for which the connection $W^{A,B}$ contains[5] two one-forms valued in finite-dimensional representations of the Lorentz subalgebra $\mathfrak{so}(1,d)$, namely the vielbein $e^a = W^{a,\bullet}$ and the spin-connection $\omega^{a,b} = W^{a,b}$. In order to describe a spin-s depth-t partially-massless field, one should consider a 1-form W taking values in the finite-dimensional irreducible representation $\mathbb{Y}_{s,t} = \begin{array}{|c|}\hline s-1 \\ \hline s-t \\ \hline\end{array}$. Upon decomposing it with respect to the Lorentz algebra, one gets a lot of auxiliary fields,

$$W^{\mathbb{Y}_{s,t}} = \{\omega^{a(s-k),b(s-m)}\}\,, \qquad \text{with} \qquad k \in \{1,2,\dots,t\}\,, \quad m \in \{t, t+1, \dots, s\}\,. \tag{2.3}$$

It is easy to construct a gauge-invariant curvature R for W, namely one simply defines it to be

$$R[W] = \nabla W + e^a \wedge \rho(P_a)\,W\,, \tag{2.4}$$

where ρ is the representation $\mathbb{Y}$ of the (anti-)de Sitter algebra.[6] This curvature is invariant under the gauge transformations generated by a 0-form ξ valued in the same representation $\mathbb{Y}$,

$$\delta_\xi W = \nabla \xi + e^a\,\rho(P_a)\,\xi\,, \tag{2.5}$$

on an (anti-)de Sitter background, i.e. defined by a vielbein e^a and spin-connection $\varpi^{a,b}$ obeying

$$\nabla e^a = 0\,, \qquad\qquad R^{ab} - e^{[a} \wedge e^{b]} = 0\,, \tag{2.6}$$

[3]This is due to the fact that partially-massless fields are closer to the massive ones. For a massive spin-s field one has to impose transversality on top of the Klein-Gordon equation, which starting from $s = 2$ requires auxiliary fields [43].

[4]Note that the frame-like description of fields arbitrary mixed-symmetry, both massless and partially-massless, has been worked out, see e.g. [50–55].

[5]Indices $A, B, \ldots = 0, \ldots, d+1$ are of $\mathfrak{g}_\Lambda$ and we can decompose them as $A = a, \bullet$, where indices $a, b, c, \ldots$ are of the Lorentz algebra.

[6]This expression can be thought of as originating from the curvature $F[A] = dA + \frac{1}{2}[A, A]$ of a connection A taking values in the algebra $\mathfrak{g}_\Lambda \Subset_\rho \mathbb{Y}$, which is the semi-direct sum of the (anti-)de Sitter algebra $\mathfrak{g}_\Lambda$ with the representation $\mathbb{Y}$, considered as an Abelian subalgebra. The component of this curvature taking values in $\mathfrak{g}_\Lambda$ is the usual curvature of the (A)dS algebra, and is assumed to vanish here, while the component in $\mathbb{Y}$ reproduces the above formula.

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where ∇ is the covariant derivative induced by ϖ and $R^{a,b} = d\varpi^{a,b} + \varpi^a{}_c \wedge \varpi^{c,b}$ is its usual Lorentz curvature 2-form. Note in particular that the second piece of this gauge transformations, the one generated by the action of the transvection generators, is algebraic (it is given by symmetrization and contraction of the background vielbein with the gauge parameters, and does not involve any derivatives).

For instance, a partially-massless spin-2 field is described in this language by a connection, taking values in $\mathbb{Y} = \square$, the fundamental (or vector) representation of the (anti-)de Sitter algebra $\mathfrak{g}_\Lambda$. Such a connection has components $W^{\square} = \{w^a, w\}$, i.e. it is composed of two 1-forms, valued in the vector and scalar representation of the Lorentz algebra respectively. Their curvature simply read

$$R^a = \nabla w^a + e^a \wedge w\,, \qquad R = \nabla w - e^a \wedge w_a\,, \tag{2.7}$$

while the gauge transformations are given by

$$\delta_{\xi,\epsilon} w^a = \nabla \xi^a + e^a\,\epsilon\,, \qquad \delta_{\xi,\epsilon} w = \nabla \epsilon - e^a\,\xi_a\,, \tag{2.8}$$

where ξ^a and ϵ are the two 0-form gauge parameters. Let us briefly review how one can recover the metric-like formulation discussed previously [44, section 5.1]. First, note that one can gauge-fix to zero the component w upon using its gauge symmetry generated by ξ^a. The residual gauge transformations (i.e. which preserve the gauge choice $w = 0$) are those generated by ϵ and $\xi_a = -\nabla_a \epsilon$, i.e.

$$\delta_\epsilon w_{a|b} = -\nabla_a \nabla_b\,\epsilon + \eta_{ab}\,\epsilon\,, \tag{2.9}$$

where $w_{b|a} = e_b^\mu\, w_\mu^c\, \eta_{ac}$. Imposing that the curvature R of w vanishes in the gauge $w = 0$ implies that the antisymmetric part of $w_{a|b}$ vanishes,

$$R|_{w=0} = 0 \qquad \Rightarrow \qquad w_{[a|b]} = 0\,. \tag{2.10}$$

This is a first sign that one can recover the symmetric rank-2 tensor subject to a two-derivative gauge transformation, which encodes the PM spin-2 field in the metric-like formulation, as the symmetric part of the 1-form w^a. Inspecting the Bianchi identities for the curvature R^a, one finds that its only possible non-trivial component is encoded by a hook, so that one can impose

$$R^a = e_b \wedge e_c\, C^{ab,c}\,, \tag{2.11}$$

where $C^{ab,c}$ is a 0-form which takes values in the irrep $\begin{smallmatrix}\square\!\square\\ \square\end{smallmatrix}$ of the Lorentz algebra. The above example is representative of the frame-like description of partially-massless field: for a spin-s and depth-t field, one can impose the zero-curvature equations

$$R^{a(s-m),b(s-n)} = 0\,, \qquad m \neq 1 \quad \text{and} \quad n \neq t\,, \tag{2.12}$$

and

$$R^{a(s-1),b(s-t)} = C^{a(s-1)c,b(s-t)d}\, e_c \wedge e_d\,, \tag{2.13}$$

where C is a 0-form, that can be thought of as a partially-massless version of the Weyl tensor. The metric-like partially-massless field can be found in the connection $e^{a(s-1)}$ valued in

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the totally symmetric irrep of the Lorentz algebra, and the above zero-curvature equations expresses the intermediate/auxiliary connections $\omega^{a(s-1),b(m)}$ with $m = 1, \dots, s-t-1$ as m derivatives of the PM field, while the last equation equates the 0-form C to a particular traceless projection of $s-t+1$ derivatives of the PM field.

One can build a gauge-invariant action from the above curvature, however, this action exhibits an intricate pattern involving the 'auxiliary connections' [44]. Let us specialize this construction to $4d$, where it is advantageous to use the two-component spinor language.

Twistor-inspired/chiral approach. The advent of twistor theory lead to a new geometrical understanding of massless fields in $4d$ in terms of holomorphic structures on a $3d$ complex manifold that is twistor space [38, 57, 58] (see also the textbooks [59–62] and, for instance, the recent review [63]). Although we will not use directly twistor theory in our description of partially-massless fields, it is very much inspired by it, and is a straightforward extension of the approach proposed for massless fields in [38, 39].

At the algebraic level, this relies on the low dimensional isomorphism $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(1,3)$. The latter relates a Lorentz vector V^a to a $\mathfrak{sl}(2,\mathbb{C})$-bi-spinor $V^{AA'}$, where both $A = 1,2$ and $A' = 1,2$ are two-component spinor indices. More generally, finite-dimensional irreducible representations of $\mathfrak{so}(1,3)$, which are mixed-symmetric traceless tensor $T^{a(m),b(n)}$ correspond to a spin-tensor carrying two groups of $m+n$ and $m-n$ totally symmetrized (un)primed indices,

$$T^{a(m),b(n)} \qquad \longleftrightarrow \qquad \big(T^{A(m+n),A'(m-n)},\ T^{A(m-n),A'(m+n)}\big)\,. \tag{2.14}$$

As usual, in the Lorentzian signature the two spin-tensors are complex conjugate of each other. In the Euclidian or split signature, they are independent real spin-tensors. Unprimed spinor indices are raised and lowered with the invariant tensor ϵ_{AB} and its inverse ϵ^{AB}, in the sense that $\epsilon^{AC}\,\epsilon_{BC} = \delta^A_B$, via

$$\xi^A = \epsilon^{AB}\,\xi_B\,, \qquad\qquad \xi_B = \xi^A\,\epsilon_{AB}\,, \tag{2.15}$$

and similarly for primed indices. In this two-component spinor language, the $\mathfrak{g}_\Lambda$-connection consists of a vierbein $e^{AA'}$, the self-dual part of the spin-connection ω^{AA}, and its anti-self-dual part $\omega^{A'A'}$. The zero-curvature equations for this connection are given by

$$R_{AA} = H_{AA}\,, \qquad R_{A'A'} = H_{A'A'}\,, \qquad \nabla e_{AA'} = 0\,, \tag{2.16}$$

with

$$R_{AA} := d\omega_{AA} + \omega_{AB}\wedge\omega^B{}_A\ , \qquad R_{A'A'} := d\omega_{A'A'} + \omega_{A'B'}\wedge\omega^{B'}{}_{A'}\ , \tag{2.17}$$

are the self-dual and anti-self-dual parts of the Lorentz curvature 2-form, and where we introduced the two-forms

$$H_{AA} := e_{AB'}\wedge e_A{}^{B'}\ , \qquad H_{A'A'} := e_{BA'}\wedge e^B{}_{A'}\ , \tag{2.18}$$

which define a basis of self-dual and anti-self-dual 2-forms respectively. There is also the 3-form basis, defined as

$$\hat{e}_{AA'} := H_{AB}\wedge e^B{}_{A'}\ . \tag{2.19}$$

In particular, the 2-forms H_{AA} and $H_{A'A'}$ verify

$$H_{AB} \wedge H_{A'B'} = 0\,, \tag{2.20}$$

and the identities

$$H_{AA} \wedge e_{AB'} = 0 \qquad \Rightarrow \qquad H_{AA} \wedge H_{AB} = 0\,, \tag{2.21}$$

which will be useful later on (for more details, see e.g. [64]).

It was shown in [39] that for massless fields, we can take the self-dual parts of the very 'last' spin-connection (by which we mean the component of the $\mathfrak{g}_\Lambda$-connection valued in the 'biggest' Lorentz Young diagram, that is, the Young diagram with the same shape as the one labelling the $\mathfrak{g}_\Lambda$-irrep) and of the Weyl tensor as our dynamical variables. Indeed, we will show that this leads to a simple action. In tensor language, the last spin-connection for a spin-s and depth-t partially-massless field is a one-form $\omega^{a(s-1),b(s-t)}$ and the Weyl tensor is of the form $C^{a(s),b(s-t+1)}$, where the indices merely indicate the symmetry type of a tensor. In the spinorial language, the self-dual components of these two fields are thus

$$\omega^{A(2s-t-1),A'(t-1)} \qquad\qquad \Psi^{A(2s-t+1),A'(t-1)}\,, \tag{2.22}$$

and their anti-self-dual cousins can be obtained via $t \to 2s-t$ for ω and $t \to 2s-t+2$ for Ψ. The chiral approach deals with one pair of such fields and ignores the duals thereof. In particular, for the spin-2 field of depth $t=2$, i.e. a partially-massless graviton, we find $\omega^{A,A'}$ and $\Psi^{A(3),A'}$. In the spin-s case, the decomposition of ω into irreducible spin-tensors reads

$$\begin{aligned} \omega^{A(2s-t-1),A'(t-1)} &= e^{A}{}_{B'}\,\Phi^{A(2s-t-2),A'(t-1)B'} + e_{B}{}^{A'}\,\Phi^{A(2s-t-1)B,A'(t-2)} \\ &\quad + e_{BB'}\Phi^{A(2s-t-1)B,A'(t-1)B'} + e^{AA'}\Phi^{A(2s-t-2),A'(t-2)}\,, \end{aligned} \tag{2.23}$$

where Φ are 0-forms. Two of these components are unphysical and can be gauged away, since the gauge transformation of ω reads

$$\begin{aligned} \delta_{\xi,\eta}\omega^{A(2s-t-1),A'(t-1)} &= \nabla\xi^{A(2s-t-1),A'(t-1)} \\ &\quad + e^{AA'}\eta^{A(2s-t-2),A'(t-2)} + e^{A}{}_{B'}\,\eta^{A(2s-t-2),A'(t-1)B'}\,, \end{aligned} \tag{2.24}$$

and contains both a differential part (the first term), and an algebraic part (the second and third terms). The latter, hereafter referred to as a *shift symmetry*, can therefore be used to gauge away the first and fourth terms in the irreducible decomposition (2.23). After this gauge fixing, the connection ω is given by

$$\omega^{A(2s-t-1),A'(t-1)} = e_{B}{}^{A'}\,\Phi^{A(2s-t-1)B,A'(t-2)} + e_{BB'}\Phi^{A(2s-t-1)B,A'(t-1)B'}\,, \tag{2.25}$$

and is subject to the residual gauge symmetry

$$\delta\Phi^{A(2s-t),A'(t-2)} = \nabla^{A}{}_{B'}\,\xi^{A(2s-t-1),A'(t-2)B'}\,, \tag{2.26a}$$

$$\delta\Phi^{A(2s-t),A'(t)} = \nabla^{AA'}\xi^{A(2s-t-1),A'(t-1)}\,, \tag{2.26b}$$

expressed in terms of its two irreducible components. Note that the gauge symmetry (2.24) is nothing but the two-component spinor translation of the gauge symmetry (2.5) in the frame-like approach, and in particular, the shift symmetry here is simply the algebraic part of the gauge symmetry of the 'last connection'.

Action. We propose the following action

$$S_{s,t}[\omega, \Psi] = \int \Psi^{A(2s-t+1),A'(t-1)}\, H_{AA} \wedge \nabla\omega_{A(2s-t-1),A'(t-1)}\,, \qquad (2.27)$$

for the description of a spin-s partially-massless field of depth-t. This action is invariant under the gauge symmetries (2.24) thanks to the property (2.21) of the background. Notice also that this action is of presymplectic AKSZ-type [65], which is not that surprising considering that the frame-like action for Gravity [66] and Conformal/Weyl Gravity [67] are also of this type, and that the relevance of this approach for higher-spin theories is established [68, 69].

Another noteworthy feature of the above action is that it is not manifestly real in the Lorentzian signature, as is the well-known cases of (self-dual) Yang-Mills theory [70] and gravity [40–42] that can be formulated in terms of chiral field variables. Nevertheless, it is worth mentioning that the use of chiral field variables does not imply that the theory is actually chiral (parity-violating) or non-unitary. This is always true for free theories that have the same degrees of freedom as their non-chiral relatives. The free action of [70] corresponds to $s = 1$, $t = 1$ of (2.27).

The equations of motion obtained from (2.27) are

$$H_{AA} \wedge \nabla\omega_{A(2s-t-1),A'(t-1)} = 0\,, \qquad H_{AA} \wedge \nabla\Psi^{A(2s-t+1),A'(t-1)} = 0\,. \qquad (2.28)$$

There are two noteworthy cases: $t = 1$ which corresponds to massless fields, and in which case the above action reproduces the one proposed in [39], and $t = s$, which corresponds to maximal depth partially-massless fields, and for which the spin-connection is balanced (meaning it has the same number of primed and unprimed indices, as opposed to the massless case where it is completely unbalanced).

These equations can be taken as a starting point to build a free differential algebra (FDA) formulation of partially-massless fields, see [44, 45, 52, 53, 55, 71]. Indeed, they can be read as expressing the fact that the first derivatives of ω and Ψ are in the kernel of an operator determined by the background self-dual 2-form H_{AA} (symmetrization for ω, contraction for Ψ). These operators are nothing but components of the presymplectic form used to build the action (2.27). The FDA is obtained by parametrizing $\nabla\omega$ and $\nabla\Psi$ as the most general elements in the kernel of this presymplectic form, i.e.

$$\nabla\omega_{A(2s-t-1),A'(t-1)} = e_A{}^{B'}\, \omega_{A(2s-t-2),A'(t-1)B'} + e_{AA'}\, \omega_{A(2s-t-2),A'(t-2)}\,, \qquad (2.29a)$$

$$\nabla\Psi_{A(2s-t+1),A'(t-1)} = e^B{}_{A'}\, \Psi_{A(2s-t+1)B,A'(t-2)} + e^{BB'}\, \Psi_{A(2s-t+1)B,A'(t-1)B'}\,, \qquad (2.29b)$$

and imposing that the resulting equations are integrable. Typically, this condition leads to constraints on the first derivatives of the components of the elements in the kernel of the symplectic form, and one should repeat the procedure (i.e. find the most general form of the first derivatives of these new fields compatible with integrability, thereby introducing new fields, and imposing once more the integrability of this equation, etc . . .). See e.g. [65] or [69, section 4] for a review. The outcome of this procedure is to build two modules of the (A)dS algebra $\mathfrak{g}_\Lambda$:

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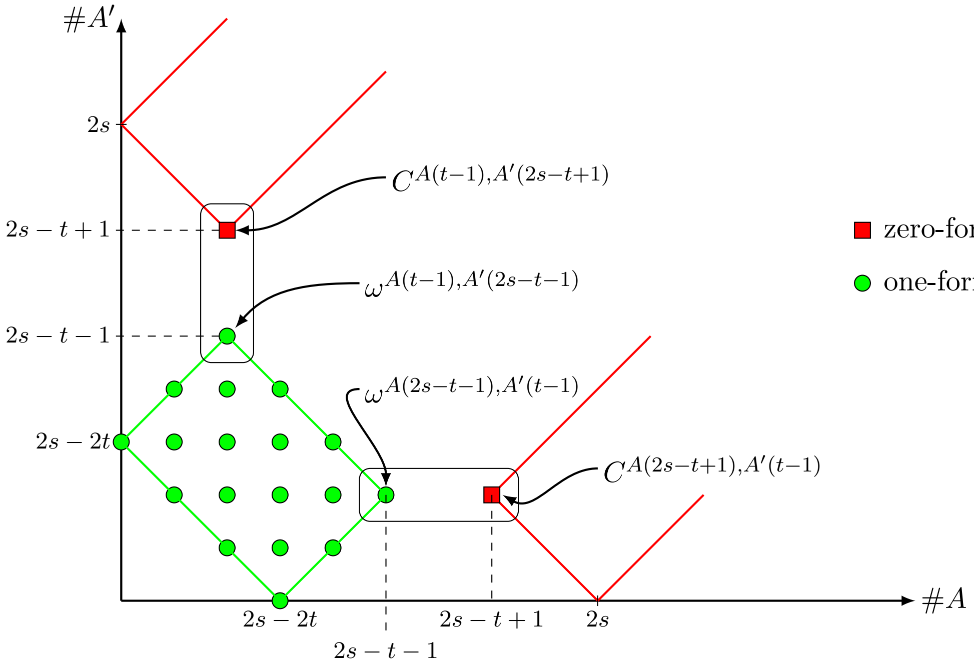

Figure 1. A diagram to show fields/coordinates involved into the description of partially-massless higher spin fields. Along the horizontal/vertical axe, we have the number of unprimed/primed indices on a spin-tensor. Components of the 1-form connection are represented by green circles, while the 0-forms (the Weyl tensor and its descendants) are represented by red rectangles. By descendants we mean the on-shell nontrivial derivatives of the Weyl tensor, which are associated with the coordinates on the on-shell jet space [60].

- A finite-dimensional one, which is spanned by the 1-forms $\omega^{A(2s-m-n),A'(n-m)}$ and their complex conjugate, with $1 \leq m \leq t$ and $t \leq n \leq s$. This corresponds to the $\mathfrak{g}_\Lambda$-module $\begin{array}{|c|}\hline s-1 \\ \hline s-t \\ \hline \end{array}$ used in the frame-like formulation;
- An infinite-dimensional one, spanned by the 0-forms $\Psi^{A(2s-t+m+n),A'(t-m+n)}$ with $n \geq 0$ and $1 \leq m \leq t$, which corresponds to the derivatives of the self-dual Weyl tensors unconstrained by equations of motion or Bianchi identities.

The pattern of connections, and descendants of the Weyl tensor, for a fixed spin-s and depth-t is illustrated in figure 1 and was already detailed in [44] (see also [50–55, 71]), while the pattern of pairs made of a connection one-form and a Weyl tensor zero-form, for a fixed spin-s and different values of the depth-t is displayed in figure 2.

Let us dwell a little on the maximal depth case $t = s$. In vector language, the last connection decomposes as

$$\omega^{a(s-1)} \simeq \boxed{\quad s \quad} \oplus \begin{array}{|c|}\hline \quad s-1 \quad \\ \hline \end{array}\!\!\begin{array}{l} \\ \end{array}_{\Box} \oplus \boxed{\quad s-2 \quad} \tag{2.30}$$

under the Lorentz group, and is subject to the algebraic symmetry

$$\delta_\epsilon \omega^{a(s-1)} = e^{\{a}\, \epsilon^{a(s-2)\}}\,, \tag{2.31}$$

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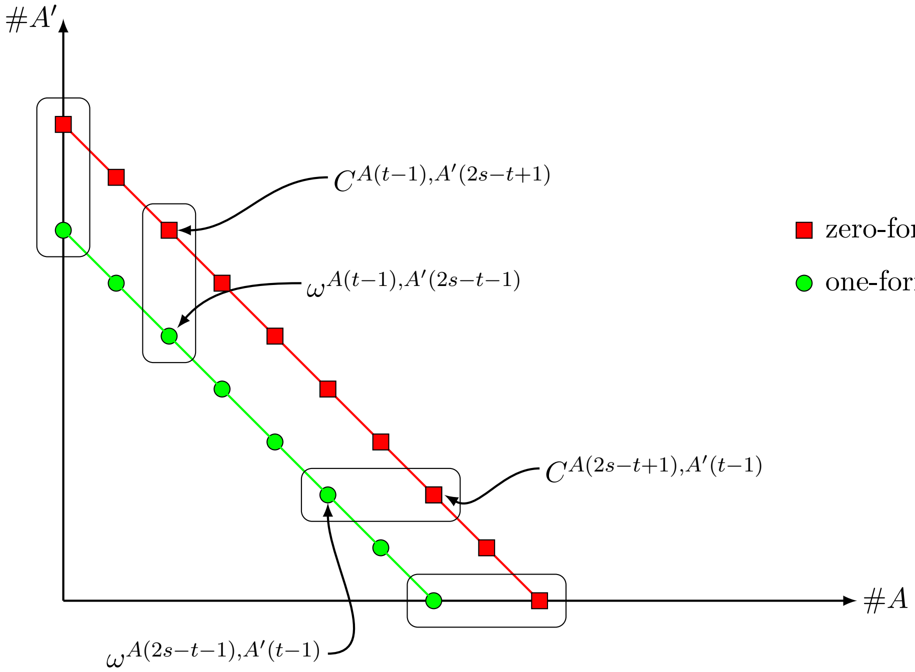

Figure 2. For a given spin-s, the fields grouped horizontally/vertically correspond to chiral/anti-chiral description of depth-t partially-massless fields. There are two descriptions for each admissible s, and t. The group on each of the axes describes massless fields in terms of (anti-)chiral variables. It is clear that extrapolation of one description beyond $t > s$ does give the other one.

where $\{\dots\}$ denotes the traceless projection of symmetrized indices. This algebraic symmetry removes the trace part $\boxed{s-2}$ in the irreducible decomposition of $\omega^{a(s-1)}$. It may however be surprising at first glance that in the two-component spinor language, one has two parameters for the algebraic symmetry of ω, namely $\eta^{A(s-2),A'(s-2)}$ and $\eta^{A(s-2),A'(s)}$. The first one simply corresponds to ϵ, converted in spinor language, but the second one appears to have no counterpart in the vector language. This is not accidental: in fact, this additional parameter has the same symmetry has the anti-self-dual part of the hook component of ω, and its rôle is simply to remove it. This is consistent with the fact that, in spinor language, ω has two irreducible components, corresponding respectively to symmetric rank-s tensor and the self-dual part of a hook tensor, and is also in accordance with the counting of degrees of freedom detailed below. Such additional symmetry is also present in the FDA form [45, 71] of Zinoviev's description of partially-massless fields [8, 72].

Massless spinning fields, described as in [38, 39], can propagate on self-dual backgrounds. This is due to the fact that the fields $\Psi^{A(2s)}$ and $\omega^{A(2s-2)}$ do not have any primed indices, hence, $\nabla^2\xi^{A(2s-2)} \equiv 0$ on a self-dual background, which ensures the gauge invariance of the action. However, partially-massless fields are always described by mixed spin-tensors, i.e. have both primed and unprimed indices. The action (2.27) as well as the equations of motion (2.28) remain consistent in Minkowski space, the difference being that the corresponding solution space is not an irreducible representation of the Poincaré group (see e.g. [50, 51, 54, 55, 73]).

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Degrees of freedom. Let us justify the main claim of the previous paragraphs, which is that the action (2.27) does describe a partially-massless spin-s and depth-t field in $4d$. To do so, we will show that the solutions of the resulting equations of motion propagate the correct number of degrees of freedom, namely $2t$ (irrespectively of the spin). In our case, the equations of motion are first order differential equations for the fields Ψ and ω. The number of physical degrees of freedom propagated by an arbitrary field, which is a solution of an involutive system of equations, is given by the formula [74]

$$N_{\text{dof}} = \tfrac{1}{2}\sum_{k=0}^{\infty} k\,(e_k - i_k - g_k)\ , \tag{2.32}$$

where e_k is the number of equations of order k in the system, i_k number of (gauge) identities of k-th order, and g_k is the number of gauge symmetry generators of order k (here, the order is the number of derivatives). Let us recall that an involutive system of order n is defined in [74] as a system of equations such that any differential consequence of these equations, of order n or less, is already a part of the system. In our case, the equation of motion for the field Ψ is given by,

$$H_{AA}\wedge\nabla\Psi^{A(2s-t+1),A'(t-1)} \propto \widehat{e}_{AB'}\nabla_A{}^{B'}\,\Psi^{A(2s-t+1),A'(t-1)} = 0\,, \tag{2.33}$$

where $\hat{e}_{AB'}$ are the basis 3-forms introduced in (2.19) above. Using it, we can write down the set of independent equations of motion as

$$E^{A(2s-t),A'(t-1)|B'} = \nabla_B{}^{B'}\,\Psi^{BA(2s-t),A'(t-1)} = 0\,, \tag{2.34}$$

and easily count that these are $e_1 = 2t(2s-t+1)$ equations of first order. The field Ψ does not have any gauge symmetry, hence $g_k = 0$ for all k. Now since the field ω has a first order gauge symmetry, the Ψ-field after integrating by parts in the action, satisfy the Bianchi identity of second order. Explicitly, this identity is given by,

$$\nabla_{FF'}E^{FA(2s-t-1),F'A'(t-1)} = 0\,, \tag{2.35}$$

which consists in $i_2 = t\,(2s-t)$ identities of the second order. Thus, the number of physical degrees of freedom described by the field Ψ is

$$N_{\text{dof}}(\Psi) = \tfrac{1}{2}\left[2t(2s-t+1) - 2t(2s-t)\right] = t\,. \tag{2.36}$$

Similarly, the equations of motion for the field ω read

$$H_{AA}\wedge\nabla\omega_{A(2s-t-1),A'(t-1)} = H_{AA}\wedge e_{DD'}\nabla^{DD'}\omega_{A(2s-t-1),A'(t-1)} = 0\,, \tag{2.37}$$

and, upon using the decomposition of ω into its irreducible components,

$$\begin{aligned}\omega^{A(2s-t-1),A'(t-1)} &= e^A{}_{B'}\,\Phi^{A(2s-t-2),A'(t-1)B'} + e_B{}^{A'}\,\Phi^{A(2s-t-1)B,A'(t-2)}\\ &\quad + e_{BB'}\Phi^{A(2s-t-1)B,A'(t-1)B'} + e^{AA'}\Phi^{A(2s-t-2),A'(t-2)}\,,\end{aligned} \tag{2.38}$$

takes the form

$$\nabla_A{}^{F'}\,\Phi_{A(2s-t),F'A'(t-1)} + \nabla_{AA'}\Phi_{A(2s-t),A'(t-2)} = 0\,. \tag{2.39}$$

These are $e_1 = t\,(2s-t+2)$ equations of first order. The gauge transformations are of first order, and generated by $g_1 = t\,(2s-t)$ parameters. Since there are no additional identities, the number of degrees of freedom propagated by ω is

$$N_{\rm dof}(\omega) = \tfrac{1}{2}\left[(2s-t+2)t-(2s-t)t\right] = t\,, \tag{2.40}$$

and hence Ψ and ω contain, in total, $2t$ physical degrees of freedom. In particular, for massless field ($t=1$), we recover 2 degrees of freedom, as expected, while for the partially-massless graviton ($t=2$), we find 4 degrees of freedom, in conformity with expectations.[7]

Note that the counting of degrees of freedom presented here applies for any values of t. In particular, when $t>s$, we see that the number of degrees of freedom keeps increasing and is larger than the one expected for a spin-s field of any depth. This is another indication that, despite the fact that the pairs of fields (ω,Ψ) can still be considered for $t>s$, and the action (2.27) still makes sense, their interpretation remains elusive and should not be related to PM fields (our proposal is that it gives two massive fields, see appendix A).

3 Interactions

Since we have a well-defined free action, the next task is to look for interacting theories. In this section, we will consider two simple types of possible interactions using the new description presented in this paper.

3.1 Yang-Mills interactions

First, we will consider Yang-Mills interactions for partially-massless fields, which are straightforward generalization of the higher spin extension of self-dual Yang-Mills theory introduced in [39], and recently revisited in [75–78], see also [79]. This type of interaction is obtained by first extending the spin-connection $\omega^{A(2s-t-1),A'(t-1)}$ and the Weyl tensor $\Psi^{A(2s-t+1),A'(t-1)}$ of a partially-massless spin-s and depth-t field to take values in a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant bilinear form[8] that we will denote by $(-,-)$. Next, we can pack up together the spin-connections for partially-massless fields of all spin and depth into a single 1-form,

$$\omega = \sum_{s=1}^{\infty}\sum_{t=1}^{s}\omega_{s,t}(x|y)\,, \qquad \omega_{s,t}(x|y) := \frac{\omega_{A(2s-t-1)A'(t-1)}}{(2s-t-1)!(t-1)!}\,y^A\dots y^A\,\bar{y}^{A'}\dots\bar{y}^{A'}\,, \tag{3.1}$$

whose curvature is defined by the usual formula

$$F = \nabla\omega + \tfrac{1}{2}\,[\omega,\omega]\,, \tag{3.2}$$

where the bracket above should be understood as the $\mathbb{C}[y,\bar{y}]$-linear extension of the Lie bracket of the Yang-Mills algebra $\mathfrak{g}$. More concretely, the Lie bracket of ω with itself is

[7]The same counting of degrees of freedom is suggested by the first step (2.29) towards the FDA form of the equations.

[8]Recall that a bilinear form is called ad-invariant if it verifies $([x,y],z) = (x,[y,z])$ for any elements $x,y,z\in\mathfrak{g}$.

given by

$$[\omega,\omega]_{s,t} = \sum_{\substack{s_1+s_2=s+1\\ t_1+t_2=t+1}} [\omega_{s_1,t_1},\omega_{s_2,t_2}]\,, \tag{3.3}$$

where the subscript (s,t) denotes the component of degree $2s-t-1$ in y and $t-1$ in $\bar{y}$. Packing up in a similar way the differential gauge parameters associated with each spin-connection into a 0-form ξ, we can define an extension of the free gauge symmetry (2.24) via

$$\delta_\xi\omega = \nabla\xi + [\omega,\xi]\,, \qquad\qquad \delta_\xi\Psi = [\Psi,\xi]\,, \tag{3.4}$$

under which the curvature transforms according to

$$\delta_\xi F = \nabla^2\xi + [F,\xi]\,, \tag{3.5}$$

where the first term can be re-written as

$$\nabla^2\xi = (H_A{}^B\, y^A\,\partial_B + H_{A'}{}^{B'}\,\bar{y}^{A'}\,\partial_{B'})\,\xi\,. \tag{3.6}$$

Similarly, we can pack up the shift symmetry parameters into a single 0-form η, and write it as

$$\delta_\eta\omega = e_{AA'}\,y^A\,(\bar{y}^{A'} + \partial^{A'})\,\eta\,, \tag{3.7}$$

so that the curvature transforms as

$$\delta_\eta F = -e_{AA'}\,y^A\,(\bar{y}^{A'} + \partial^{A'})\,(\nabla\eta + [\omega,\eta])\,, \tag{3.8}$$

since the vierbein is torsionless and does not take values in the Lie algebra $\mathfrak{g}$. We will consider the action

$$\begin{aligned} S_{PMYM}[\omega,\Psi] &= \langle\Psi \mid \tfrac{1}{2}\,H_{AA}\,y^A y^A \wedge F\rangle \\ &:= \sum_{1\le t\le s} \tfrac{1}{(2s-t-1)!(t-1)!} \int \big(\Psi^{A(2s-t+1),A'(t-1)}, H_{AA}\wedge F_{A(2s-t-1),A'(t-1)}\big)\,, \end{aligned} \tag{3.9}$$

which defines a complete interacting theory for partially-massless fields. The interactions are of Yang-Mills type. This action is invariant under shift symmetry since its variation under this transformation will produce a term $H_{AA}\wedge e_{AB'}=0$, as can be seen from (3.8). Its variation under the gauge transformations (3.4) is given by

$$\delta_\xi S_{PMYM} = \langle[\Psi,\xi] \mid \tfrac{1}{2}\,H_{AA}\,y^A y^A\wedge F\rangle + \langle\Psi \mid \tfrac{1}{2}\,H_{AA}\,y^A y^A \wedge(\nabla^2\xi + [F,\xi])\rangle = 0\,, \tag{3.10}$$

and vanishes due to the fact that the term $\nabla^2\xi$ produces $H_{AA}\wedge H_{AB} = 0 = H_{AA}\wedge H_{A'B'}$ according to (3.6), and the two remaining terms cancel one another due to the ad-invariance of the bilinear form on $\mathfrak{g}$.

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3.2 Current interactions

Consider the functional

$$S_{\text{int}}[\omega, \Psi] = \int T^{A(2s-t),A'(t)}(\Psi)\, \omega_{A(2s-t-1),A'(t-1)}\, \hat{e}_{AA'} \tag{3.11}$$

where the spin-tensor $T^{A(2s-t+1),A'(t)}(\Psi)$ is a 0-form built out of Weyl tensors of some (partially-)massless fields (of possibly different spins and depths), which verifies

$$\nabla_{BB'} T^{A(2s-t)B,A'(t-1)B'}(\Psi) \approx 0\,, \tag{3.12}$$

where the symbol $\approx$ signifies that the spin-tensor $T(\Psi)$ is divergenceless only on-shell. This term is invariant under the shift symmetry, as a consequence of the fact that

$$e_{AA'} \wedge \hat{e}_{BB'} = -\tfrac{1}{4}\, \epsilon_{AB}\, \epsilon_{A'B'}\, \text{vol} \qquad \Rightarrow \qquad e_{AA'} \wedge \hat{e}_{AB'} = 0\,, \tag{3.13}$$

where 'vol' denotes a volume form on the background, and the fact that Ψ is assumed to be inert under this symmetry. Under the differential gauge symmetry, the variation of this term reads

$$\delta_\xi S_{\text{int}}[\omega, \Psi] = \int T^{A(2s-t),A'(t)}(\Psi)\, \nabla \xi_{A(2s-t-1),A'(t-1)}\, \hat{e}_{AA'} \tag{3.14a}$$

$$= -\int \nabla^{BB'}\, T^{A(2s-t),A'(t)}(\Psi)\, \xi_{A(2s-t-1),A'(t-1)}\, e_{BB'}\, \hat{e}_{AA'} \tag{3.14b}$$

$$= \tfrac{1}{4} \int \nabla_{BB'} T^{A(2s-t-1)B,A'(t-1)B'}(\Psi)\, \xi_{A(2s-t-1),A'(t-1)}\, \text{vol} \approx 0\,, \tag{3.14c}$$

and vanishes on-shell. It therefore provides a good starting point to construct interactions for partially-massless fields.

Indeed, divergenceless spin-tensors are fairly easy to construct out of the Weyl tensors of a pair of massless fields. Consider for instance the Bel-Robinson tensor

$$T_{abcd} = \tfrac{1}{4}\, \left(C_a{}^p{}_b{}^q C_{cpdq} + *C_a{}^p{}_b{}^q * C_{cpdq} \right)\,, \tag{3.15}$$

where C_{abcd} is the gravitational Weyl tensor and $*$ is the Hodge dual operator, i.e. $*C_{abcd} = \epsilon_{ab}{}^{pq} C_{pqcd}$. This tensor is divergenceless as a consequence of Einstein's equation in vacuum. In spinor notations, this tensor takes an especially simple form, namely it is given by the product of the self-dual and anti-self-dual Weyl tensor,

$$T_{A(4),A'(4)} = \Psi_{A(4)}\, \Psi_{A'(4)}\,, \tag{3.16}$$

and suggests the generalization (see [80] for a complete set of currents)

$$T_{A(2s_1),A'(2s_2)} = \Psi_{A(2s_1)}\, \Psi_{A'(2s_2)}\,, \tag{3.17}$$

given by the product of the Weyl tensors of two massless fields of spin s_1 and s_2. This spin-tensor will be divergence-free as a consequence of the equation of motion

$$\nabla^B{}_{B'}\, \Psi_{A(2s_1-1)B} \approx 0\,, \qquad \nabla_B{}^{B'}\, \Psi_{A'(2s_2-1)B'} \approx 0\,, \tag{3.18}$$

for these Weyl tensors.

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We will consider the one-parameter family of actions

$$S[\omega,\Psi] = S_{\rm free}[\omega,\Psi] + \alpha\, S_{\rm int}[\omega,\Psi]\,, \qquad \alpha \in \mathbb{C}\,, \tag{3.19}$$

whose first piece,

$$\begin{aligned} S_{\rm free}[\omega,\Psi] = \int \Psi^{A(2s-t)}\, H_{AA} \wedge \nabla\omega_{A(2s-t-2)} + \Psi^{A'(t)}\, H_{A'A'} \wedge \nabla\omega_{A'(t-2)} \\ + \Psi^{A(2s-t+1),A'(t-1)}\, H_{AA} \wedge \nabla\omega_{A(2s-t-1),A'(t-1)}\,, \end{aligned} \tag{3.20}$$

is the sum of the free actions for the massless fields of spin $s-\frac{t}{2}$ and $\frac{t}{2}$ as well as for the partially-massless field of spin-s and depth-t, and the second piece is the current interaction

$$S_{\rm int}[\omega,\Psi] = \int \Psi^{A(2s-t)}\, \Psi^{A'(t)}\, \hat{e}_{AA'} \wedge \omega_{A(2s-t-1),A'(t-1)} \tag{3.21}$$

made out of the current associated with the previous pair of massless fields and the partially-massless field. Note that we will restrict ourselves to bosonic fields, and hence will assume that t is even. As already argued before, all of these pieces are invariant under shift symmetry. Moreover, the free action is invariant under the differential gauge symmetry

$$\delta_\epsilon \omega_{A(2s-t-2)} = \nabla\epsilon_{A(2s-t-2)}\,, \qquad \delta_\epsilon \omega_{A'(t-2)} = \nabla\epsilon_{A'(t-2)}\,, \tag{3.22}$$

for the massless fields, and

$$\delta_\xi \omega_{A(2s-t-1),A'(t-1)} = \nabla\xi_{A(2s-t-1),A'(t-1)}\,, \tag{3.23}$$

for the partially-massless field. Under this last gauge transformation, the variation of the current interaction term reads

$$\delta_\xi S_{\rm int}[\omega,\Psi] = \int \nabla\big(\Psi^{A(2s-t)}\, \Psi^{A'(t)}\big)\, \hat{e}_{AA'}\, \xi_{A(2s-t-1),A'(t-1)}\,, \tag{3.24}$$

and vanishes only on-shell as explained before. It can be compensated off-shell by deforming the gauge symmetry of the pair of massless fields as follows,

$$\delta_\xi \omega_{A(2s-t-2)} = +\tfrac{3}{2}\,\alpha\, \Psi^{A'(t)}\, e^B{}_{A'}\, \xi_{A(2s-t-2)B,A'(t-1)}\,, \tag{3.25a}$$

$$\delta_\xi \omega_{A'(t-2)} = -\tfrac{3}{2}\,\alpha\, \Psi^{A(2s-t)}\, e_A{}^{B'}\, \xi_{A(2s-t-1),A'(t-2)B'}\,, \tag{3.25b}$$

i.e. with terms depending on the gauge parameter of the partially-massless field. The variation of the free actions for the massless fields under this modification of their gauge symmetry then reads

$$\begin{aligned} \delta_\xi S_{\rm free}[\omega,\Psi] = -\tfrac{3}{2}\,\alpha \int \nabla\Psi^{A(2s-t)}\, \Psi^{A'(t)}\, H_{AA}\, e^B{}_{A'}\, \xi_{A(2s-t-2)B,A'(t-1)} \\ +\tfrac{3}{2}\,\alpha \int \Psi^{A(2s-t)}\, \nabla\Psi^{A'(t)}\, H_{A'A'}\, e_A{}^{B'}\, \xi_{A(2s-t-1),A'(t-2)B'}\,, \end{aligned} \tag{3.26}$$

which, upon using

$$H_{AA}\, e^B{}_{A'} = +\tfrac{2}{3}\, \hat{e}_{AA'}\, \delta_A^B\,, \qquad H_{A'A'}\, e_A{}^{B'} = -\tfrac{2}{3}\, \hat{e}_{AA'}\, \delta_{A'}^{B'}, \tag{3.27}$$

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can be brought to the form

$$\delta_\xi S_{\text{free}}[\omega,\Psi] = -\alpha \int \nabla \left(\Psi^{A(2s-t)}\,\Psi^{A'(t)}\right) \hat{e}_{AA}\,\xi_{A(2s-t-1),A'(t-1)}\,, \tag{3.28}$$

so that the full action (3.19) is gauge invariant. Note that the deformations (3.25) of the gauge symmetries are Abelian, which is not the case for the current interactions in the non-chiral formulation. A straightforward generalization of these current interactions is to take advantage of other conserved currents that involve derivatives, see e.g. [80]. Schematically they read $J_{2s_1+k,2s_2+k} \sim \Psi_{2s_1}\nabla^k\overline{\Psi}_{2s_2}$. In all these cases, except for $s_1 = s_2 = 0$, the action does not require any higher order corrections.

Note also that this type of interaction is simply a Noether coupling, which is similar to the one explored in [37]. The spectrum of the two resulting theories are however different: here, we find interactions between a partially-massless field of spin-s and even depth-t, and two massless fields of spin $s - \frac{t}{2}$ and $\frac{t}{2}$, whereas the interacting theory constructed in [37] involves only partially-massless spin-2 fields.

4 Discussion and conclusions

We have studied the simplest types of interactions: Yang-Mills and current ones. It would be interesting to classify all possible interactions within the new approach to partially-massless fields advocated in the present paper. For example, there should exist partially-massless theories featuring gravitational interactions. Another important omission is to have genuine non-Abelian higher spin higher derivative interactions. Such interactions, as different from, say, the Yang-Mills ones, introduce nontrivial constraints that fix the spectrum of a theory together with all the couplings.

The elephant in the room is twistor theory, which played an important, but silent, rôle in the paper. Indeed, the twistor approach directly leads to field variables $\Psi^{A(2s)}$ and $\omega^{A(2s-2)}$ for massless fields [38]. This was the starting point of our generalization to partially-massless fields. However, the original twistor formulation of partially-massless fields seems to be missing at the moment. It would be interesting to bridge this gap.

At least for the purely massless case there exists a complete, local higher spin gravity — Chiral Theory [81–84], which in addition to Yang-Mills and gravitational interactions incorporates genuine higher spin interactions. The theory admits any value of the cosmological constant, including zero. As was shown in [85], Chiral Theory has two contractions where the scalar field can be dropped while either Yang-Mills or gravitational interactions are kept (no genuine higher spin interactions are present). These two contractions have simple covariant actions [39] and twistor origin [75–77]. Within AdS/CFT duality, Chiral Theory should be dual to a subsector of Chern-Simons matter theories [86].

In view of the facts collected here-above, it looks plausible that there exist (Chiral) higher spin gravities with partially-massless fields in the spectrum [86]. These theories should admit contractions that feature either Yang-Mills or gravitational interactions, the former of which are considered in the present paper. Within AdS/CFT duality, such theories should be dual to a subsector of isotropic (Chern-Simons) Lifshitz CFT's [13], i.e. of vector models with higher-derivative kinetic terms.[9]

Lastly, it would be interesting to explore a family of deformations of the actions proposed in the paper via the Ψ^2-terms. Such deformation mimics the well-known result on how Yang-Mills theory can be represented as a deformation of the self-dual Yang-Mills theory [70]: $\Psi F(\omega)$-type actions need to be completed with Ψ^2-terms. This idea can be interesting already for free fields, resulting in a new second order action for partially-massless fields, which is still simpler than its cousins in terms of non-chiral field variables. For massless fields the Ψ^2-deformation was also shown to give higher spin theories with nontrivial scattering already in flat space [77].

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A Beyond maximal depth

As is clear from the discussion in section 2, the action (2.27) and equations of motion (2.28) are formally well-defined beyond the maximal depth $t = s$. Moreover, the number of physical degrees of freedom still follows the $2t$-track. While it is beyond the scope of the present paper to analyze the $t > s$ case in detail, let us make few remarks.

For $t = s + 1$, we are presented with the puzzle that the 0-form $\Psi^{A(s),A'(s)}$ is balanced, and hence in vector language corresponds to a symmetric tensor. It therefore cannot be related to any Weyl tensor, since the latter are always valued in two-row diagrams. For $t = s+2, \dots, 2s-1$, let us define $t = 2s-\tau$, with $\tau = 1, \dots, s-2$, so that the pairs of fields in these cases take the forms $(\omega^{A(\tau-1),A'(2s-\tau-1)}, \Psi^{A(\tau+1),A'(2s-\tau-1)})$. In this parametrization, the 1-form ω seems like the anti-self-dual part of the last connection for a spin-s field of depth-τ, but the 0-form does not have the required symmetry to be considered as the corresponding Weyl tensor. This can be traced back to the fact that we used the self-dual basis 2-forms H_{AA} in the action to contract the 0-form Ψ. Consequently, the number of unprimed indices in ω and Ψ differs by 2, but when crossing the boundary $t = s + 1$, this difference is now the source of the mismatch between the pairs of indices for them to be identified with the anti-self-dual part of the last connection and Weyl tensor for a partially-massless field.

[9] Chern-Simons extension of these models have not been explored so far. It also remains unclear if the $3d$ bosonization duality can be extended to these models.

More importantly, the equations of motion obtained in these cases do not describe the propagation of a partially-massless field: one can check that the first few descendants of the Weyl tensor which are not constrained by Bianchi identities do not generate the usual module of a PM anti-self-dual Weyl tensor. Indeed, consider a 0-form $\Psi^{A(t-1),A'(2s-t+1)}$ where the parametrization of its indices suggests that it corresponds to the anti-self-dual part of the Weyl tensor of a spin-s and depth-t PM field, subject to the equation of motion

$$H^{BB}\,\nabla\Psi_{A(t-3)BB,A'(2s-t-1)} \approx 0\,. \tag{A.1}$$

Then, one finds

$$\nabla\Psi_{A(t-1),A'(2s-t+1)} = e^{B}{}_{A'}\;\Psi_{A(t-1)B,A'(2s-t)} + e^{BB'}\;\Psi_{A(t-1)B,A'(2s-t+1)B'}\,, \tag{A.2}$$

instead of

$$\nabla\Psi_{A(t-1),A'(2s-t+1)} = e_{A}{}^{B'}\;\Psi_{A(t-2),A'(2s-t+1)B'} + e^{BB'}\;\Psi_{A(t-1)B,A'(2s-t+1)B'}\,, \tag{A.3}$$

as would be expected for the anti-self-dual part of a spin-s and depth-t Weyl tensor. One can notice that, though the second term on the right hand side of these two expressions are identical, the first one is not. In vector language, the expected spectrum of 0-forms is given by Young diagrams of the Lorentz group of the form

$$\begin{array}{|c|c|c|}\hline \multicolumn{2}{|c|}{s} & n \\ \hline s-t+1 & m & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\,, \tag{A.4}$$

with $n \geq 0$ and $m = 0,\dots,t-1$. This simply corresponds to the fact that the derivatives of the Weyl tensor that are unconstrained by equations of motion and Bianchi identities are those projected in the first two rows of the Weyl tensor Young diagram (in arbitrary number in the first row, or only up to $t-1$ in the second row). The equation (A.2) is not compatible with this because the two 0-forms appearing on the right hand side correspond to the diagrams

$$\begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{s} \\ \hline s-t & \times \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|}\hline s & \\ \hline s-t+1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{A.5}$$

so that in particular, the first diagram is unexpected (see [44, 71]), due to the fact that a box has been removed in the second row (crossed hereabove) instead of being added. Due to this early departure in the descendants of Ψ, the whole module generated by the infinite tower of 0-form required to build an FDA will not correspond to that of a PM Weyl tensor. Once again, this can be traced back to the fact that the expected equations (A.3) is the parametrization of a generic element in the kernel of the symplectic form determined by $H_{A'A'}$, i.e. it is a solution of $H^{B'B'}\Psi_{A(t-1),A'(2s-t-1)B'B'} \approx 0$.

A possible scenario would be that this system, for $t = s+k$ and $k = 1,\dots,s-1$, describes a reducible representation of $\mathfrak{g}_\Lambda$, composed of two massive fields of spin-s and $k-1$. A trivial, but necessary, check is that the counting of degrees of freedom is consistent, since $2t = 2s+1+2(k-1)+1$. A more significant hint, which motivates our conjecture, is that the spectrum of 0-forms in this case, represented in figure 3, agrees with this proposal.

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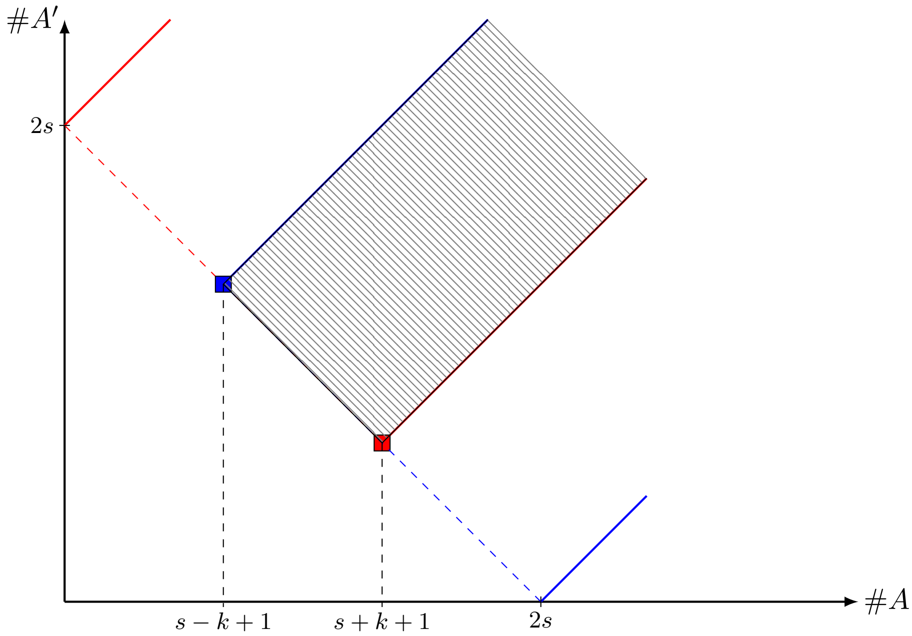

Figure 3. In blue, the region covered by descendants of $\Psi^{A(s-k+1),A'(s+k-1)}$, in red the descendants of $\Psi^{A(s+k-1),A'(s-k+1)}$ and in gray the overlap between these two regions.

Indeed, when the depth t goes beyond s, the two strips of 0-forms start overlapping. The whole region covered by these strips corresponds to the spectrum of 0-forms of a massive spin-s field [71], when each 0-form appears with multiplicity 1. The overlapping region could similarly be interpreted as the collection of 0-forms describing a massive spin-$(k-1)$ field, due to the width of this strip, but that would be represented by spin-tensors of higher ranks than expected. In other words, this massive spin-$(k-1)$ field could appear in our system as a spin-tensor, which, due to some equation of motion, should be expressed as derivative of a lower rank spin-tensor, the latter being the genuine massive spin-$(k-1)$ field. Note that this is to be taken, for the time being, only as a proposal since proving rigorously the above statement would go beyond the scope of this paper, and is left for potential future work.

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