

(Weyl-)Fefferman-Graham asymptotic symmetries

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Based on [L. Ciambelli, AD, R. Ruzziconi, C. Zwickel (2308.15480)]

I. Plan and Motivations

II. Fefferman-Graham

III. Weyl-Fefferman-Graham

IV. Summary and Future Possibilities

I. Plan and Motivations 1/2

- Probe the AdS/CFT correspondence

Study of the classical phase space of **asymptotically AdS gravity**

Asymptotic symmetries of AdS \leftrightarrow global symmetries of CFT

- Corner proposal [Donnelly-Freidel '16, Speranza '18, Ciambelli-Leigh '21, ...]

Gravitational theory \leftarrow **charges** and their algebra at **corners**

Organising principle for quantum observables

- Focus on **3D gravity**: no propagation of degrees of freedom

Presence of black holes [Banados-Teitelboim-Zanelli '92]

Asympt. symmetry enhancement w.r.t. vacuum isometries [Brown-Henneaux '86]

I. Plan and Motivations 2/2

- Select the allowed metric fluctuations at infinity

No requirement to fix any particular gauge but it is often convenient

For example: **Fefferman-Graham, Bondi gauge** [Starobinsky '83, Fefferman-Graham '85]
[Bondi-van der Burg-Metzner '62, Sachs '62]

- **Charged diffeomorphisms** map inequivalent physical configurations

Symmetries used to gauge-fix can be charged (see, e.g., [Geiller-Goeller-Zwikel '21])

- In this talk: **Weyl-Fefferman-Graham gauge** [Ciambelli-Leigh '19, Jia-Karydas '21]

Interesting implications for holography

I. Plan and Motivations

II. Fefferman-Graham

III. Weyl-Fefferman-Graham

IV. Summary and Future Possibilities

II. Fefferman-Graham gauge in 3D: definition

- Useful **gauge fixing** for holography: [Fefferman-Graham '85]

Any asymptotically AdS₃ space can be written near the boundary as ($\ell = 1$)

$$ds_{\text{AdS}}^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{d\rho^2}{\rho^2} + h_{ij}(\rho, x) dx^i dx^j$$

ρ is a spacelike coordinate s.t. $\rho = 0$ locates the bdy

x^i are bdy coords: $x^1 = t$ is timelike and $x^2 = \theta$ is spacelike

- Boundary geometry** \rightarrow leading order in the asymptotic expansion

$$h_{ij}(\rho, x) = \frac{1}{\rho^2} h_{ij}^{(0)}(x) + \mathcal{O}(1)$$

- Induced metric** on the bdy may be defined if $g \rightarrow \Omega^2 g$ [Penrose '63]

Ambiguity: $\Omega \rightarrow \Omega' = e^\sigma \Omega \Rightarrow h^{(0)} \rightarrow h_\sigma^{(0)} = e^{2\sigma} h^{(0)}$, σ indep. of ρ

Bulk metric induces a **bdy conformal class** $[h^{(0)}]$ [Rooman-Spindel '00]

II. FG gauge in 3D: solution space

- **Finite** FG expansion in 3D:

$$h_{ij}(\rho, x) = \rho^{-2} h_{ij}^{(0)}(x) + h_{ij}^{(2)}(x) + \rho^2 h_{ij}^{(4)}(x)$$

with

$$h_{ij}^{(4)} = \frac{1}{4} h_{ik}^{(2)} h_{(0)}^{kl} h_{lj}^{(2)}, \quad h_{(0)}^{ij} h_{ij}^{(2)} = -\frac{1}{2} R^{(0)}, \quad D_{(0)}^i h_{ij}^{(2)} = -\frac{1}{2} \partial_j R^{(0)}$$

- **Holographic stress-energy tensor:** [Henningson-Skenderis '98, Balasubramanian-Kraus '99]

$$T_{ij} = \frac{1}{8\pi G} \left(h_{ij}^{(2)} + \frac{1}{2} h_{ij}^{(0)} R^{(0)} \right)$$

such that ($c = \frac{3\ell}{2G}$ is the BH central charge)

$$T_i{}^i = \frac{c}{24\pi} R^{(0)}, \quad D_i^{(0)} T^{ij} = 0$$

- Bdy **conformally-flat** parametrization: [Alessio-Barnich-Ciambelli-Mao-Ruzziconi '20]

$$h_{ij}^{(0)}(x) = e^{2\phi(x)} \eta_{ij}$$

II. FG gauge in 3D: symmetries

- Residual symmetries:** $(\mathcal{L}_\xi g_{\rho\rho} = 0, \mathcal{L}_\xi g_{\rho i} = 0, \mathcal{L}_\xi h_{ij} = \mathcal{O}(\rho^{-2}))$

$$\xi^\rho = \rho\omega + \mathcal{O}(\rho^3), \quad \xi^\pm = Y^\pm + \rho^2\zeta^\pm + \mathcal{O}(\rho^4)$$

where $(x^\pm = \theta \pm t)$

$$\begin{aligned}\omega(x^+, x^-) &= -\sigma + \frac{1}{2}\partial_i Y^i + Y^i \partial_i \phi, \\ \zeta^\pm(x^+, x^-) &= e^{-2\phi} \left[\partial_\mp \sigma - (Y^i \partial_i + \partial_\mp Y^\mp) \partial_\mp \phi - \frac{1}{2} \partial_\mp \partial_i Y^i \right]\end{aligned}$$

- Modified Lie bracket:** [Barnich-Troessaert '10]

$$[\xi_1, \xi_2]_M := [\xi_1, \xi_2] - \delta_{\xi_1} \xi_2 + \delta_{\xi_2} \xi_1$$

Residual algebra: $\text{Witt} \oplus \text{Witt} \oplus \mathfrak{u}(1)$

$$\hat{Y}^\pm = Y_1^\pm \partial_\pm Y_2^\pm - Y_2^\pm \partial_\pm Y_1^\pm, \quad \hat{\sigma} = 0$$

- Gauge transformations:**

$$\delta_\xi l_\pm = Y^\pm \partial_\pm l_\pm + 2l_\pm \partial_\pm Y^\pm - \frac{1}{2} \partial_\pm^3 Y^\pm, \quad \delta_\xi \phi = \sigma$$

II. FG gauge in 3D: holographic renormalization

- **Renormalized action:** [Henningson-Skenderis '98, Papadimitriou-Skenderis '05]

$$S_{ren} = \frac{1}{16\pi G} \int d^3x (R + 2) + \frac{1}{8\pi G} \int d^2x \sqrt{-\gamma} (K - 1) \\ + \frac{\rho^2 \log \rho}{16\pi G} \int d^2x \sqrt{-\gamma} R^{(0)}$$

where $n_\mu = -\sqrt{-\gamma} \delta_\mu^\rho$, $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and $K = g^{\mu\nu} \nabla_\mu n_\nu$

- On-shell variation of the renormalized action:

$$\delta S_{ren} \approx \frac{1}{2} \int d^2x \sqrt{-h^{(0)}} T^{ij} \delta h_{ij}^{(0)} = \frac{c}{24\pi} \int d^2x \sqrt{-h^{(0)}} R^{(0)} \delta\phi$$

Under Weyl diffeos: presence of **Weyl anomaly** ($\delta_\sigma h_{ij}^{(0)} = 2\sigma h_{ij}^{(0)}$)

$$\delta_\sigma S_{ren} = \int d^2x \sqrt{-h^{(0)}} \mathcal{A} \sigma, \quad \mathcal{A} = \frac{c}{24\pi} R^{(0)}$$

II. FG gauge in 3D: charges

- Renormalized symplectic current:

$$\delta S_{ren} \approx \int d^2x \Theta_{ren}, \quad \omega_{ren} = \delta \Theta_{ren} = \frac{1}{2} \delta \left(\sqrt{-h^{(0)}} T^{ij} \right) \wedge \delta h_{ij}^{(0)}$$

- Surface charges: finite, integrable, non conserved

$$Q_\xi = -\frac{1}{8\pi G} \int_0^{2\pi} d\theta \left(\ell_+ Y^+ - \ell_- Y^- + \phi \partial_t \sigma - \sigma \partial_t \phi \right)$$

- Charge algebra: $\{Q_{\xi_1}, Q_{\xi_2}\} = \delta_{\xi_2} Q_{\xi_1} = Q_{[\xi_1, \xi_2]_M} + \mathcal{K}[\xi_1, \xi_2]$

\hookrightarrow Virasoro \oplus Virasoro \oplus Weyl ($Y^\pm \sim e^{inx^\pm}$, $\sigma \sim e^{ipx^+} e^{iqx^-}$)

$$\{Q_{\xi_n^{Y^\pm}}, Q_{\xi_m^{Y^\pm}}\} = i(n-m)Q_{\xi_{n+m}^{Y^\pm}} - im^3 \frac{c}{12} \delta_{n+m,0}$$

$$\{Q_{\xi_{pq}^{\sigma}}, Q_{\xi_{rs}^{\sigma}}\} = -i(r-q) \frac{c}{3} e^{2i(q+s)t} \delta_{p+r, q+s}$$

II. FG gauge in 3D: Penrose-Brown-Henneaux transfos

- **Issue:** Under a Weyl transformation on the boundary [Henningson-Skenderis '98]

$$\rho \rightarrow \rho' = \frac{\rho}{\mathfrak{B}(x)}, \quad x^i \rightarrow x'^i = x^i + \xi^i(\rho, x)$$

↪ FG expansion does not transform Weyl-covariantly

- **Solution:** relax the FG ansatz to the **WFG** gauge [Ciambelli-Leigh '19]

$$ds_{\text{AdS}}^2 = \left(\frac{d\rho}{\rho} - k_i(\rho, x) dx^i \right)^2 + h_{ij}(\rho, x) dx^i dx^j, \quad k_i(\rho, x) = k_i^{(0)}(x) + \mathcal{O}(\rho^2)$$

↪ $h_{ij}^{(0)}(x)$ is the bdy metric and $k_i^{(0)}(x)$ is a bdy **Weyl connection**

- (1) Weyl rescalings at the bdy ← purely radial bulk diffeos
- (2) Full Weyl geometry at the bdy

- **Question:** Is the Weyl structure associated with an asymptotic symmetry?

I. Plan and Motivations

II. Fefferman-Graham

III. **Weyl-Fefferman-Graham**

IV. Summary and Future Possibilities

III. Weyl-Fefferman-Graham gauge in 3D: PBH transfos

- **Reminder:** WFG ansatz

$$ds_{\text{AdS}}^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(\frac{d\rho}{\rho} - k_i(\rho, x) dx^i \right)^2 + h_{ij}(\rho, x) dx^i dx^j$$

- Under radial diffeos inducing **bdy Weyl transfos:** WFG is preserved

$$\rho \rightarrow \rho' = \frac{\rho}{\mathfrak{B}(x)}, \quad x^i \rightarrow x'^i = x^i$$

↪ radial expansion transforms **Weyl-covariantly**

$$k_i^{(2n)}(x) \rightarrow k_i^{(2n)}(x) \mathfrak{B}(x)^{2n} - \delta_{n,0} \partial_i \ln \mathfrak{B}(x),$$

$$h_{ij}^{(2n)}(x) \rightarrow h_{ij}^{(2n)}(x) \mathfrak{B}(x)^{2n-2}$$

- The leading term in k_i transforms as a **Weyl connection**

$$k_i^{(0)} \rightarrow k_i^{(0)} - \partial_i \ln \mathfrak{B}$$

III. WFG gauge in 3D: geometry 1/2

- Choice of **dual form and vector basis** for the WFG ansatz:

$$E^\rho = \frac{d\rho}{\rho} - k_i(\rho, x)dx^i, \quad E^i = dx^i,$$

$$E_\rho = \rho\partial_\rho \equiv D_\rho, \quad E_i = \partial_i + \rho k_i(\rho, x)\partial_\rho \equiv D_i$$

- Coefficients of bulk **Levi-Civita connection** ∇ in $\{D_\rho, D_i\}$:

$$\nabla_{D_i} D_j = \Gamma_{ij}^k D_k + \Gamma_{ij}^\rho D_\rho$$

\hookrightarrow at leading order:

$$(\Gamma^{(0)})_{ij}^k = \frac{1}{2} h_{(0)}^{kl} \left[(\partial_i - 2k_i^{(0)})h_{jl}^{(0)} + (\partial_j - 2k_j^{(0)})h_{il}^{(0)} - (\partial_l - 2k_l^{(0)})h_{ij}^{(0)} \right]$$

\hookrightarrow torsion-free connection with Weyl metricity [G. B. Folland '70]

III. WFG gauge in 3D: geometry 2/2

- **Induced connection** $\nabla^{(0)}$: on the boundary metric

$$\nabla_i^{(0)} h_{jk}^{(0)} = 2k_i^{(0)} h_{jk}^{(0)}$$

- **Weyl covariant connection** $\hat{\nabla}^{(0)}$: on a generic Weyl-weight ω_T tensor

$$\hat{\nabla}_i^{(0)} T \equiv \nabla_i^{(0)} T + \omega_T k_i^{(0)} T$$

$\hookrightarrow \hat{\nabla}^{(0)}$ is metric and $\hat{\nabla}_i^{(0)} T$ is Weyl covariant

- All geometric quantities built with $\hat{\nabla}^{(0)}$ are Weyl covariant
- **Conclusion**: WFG is equipped with a **full Weyl geometry**, a metric $h_{ij}^{(0)}$ and a Weyl connection $k_i^{(0)}$ at the bdy
- Motivated in the ambient construction [Jia-Karydas-Leigh '23]

III. WFG gauge in 3D: solution space and symmetries

- **Infinite** WFG expansion in 3D:

$$h_{ij} = \rho^{-2} h_{ij}^{(0)} + h_{ij}^{(2)} + \rho^2 h_{ij}^{(4)} + \mathcal{O}(\rho^6), \quad k_i = k_i^{(0)} + \rho^2 k_i^{(2)} + \mathcal{O}(\rho^4)$$

where $(x^\pm = \theta \pm t, K_i^{(0)} = k_i^{(0)} - \partial_i \phi, \partial_\pm \ell_\mp = 0)$

$$h_{\pm\pm}^{(0)} = 0, \quad h_{\pm\pm}^{(2)} = \ell_\pm - (K_\pm^{(0)})^2 - \partial_\pm K_\pm^{(0)}, \quad \text{etc.}$$

$$h_{+-}^{(0)} = \frac{1}{2} e^{2\phi}, \quad h_{+-}^{(2)} = -\frac{1}{2} (\partial_- K_+^{(0)} + \partial_+ K_-^{(0)})$$

- **Residual symmetries:** $\text{Witt} \oplus \text{Witt} \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$

$$\xi^\rho = \rho \omega + \mathcal{O}(\rho^3), \quad \xi^\pm = Y^\pm + \rho^2 \zeta^\pm + \mathcal{O}(\rho^4)$$

where $(H_i^{(0)} = h_i^{(0)} - \partial_i \sigma, \omega = -\sigma + \frac{1}{2} \partial_i Y^i + Y^i \partial_i \phi)$

$$\zeta^\pm = e^{-2\phi} \left(K_\mp^{(0)} \partial_\mp Y^\mp - H_\mp^{(0)} - \frac{1}{2} \partial_\mp \partial_i Y^i + Y^i \partial_i K_\mp^{(0)} \right)$$

III. WFG gauge in 3D: holographic renormalization

- **Renormalized action:** ($n_\mu = -\sqrt{-\gamma}\delta_\mu^\rho$, $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and $K = g^{\mu\nu}\nabla_\mu n_\nu$)

$$S_{ren} = \frac{1}{16\pi G} \int d^3x (R + 2) + \frac{1}{8\pi G} \int d^2x \sqrt{-\gamma} (K - 1) \\ + \frac{\rho^2 \log \rho}{16\pi G} \int d^2x \sqrt{-\gamma} \hat{R}^{(0)} + \frac{1}{16\pi G} \int d^2x \sqrt{-\gamma} k_i \gamma^{ij} k_j$$

- **Renormalized presymplectic potential:**

$$\Theta_{ren} = -\sqrt{-h^{(0)}} \left(\frac{1}{2} T^{ij} \delta h_{ij}^{(0)} - J^i \delta k_i^{(0)} \right)$$

where

$$T^{ij} = -\frac{2}{\sqrt{-h^{(0)}}} \frac{\delta S_{ren}}{\delta h_{ij}^{(0)}} \approx \frac{1}{8\pi G} \left(h_{(2)}^{ij} + \frac{1}{2} h_{(0)}^{ij} R^{(0)} + \hat{\nabla}_{(0)}^{(i} k_{(0)}^{j)} \right), \\ J^i = \frac{1}{\sqrt{-h^{(0)}}} \frac{\delta S_{ren}}{\delta k_i^{(0)}} \approx \frac{1}{8\pi G} k_i^{(0)}$$

III. WFG gauge in 3D: Ward identities and charges

- Under **bdy diffeos**: $(\delta_\xi h_{ij}^{(0)} = \mathcal{L}_Y h_{ij}^{(0)} - 2\omega h_{ij}^{(0)}, \delta_\xi k_i^{(0)} = \mathcal{L}_Y k_i^{(0)} - \partial_i \omega - 2\zeta_i)$

$$\nabla_i^{(0)} T^i_j = J^j f_{ij}^{(0)} + \nabla_i^{(0)} J^j k_j^{(0)}$$

where

$$f_{ij}^{(0)} = \nabla_i^{(0)} k_j^{(0)} - \nabla_j^{(0)} k_i^{(0)}$$

- Under **bdy Weyl transfos**: Weyl-covariant Weyl anomaly

$$T^i_i + \hat{\nabla}_i^{(0)} J^i = \frac{c}{24\pi} \hat{R}^{(0)}$$

- **Surface charges**: finite, integrable, non conserved

$$Q_\xi = -\frac{1}{8\pi G} \int_0^{2\pi} d\theta \left(\ell_+ Y^+ - \ell_- Y^- + \phi \partial_t \sigma - \sigma \partial_t \phi \right)$$

↔ The **Weyl connection** is **not** associated with an **asymptotic symmetry**!

III. WFG gauge in 3D: Chern-Simons

- Isometry algebra of $\text{AdS}_3 = \mathfrak{so}(2,2)$

$$[M_a, M_b] = \epsilon_{abc} M^c, \quad [M_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \left(\frac{G}{\ell}\right)^2 \epsilon_{abc} M^c$$

- Dreibein or first order formalism ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$)

$$g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b, \quad de^a + \epsilon^{abc} \omega_b \wedge e_c = 0$$

- **Einstein gravitation** \rightarrow **Chern-Simons** gauge theory ($\mathcal{A} = \frac{1}{G} e^a P_a + \omega^a M_a$)
[Achucarro-Townsend '86, Witten '88]

$$S_{EH} = \frac{1}{16\pi} \int \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)$$

- **Surface charges** for WFG: **Weyl connection** \rightarrow **asymptotic symmetry!**

$$Q_\xi = -\frac{1}{8\pi G} \int_0^{2\pi} d\theta \left(\ell_+ Y^+ - \ell_- Y^- - \phi(h_t^{(0)} - \partial_t \sigma) + \sigma(k_t^{(0)} - \partial_t \phi) \right)$$

III. WFG gauge in 3D: finite boundary term

- **Option 1:** Add a **finite bdy counterterm** to the action

$$\bar{S}_{ren} = S_{ren} + S_o, \quad S_o = \int d^2x L_o[h_{ij}^{(0)}, k_i^{(0)}]$$

with the following choice

$$L_o = \lim_{\rho \rightarrow 0} \left[\frac{1}{16\pi G} k_i \gamma^{ij} \partial_j \sqrt{-\gamma} \right] = \frac{1}{16\pi G} k_i^{(0)} h_{(0)}^{ij} \partial_j \sqrt{-h^{(0)}}$$

- **Presymplectic potential:**

$$\bar{\Theta}_{ren} = -\sqrt{-h^{(0)}} \left(\frac{1}{2} \bar{T}^{ij} \delta h_{ij}^{(0)} - \bar{J}^i \delta k_i^{(0)} \right)$$

where $(J^i = \frac{1}{8\pi G} k_{(0)}^i)$

$$\bar{T}^{ij} = T^{ij} + J^{(i} \partial^{j)} \sqrt{-h^{(0)}} + \frac{1}{2} h_{(0)}^{ij} \nabla_k^{(0)} J^k,$$

$$\bar{J}^i = J^i + \frac{1}{16\pi G} \partial^i \log \sqrt{-h^{(0)}}$$

III. WFG gauge in 3D: finite corner term

- **Option 1:** Add a **finite corner counterterm** to the action [McNees-Zwikel '23]

$$\tilde{S}_{ren} = S_{ren} + S_C, \quad S_C = \int d^2x \partial_i L_C^i[h_{ij}^{(0)}, k_i^{(0)}]$$

with the following choice

$$L_C^i = \lim_{\rho \rightarrow 0} \left[-\frac{1}{16\pi G} \sqrt{-\gamma} \gamma^{ij} k_j \right] = -\frac{1}{16\pi G} \sqrt{-h^{(0)}} h_{(0)}^{ij} k_j^{(0)}$$

- **Presymplectic potential:**

$$\tilde{\Theta}_{ren} = -\sqrt{-h^{(0)}} \left(\frac{1}{2} \tilde{T}^{ij} \delta h_{ij}^{(0)} - J^i \delta K_i^{(0)} \right)$$

where

$$\tilde{T}^{ij} = T^{ij} + \frac{1}{2} h_{(0)}^{ij} \hat{\nabla}_k^{(0)} J^k, \quad K_i^{(0)} = k_i^{(0)} - \frac{1}{2} \partial_i \ln \sqrt{-h^{(0)}}$$

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IV. Summary

Main goal:

- Explore the charges of 3D gravity in **Weyl-Fefferman-Graham** gauge

Results:

- Restored Weyl covariance of the boundary
- Residual symmetries, surface charges and anomalies
- FG fixing can constrain the physical solution space

Future possibilities:

- Explicit diffeomorphisms, synthesis of other gauge relaxation proposals
[Grumiller-Riegler '16, Geiller-Goeller-Zwikel '21, Campoleoni-AD-Ciambelli-Marteau-Petropoulos-Ruzziconi '22]
- Extension to higher dimensions [Ciambelli-Leigh '19, Petkou-Petropoulos-Rivera-Siampos '22, Campoleoni-AD-Pekar-Petropoulos-Rivera-Vilatte '23]
- Coupling to higher spins [Campoleoni-Fredenhagen-Pfenninger-Theisen '10, Li-Theisen '15, WIP with A. Campoleoni]

IV. Summary



Magritte style painting of Weyl, Fefferman and Graham at PI. Created with the assistance of DALL-E 2.

Thank you for listening!