HOMOTOPY CARTAN CALCULUS AND INNER DEFORMATIONS OF A_∞-ALGEBRAS

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ABSTRACT. We consider inner deformations of families of A_∞-algebras. With the help of noncommutative Cartan's calculus, we prove the invariance of Hochschild (co)homology under inner deformations. The invariance also holds for cyclic cohomology classes that satisfy some additional conditions. Applications to dg-algebras and QFT problems are briefly discussed.

1. INTRODUCTION

Strong homotopy algebras, mostly L_∞, but A_∞ too, have a long track record of applications in physics. Every gauge theory, for one, leads naturally to an L_∞-algebra within the Batalin–Vilkovisky formalism [1, 2]. A_∞/L_∞-algebras are at the heart of open/closed string field theory [3, 4, 5]. It is hardly possible to overestimate the organizing role of L_∞-algebra language in the final solution of the deformation quantization problem [6].

Our motivation for this work stems from the two recent incarnations of L_∞-algebras: (i) there is a large class of integrable models [7] relied on L_∞-algebras; (ii) L_∞-algebras occur as generalized global symmetries of certain conformal field theories (CFT). In both cases the relevant L_∞-algebras originate from A_∞ ones through the symmetrization map (see Theorem 2.9, below). The study of observables and invariants in the models of (i) and (ii) calls for algebraic techniques we develop in the present paper. Let us elaborate on item (ii), which is, perhaps, the first occurrence of L_∞ structures as a kind of global symmetry in physics. In quantum field theory (QFT), by an (infinitesimal) symmetry one usually means a Lie algebra action on a set of fields or operators which form, thereby, its module. One may regard these data as a very degenerate L_∞-algebra with bilinear structure maps only. For example, this situation is realized for Chern–Simons matter vector models in the large-N limit. What happens when one departs from this limit is that the Lie algebra and its module deform into an L_∞-algebra, providing thus mathematical grounds [8] to what was dubbed *slightly broken higher-spin symmetry* in [9]. One application of the present paper is construction of various invariants, e.g. correlation functions, in such models [10].

All A_∞-algebras underlying the L_∞-algebras of items (i) and (ii) share one important property: each of them is obtained by *inner deformation* of a family of dg-algebras. The concept of inner deformation was introduced in our recent paper [11]. In that paper, we show how to construct a minimal A_∞-algebra starting from *any* one-parameter family of dg-algebras A. The

resulting A_∞-structure extends the binary multiplication in A by higher structure maps[1]. The construction is algorithmic and requires no extra data, hence the name. From the physical viewpoint, adding a coherent set of higher products is equivalent to inclusion of interaction in a free gauge model. One may wonder what happens to the (co)homology of A, regarded as a graded associative algebra, upon inner deformations. In Sec. 7, we answer this question by showing that every inner deformation of an A_∞-algebra extends to a deformation of its Hochschild (co)cycles (Theorem 7.1). In other words, Hochschild (co)homology is invariant under inner deformations of families. Furthermore, the statement holds true for some classes of cyclic (co)homology, as is show in Sec. 8.

Our proofs make significant use of the noncommutative Cartan's calculus up to homotopy, as developed in Refs. [12, 13, 14, 15, 16]. Therefore, in the next Sec. 2-5, we recall relevant definitions and constructions. Guidance is also given, whenever necessary or helpful, on further reading. The original results of the paper are exposed in Sec. 6-8, where we consider deformation flows on families of A_∞-algebras and their (co)homology.

2. Gerstenhaber Algebras up to Homotopy

Throughout the paper we work over a fixed ground field k of characteristic zero. All unadorned tensor products $\otimes$ and Hom's are defined over k. When dealing with graded vector spaces and algebras, we tend to write down all sign factors explicitly. However, one can easily keep track of all signs using the Koszul rule: whenever two graded objects a and B are swapped, the factor of $(-1)^{|a||B|}$ appears.

Given a pair of cochain complexes (C, δ) and (K, ∂), we say that a homomorphism $f : C^{\otimes n} \to K$ of degree $|f| \in \mathbb{Z}$ is a $\delta\partial$-morphism if

$$\partial f = (-1)^{|f|} f \delta_n \,,$$

where

$$\delta_n(a_1 \otimes \cdots \otimes a_n) = \sum_{k=1}^n (-1)^{|a_1| + \cdots + |a_{k-1}|} a_1 \otimes \cdots \otimes \delta a_k \otimes \cdots \otimes a_n$$

is the standard differential in the n-th tensor power of the complex C. Clearly, a $\delta\partial$-morphism of degree zero is just a cochain transformation.

A $\delta\partial$-morphism $f : C^{\otimes n} \to K$ is said to be *homotopic to zero* (in writing $f \simeq 0$) if there exists a homomorphism $g : C^{\otimes n} \to K$ of degree $|g| = |f| - 1$ such that

$$f = \partial g - (-1)^{|g|} g \delta_n \,.$$

In the special case that $C = K$ and $\delta = \partial$ we speak about δ-*morphisms*.

Definition 2.1. A *Gerstenhaber algebra up to homotopy* is a cochain complex (G, δ) composed of k-vector spaces and endowed with a pair of binary multiplication operations: a cup product $\cup$ of degree 1 and a bracket $[\ ,\ ]$

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[1]Therefore it is not just a minimal model of A.

of degree 0. In addition to k-bilinearity the multiplication operations are supposed to satisfy the following set of axioms:

(hG1) $\delta(a \cup b) = \delta a \cup b + (-1)^{|a|+1} a \cup \delta b$,

(hG2) $\delta[a, b] = [\delta a, b] + (-1)^{|a|}[a, \delta b]$,

(hG3) $(a \cup b) \cup c \simeq a \cup (b \cup c)$,

(hG4) $a \cup b \simeq (-1)^{(|a|+1)(|b|+1)} b \cup a$,

(hG5) $[a, b] \simeq -(-1)^{|a||b|}[b, a]$,

(hG6) $[a, [b, c]] \simeq [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$,

(hG7) $[a, b \cup c] \simeq [a, b] \cup c + (-1)^{|a|(|b|+1)} b \cup [a, c]$

for all homogeneous elements $a, b, c \in G$.

In what follows, we will abbreviate the verbose term 'Gerstenhaber algebra up to homotopy' to hG-algebra.

The first two axioms imply that either multiplication operation defines a δ-morphism from $G \otimes G$ to G. More precisely, the cup product gives the δ-morphism $m(a, b) = (-1)^{|a|-1} a \cup b$, while the bracket defines a δ-morphism as it is[2]. Then, relations (hG3,4) say that the cup product is associative and graded commutative up to homotopy. For example, unfolding the term 'up to homotopy' for (hG3) we get

$$(2.1) \qquad (a \cup b) \cup c - a \cup (b \cup c)$$

$$= \delta A(a, b, c) + A(\delta a, b, c) + (-1)^{|a|} A(a, \delta b, c) + (-1)^{|a|+|b|} A(a, b, \delta c)$$

for some homomorphism $A : G \otimes G \otimes G \to G$ of degree 1. Axioms (hG5,6) identify the bracket as a Lie bracket up to homotopy. Finally, relation (hG7) expresses compatibility between the two multiplication operations (the derivation property up to homotopy).

Example 2.2. Every cochain complex (C, δ) of k-vector spaces is an hG-algebra whose cup product and bracket are equal to zero.

Example 2.3. Every commutative dg-algebra (A, δ) gives rise to an hG-algebra on $G = A[1]$ w.r.t. the cup product $a \cup b = a \cdot b$ and the zero bracket[3]. Similarly, one can think of each dg-Lie algebra (L, δ) as an hG-algebra with zero cup product.

Example 2.4. If $\delta = 0$, the above definition of an hG-algebra reduces to that of a *Gerstenhaber algebra*, see [17]. This is defined by Axioms (hG3-7) with $\simeq$ replaced by the strict equality. In the general case, both the cup product and the bracket pass through the δ-cohomology and induce a Gerstenhaber algebra structure on $H(G)$.

A prototypical example of Gerstenhaber algebras is the algebra of polyvector fields $\mathcal{V}(M)$ on a smooth manifold M. If $\mathcal{V}^n(M)$ is the space of n-vector fields, then we set $G^{n-1} = \mathcal{V}^n(M)$. The role of cup product is

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[2]Notice that the graded Leibniz rules (hG1) and (hG7) are compatible with Koszul's sign convention if one thinks of $\cup$ as an object of degree 1.

[3]By definition, $A[m]$ is a graded vector space with $A[m]_n = A_{n+m}$.

played here by the wedge product on polyvectors and the Lie bracket is given by the Schouten–Nijenhuis bracket.

In the examples above, all Axioms (hG1-7) were satisfied in the strong sense, and not just 'up to homotopy'. To come up with genuine examples of hG-algebras, we need to introduce the notions of A_∞- and L_∞-algebras. These are defined as follows.

Let $V = \bigoplus V_n$ be a $\mathbb{Z}$-graded vector space over k. Denote by $C_\bullet(V)$ the linear space of the reduced tensor algebra

$$TV = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

and let $C^\bullet(V)$ denote the space of homomorphisms $\mathrm{Hom}(TV, V)$. The latter is known to carry the structure of a graded Lie algebra w.r.t. the *Gerstenhaber bracket*:

$$(2.2) \qquad [A, B] = A \circ B - (-1)^{|A||B|} B \circ A,$$

where

$$(2.3) \qquad (A \circ B)(a_1, a_2, \ldots, a_{m+n-1})$$

$$= \sum_{k=0}^{n-1} (-1)^{|B|(|a_1|+\cdots+|a_k|)} A(a_1, \ldots, a_k, B(a_{k+1}, \ldots, a_{k+m}), \ldots, a_{m+n-1})$$

for all homogeneous $A \in C^n(V)$ and $B \in C^m(V)$. Here $|A|$ denotes the degree of A as a homomorphism of graded vector spaces. It follows from the definition that $[A, A] = 2A \circ A$ for every odd A.

Definition 2.5. An A_∞-algebra structure on a graded vector space $V = \bigoplus V_n$ is given by a Maurer–Cartan element m of the graded Lie algebra $(C^\bullet(V), [\ ,\ ])$, that is,

$$(2.4) \qquad m \circ m = 0, \qquad |m| = 1.$$

The pair (V, m) is called an A_∞-*algebra*.

By definition, each A_∞-structure m is given by an (infinite) sum

$$(2.5) \qquad m = m_1 + m_2 + m_3 + \ldots$$

of multi-linear maps $m_n \in C^n(V)$ of degree 1. It follows from (2.4) that the first structure map $m_1 : V \to V$ squares to zero; and hence, it makes V into a cochain complex with differential $\delta = m_1$. An A_∞-algebra is called *minimal* if $m_1 = 0$. The second structure map $m_2 : V \otimes V \to V$ defines the multiplication operation $a \cup b = (-1)^{|a|-1} m_2(a, b)$, which is a cochain transformation of the complexes, i.e., δ differentiates the cup product by the Leibniz rule. The product obeys Rel. (2.1) with $A(a, b, c) = (-1)^{|b|-1} m_3(a, b, c)$, meaning associativity up to homotopy. Thus, we arrive at the following statement.

Proposition 2.6. *Let (V, m) be an A_∞-algebra with m_2 obeying the relation*

$$(2.6) \qquad m_2(a, b) = -(-1)^{|a||b|}m_2(b, a)$$

for all homogeneous $a, b \in V$. Then V is a hG-algebra with

$$\delta a = m_1(a), \qquad a \cup b = (-1)^{|a|-1}m_2(a, b), \quad \text{and} \quad [a, b] = 0.$$

The skew-symmetry condition (2.6) admits a coherent extension to all higher structure maps leading to the notion of a C_∞-*algebra*, see e.g. [18], [19]. Therefore, each C_∞-algebra generates an hG-algebra with trivial bracket.

Like A_∞-algebras, L_∞-algebras can be expressed in terms of n-ary operations subject to a coherent set of quadratic relations.

Definition 2.7. An L_∞-algebra structure on a graded vector space $V = \bigoplus V_n$ is given by a sequence of homomorphisms $l_n : V^{\otimes n} \to V$, $n = 1, 2, \ldots$, of degree 1 that are graded symmetric,

$$l_n(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = (-1)^{|a_i||a_{i+1}|}l_n(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n),$$

and satisfy the 'generalized Jacobi identities':

$$\sum_{i+n=m} \sum_{\sigma \in \text{Sh}(i,n)}(-1)^\kappa l_{n+1}(l_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), a_{\sigma(i+1)}, \ldots, a_{\sigma(i+n)}) = 0$$

for all $m = 1, 2, \ldots$ Here $(-1)^\kappa$ is the Koszul sign resulting from permutations of homogeneous vectors $a_1, \ldots, a_n \in V$ and the second sum is over all (i, n)-shuffles, that is, permutations $\sigma \in S_{i+n}$ satisfying $\sigma(k) < \sigma(k+1)$ for $k \neq i$.

It follows from the definition that l_1 is a differential of degree 1, i.e., $l_1^2 = 0$. The second structure map l_2 endows $V[-1]$ with the skew-symmetric bracket $[a, b] = (-1)^{|a|+1}l_2(a, b)$, which is differentiated by l_1 and obeys the Jacobi identity up to homotopy. Explicitly,

$$[[a, b], c] + (-1)^{(|a|+1)(|b|+|c|)}[[b, c], a] + (-1)^{(|c|+1)(|a|+|b|)}[[c, a], b]$$

$$= \delta J(a, b, c) + J(\delta a, b, c) + (-1)^{|a|}J(a, \delta b, c) + (-1)^{|a|+|b|}J(a, b, \delta c),$$

where $\delta a = l_1(a)$ and $J(a, b, c) = (-1)^{|b|+1}l_3(a, b, c)$. As with A_∞-algebras, one can combine all the structure maps $\{l_n\}$ into a single non-homogeneous map $l = l_1 + l_2 + l_3 + \cdots \in C^\bullet(V)$ of degree 1.

Proposition 2.8. *Every L_∞-algebra (V, l) gives $G = V[-1]$ the structure of an hG-algebra with*

$$\delta a = l_1(a), \qquad [a, b] = (-1)^{|a|+1}l_2(a, b), \quad \text{and} \quad a \cup b = 0$$

for all $a, b \in V$.

The next theorem states that full symmetrization of an A_∞-algebra gives an L_∞-algebra, see e.g. [20, Th. 3.1], [21].

Theorem 2.9. *Every A_∞-algebra (V, m) induces an L_∞-structure on the space V with*

$$(2.7) \qquad l_n(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (-1)^\kappa m_n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \qquad \forall n > 0,$$

$(-1)^\kappa$ *being the Koszul sign of the permutation σ.*

For $n = 2$, Eq. (2.7) takes the form

$$(2.8) \qquad\qquad l_2(a, b) = m_2(a, b) + (-1)^{|a||b|} m_2(b, a).$$

Comparing this with Eq. (2.6), we arrive at the following alternatives that are true for every A_∞-algebra (V, m) with $m_2 \neq 0$: Either V is an hG-algebra with a non-trivial cup product and the zero bracket (Prop. 2.6) or $V[-1]$ is an hG-algebra with a non-trivial bracket and the cup product identically zero (Prop. 2.8, Eq. (2.8)).

Both the A_∞- and L_∞-algebras form categories, whose morphisms are most concisely defined in terms of coderivations, see Sec. 5.

As the last example of hG-algebras we mention, without going into detail, a *homotopy Gerstenhaber algebra* on the Hochschild complex of an associative algebra [22], see Example 3.3 below.

3. Homotopy Cartan Calculus

If (K, ∂) is a cochain complex of k-vector spaces, then the space $\mathrm{End}(K)$ has the structure of a dg-algebra w.r.t. the composition of endomorphisms and the differential $\hat\partial$ defined by

$$\hat\partial A = [\partial, A] \qquad \forall A \in \mathrm{End}(K).$$

Definition 3.1. A *homotopy Cartan precalculus* over an hG-algebra $(G, \delta, \cup, [\ ,\ ])$ is a cochain complex of vector spaces (K, ∂) together with a pair of k-linear homomorphisms

$$(3.1) \qquad\qquad i : G \to \mathrm{End}(K), \qquad L : G \to \mathrm{End}(K)$$

of degree 1 and 0, respectively, with the following properties:

(hC1) $i_{\delta a} = -\hat\partial i_a$,
(hC2) $L_{\delta a} = \hat\partial L_a$,
(hC3) $i_a i_b \simeq i_{a \cup b}$,
(hC4) $[L_a, L_b] \simeq L_{[a,b]}$,
(hC5) $[i_a, L_b] \simeq i_{[a,b]}$,
(hC6) $L_a i_b - (-1)^{|a|} i_a L_b \simeq L_{a \cup b}$

for all $a, b \in G$. The first two conditions say that the maps (3.1) are $\delta\hat\partial$-morphisms and L is a cochain transformation.

The name 'homotopy Cartan precalculus' comes from a special, albeit important, example where both the differentials δ and ∂ are zero (no homotopy). In that case, the remaining nontrivial Axioms (hC3-6) take the form of equalities satisfied by the operators of Lie derivative and interior product

in the algebra of differential forms $\Lambda(M)$ on a smooth manifold M; in so doing, G is identified with the Gerstenhaber algebra of polyvector fields $\mathcal{V}(M)$ (Example 2.4), while $K^{-n} = \Lambda^n(M)$. One more important ingredient of the conventional Cartan's calculus is the exterior differential d, which enters the 'Cartan magic formula'. Its natural extension to the 'up-to-homotopy' setting leads us to the following definition.

Definition 3.2. A homotopy Cartan precalculus is called *calculus*, if there exists an operator $d : K \to K$ of degree -1 such that

(hC7) $d\partial = -\partial d$,
(hC8) $d^2 \simeq 0$,
(hC9) $di_a + (-1)^{|a|} i_a d \simeq -L_a$.

Axiom (hC7) says that d is a ∂-morphism of K. Together Axioms (hC7-9) and (hC1) give one more familiar relation

$$(3.2) \qquad dL_a - (-1)^{|a|} L_a d \simeq 0.$$

As a result, all the maps i, L, and d pass through δ- and $\hat{\partial}$-cohomology, inducing the conventional Cartan's calculus relations in $H(\mathrm{End}(K))$ for the Gerstenhaber algebra $H(G)$.

In the sequel, we abbreviate 'homotopy Cartan (pre)calculus' to hC-(pre)calculus.

Example 3.3. Let A be a graded associative algebra and let $V = A[1]$. The spaces $C_\bullet(V)$ and $C^\bullet(V)$ are identified, respectively, with the spaces of Hochschild chains and cochains of the algebra A with coefficients in itself. The corresponding Hochschild differentials are given by

$$(\delta D)(a_0, \ldots, a_n)$$
$$= (-1)^{|D| + \bar{a}_0 + \cdots + \bar{a}_{n-1}} D(a_0, \ldots, a_{n-1}) a_n + (-1)^{(|D|+1)\bar{a}_0} a_0 D(a_1, \ldots, a_n)$$
$$- \sum_{i=0}^{n-1} (-1)^{|D| + \bar{a}_0 + \cdots + \bar{a}_i} D(a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n)$$

for all $D \in C^n(V)$ and

$$\partial(a_0 \otimes a_1 \otimes \cdots \otimes a_n)$$
$$= (-1)^{\bar{a}_n(\bar{a}_0 + \cdots + \bar{a}_n + 1)} a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$
$$+ \sum_{i=0}^{n-1} (-1)^{\bar{a}_0 + \cdots + \bar{a}_i} a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n.$$

Here $\bar{a} = |a| - 1$ is the degree of $a \in A$ as an element of $V = A[1]$. Notice that $\delta D = [m_2, D]$, where $m_2(a, b) = (-1)^{\bar{a}} ab$.

As was first observed in [17], the cochain complex $C^\bullet(V)$ carries the structure of an hG-algebra for the Gerstenhaber bracket (2.2) and the cup product

$$(D \cup E)(a_1, \ldots, a_{k+l})$$
$$= (-1)^{(|E|+1)(\bar{a}_1 + \cdots + \bar{a}_k)} D(a_1, \ldots, a_k) E(a_{k+1}, \ldots, a_{k+l})$$

for all $D \in C^k(V)$ and $E \in C^l(V)$.

Natural action of Hochschild cochains on chains gives rise to an hC-precalculus with

$$(3.3) \qquad\qquad\qquad i_D(a_0 \otimes \cdots \otimes a_n)$$

$$= (-1)^{(\bar{a}_0 + \cdots + \bar{a}_n)(\bar{a}_{n-k+1} + \cdots + \bar{a}_n) + |D| - 1} D(a_{n-k+1}, \ldots, a_n) a_0 \otimes a_1 \otimes \cdots \otimes a_{n-k}$$

and

$$(3.4) \qquad\qquad\qquad L_E(a_0 \otimes \cdots \otimes a_n)$$

$$= \sum_{i=0}^{n-k} (-1)^{|E|(\bar{a}_0 + \cdots + \bar{a}_i)} a_0 \otimes \cdots \otimes a_i \otimes E(a_{i+1}, \ldots, a_{i+k}) \otimes a_{i+k+1} \otimes \cdots \otimes a_n$$

$$+ \sum_{i=n-k}^{n} (-1)^{(\bar{a}_0 + \cdots + \bar{a}_n + 1)(\bar{a}_{i+1} + \cdots + \bar{a}_n)} E(a_{i+1}, \ldots, a_n, a_0, \ldots, a_{i+k-n-1})$$

$$\otimes a_{i+k-n} \otimes \cdots \otimes a_i$$

for all $D, E \in C^k(V)$. Direct verification shows that the operators (3.3) and (3.4) obey Axioms (hC3,4) with strict equalities.

For a unital algebra A, one can also define the normalized Hochschild complexes $\bar{C}^\bullet(V)$ and $\bar{C}_\bullet(V)$ together with the Connes–Rinehart boundary operator

$$B(a_0 \otimes \cdots \otimes a_n)$$

$$(3.5) \qquad = \sum_{i=0}^{n} (-1)^{(\bar{a}_0 + \cdots + \bar{a}_i)(\bar{a}_{i+1} + \cdots + \bar{a}_n)} e \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_i \,.$$

Here e is the unit of A, $a_0 \in A$, and $a_1, \ldots, a_n \in \bar{A} = A/ke$. This makes the above hC-precalculus into an hC-calculus with differential $d = B$. Notice that $d^2 = 0$ and the l.h.s. of Rel. (3.2) vanishes in the usual sense. For more detail, see [12], [23], [24, Sec. 3].

4. hG-ALGEBRA STRUCTURE AND hC-CALCULUS ON THE HOCHSCHILD COMPLEX OF AN A_∞-ALGEBRA

In this section, we extend Example 3.3 to an arbitrary A_∞-algebra (V, m). First of all, the Gerstenhaber bracket (2.2) makes $C^\bullet(V)$ into a dg-Lie algebra with the differential given by the adjoint action of m, i.e., $\delta A = [m, A]$. To define a cup product on $C^\bullet(V)$, we need the construction of *braces* [25], [13], [22].

Definition 4.1. For homogeneous elements $A, A_1, \ldots, A_m \in C^\bullet(V)$ and $a_1, \ldots, a_n \in V$, define the braces $A\{A_1, \ldots, A_m\} \in C^\bullet(V)$ by the formula

$$A\{A_1, \ldots, A_m\}(a_1, \ldots, a_n)$$

$$= \sum_{0 \le k_1 \le \cdots \le k_m \le n} (-1)^\kappa A(a_1, \ldots, a_{k_1}, A_1(a_{k_1+1}, \ldots),$$

$$\ldots, a_{k_m}, A_m(a_{k_m+1}, \ldots), \ldots, a_n) \,,$$

where $\kappa = \sum_{i=1}^m |A_i| \sum_{j=1}^{k_i} |a_j|$. It is assumed that $A\{\varnothing\} = A$.

It follows from the definition that $A\{A_1\} = A \circ A_1$ and one may think of the braces as a generalization of the composition product (2.3). The braces obey the *higher pre-Jacobi identities* [22]:

$$A\{A_1, \ldots, A_m\}\{B_1, \ldots, B_n\}$$

$$(4.1) \qquad = \sum_{AB\text{-shuffles}} (-1)^\kappa A\{B_1, \ldots, B_{k_1}, A_1\{B_{k_1+1}, \ldots\},$$

$$\ldots, B_{k_m}, A_m\{B_{k_m+1}, \ldots\}, \ldots, B_n\},$$

where $\kappa = \sum_{i=1}^m |A_i| \sum_{j=1}^{k_i} |B_j|$. Here summation is over all shuffles of the A's and B's (i.e., the order of elements in either group is preserved under permutations) and the case of empty braces $A_k\{\varnothing\}$ is not excluded.

In [13], Getzler shows that every A_∞-structure m on V induces an A_∞-structure M on $C^\bullet(V)$. The latter is given by

$$M_0(\varnothing) = 0\,,$$

$$(4.2) \qquad M_1(A) = m \circ A - (-1)^{|A|} A \circ m\,,$$

$$M_k(A_1, \ldots, A_k) = m\{A_1, \ldots, A_k\}\,, \qquad k > 1\,.$$

Indeed, applying (4.1) one finds

$$(4.3) \qquad (M \circ M)(A_1, \ldots, A_n) = (m \circ m)\{A_1, \ldots, A_n\} = 0\,.$$

Notice that

$$(4.4) \qquad \delta A = [m, A] = M_1(A)\,.$$

Proposition 4.2. *The dg-Lie algebra* $(C^\bullet(V), \delta, [\ ,\ ])$ *is an hG-algebra for the cup product*

$$(4.5) \qquad A \cup B = (-1)^{|A|-1} M_2(A, B)\,.$$

Proof. By the definition of an A_∞-structure,

$$(4.6) \qquad M_1 \circ M_1 = 0\,, \qquad [M_1, M_2] = 0\,, \qquad M_2 \circ M_2 = -[M_1, M_3]\,.$$

The first relation identifies the operator (4.4) as differential, i.e., $\delta^2 = 0$. Then the second equality amounts to Axiom (hG1) of Definition 2.1. Axiom (hG3) – associativity up to homotopy – is equivalent to the third relation. The commutativity up to homotopy (hG4) follows from the identity

$$M_2(A, B) + (-1)^{|A||B|} M_2(B, A)$$

$$= -M_1(A \circ B) + M_1(A) \circ B + (-1)^{|A|} A \circ M_1(B)\,,$$

which is a consequence of the pre-Jacobi identities (4.1). The same pre-Jacobi identities lead to the equality

$$[A, M_2(B,C)] - (-1)^{|A|}M_2([A,B],C) - (-1)^{|A|(|B|+1)}M_2(B,[A,C])$$

$$= (-1)^{|A|}\Big(M_1(A\{B,C\}) - M_1(A)\{B,C\}$$

$$-(-1)^{|A|}A\{M_1(B),C\} - (-1)^{|A|+|B|}A\{B,M_1(C)\}\Big),$$

which is equivalent to Axiom (hG7). $\square$

Now we would like to equip the hG-algebra above with an hC-calculus following the pattern of Example 3.3. To this end, let us consider the graded associative algebra of endomorphisms of $C_\bullet(V)$ as an A_∞-algebra $\mathrm{End}(C_\bullet(V))[1]$.

Theorem 4.3 ([13]). *There is a morphism from the A_∞-algebra $(C^\bullet(V), M)$ to $\mathrm{End}(C_\bullet(V))[1]$.*

By definition, the morphism in question is given by a map

$$I : TC^\bullet(V) \to \mathrm{End}(C_\bullet(V))[1]$$

obeying the sequence of relations

$$\sum_{k=0}^{n}(-1)^{|A_1|+\cdots+|A_k|}I(A_1,\ldots,A_k)I(A_{k+1},\ldots,A_n) = -(I \circ M)(A_1,\ldots,A_n).$$

For $n = 0, 1, 2$ we get

$$I(\varnothing)I(\varnothing) = 0, \qquad [I(\varnothing), I(A)] = -I(M_1(A)),$$

$$[I(\varnothing), I(A,B)] + (-1)^{|A|}I(A)I(B)$$

$$= -I(M_1(A), B) - (-1)^{|A|}I(A, M_1(B)) - I(M_2(A,B)).$$

Hence we will satisfy Axiom (hC3) of Definition 3.1, if set

$$\partial = I(\varnothing), \qquad i_A = I(A).$$

Here is an explicit expression for the morphism I:

(4.7)
$$I(\varnothing)(a_0,\ldots,a_n)$$

$$= \sum_{l=1}^{\infty}\sum_{j=n-l+1}^{n}(-1)^{(|a_0|+\cdots+|a_j|)(|a_{j+1}|+\cdots+|a_n|)}(m_l(a_{j+1},\ldots,a_n,a_0,\ldots),\ldots,a_j)$$

$$+ \sum_{l=1}^{\infty}\sum_{j=0}^{n-l}(-1)^{|a_0|+\cdots+|a_j|}(a_0,\ldots,m_l(a_{j+1},\ldots,a_{j+l}),\ldots,a_n)$$

(a_0 is inside m_l in the first double sum) and

(4.8)
$$I(A_1,\ldots,A_k)(a_0,\ldots,a_n)$$

$$= \sum_{l=k+1}^{\infty} \sum_{j_0,\ldots,j_k} (-1)^{(|a_0|+\cdots+|a_{j_0}|)(|a_{j_0+1}|+\cdots+|a_n|)+\sum_{i=1}^k |A_i|(|a_0|+\cdots+|a_{j_i}|)}$$

$$\times (m_l(a_{j_0+1},\ldots,A_1(a_{j_1+1},\ldots),\ldots,A_k(a_{j_k+1},\ldots),\ldots,a_n,a_0,\ldots),\ldots,a_{j_0})$$

for $k > 0$. The sum is taken over all possible values of $j_0,\ldots,j_k$ such that a_0 is to the right of A_k and both are inside m_l. Hereinafter we denote by $(a_0,\ldots,a_n)$ the chain $a_0 \otimes \cdots \otimes a_n$ of $C_n(V)$.

In order to define the Lie derivative operation, consider the map

$$L : TC^\bullet(V) \to \mathrm{End}(C_\bullet(V))$$

given by

(4.9) $$L(A_1,\ldots,A_k)(a_0,\ldots,a_n) = \sum_{j_1,\ldots,j_k} (-1)^{\sum_{i=1}^k |A_i|(|a_0|+\cdots+|a_{j_i}|)}$$

$$\times (a_0,\ldots,A_1(a_{j_1+1},\ldots),\ldots,A_k(a_{j_k+1},\ldots),\ldots,a_n)$$

$$+ \sum_{i=1}^k \sum_{j_1,\ldots,j_k} (-1)^{(|A_2|+\cdots+|A_i|)(|A_{i+1}|+\cdots+|A_k|)+(|a_0|+\cdots+|a_{j_1}|)(|a_{j_1+1}|+\cdots+|a_n|)}$$

$$\times (-1)^{\sum_{l=2}^i |A_l|(|a_0|+\cdots+|a_{j_l}|+|a_{j_1+1}|+\cdots+|a_n|)+\sum_{l=i+1}^k |A_l|(|a_{j_1+1}|+\cdots+|a_{j_l}|)}$$

$$\times (A_1(a_{j_1+1},\ldots,A_{i+1}(a_{j_1+1},\ldots),\ldots,A_k(a_{j_k+1},\ldots),\ldots,a_0,\ldots),$$

$$\ldots,A_2(\ldots),\ldots,A_i(\ldots),\ldots,a_{j_1}).$$

Summation is taken over all possible values $j_1,\ldots,j_k$ subject to the following condition: in the first single sum a_0 is to the left of A_1, while in the second double sum a_0 is inside A_1 and to the right of A_k. For a single cochain $A \in C^p(V)$, Eq. (4.9) simplifies to

$$L(A)(a_0,\ldots,a_n) =$$

$$\sum_{j=0}^{n-p} (-1)^{|A|(|a_0|+\cdots+|a_j|)} (a_0,\ldots,A(a_{j+1},\ldots,a_{j+p}),\ldots,a_n)$$

$$+ \sum_{j=n-p+1}^{n} (-1)^{(|a_0|+\cdots+|a_j|)(|a_{j+1}|+\cdots+|a_n|)} (A(a_{j+1},\ldots,a_0,\ldots,a_{j+p-n-1}),\ldots,a_j).$$

Comparing the last expression with (4.7), we conclude that $\partial = L(m)$.

In [13], Getzler proved the following identity for I and L:

(4.10) $$L \circ M = IL - LI$$

Explicitly,

$$\sum_{k=1}^{\infty} \sum_{j=0}^{n-k} (-1)^{|A_1|+\cdots+|A_j|} I(A_1,\ldots,A_j,M_k(A_{j+1},\ldots A_{j+k}),\ldots,A_n)$$

$$= \sum_{j=0}^{n-1} I(A_1,\ldots,A_j)L(A_{j+1},\ldots,A_n)$$

$$-\sum_{j=1}^{n}(-1)^{|A_1|+\cdots+|A_j|}L(A_1,\ldots,A_j)I(A_{j+1},\ldots,A_n)$$

for $n = 1, 2, \ldots$ In particular,

$$I(\varnothing)L(A) - (-1)^{|A|}L(A)I(\varnothing) = L(M_1(A))$$

and

$$I(A)L(B) - (-1)^{|A|}L(A)I(B) + [I(\varnothing), L(A, B)]$$

$$= L(M_1(A), B) + (-1)^{|A|}L(A, M_1(B)) + L(M_2(A, B)).$$

Therefore, we can satisfy Axioms (hC2) and (hC6) by setting $L_A = L(A)$. The remaining two Axioms (hC4) and (hC5) can now be verified directly. As shown in [13, Lem. 2.3], the assignment $A \mapsto L_A$ defines a homomorphism of graded Lie algebras, that is,

(4.11)
$$[L_A, L_B] = L_{[A,B]}.$$

Axiom (hC5) can be written as

$$(-1)^{|A|}[L_A, i_B] = i_{[A,B]} + \hat{\partial}T(A, B) - T(\delta A, B) - (-1)^{|A|}T(A, \delta B),$$

or equivalently,

(4.12)
$$i_B L_A - (-1)^{|A|(|B|+1)}L_A i_B = i_{[B,A]}$$

$$+(-1)^{|A||B|}\Big(T(\delta A, B)+(-1)^{|A|}T(A, \delta B)-\partial T(A, B)+(-1)^{|A|+|B|}T(A, B)\partial\Big),$$

where the Gel'fand–Daletskii–Tsygan homotopy $T(A, B)$ is given by

(4.13)
$$T(A, B)(a_0, \ldots, a_n)$$

$$= \sum_{j,i}(-1)^{\kappa}(A(a_{j+1}, \ldots, B(a_{i+1}, \ldots, a_{q+1}), \ldots, a_0, \ldots), \ldots, a_j).$$

Here $(-1)^{\kappa}$ is the standard Koszul sign and the sum is taken over all i and j such that a_0 is inside A but to the right of B. (As above, all permutations preserve the cyclic order of a's.) In a slightly different notation an explicit expression for $T(A, B)$ was given in [13] and [24, Ex. 3.13]; in the former paper it is denoted by ρ.

The above operations of Lie derivative L_A and contraction i_A can further be supplemented with a differential d to give an example of hC-calculus. To this end, one adds a new element e of degree -1 and defines the augmented vector space $\tilde{V} = V \oplus ke$. The A_∞-structure m extends to $\tilde{V}$ by setting

$$m_2(e, a) = (-1)^{|a|}m_2(a, e) = a, \qquad m_2(e, e) = e, \qquad m_k(\ldots, e, \ldots) = 0$$

for all $k \neq 2$ and $a \in V$. The space $\bar{C}_n(V)$ of normalized Hochschild n-chains is spanned by the tensor products $a_0 \otimes a_1 \otimes \cdots \otimes a_n$, where $a_0 \in \tilde{V}$ and $a_1, \ldots, a_n \in V$. The differential d, being independent of a particular A_∞-structure on V, is given by the Connes–Rinehart operator (3.5). Since the symbol d is usually overloaded, we will denote this differential by B,

which is more standard in the case of $\bar{C}_\bullet(V)$. Axioms (hC8,9) of Definition 3.2 take now the form

$$B^2 = 0\,, \qquad [B, i_A] = -L_A - \hat{\partial}S_A - S_{\delta A}\,,$$

where

$$S_A(a_0, \ldots, a_n) = \sum_{j \le i} (-1)^\kappa (e, a_{j+1}, \ldots, A(a_{i+1}, \ldots, a_{i+p}), \ldots, a_0, \ldots, a_j)$$

and the Koszul sign is given by

$$\kappa = (|a_0| + \cdots + |a_j|)(|a_{j+1}| + \cdots + |a_n|) + |A|(|a_{j+1}| + \cdots + |a_i|)\,.$$

The sum is taken over all cyclic permutations of a's such that a_0 appears to the right of $A \in C^p(V)$.

5. Coderivations vs. Lie derivatives

Recall that, in addition to the associative algebra structure, the space $C_\bullet(V) = TV$ carries the structure of a coassociative coalgebra with respect to the (reduced) coproduct $\Delta : TV \to TV \otimes TV$,

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)\,.$$

Coassociativity is expressed by the relation $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

A linear map $D : TV \to TV$ is a *coderivation*, if it obeys the co-Leibniz rule

$$\Delta D = (D \otimes 1 + 1 \otimes D)\Delta\,.$$

The space of coderivations is known to be isomorphic to the space $C^\bullet(V) = \mathrm{Hom}(TV, V)$, so that any homomorphism $A : TV \to V$ induces a coderivation $D_A : TV \to TV$ and vice versa: if $A \in C^m(V)$, then

$$D_A(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-m+1} (-1)^{|A|(|a_1| + \cdots + |a_{i-1}|)} a_1 \otimes \cdots \otimes a_{i-1}$$
$$\otimes A(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_n$$

for $n \ge m$ and zero otherwise. This allows one to interpret the Gerstenhaber bracket (2.2) as the commutator of two coderivations. More precisely, the assignment $A \mapsto D_A$ defines a homomorphism from the Lie algebra $C^\bullet(V)$ for the Gerstenhaber bracket (2.2) to the Lie algebra of coderivations:

(5.1) $$[D_A, D_B] = D_{[A,B]} \qquad \forall A, B \in C^\bullet(V)\,.$$

Thus, we have two representations of the graded Lie algebra $(C^\bullet(V), [\,,\,])$ in the space $C_\bullet(V)$: by Lie derivatives (4.11) and by coderivations (5.1). Neither representation is irreducible. To describe invariant subspaces, we introduce the generator λ of cyclic permutations,

$$\lambda(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^{|a_0|(|a_1| + \cdots + |a_n|)}(a_1 \otimes \cdots \otimes a_n \otimes a_0)\,,$$

and let $N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n$ denote the associated norm map. Since $\lambda^{n+1} = 1$, we have $N(1 - \lambda) = (1 - \lambda)N = 0$ on $C_n(V)$. The operators N and $1 - \lambda$ intertwine the L- and D-representations of the Lie algebra $C^\bullet(V)$, that is,

$$(5.2) \qquad NL_A = D_A N, \qquad L_A(1 - \lambda) = (1 - \lambda)D_A \qquad \forall A \in C^\bullet(V).$$

It follows from these relations that the subspace $\mathrm{Im}(1 - \lambda) = \ker N \subset C_\bullet(V)$ is invariant under the action of L_A, while $\mathrm{Im}N = \ker(1 - \lambda) \subset C_\bullet(V)$ is an invariant subspace for D_A.

If (V, m) is an A_∞-algebra, then the coderivation D_m squares to zero and makes the coassociative coalgebra (TV, Δ) into a codifferential coalgebra (TV, Δ, D_m). This allows one to define morphisms of A_∞-algebras (V, m) and (V', m') as homomorphisms $h : TV \to TV'$ of the corresponding codifferential coalgebras.

The same language of coderivations applies to L_∞-algebras and their morphisms [26]. This time one starts with the symmetric algebra SV of a graded space V. This is a subcoalgebra $SV \subset TV$ spanned by symmetric tensors. Upon restriction to SV one gets the coproduct

$$\Delta(a_1 \vee \cdots \vee a_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(i, n-i)} (-1)^\kappa (a_{\sigma(1)} \vee \cdots \vee a_{\sigma(i)}) \otimes (a_{\sigma(i+1)} \vee \cdots \vee a_{\sigma(n)}).$$

An L_∞-structure on V can now be defined as a coderivation of (SV, Δ) that has degree one and squares to zero. Again, it is easy to see that each coderivation D of (SV, Δ) (and hence, L_∞-structure) defines and is defined by some element of $\mathrm{Hom}(SV, V)$. Explicitly,

$$D_A(a_1 \vee \cdots \vee a_n) = \sum_{\sigma \in \mathrm{Sh}(i, n-i)} (-1)^\kappa A(a_{\sigma(1)} \vee \cdots \vee a_{\sigma(i)}) \vee a_{\sigma(i+1)} \cdots \vee a_{\sigma(n)}$$

for all $A \in \mathrm{Hom}(S^i V, V)$. The generalized Jacobi identities of Definition 2.7 are equivalent to the condition $D_l^2 = 0$. Thus, there is a one-to-one correspondence between L_∞-structures on V and codifferentials on the coalgebra (SV, Δ). This makes possible to define the morphisms of L_∞-algebras as the homomorphisms of the corresponding codifferential coalgebras (SV, Δ, D_l). Theorem 2.9 provides then a functor from the category of A_∞-algebras to the category of L_∞-algebras, the *symmetrization map*.

For every A_∞-algebra (V, m) the operator of Lie derivative $\partial = L_m$ makes the space $C_\bullet(V)$ into a complex, which we denote by $C_\bullet(V, m)$. The complex $C_\bullet(V, m)$ is known as the Hochschild chain complex of the A_∞-algebra (V, m) with coefficients in itself. We let denote its homology groups by $HH_\bullet(V, m)$. For general definitions of A_∞-(bi)modules and A_∞-(co)homology we refer the reader to [27], [28, Sec. 3], [19]. It follows from the second relation in (5.2) that the subspace $\mathrm{Im}(1 - \lambda) \subset C_\bullet(V)$ is a subcomplex of $C_\bullet(V, m)$. One may regard the corresponding quotient complex

$$C_\bullet^\lambda(V, m) = C_\bullet(V, m)/\mathrm{Im}(1 - \lambda)$$

as an A_∞ generalization of the cyclic chain complex of an associative algebra [29]. Its homology will be denoted by $HC_\bullet^\lambda(V, m)$. Passing to the dual space $C_\bullet(V)^*$, one can also define the corresponding cochain complexes $C^\bullet(V, m)$ and $C_\lambda^\bullet(V, m)$ of an A_∞-algebra (V, m). The latter consists of cochains $S : TV \to k$ that obey the cyclicity condition $S(1 - \lambda) = 0$. We denote the cohomology groups of these complexes by $HH^\bullet(V, m)$ and $HC_\lambda^\bullet(V, m)$, respectively.

Remark 5.1. A word about terminology. In the above notation for the (co)chain complexes and their (co)homology groups, we use the label $\bullet$ just to distinguish between the spaces TV and $\mathrm{Hom}(TV, k)$, i.e., the Hochschild chains and cochains. The usual simplicial degree associated with the number of factors in tensor products $a_0 \otimes \cdots \otimes a_n$ is not essential for our considerations as the differential ∂ is not generally homogeneous relative to it. Since $|\partial| = 1$, all the aforementioned complexes are *cochain* complexes with respect to the $\mathbb{Z}$-degree, though this is not indicated in their notation explicitly. However, as in the case of associative algebras, we still call the elements of $C_\bullet(V, m)$ chains (cycles) and the elements of $C^\bullet(V, m)$ cochains (cocycles).

In much the same way one can define the (co)homology theory for L_∞-algebras with various coefficients. In particular, the complex (SV, D_l) above computes the homology of an L_∞-algebra with trivial coefficients. For further references, we also introduce the corresponding cochain complex $(SV)^*$ and denote its cohomology groups by $H^\bullet(V, l)$. A detailed discussion of L_∞-modules and L_∞-(co)homology can be found in [20], [30].

6. FAMILIES OF A_∞-ALGEBRAS AND THEIR INNER DEFORMATIONS

Let (V, m) be formal n-parameter deformation of an A_∞-algebra (V, m_0), that is, an A_∞-structure m is given by a Maurer–Cartan element $m \in C^\bullet(V)[[t_1, \ldots, t_n]]$ such that $m|_{t=0} = m_0$. For more generality we allow the deformation parameters t_i to carry *even* $\mathbb{Z}$-degrees. We will refer to (V, m) as a *family of A_∞-algebras*. According to Proposition 4.2, the family (V, m) gives rise to the family of hG-algebras $G_t = \big(C^\bullet(V)[[t_1, \ldots, t_n]], \delta, [\ , \ ], \cup\big)$. Let us denote

$$m_{(i_1 i_2 \cdots i_k)} = \frac{\partial^k m}{\partial t_{i_1} \partial t_{i_2} \cdots \partial t_{i_k}} \in C^\bullet(V)[[t_1, \ldots, t_n]] \, .$$

It is clear that $|m_{(i_1 i_2 \cdots i_k)}| = 1 - |t_{i_1}| - \ldots - |t_{i_k}|$. We will also use this notation for partial derivatives of other cochains. Differentiating the defining relation $m \circ m = 0$ w.r.t. the parameters, we get

$$(6.1) \qquad\qquad [m, m_{(i)}] = \delta m_{(i)} = 0 \, .$$

Hence, $m_{(i)}$ is a δ-cocycle representing a cohomology class of $H^{1-|t_i|}(G_t)$. As discussed in Sec. 2, the bracket and the cup product pass through the cohomology and make the graded space $H^\bullet(G_t)$ into a Gerstenhaber algebra. Let $\mathcal{I} = \bigoplus \mathcal{I}^p$ denote the subalgebra of G_t generated by the cocycles $m_{(i)}$.

Passing to the δ-cohomology, we also define the subalgebra $H^\bullet(\mathcal{I})$ of the Gerstenhaber algebra $H^\bullet(G_t)$.

Proposition 6.1. *The Gerstenhaber bracket on G_t induces the trivial Lie bracket on $H^\bullet(\mathcal{I})$.*

Proof. Differentiating identity (6.1) once again, we get

$$(6.2) \qquad\qquad -[m_{(i)}, m_{(j)}] = [m, m_{(ij)}] = \delta m_{(ij)}\,.$$

Hence, the bracket $[m_{(i)}, m_{(j)}]$ is a δ-coboundary. By Axiom (hG7) this conclusion extends to arbitrary cup products of $m_{(i)}$'s.

$\square$

Thus, the algebra $H^\bullet(\mathcal{I})$ is generated by cup products of the partial derivatives $m_{(i)}$, so that the general element of $H^\bullet(\mathcal{I})$ is represented by the cup polynomial

$$(6.3) \qquad\qquad \Psi = \sum_{l=0}^{L} c^{j_1\cdots j_l} m_{(j_1)} \cup m_{(j_2)} \cup \cdots \cup m_{(j_l)}\,,$$

where $c^{j_1\cdots j_l} \in k[[t_1, \ldots, t_n]]$. Since all t's are supposed to be even, the graded associative algebra $H^\bullet(\mathcal{I})[-1]$ is commutative in the usual sense and we may assume the coefficients $c^{j_1\cdots j_l}$ to be fully symmetric in permutations of indices. It should be borne in mind that the family of A_∞-structures m enters the polynomial (6.3) both through the partial derivatives $m_{(j)}$ and through the cup products.

Theorem 6.2 ([11])**.** *Let $m \in C^\bullet(V)[[t_1, \ldots, t_n]]$ be an n-parameter family of A_∞-structures and let Ψ be a cocycle representing an element of $\mathcal{I}^p$. Then one can define an $(n+1)$-parameter family of A_∞-structures $\tilde{m} \in C^\bullet(V)[[t_0, t_1, \ldots, t_n]]$ as a unique formal solution to the differential equation*

$$(6.4) \qquad\qquad \tilde{m}_{(0)} = \Psi[\tilde{m}]$$

subject to the initial condition $\tilde{m}|_{t_0=0} = m$. Here the new formal parameter t_0 has degree $1 - p$.

Proof. [4] Suppose the cocycle Ψ is given by a polynomial of the form (6.3). For definiteness we assume that all cup products in Ψ are performed from right to left. Clearly, Eq.(6.4) has a unique formal solution that starts as

$$\tilde{m}(t_0) = m + t_0 \Psi[m] + O(t_0^2)\,.$$

Writing identity (4.3) for $m = \tilde{m}$, $A_1 = \tilde{m}_{(j)}$, and $A_2 = A$ we get

$$(6.5) \quad [\tilde{m}, \tilde{m}_{(j)} \cup A] = (\tilde{m} \circ \tilde{m})_{(j)} \cup A + \tilde{m}_{(j)} \cup [\tilde{m}, A] + (\tilde{m} \circ \tilde{m})\{\tilde{m}_{(j)}, A\}\,.$$

---

[4]We provide the proof because that in [11] contains an unfortunate misprint.

Here we also used the definition (4.5). On the other hand, differentiating the cochain $\tilde{m} \circ \tilde{m}$ w.r.t. t_0 we obtain

$$(\tilde{m} \circ \tilde{m})_{(0)} = [\tilde{m}, \tilde{m}_{(0)}] = [\tilde{m}, \Psi[\tilde{m}]].$$

Repeated use of identity (6.5) allows us to bring the last equality into the form

$$(\tilde{m} \circ \tilde{m})_{(0)} = \sum_{l=0}^{L} c^{j_1 \cdots j_l} \left[\sum_{k=1}^{l} \tilde{m}_{(j_1)} \cup \cdots \cup (\tilde{m} \circ \tilde{m})_{(j_k)} \cup \cdots \cup \tilde{m}_{(j_l)} \right.$$

$$+(\tilde{m} \circ \tilde{m})\{\tilde{m}_{(j_1)}, \tilde{m}_{(j_2)} \cup \cdots \cup \tilde{m}_{j_l}\}$$

$$\left. + \sum_{k=1}^{l-2} \tilde{m}_{(j_1)} \cup \cdots \cup \tilde{m}_{(j_k)} \cup (\tilde{m} \circ \tilde{m})\{\tilde{m}_{(j_{k+1})}, \tilde{m}_{(j_{k+2})} \cup \cdots \cup \tilde{m}_{(j_l)}\} \right].$$

(Cup multiplication is performed from right to left.) We see that $\tilde{m} \circ \tilde{m}$ satisfies a linear differential equation. With account of the initial condition $\tilde{m} \circ \tilde{m}|_{t_0=0} = m \circ m = 0$ this means $\tilde{m} \circ \tilde{m} = 0$. Hence, $\tilde{m}$ defines an $(n+1)$-parameter family of A_∞-structures.

$\square$

We call the deformations of Theorem 6.2 the *inner deformations* of families of A_∞-algebras. Each inner deformation of an A_∞-algebra defines then a deformation of the associated hG-algebra G_t.

Remark 6.3. As a small interlude let us explain in more detail some physical motivations for the constructions discussed above and below. Consider a collection of form-fields Φ^A of various degrees living on a space-time manifold M. Regarding these forms as dynamical fields, we can impose on them the equations of motion

$$(6.6) \qquad d\Phi^A = \sum_{n \geq 2} l^A_{A_1 \cdots A_2} \Phi^{A_1} \wedge \ldots \wedge \Phi^{A_n},$$

d being the exterior differential. Formal integrability of the field equations (stemming from $d^2 = 0$) requires the l's to be structure constants of some minimal L_∞-algebra on the graded vector space dual to the space of form-fields Φ^A. It is not hard to see that every system of PDE can be brought to the form (6.6) for a suitable (perhaps infinite) collection of form-fields Φ^A. Since the L_∞-algebra is minimal, system (6.6) admits a consistent truncation setting $l^A_{A_1 \cdots A_n} = 0$ for all $n > 2$. The resulting theory is completely determined by a graded Lie algebra L with structure constants $l^A_{A_1 A_2}$. In physical terms, one may view the truncated system as a free field theory, regarding all the higher terms in r.h.s. of (6.6) as interaction. Usually, the Lie algebra L, being determined by fundamental symmetries of the theory, is known in advance. Furthermore, in many interesting cases it comes from the commutator of an associative algebra A. The main problem is then to reconstruct all higher structure constants associated with the interaction

vertices in (6.6). From the algebraic viewpoint, this amounts to deformation of the Lie algebra L to a minimal L_∞-algebra by adding a coherent set of higher multi-brackets. One can also do this at the level of A_∞-algebras: first deform A to a minimal A_∞-algebra and then apply the symmetrization map (2.7). It is significant that even at the free level the field equations may depend on some free parameters like masses of particles, mixing angles, etc. The machinery of inner deformations suggests then a systematic way to convert this dependence into higher structure maps of a minimal A_∞-algebra, that is, into interaction vertices.

Given a system of PDE, one can wounder about its characteristic cohomology [31, 32]. Under certain assumptions [33] the characteristic cohomology of (6.6) is generated by on-shell closed differential forms

$$(6.7) \qquad J = \sum_n J_{A_1 \cdots A_n} \Phi^{A_1} \wedge \ldots \wedge \Phi^{A_n} \,,$$

meaning that $dJ = 0$ whenever Φ^A satisfy the equations of motion (6.6). In case $\deg J = \dim M - 1$, the form J defines a usual conserved current, whose integral over a closed hypersurface in M gives a conserved charge; the forms (6.7) of degree less than $\dim M - 1$ define the so-called lower-degree conservation laws, see e.g. [34, 35]. One may check that the form (6.7) is on-shell closed iff its structure constants $J_{A_1 \cdots A_n}$ define and are defined by a cocycle of L_∞-algebra cohomology with trivial coefficients. If the L_∞-algebra was obtained by an inner deformation of a graded Lie algebra L (or the underlying associative algebra A), one can try to extend this deformation to L_∞-cohomology (resp. A_∞-cohomology). It is this problem that we address in the rest of this paper.

7. Inner deformations of Hochschild (co)cocycles

In this section, we show that all inner deformations of A_∞-algebras are accompanied by deformations of their Hochschild (co)cycles. Let us start with some auxiliary identities. Repeated application of Axioms (hC1-6) gives

$$L_{m_{(j_1)} \cup \cdots \cup m_{(j_n)}} \simeq L_{m_{(j_1)}} i_{m_{(j_2)} \cup \cdots \cup m_{(j_n)}} + i_{m_{(j_1)}} L_{m_{(j_2)} \cup \cdots \cup m_{(j_n)}}$$

$$\simeq \sum_{k=1}^{n} i_{m_{(j_1)}} \cdots i_{m_{(j_{k-1})}} L_{m_{(j_k)}} i_{m_{(j_{k+1})}} \cdots i_{m_{(j_n)}}$$

$$\simeq \sum_{k=1}^{n} \Big(i_{m_{(j_1)}} \cdots i_{m_{(j_{k-1})}} i_{m_{(j_{k+1})}} L_{m_{(j_k)}} i_{m_{(j_{k+2})}} \cdots i_{m_{(j_n)}}$$

$$- i_{m_{(j_1)}} \cdots i_{m_{(j_{k-1})}} i_{[m_{(j_k)}, m_{(j_{k+1})}]} i_{m_{(j_{k+2})}} \cdots i_{m_{(j_n)}} \Big)$$

$$\simeq \sum_{k=1}^{n} \Big(i_{m_{(j_1)}} \cdots i_{m_{(j_{k-1})}} i_{m_{(j_{k+1})}} \cdots i_{m_{(j_n)}} L_{m_{(j_k)}}$$

$$-\sum_{l=1}^{n-k}[L_m, i_{m_{(j_1)}}\cdots i_{m_{(j_{k-1})}}i_{m_{(j_{k+1})}}\cdots i_{m_{(j_{k+l-1})}}i_{m_{(j_k,j_{k+l})}}i_{m_{(j_{k+l+1})}}\cdots i_{m_{(j_n)}}]\Big).$$

Here we also used Eq. (6.2). Replacing the monomial $m_{(j_1)} \cup \cdots \cup m_{(j_n)}$ above with a general cup polynomial (6.3) and restoring the strict equality sign, we can write

$$(7.1) \qquad\qquad L_\Psi = [L_m, W] + \sum_{j=1}^n V^j L_{m_{(j)}}$$

for some $W = W[m] \in C^\bullet(V)[[t_1,\ldots,t_n]]$ and[5]

$$V^j = \sum_{l=1}^L l c^{j_1\ldots j_{l-1}j} i_{m_{(j_1)}}\cdots i_{m_{(j_{l-1})}}.$$

It is important that $[L_m, V^j] = 0$.

Suppose now that deformation generated by the flow (6.4) extends to deformation of the corresponding Hochschild cycles, that is, for every cycle $\alpha \in C_\bullet(V, m)$ there exists a cycle $\tilde\alpha \in C_\bullet(V, \tilde m)$ such that $\tilde\alpha|_{t_0=0} = \alpha$. Differentiating the boundary $\tilde\partial\tilde\alpha = L_{\tilde m}\tilde\alpha$ w.r.t. t_0 we get

$$(\tilde\partial\tilde\alpha)_{(0)} = (L_{\tilde m}\tilde\alpha)_{(0)} = L_{\tilde m_{(0)}}\tilde\alpha + L_{\tilde m}\tilde\alpha_{(0)} = L_\Psi\tilde\alpha + L_{\tilde m}\tilde\alpha_{(0)},$$

where $\Psi = \Psi[\tilde m]$. With account of (7.1) this is equivalent to

$$(7.2) \qquad (\tilde\partial\tilde\alpha)_{(0)} = L_{\tilde m}W\tilde\alpha + WL_{\tilde m}\tilde\alpha + \sum_{j=1}^n V^j L_{m_{(j)}}\tilde\alpha + L_{\tilde m}\tilde\alpha_{(0)}$$

for some $W = W[\tilde m]$. The r.h.s of this equality suggests to impose the following differential equation on $\tilde\alpha$:

$$(7.3) \qquad\qquad \tilde\alpha_{(0)} = \sum_{j=1}^n V^j \tilde\alpha_{(j)} - W\tilde\alpha.$$

Then Eq. (7.2) takes the form

$$(7.4) \qquad\qquad (\tilde\partial\tilde\alpha)_{(0)} = W(\tilde\partial\tilde\alpha) + \sum_{j=1}^n V^j(\tilde\partial\tilde\alpha)_{(j)}.$$

We see that once a chain $\tilde\alpha$ satisfies Eq. (7.3), its boundary $\tilde\partial\tilde\alpha$ obeys the linear homogeneous equation (7.4). For the zero initial condition $\tilde\partial\tilde\alpha|_{t_0=0} = \partial\alpha = 0$, this means that $\tilde\alpha$, being uniquely defined by Eq. (7.3), is a cycle of $C_\bullet(V, \tilde m)$. We thus arrive at the main result of this paper.

Theorem 7.1. *Let $\tilde m$ be an inner deformation of a family of A_∞-structures m. Then each Hochschild cycle α of $C_\bullet(V, m)$ deforms to a cycle $\tilde\alpha$ of $C_\bullet(V, \tilde m)$.*

---

[5]Recall that $c^{j_1\ldots j_l}$ are symmetric in upper indices.

This means stability of Hochcshild (co)homology under inner deformations of A_∞-algebras. Let us write down a more explicit formula for the flow (7.3) in a special case of quadratic deformation flow (6.4), namely,

$$\tilde{m}_{(0)} = \tilde{m}_{(1)} \cup \tilde{m}_{(2)} \,, \tag{7.5}$$

$\tilde{m}(0, t_1, t_2) = m(t_1, t_2)$ being a two-parameter family of A_∞-structures. The associated flow (7.3) on chains takes then the form

$$\tilde{\alpha}_{(0)} = i_{\tilde{m}_{(1)}} \tilde{\alpha}_{(2)} + i_{\tilde{m}_{(2)}} \tilde{\alpha}_{(1)} + \big(i_{\tilde{m}_{(1,2)}} + T(\tilde{m}_{(1)}, \tilde{m}_{(2)}) - L(\tilde{m}_{(1)}, \tilde{m}_{(2)})\big)\tilde{\alpha},$$

where the endomorphisms L and T are defined by Eqs. (4.9) and (4.13).

Since $\mathcal{I}$ consists of δ-cocycles, each $\Psi \in \mathcal{I}$ generates an endomorphism of the $k[[t_1, \ldots, t_n]]$-vector space $HH_\bullet(m, V)$. At the level of chains, the endomorphism is defined by the assignment $\alpha \mapsto i_\Psi \alpha$ for all $\alpha \in C_\bullet(V, m)$. This makes the space of Hochschild homology into a module over the commutative algebra $H^\bullet(\mathcal{I})[-1]$. Generally, the algebra $H^\bullet(\mathcal{I})[-1]$ is not free, and the δ-cohomology classes of some polynomials (6.3) may vanish identically, thereby defining relations in $H^\bullet(\mathcal{I})[-1]$.

Theorem 7.2. *Every relation* $[\Psi] = 0$ *in* $H^\bullet(\mathcal{I})[-1]$ *gives rise to an endomorphism* h_Ψ *of the space* $HH_\bullet(V, m)$.

Proof. If the polynomial (6.3) defines a relation, then $\Psi = \delta U$ for some $U \in C^\bullet(V)[[t_1, \ldots, t_n]]$. Let α be a cycle of $C_\bullet(V, m)$. By (4.11) and (7.1)

$$\begin{aligned}
0 &= L_{\Psi - \delta U}\alpha = L_\Psi \alpha - [L_m, L_U]\alpha \\
&= [L_m, W - L_U]\alpha + \sum_{j=1}^n V^j L_{m_{(j)}}\alpha \\
&= L_m\Big(W\alpha - L_U\alpha - \sum_{j=1}^n V^j \alpha_{(j)}\Big).
\end{aligned}$$

Hence, the expression in the round brackets is a cycle and we can define the desired endomorphism h_Ψ by the formula

$$h_\Psi[\alpha] = \Big[W\alpha - L_U\alpha - \sum_{j=1}^n V^j \alpha_{(j)}\Big].$$

$$\square$$

Thus, the algebra $H(\mathcal{I})[-1]$ gives rise to a set of operations on $HH_\bullet(V, m)$, both through the action of its generators and through relations among them.

Theorems 7.1 and 7.2 have obvious counterparts for the cochain complex $C^\bullet(V, m)$. In particular, if S is a cocycle of $C^\bullet(V, m)$ and the inner deformation is generated by the quadratic flow (7.5), then the flow (7.3) is replaced with the following one:

$$\tilde{S}_{(0)} = \tilde{S}_{(1)} i_{m_{(2)}} + S_{(2)} i_{m_{(1)}} + \tilde{S}\big(i_{m_{(1,2)}} + L(m_{(1)}, m_{(2)}) + T(m_{(2)}, m_{(1)})\big)\,.$$

It is easy to see that $\tilde{S}\tilde{\partial} = 0$ whenever $S\partial = 0$.

8. An application to dg-algebras

By way of illustration, let us apply the above machinery of inner deformations to the case of dg-algebras. Recall that a dg-algebra is given by a triple $(A, \cdot, d)$, where $A = \bigoplus A_n$ is a $\mathbb{Z}$-graded vector space over k endowed with an associative dot product and a differential $d : A_n \to A_{n-1}$. We also assume the dg-algebra A to enjoy a closed trace, i.e., a linear map $\mathrm{Tr} : A \to k$ obeying the two conditions

$$(8.1) \qquad \mathrm{Tr}(ab) = (-1)^{|a||b|}\mathrm{Tr}(ba)\,, \qquad \mathrm{Tr}(da) = 0\,.$$

For our purposes, it is convenient to regard the graded space $V = A[1]$ as an A_∞-algebra with $m = m_2$, where $m_2(a,b) = (-1)^{|a|-1}ab$. (There is no way to include d into the A_∞-structure m, as it has degree -1; notice, however, that $[m, d] = 0$.) Extending the trace functional from A to the whole tensor algebra TA by zero, we can write Rels. (8.1) as $\mathrm{Tr}L_m = 0$ and $\mathrm{Tr}L_d = 0$. Define the sequence of multi-linear maps on V by setting

$$S_0 = \mathrm{Tr}\,, \qquad S_{k+1} = S_k i_d\,, \qquad k = 0, 1, 2, \ldots,$$

or, more explicitly,

$$(8.2) \qquad S_k(a_0, a_1, \ldots, a_k) = \mathrm{Tr}(a_0 da_1 \cdots da_k)\,.$$

We claim that all S_k's are cocycles of the Hochschild complex $C^\bullet(V, m)$. Indeed,

$$S_k \partial = \mathrm{Tr}(i_d)^k \partial = \mathrm{Tr}L_m(i_d)^k = 0\,.$$

Here we used (hC1). A more familiar form of the cocycle condition above is

$$(8.3) \qquad \begin{aligned} &\sum_{j=0}^{k}(-1)^{\bar{a}_0 + \cdots + \bar{a}_j}S_k(a_0, \ldots, a_j a_{j+1}, \ldots, a_{k+1}) \\ &+ (-1)^{\bar{a}_{k+1}(\bar{a}_0 + \cdots + \bar{a}_k + 1)}S_k(a_{k+1}a_0, a_1, \ldots, a_k) = 0\,, \end{aligned}$$

where $\bar{a} = |a| - 1$ is the degree of a as an element of $V = A[1]$. Using the explicit expression (8.2), one can easily verify the cyclicity property

$$(8.4) \qquad S_k(a_0, a_1, \ldots, a_k) = (-1)^{\bar{a}_0(\bar{a}_1 + \cdots + \bar{a}_k)}S_k(a_1, \ldots, a_k, a_0)\,.$$

Together Rels. (8.3) and (8.4) identify S_k as a cyclic cocycle representing a class of $HC_\lambda^k(A)$.

Suppose now that the dg-algebra A admits a formal deformation $A_t = A[[t]]$, the parameter t being of degree zero. It is known that neither cyclic nor Hochschild (co)homology is homotopy invariant. Nevertheless, we assume that the trace (8.1), representing a class of $HC_\lambda^0(A) = HH^0(A)$, survives deformation, so that the cyclic cocycles (8.2) make sense for the deformed dg-algebra A_t as well.

The results of the previous section suggest that a kind of homotopy invariance does take place for the Hochschild (co)homology, if we restrict ourselves to inner deformations of families. Although there is no way to define inner deformations of A_t as an associative algebra, such deformations are possible

in a larger category of A_∞-algebras [11]. The construction goes as follows. First, we introduce an auxiliary formal variable v of degree two and define the dg-algebra $A_t[[v]]$, just extending the original differential and product by $k[[v]]$-linearity. Multiplying the differential d by v gives then a differential of degree one on $A_t[[v]]$. This allows us to regard $A_t[[v]]$ as a two-parameter family of A_∞-algebras with $m = m_1 + m_2$, where

$$m_1(a) = vda, \qquad m_2(a,b) = (-1)^{\bar{a}}ab.$$

The corresponding algebra of inner deformations $\mathcal{I}$ is generated by the co-cycles $m_{(t)}$ and $m_{(v)} = d$. Now we define the sequence of cocycles

(8.5) $$\Psi_0 = m_{(t)}, \qquad \Psi_{n+1} = d \cup \Psi_n, \qquad n = 0, 1, 2, \dots,$$

all are of degree one. According to Theorem 6.2, each cocycle Ψ_n gives rise to the deformation flow $\tilde{m}_{(s)} = \Psi_n[\tilde{m}]$ on the space of A_∞-structures, with s being a new formal variable of degree zero. Since the parameter v plays an auxiliary role in our construction, we can finally put it to zero to get a family of A_∞-structures $\bar{m} = \tilde{m}|_{v=0}$ on V parameterized by t and s; both the parameters are of degree zero. The family starts as

(8.6) $$\bar{m} = m_2 + s\bar{\Psi}_n + O(s^2),$$

where

$$\bar{\Psi}_n(a_0, a_1, \dots, a_{n+1}) = (-1)^{n+\bar{a}_n}da_0 \cdots da_{n-1}(a_n \diamond a_{n+1})$$

and $\diamond$ stands for the t-derivative of the dot product in A_t. By construction, $\bar{\Psi}_n$ is a Hochschild cocycle representing a class of $HH^{n+2}(A_t, A_t)$. If the class happens to be nontrivial, then (8.6) defines a nontrivial deformation of the associative algebra A_t in the category of A_∞-algebras[6]. Notice that the resulting A_∞-structure $\bar{m}$ is minimal, for $\bar{m}_1 = 0$. We denote the corresponding differential by $\bar{\partial} = L_{\bar{m}}$.

As shown in Sec. 7, every inner deformation of an A_∞-algebra extends to Hochschild (co)cycles.

Proposition 8.1. *The inner deformation (8.6) generates the following first-order deformation of the cocycles (8.2):*

(8.7) $$\bar{S}_k = S_k + s(S_{k+n})_{(t)} + O(s^2).$$

Proof. Computation is somewhat simplified due to specific properties of the cocycles (8.2). First of all, we have

$$S_k L_{\bar{\Psi}_{n+1}} = S_k L_{d \cup \bar{\Psi}_n} = S_k(i_d L_{\bar{\Psi}_n} + L_d i_{\bar{\Psi}_n} + [\partial, L(d, \bar{\Psi}_n)]).$$

Actually, the last two terms on the right vanish. It is immediate that

$$S_k L_d = \mathrm{Tr}L_d(i_d)^k = 0, \qquad S_k\partial = 0.$$

---

[6]For Ψ_0, the deformation boils down to the trivial shift $t \mapsto t + s$.

The remaining term $S_k L(d, \bar{\Psi}_n)\partial$ of the commutator requires more work. Since d is a unary map, it follows from the definition (4.9) that

$$L(d, \bar{\Psi}_n)(a_0, \ldots, a_p)$$

(8.8)
$$= \sum_{0 \le i < j \le p-n-1} (-1)^\kappa (a_0, \ldots, da_i, \ldots, \bar{\Psi}_n(a_j, \ldots, a_{j+n+1}), \ldots, a_p),$$

whence

$$S_k L(d, \bar{\Psi}_n)(a_0, \ldots, a_{k+n+1})$$

$$= \mathrm{Tr}\Big(d \sum_{j=1}^{k} (-1)^\kappa a_0 da_1 \cdots d\bar{\Psi}_n(a_j, \ldots, da_{j+n+1}) \cdots da_{k+n+1}\Big) = 0.$$

Therefore,

$$S_k L_{\bar{\Psi}_{n+1}} = S_{k+1} L_{\bar{\Psi}_n} = \ldots = S_{k+n+1} L_{\bar{\Psi}_0}.$$

Now, verifying the cocycle condition for (8.7), we find

$$\begin{aligned}
\bar{S}_k \bar{\partial} &= \big(S_k + s(S_{k+n})_{(t)} + O(s^2)\big)\big(L_{m_2} + s L_{\bar{\Psi}_n} + O(s^2)\big) \\
&= s\big((S_{k+n})_{(t)} L_{m_2} + S_k L_{\bar{\Psi}_n}\big) + O(s^2) \\
&= s\big((S_{k+n})_{(t)} L_{m_2} + S_{k+n} L_{\bar{\Psi}_0}\big) + O(s^2) \\
&= s(S_{k+n} L_{m_2})_{(t)} + O(s^2) = O(s^2).
\end{aligned}$$

The proof is complete.   □

Notice that the first-order deformation (8.7) enjoys cyclicity (8.4) as well. It is not clear whether this property holds true in higher orders in s. Should this be the case the deformed cocycles $\bar{S}_k$ would define elements of the cyclic cohomology group $HC_\lambda^\bullet(V, \bar{m})$.

As mentioned in the Introduction, inner deformations of A_∞ algebras play a crucial role in construction of a new class of classical integrable theories [7] and in applications to Chern–Simons matter theories and $3d$ bosonization duality [8]. In this context, the most interesting are deformations of dg-algebras that are generated by the first Hochschild cocycle Ψ_1 of (8.5). Furthermore, it is demanded that the deformed cocycles (8.7) be cohomologous to cyclic ones to be interpreted in terms of correlators of higher-spin currents. In the rest of this paper, we show that the inner deformation generated by Ψ_1 does respect cyclicity under the two assumptions:

(I) $A_t = \bigoplus_{n \in \mathbb{Z}} (A_t)_n$ is a family of unital dg-algebras with $(A_t)_n = 0$ for all $n > 1$,

(II) the differential $d : (A_t)_n \to (A_t)_{n-1}$ is independent of t, i.e., $d_{(t)} = 0$.

Again, this is enough for the applications mentioned above. Even with these technical restrictions we need some preparation.

The main difficulty here is that the operator $1 - \lambda$ features no good algebraic properties in hG-calculus and the cyclicity of deformed cochains is

hard to control. Fortunately, there is an equivalent definition of cyclic coho-
mology, which fits perfectly well with hG-calculus. Following [28], [13], we
introduce a new formal variable u of degree 2 and endow the space $\bar{C}_\bullet(V)[u]$
of polynomials in u with the differential $D = \partial - uB$, where B is the Connes–
Rinehart operator (3.5). Regarding $\bar{C}_\bullet(V)[u]$ as a $k[u]$-module, we define
the dual cochain complex $\bar{C}_\bullet(V)^*[u]$. Assumption (I) above implies that the
space V is non-positively graded. As a consequence, the complex $\bar{C}_\bullet(V)^*[u]$
is concentrated in non-negative degrees and each p-cochain $c \in \bar{C}_\bullet(V)^*[u]$ is
given by a polynomial

$$c = c_p + c_{p-2}u + \cdots + c_{p-2q}u^q, \qquad |c_i| = i,$$

of degree $q \leq p/2$. We denote the cohomology of this complex by $HC^\bullet_{[u]}(V, m)$.
The following isomorphism is well known:

$$(8.9) \qquad\qquad HC^\bullet_{[u]}(V, m) \simeq HC^\bullet_\lambda(V, m).$$

See e.g. [36], [37]. The isomorphism allows us to work entirely in terms of
hG-calculus' operations. It is natural to think of D as a perturbation of the
Hochschild differential ∂. We also introduce the operator $I_A = i_A - uS_A$, a
perturbation of the contraction i_A. Then, the following relations hold:

(1) $D^2 = 0$,

(2) $[D, L_A] = L_{\delta A}$,

(2) $uL_A = [D, I_A] + I_{\delta A}$,

(3) $L_{A \cup B} = L_A I_B - (-1)^{|A|} I_A L_B$
$\qquad\qquad - (-1)^{|A|}[D, L(A, B)] + (-1)^{|A|} L(\delta A, B) + L(A, \delta B)$,

(4) $(-1)^{|A|}[L_A, I_B] = I_{[A,B]} + [D, T(A, B)] - T(\delta A, B) - (-1)^{|A|} T(A, \delta B)$,

(5) $uL(A, B) = (-1)^{|A|-1} I_{A \cup B} + (-1)^{|A|} I_A I_B + [D, I(A, B)]$
$\qquad\qquad\qquad\qquad + I(\delta A, B) + (-1)^{|A|} I(A, \delta B)$.

The reader can verify these identities either directly, using the definitions
of Sec. 4, or consult the papers [13], [24]. Comparing Rels. (1-5) with
Axioms (hC1-6), we see that the operations D, L, and I do not define
a precalculus because of the 'curvature' terms uL_A and $uL(A, B)$ in (2)
and (5). Nonetheless, these relations appear to be very helpful in practical
calculations, as we will see shortly.

Given a family of A_∞-structures $m = vd + m_2$ as above, we consider the
inner deformation generated by the cocycle

$$\Psi_1 = m_{(v)} \cup m_{(t)} = d \cup m_2'.$$

Hereinafter the prime stands for the derivative with respect to t. Notice
that the r.h.s. of the equation $\tilde{m}_{(s)} = \Psi_1[\tilde{m}]$ does not depend on v. As a
result, its formal solution has the form

$$\tilde{m} = vd + \bar{m} = m + s(d \cup m_2') + O(s^2),$$

where the cochain $\bar{m}$ is independent of v either. It follows from $[\tilde{m}, \tilde{m}] = 0$ that $\bar{m} = \bar{m}(t, s)$ is a two-parameter family of minimal A_∞-structures which commutes with the differential d, i.e.,

$$[\bar{m}, \bar{m}] = 0, \qquad [d, \bar{m}] = 0.$$

We are interested in cocycles $\hat{S} \in \bar{C}_\bullet(V)^*[u]$ of the differential $\bar{D} = \bar{\partial} - uB$.

Proposition 8.2. *With notation and assumptions above the deformation flow*

(8.10) $$\bar{m}_{(s)} = d \cup \bar{m}', \qquad \hat{S}_{(s)} = \hat{S}' I_d - \hat{S} L(d, \bar{m}')$$

preserves the conditions

(8.11) $$\hat{S}\bar{D} = 0, \qquad \hat{S}L_d = 0.$$

In other words, the flow maps the space of L_d-closed $\bar{D}$-cocycles into itself.

Proof. Differentiating the cochain $\hat{S}\bar{D}$, we get

$$\begin{aligned}(\hat{S}\bar{D})_{(s)} &= \hat{S}_{(s)}\bar{D} + \hat{S}\bar{D}_{(s)} = \hat{S}_{(s)}\bar{D} + \hat{S}L_{\bar{m}_{(s)}} = \hat{S}_{(s)}\bar{D} + \hat{S}L_{d \cup \bar{m}'} \\ &= \hat{S}_{(s)}\bar{D} + \hat{S}(L_d I_{\bar{m}'} + I_d L_{\bar{m}'} + [\bar{D}, L(d, \bar{m}')]).\end{aligned}$$

Since d is a unary operator, $T(d, \bar{m}') = 0$. Furthermore, $[d, \bar{m}'] = [d, \bar{m}]' = 0$, by assumption (II). Combining these observations with the flow (8.10) for $\hat{S}$, we can proceed as follows

(8.12) $$\begin{aligned}(\hat{S}\bar{D})_{(s)} &= \hat{S}_{(s)}\bar{D} + \hat{S}(L_d I_{\bar{m}'} + L_{\bar{m}'} I_d + [\bar{D}, L(d, \bar{m}')]) \\ &= \hat{S}' I_d \bar{D} + \hat{S}L_d I_{\bar{m}'} + \hat{S}L_{\bar{m}'} I_d + \hat{S}\bar{D}L(d, \bar{m}') \\ &= \hat{S}'\bar{D}I_d + u\hat{S}'L_d + \hat{S}L_d I_{\bar{m}'} + \hat{S}L_{\bar{m}'} I_d + \hat{S}\bar{D}L(d, \bar{m}') \\ &= (\hat{S}\bar{D})' I_d + u(\hat{S}L_d)' + (\hat{S}L_d)I_{\bar{m}'} + (\hat{S}\bar{D})L(d, \bar{m}').\end{aligned}$$

In the same way we find

$$(\hat{S}L_d)_{(s)} = \hat{S}_{(s)}L_d = \hat{S}' I_d L_d - \hat{S}L(d, \bar{m}')L_d = \hat{S}' L_d I_d - \hat{S}L(d, \bar{m}')L_d.$$

The operator $L(d, \bar{m}')$ acts according to Eq. (8.8) with $\bar{\Psi}_n$ replaced by $\bar{m}'$. Since the differential d commutes with $\bar{m}'$, one can easily see that $[L(d, \bar{m}'), L_d] = 0$. Therefore, we can write

(8.13) $$(\hat{S}L_d)_{(s)} = (\hat{S}L_d)' I_d - (\hat{S}L_d)L(d, \bar{m}').$$

Taken together Eqs. (8.12) and (8.13) imply the invariance of the algebraic constraints (8.11) under the flow (8.10). $\qquad\square$

Returning to the Hochschild cocycles (8.2) of the dg-algebra A, one can see that all of them are annihilated by the Connes–Rinehart operator, $S_k B = 0$, and hence, by the differential $D = \partial - uB$. Furthermore, they are annihilated by the Lie derivative L_d as well. By Proposition 8.2, the deformed cocycles $\hat{S}_k$ define some cohomology classes of $HC^\bullet_{[u]}(V, \bar{m})$. It is apparent

that the leading term $\bar{S}_k = \hat{S}_k|_{u=0}$ coincides with the Hochschild cocycle (8.7) and should be cohomologous to a cyclic cocycle of $C^\bullet_\lambda(V, \bar{m})$ due to the isomorphism (8.9). In other words, the cohomology classes of $\bar{S}_k$ belong to the preimage of the canonical homomorphism $I : HC^\bullet_\lambda(V, \bar{m}) \to HH^\bullet(V, \bar{m})$. Let us denote a cyclic representative of the class $[\bar{S}_k] \in HC^\bullet_\lambda(V, \bar{m})$ by $\bar{S}_k^\lambda$. By definition, $\bar{S}_k^\lambda(1 - \lambda) = 0$.

Now let $\bar{l}$ be the L_∞-structure on V associated with the A_∞-structure $\bar{m}$ according to Theorem 2.9. It is easy to see that the symmetrization map

$$(8.14) \qquad (\vartheta S)(a_0 \vee \ldots \vee a_n) = \sum_{\sigma \in S_n} (-1)^\kappa S(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

commutes with the differentials $D_{\bar{l}}$ in SV and $\bar{\partial}$ in $C^\lambda_\bullet(V, \bar{m})$, inducing thus a homomorphism in cohomology:

$$\vartheta^* : HH^\bullet_\lambda(V, \bar{m}) \to H^\bullet(V, \bar{l}) \,.$$

Notice that the map ϑ is well-defined as the cochain S enjoys cyclicity.

Applying the symmetrization map (8.14) to the cyclic cocycles $\bar{S}_k^\lambda$ yields the sequence of $D_{\bar{l}}$-cocycles $\Phi_k = \vartheta(\bar{S}_k^\lambda)$, which represent some classes of L_∞-cohomology $H^\bullet(V, \bar{l})$, see Sec. 5. By definition, each cocycle Φ_k is given by a collection of multi-linear maps $\Phi_k^n : S^{n+1}V \to k$ with $n \geq k$. The functions $\Phi_k^n(a_0, \ldots, a_n)$, being symmetric, are completely specified by their values $\Phi_k^n(a, \ldots, a)$ on coinciding arguments $a \in V$. This allows us to encode the cocycle $\Phi_k = \{\Phi_k^n\}_{n=k}^\infty$ by a single function on V:

$$(8.15) \qquad \Phi_k(a) = \sum_{n=k}^\infty \Phi_k^n(\underbrace{a, \ldots, a}_{n+1}) \,.$$

Similarly, one can define a generating function for the L_∞-structure $\bar{l}$. One just regards V as a graded manifold and endow it with the homological vector field Q whose action on 'coordinate functions' $a \in V$ is given by

$$Qa = \sum_{n=2}^\infty \bar{l}_n(\underbrace{a, \ldots, a}_{n}) \,.$$

In this geometric language the cocycle condition $\Phi_k D_{\bar{l}} = 0$ amounts to the Q-invariance of the function (8.15), i.e., $Q\Phi_k = 0$.

We conclude this paper by presenting an example where the deformation flow Ψ_1 for the cocycles Φ_k can be integrated explicitly. Let $A = A_0 \oplus A_1$ be a one-parameter family of dg-algebras with trace. Suppose that the differential $d : A_1 \to A_0$ defines an isomorphism of vector spaces and the trace $\mathrm{Tr} : A \to k$ vanishes on A_0. As is shown in [10], the cocycles (8.15) are given then by the formula

$$(8.16) \qquad \Phi_k(a) = \sum_{n=0}^\infty \frac{1}{k+n+1} \mathrm{Tr}(a \underbrace{da \cdots da}_{k+n}) \Big|_{s^n} \qquad \forall a \in A_1 \,,$$

cf. Eq.(8.7). In this expression, the parameter t, which enters both the trace and the product, is replaced with $t + s$ and the symbol $\rfloor_{s^n}$ instructs one to take only the terms that are proportional to s^n. We can also assemble all $\Phi_k(a)$ into a single generating function $\Phi(a) = \sum_{k \geq 0} \nu^{-k-1} \Phi_k(a)$ with an auxiliary parameter ν to find

$$(8.17) \qquad \Phi(a) = \text{P.p.} \,\text{Tr}\big(d^{-1} \ln(1 - \nu^{-1} da)\big)\Big|_{t \to t + s\nu}.$$

Here P.p. stands for the principal part of the Laurent series in ν.

The existence of compact expression (8.17) looks quite surprising because no simple formula is known for the deformed A_∞-structure $\bar{m}$. Note also a striking similarity of (8.17) with the partition function of a one-loop exact QFT. In the applications to Chern–Simons matter vector models and three-dimensional bosonization duality, one identifies V with the space of conformally invariant tensor currents [10]. Conceivably all these might point to a deeper relationship of the cocycles (8.16) with one-loop exact QFT models yet to be found.

Acknowledgements

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