Higher-Spin Self-Dual Yang-Mills and Gravity from the twistor space

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Abstract

We lift the recently proposed theories of higher-spin self-dual Yang-Mills (SDYM) and gravity (SDGR) to the twistor space. We find that the most natural room for the twistor formulation of these theories is not in the projective, but in the full twistor space, which is the total space of the spinor bundle over the 4-dimensional manifold. In the case of higher-spin extension of the SDYM we prove an analogue of the Ward theorem, and show that there is a one-to-one correspondence between the solutions of the field equations and holomorphic vector bundles over the twistor space. In the case of the higher-spin extension of SDGR we show show that there is a one-to-one correspondence between solutions of the field equations and Ehresmann connections on the twistor space whose horizontal distributions are Poisson, and whose curvature is decomposable. These data then define an almost complex structure on the twistor space that is integrable.

1 Introduction

Despite the abundance of no-go theorems against theories with massless higher-spin fields (Higher spin gravities or HiSGRA) there are several examples that bypass these pitfalls: $3d$ models with topological (partially-)massless and conformal higher-spin fields [1–8]; Chiral HiSGRA [9–13]; higher-spin extensions of $4d$ conformal gravity [14–16]; IKKT-based theories [17–19]; collective dipole [20, 21].[1] Out of these examples only Chiral HiSGRA is a local

[1]The latter two are strictly speaking not local field theories. Nevertheless, they come equipped with well-defined prescriptions to compute e.g. holographic correlators.

field theory with propagating massless fields, which makes it an interesting playground where the usual QFT and AdS/CFT methods apply, especially in view of the recent developments such as one-loop UV-finiteness [12, 13, 22] and fully covariant equations of motion of Chiral HiSGRA for any value of the cosmological constant [23–26], including zero.

Chiral HiSGRA admits two simple consistent truncations where either Yang-Mills or gravitational interactions are retained and the scalar field is dropped, which was shown in [27] in the light-cone gauge. These two truncations can be thought of as higher-spin extensions of the self-dual Yang-Mills (SDYM) and self-dual Gravity (SDGR) theories, HS-SDYM and HS-SDGR, respectively. In [28] two of the present authors have constructed the covariant formulations of these HS-SDYM and HS-SDGR. The question left unanswered was whether there exist a twistor space description of these theories. An attempt in this direction using the standard projective twistor space was made in [18]. Another attempt at a twistor description of higher spins is a recent paper [29], where a twistor description of the full chiral higher-spin gravity is contemplated. For applications of twistor techniques to conformal HiSGRA see [30, 31].

This work is a natural continuation of [28] in that it gives an answer to the question of the twistor space description of self-dual higher-spin theories. Our main new insight is that the appropriate room for the higher-spin (self-dual) theories is not in the projective twistor space, but rather in the full spinor bundle over the four-dimensional manifold in question. The usual SDYM and SDGR then arise as the subsectors of the higher-spin extensions that descend to the projective twistor space.

In the case of HS-SDYM we provide both a description of the twistor space lift of the theory, as well as a statement in the opposite direction, i.e. an analog of the Ward theorem. Our main statement can be formulated as the following theorem.

Let $S' \xrightarrow{\pi} \mathrm{S}^4$ be the bundle of (primed) spinors on S^4, the twistor space $\mathbb{T} = (\mathbb{C}^4)^* \xrightarrow{\pi} \mathrm{S}^4$ is obtained by deleting from S' the zero section. Let U be an open set of S^4 and let $V = \pi^*(U)$ be the corresponding open set of $\mathbb{T}$.

Theorem A. There is a one-to-one correspondence between solutions of the higher-spin self-dual Yang-Mills equations on $U \subset S^4$ (up to a gauge) and holomorphic bundles $\mathcal{E} \to V$ such that the restriction of $\mathcal{E}$ along each of the fibres of $V \to U$ is trivial.

This theorem can be viewed as the higher-spin analogue of the Ward theorem.

In the case of HS-SDGR we have the following statement. Let $S' \xrightarrow{\pi} M^4$ be the bundle of (primed) 2-component spinors on a 4-manifold M^4, the twistor space $\mathcal{T} \xrightarrow{\pi} M^4$ is obtained by deleting from S' the zero section. Let $\mathcal{V} \subset T\mathcal{T}$ be the vertical distribution. Let U be an open set of M^4 and let $V = \pi^*(U)$ be the corresponding open set of $\mathcal{T}$. We define the horizontal distributions on TV to be those in the kernel of the projection $P\colon T\mathcal{T} \to \mathcal{V}$

$$P = \tau^{A'}\frac{\partial}{\partial \pi^{A'}} + \hat{\tau}^{A'}\frac{\partial}{\partial \hat{\pi}^{A'}}, \qquad \tau^{A'} := d\pi^{A'} + \mathcal{A}^{A'}(x, \pi, \hat{\pi})\,.$$

These are parametrised by the Ehresmann connection $\mathcal{A}^{A'}(x, \pi, \hat{\pi})$. Here A' is the 2-component spinor index, $\pi^{A'}$ is the fibre coordinate and $\hat{\pi}^{A'}$ is constructed by $\pi^{A'}$ using the hat operation that is available in the Euclidean signature and squares to minus the identity. The connection $\mathcal{A}^{A'}(x, \pi, \hat{\pi})$ is a one-form on M^4. The twistor space naturally is a Poisson manifold with Poisson structure

$$\epsilon^{B'A'}\frac{\partial}{\partial \pi^{A'}}\frac{\partial}{\partial \pi^{B'}} + \epsilon^{B'A'}\frac{\partial}{\partial \hat{\pi}^{A'}}\frac{\partial}{\partial \hat{\pi}^{B'}}\,, \tag{1.1}$$

where $\epsilon^{A'B'}$ is the inverse of the volume form on the $(\mathbb{C}^2)^*$ fibres.

Theorem B. For horizontal distributions that are infinitesimal symmetries of the Poisson structure (1.1) the Ehresmann connection $\mathcal{A}^{A'}$ has a potential: $\mathcal{A}^{A'} = -\epsilon^{A'B'}\partial_{B'}A$, where $A = A(x,\pi)$. In particular, the Ehresmann connection $\mathcal{A}^{A'}$ of Poisson horizontal distributions is independent of $\hat{\pi}$. Furthermore, its curvature 2-form has also a potential $\mathcal{F}^{A'} = -\epsilon^{A'B'}\partial_{B'}F$, where $F = F(x,\pi) = dA + (1/2)\{A, A\}$ and $\{\cdot,\cdot\}$ is the Poisson bracket given by (1.1). There is a one-to-one correspondence (up to a gauge) between solutions of the higher-spin self-dual gravity equations on U and Poisson horizontal distributions on TV whose curvature potential F is decomposable $F \wedge F = 0$. What is more, the two simple factors of F define, together with the 1-forms $\tau^{A'}$, an almost complex structure on V that is integrable.

In both the cases of HS-SDYM and HS-SDGR we explain the geometric origin of the subtle invariances that the field equations possess. These are seen to come from the diffeomorphisms of the twistor space.

The organisation of this paper is very simple: in sections 2 and 3 we discuss (higher-spin extensions) of self-dual Yang-Mills theory and self-dual gravity, respectively.

2 Higher-spin self-dual Yang-Mills in twistor space

Let $P \to S^4$ be a $GL(N,\mathbb{C})$-principal bundle and let $E \to S^4$ be some associated bundle.

2.1 HS-SDYM equations: Spacetime equations

We follow [28] in this subsection. The "Higher-spin YM potential" is a (formal) sum of Lie algebra-valued one-forms with different numbers of primed spinor indices

$$\mathbb{A} = \sum_{s=1}^{\infty} A^{A'(2s-2)} = A + A^{A'(2)} + A^{A'(4)} + \dots \tag{2.1}$$

where all fields take values in the Lie algebra of $\mathrm{GL}(N,\mathbb{C})$ and are also one-forms. The first term in the sum on the right-hand side is the usual Yang-Mills gauge potential – a Lie algebra-valued one-form. The notation $A'(n)$ is standard in the higher-spin literature, and denotes n different primed indices that are symmetrized. Thus,

$$A'(n) \equiv (A'_1 \dots A'_n)\,.$$

For any

$$\xi = \sum_{s=1}^{\infty} \xi^{A'(2s-2)} = \xi + \xi^{A'(2)} + \xi^{A'(4)} + \dots \tag{2.2}$$

taking values in $\mathfrak{gl}(N,\mathbb{C})$, the "higher-spin gauge transformations" are

$$\delta_\xi \mathbb{A} = \sum_{s=1}^{\infty} d\xi^{A'(2s-2)} + \sum_{s=1}^{\infty}\sum_{\tilde{s}=1}^{\infty}([A,\xi])^{A'(2s+2\tilde{s}-4)}\,. \tag{2.3}$$

It is useful to write the first couple of terms in this series explicitly

$$\delta_\xi A = (d\xi + [A,\xi]) + (d\xi^{A'(2)} + [A,\xi^{A'(2)}] + [A^{A'(2)},\xi]) + \\ (d\xi^{A'(4)} + [A,\xi^{A'(4)}] + [A^{A'(4)},\xi] + [A^{A'(2)},\xi^{A'(2)}]) + \dots \tag{2.4}$$

The first term in this sum is the usual gauge transformation of Yang-Mills field. It is important to note that in the last term in the second line the spinor indices of $A^{A'(2)}$ and $\xi^{A'(2)}$ are assumed to be symmetrized, so that only the terms with totally symmetric spinor indices arise. This can easily be implemented by defining generating functions

$$A(\pi) = \sum_{s=1}^{\infty} A^{A'(2s-2)}\,\pi_{A'}\dots\pi_{A'}\,, \qquad\qquad \xi(\pi) = \sum_{s=1}^{\infty} \xi^{A'(2s-2)}\pi_{A'}\dots\pi_{A'}$$

with the help of an auxiliary commuting variable $\pi_{A'}$.

The corresponding "field strength" is defined as

$$F = \sum_{s=1}^{\infty} dA^{A'(2s-2)} + \sum_{s=1}^{\infty}\sum_{\tilde{s}=1}^{\infty} \frac{1}{2}\,[A\wedge A]^{A'(2s+2\tilde{s}-4)}\;. \tag{2.5}$$

Again, the spinor indices in a product expression are always symmetrized to produce only the terms with totally symmetric spinor indices. Restricting to the first terms in the above sum we recover the usual Yang-Mills curvature.

The higher-spin self-dual Yang-Mills equations are

$$F\big|_{ASD} = 0\,. \tag{2.6}$$

To give this more concrete meaning we recall that the ASD projection of a 2-form is computed by converting the spacetime indices into the spinor ones $\mu \to MM'$. A 2-form B is then split into its SD and ASD parts as follows

$$B_{MM'NN'} = \frac{1}{2}B_{(ME'N)}{}^{E'}\epsilon_{M'N'} + \frac{1}{2}B_{E(M'}{}^{E}{}_{N')}\epsilon_{MN}\,. \tag{2.7}$$

The second terms is the ASD part of the 2-form B. Thus, in (2.6) there is a pair of primed spinor indices coming from the projection of the 2-form onto its ASD part, as well as the primed spinor indices that are labels of the different summands in F. It is assumed that all the primed spinor indices are symmetrized in (2.6). Let us write down the first few equations contained in (2.6) explicitly. Thus,

$$d_E{}^{(A'_1}A^{EA'_2)} + \frac{1}{2}[A_E{}^{(A'_1}, A^{EA'_2)}] = 0 \tag{2.8}$$

is the usual SDYM equation. The next equation has four free primed spinor indices and reads

$$d_E{}^{(A'_1}A^{EA'_2A'_3A'_4)} + [A_E{}^{(A'_1}, A^{EA'_2A'_3A'_4)}] = 0\,. \tag{2.9}$$

Because the spinor index that comes from the one-form index is always symmetrized with the other primed spinor indices in the above equations, the arising equations enjoy an extra invariance under shift symmetry:

$$\delta A = dx^{BB'}\sum_{s=2}^{\infty}\delta_{B'}{}^{A'}\eta_B{}^{A'(2s-4)} = dx^{B(A'_1}\eta_B^{A'_2)} + \dots \tag{2.10}$$

As opposed to the "higher-spin gauge transformations", which reduce to usual gauge transformations when restricted to the $s=1$ terms, the shift invariance is a genuine new feature of the higher-spin equations.

2.2 Geometrical realisation in twistor space

Let $S' \xrightarrow{\pi} \mathrm{S}^4$ be the bundle of (primed) spinors on S^4. We will denote the primed spinor that is a coordinate along the fibre by $\pi_{A'}$. The twistor space $\mathbb{T} = (\mathbb{C}^4)^*$ is obtained by deleting from S' the zero section. The Euclidean spinors admit a hat operator. This is an anti-linear operator that maps primed spinors to primed spinors and squares to minus the identity. We will make extensive use of the following identities:

$$\pi_{B'}\hat{\pi}^{A'} - \hat{\pi}_{B'}\pi^{A'} = \langle \pi\hat{\pi}\rangle \, \delta_{B'}{}^{A'} \,, \tag{2.11}$$

where we introduced

$$\left(\pi_{D'}\hat{\pi}^{D'}\right) := \langle \pi\hat{\pi}\rangle \,. \tag{2.12}$$

The complex structure on $\mathbb{T}$ that we use is as follows. The basis of $(0,1)$ 1-forms is given by

$$d\hat{\pi}^{A'}, \qquad \hat{\pi}_{A'}dx^{AA'} \,, \tag{2.13}$$

with dual vector fields

$$\frac{\partial}{\partial\hat{\pi}^{A'}}\,, \qquad -\frac{\pi^{A'}\partial_{AA'}}{\langle \pi\hat{\pi}\rangle}\,. \tag{2.14}$$

Making use of the identity (2.11), the projection of $A_{AA'}dx^{AA'}$ on its $(1,0)$ and $(0,1)$ parts are

$$A_{AA'}dx^{AA'}\big|_{1,0} = \frac{A_{AA'}\hat{\pi}^{A'}}{\langle \pi\hat{\pi}\rangle}\, dx^{AB'}\pi_{B'}, \qquad A_{AA'}dx^{AA'}\big|_{0,1} = -\frac{A_{AA'}\pi^{A'}}{\langle \pi\hat{\pi}\rangle}\, dx^{AB'}\hat{\pi}_{B'}\,.$$

2.2.1 Higher-Spin Yang-Mills fields on twistor space

The higher-spin Yang-Mills potential (2.1) that was introduced as a formal sum of Lie algebra-valued one-forms with a different number of spinor indices naturally arises from the following field on the twistor space,

$$\mathcal{A}(x,\pi) := \sum_{s=1}^{\infty} A(x)^{A'(2s-2)}\pi_{A'(2s-2)}. \tag{2.15}$$

This reinterprets the connection as a gauge potential in the pullback bundle $\pi^*E \to \mathbb{T}$, which is a gauge bundle where the base is now the full twistor space $\mathbb{T}$, and not just its projectivised version as in the standard construction. One can then readily see that, modulo the terms involving $d\pi_{A'}$, the higher-spin gauge transformations (2.3) coincide with the usual gauge transformations in the twistor space

$$\delta_\xi \mathcal{A} = d\xi + [\mathcal{A}, \xi] \,,$$

and the higher-spin field strength (2.5) coincides with the usual curvature 2-form

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \,,$$

again modulo the terms involving $d\pi_{A'}$. In turn, the HS-SDYM field equations are clearly related to

$$\mathcal{F}\big|_{0,2} = 0 \,,$$

again modulo the terms along the fibre direction. We now develop a holomorphic bundle interpretation that clarifies all these issues.

2.2.2 Holomorphic bundle interpretation

Just as in the usual Ward correspondence one can introduce the $(0,1)$ part of the gauge field

$$a := -\sum_{s=1}^{\infty}\left(A^{A'(2s-2)}\pi_{A'(2s-2)}\right)\Big|_{0,1} = \frac{dx^{AB'}\hat{\pi}_{B'}}{\langle\pi\hat{\pi}\rangle}\sum_{s=1}^{\infty}A_A{}^{A'(2s-1)}\pi_{A'(2s-1)}\,. \tag{2.16}$$

Note that $2s-2$ of the primed indices on $A_A{}^{A'(2s-1)}$ are those that existed already in (2.1), while an additional primed index arises when the one-form index of the gauge potential is converted into a pair of unprimed and primed spinors.

Note that the projection onto the part of the higher-spin gauge potential (2.1) that is invariant under (2.10) has already occurred here, because the primed spinor index that came from the one-form index became symmetrized with the other primed indices of the gauge potentials. Indeed, under the shift symmetry (2.10) the higher-spin potential (2.15) is shifted by a $(1,0)$ term and therefore (2.16) is invariant.

This defines a differential operator on $\pi^*E\to\mathbb{T}$

$$\overline{D}:\left|\begin{array}{ccc}\Gamma\left[\pi^*E\right] & \to & \Omega^{0,1}\left(\mathbb{T},\pi^*E\right)\\ \Phi & \mapsto & \left(\bar{\partial}+a\right)\Phi\,,\end{array}\right.$$

which maps sections of the bundle π^*E into $(0,1)$ forms on $\mathbb{T}$ valued in the space of sections. The HS-SDYM field equations are then the conditions for $\pi^*E\to\mathbb{T}$ to be holomorphic:

$$\bar{\partial}a+\frac{1}{2}[a,a]=0\,. \tag{2.17}$$

Seeing this is an exercise. Some key steps of this computation, which illustrate how it works, are as follows. First, we have

$$d\left(\frac{dx^{AB'}\hat{\pi}_{B'}}{\langle\pi\hat{\pi}\rangle}\right) = -\frac{dx^{AB'}\wedge d\hat{\pi}_{B'}}{\langle\pi\hat{\pi}\rangle} - d\langle\pi\hat{\pi}\rangle\wedge\frac{dx^{AB'}\hat{\pi}_{B'}}{\langle\pi\hat{\pi}\rangle^2}\,. \tag{2.18}$$

We need to project it onto the $(0,2)$ part. This is done by inserting the identity (2.11) in the first term, and then keeping only the $\pi d\hat{\pi}$ part in the second term. This gives

$$d\left(\frac{dx^{AB'}\hat{\pi}_{B'}}{\langle\pi\hat{\pi}\rangle}\right)\Big|_{0,2} = \frac{dx^{AB'}\hat{\pi}_{B'}\wedge\pi^{C'}d\hat{\pi}_{C'}}{\langle\pi\hat{\pi}\rangle^2} - \pi_{C'}d\hat{\pi}^{C'}\wedge\frac{dx^{AB'}\hat{\pi}_{B'}}{\langle\pi\hat{\pi}\rangle^2}=0\,. \tag{2.19}$$

Thus, applying the exterior derivative to the potential (2.16) and keeping the projection onto the $(0,2)$ part gives

$$\frac{\Sigma^{B'C'}\hat{\pi}_{B'}\hat{\pi}_{C'}}{\langle\pi\hat{\pi}\rangle^2}\sum_{s=1}^{\infty}\partial_A{}^{D'}A^{AA'(2s-1)}\pi_{A'(2s-1)}\pi_{D'} \tag{2.20}$$

where we used

$$(dx^{AB'}\hat{\pi}_{B'})\wedge(dx^{BC'}\hat{\pi}_{C'}) = \epsilon^{AB}\Sigma^{B'C'}\hat{\pi}_{B'}\hat{\pi}_{C'}\,,\qquad \Sigma^{B'C'} := \frac{1}{2}dx_A{}^{B'}\wedge dx^{AC'}\,. \tag{2.21}$$

Similarly, taking the $[a,a]$ we get

$$[a,a] = \frac{\Sigma^{B'C'}\hat{\pi}_{B'}\hat{\pi}_{C'}}{\langle\pi\hat{\pi}\rangle^2}\sum_{s=1}^{\infty}\sum_{\tilde{s}=1}^{\infty}[A_A{}^{A'(2s-1)},A^{AA'(2\tilde{s}-1)}]\pi_{A'(2s-1)}\pi_{A'(2\tilde{s}-1)}\,. \tag{2.22}$$

Adding the two parts of (2.17) we get the higher-spin SDYM equations

$$\partial_A{}^{A'} A^{AA'(2s-1)} + \frac{1}{2}\sum_{s=1}^{\infty}\sum_{\tilde{s}=1}^{\infty}[A_A{}^{A'(2s-1)}, A^{AA'(2\tilde{s}-1)}] = 0\,. \tag{2.23}$$

Let us make a final important remark: while (2.15) clearly receives the interpretation of a connection on the whole of $\pi^*E \to S'$ (not just on $\pi^*E \to \mathbb{T}$), one however sees from (2.16) that, for every fixed x, $a(x, \lambda\pi)$ does not have a well defined limit as $|\lambda| \to 0$ and thus does not continuously extend on S'. This stems from the fact that there is no complex structure defined along the zero section of $S' \to S^4$: the projection

$$A_{AA'}dx^{AA'}\big|_{0,1} = -\frac{A_{AA'}\pi^{A'}}{\pi_{D'}\hat{\pi}^{D'}}\, dx^{AB'}\hat{\pi}_{B'}.$$

can only be continuous at $\pi = 0$ if $A_{AA'} = 0$. So, it is important to remove the zero section from the total bundle of spinors to be able to give the twistor interpretation to the whole construction.

2.3 A higher-spin Ward correspondence

2.3.1 Correspondence

Let $S' \xrightarrow{\pi} \mathrm{S}^4$ be the bundle of spinors on S^4, the twistor space $\mathbb{T} = (\mathbb{C}^4)^* \xrightarrow{\pi} \mathrm{S}^4$ is obtained by deleting from S' the zero section. Let U be an open set of S^4 and let $V = \pi^*(U)$ be the corresponding open set of $\mathbb{T}$.

Theorem 2.1. *There is a one-to-one correspondence between solutions of the higher-spin self-dual Yang-Mills equations on $U \subset S^4$ (up to a gauge) and holomorphic bundles $\mathcal{E} \to V$ such that the restriction of $\mathcal{E}$ along each of the fibres of $V \to U$ is trivial.*

This theorem should be understood as a "higher-spin Ward theorem", generalizing the classical work [32] to higher spins: Holomorphic bundles on twistor space $\mathbb{T}$ correspond to higher-spin self-dual Yang-Mills solutions. The geometrical restriction to bundles which descend to *projective* twistor space $\mathrm{P}\mathbb{T}$ amounts to restricting the higher-spin fields to the usual self-dual Yang-Mills.

2.3.2 Proof

Let $\mathcal{E} \to V$ be a holomorphic bundle satisfying the requirements of the theorem. It can equivalently be represented by a differential operator

$$\overline{D}\colon \left| \begin{array}{ccc} \Gamma[\mathcal{E}] & \to & \Omega^{0,1}[\mathcal{E}] \\ \Phi & \mapsto & \left(\bar{\partial} + a\right)\Phi \end{array} \right.$$

with vanishing "curvature"

$$\overline{D}^2 = \bar{\partial}a + \tfrac{1}{2}[a, a] = 0\,.$$

In a local patch, we can write

$$a = a_A\, dx^{AB'}\hat{\pi}_{B'} + \tilde{a}_{A'}\, d\hat{\pi}^{A'}\,,$$

and the curvature then is

$$\overline{D}^2 = \left(\frac{\partial}{\partial\hat{\pi}^{A'}}\tilde{a}_{B'} + \tfrac{1}{2}[\tilde{a}_{A'},\tilde{a}_{B'}]\right) d\hat{\pi}^{A'} \wedge d\hat{\pi}^{B'} \tag{2.24}$$
$$+ \left(\frac{\partial}{\partial\hat{\pi}^{A'}}a_A + \frac{\pi_{A'}}{\langle\pi\hat{\pi}\rangle}a_A + \frac{\pi^{C'}}{\langle\pi\hat{\pi}\rangle}\partial_{AC'}\tilde{a}_{A'} + [\tilde{a}_{A'}, a_A]\right) d\hat{\pi}^{A'} \wedge dx^{AB'}\hat{\pi}_{B'}$$
$$+ \left(-\frac{\pi^{C'}}{\langle\pi\hat{\pi}\rangle}\partial_{AC'}a_B + \tfrac{1}{2}[a_A, a_B]\right) dx^{AA'}\hat{\pi}_{A'} \wedge dx^{BB'}\hat{\pi}_{B'}\,.$$

The vanishing of the first line means that the restriction of $\mathcal{E} \to V$ along each of the $(\mathbb{C}^2)^*$ fibres is holomorphic. This is because the restriction of $\overline{D}$ to the fibres is

$$d\hat{\pi}^{A'}\left(\frac{\partial}{\partial\pi^{A'}} + \tilde{a}_{A'}\right).$$

By hypothesis this bundle is trivial, therefore one can choose[2] a trivialisation such that $\tilde{a}_{A'} = 0$. Within this assumption, the second line of the curvature (2.24) can be rewritten as

$$\hat{\pi}^{A'}\frac{\partial}{\partial\hat{\pi}^{A'}}a_A = -a_A, \qquad \pi^{A'}\frac{\partial}{\partial\hat{\pi}^{A'}}a_A = 0\,.$$

Introducing

$$\mathcal{A}_A(x,\pi) := \langle\pi\hat{\pi}\rangle a_A$$

these are equivalent to

$$\frac{\partial}{\partial\hat{\pi}^{A'}}\mathcal{A}_A = 0\,.$$

Therefore $\mathcal{A}_A$ is a holomorphic function along the $(\mathbb{C}^2)^*$ fibres of $V \to U$. However, by Hartogs's extension theorem there do not exist isolated singularities in complex dimension $n \geq 2$ and therefore for every $x \in U$, $\mathcal{A}_A(x,\pi)$ can be holomorphically extended to the whole of the $\mathbb{C}^2$ fibres. In particular we must have

$$\mathcal{A}_A(x,\pi,\hat{\pi}) = \sum_{s=1}^{\infty} A_A{}^{A'(2s-1)}(x)\pi_{A'(2s-1)}.$$

Strictly speaking, we should sum over all spins in the above sum, both integers and half-integers. However, one can easily restrict the sum to integer spins only by requiring A to be invariant under parity on $\mathbb{C}^2$. This reproduces the ansatz for the connection (2.15). The vanishing of the third line (2.24) is then equivalent to the higher-spin self-dual Yang-Mills equations. □

3 Higher-Spin Self-Dual Gravity in twistor space

Let $P \to M^4$ be a $SU(2)$-principal bundle on M^4. Let $S' \xrightarrow{\pi} M$ be the bundle of spinors, defined as the associated bundle for the fundamental representation of $SU(2)$. The twistor space $\mathcal{T}$ is obtained by deleting from S' the zero section.

[2]Note that as opposed to the usual SDYM correspondence where one can always make this choice of gauge, this here really makes use of the first of the field equations (2.24).

3.1 HS-SDGR equations: Spacetime equations

3.1.1 HS-SDGR equations $\Lambda \neq 0$

The "Higher-spin gravity potential" is best described using its generating functional, as in [28]. We define

$$A = \sum_{n=2}^{\infty} \frac{1}{n!} A^{A'(n)} \pi_{A'_1} \dots \pi_{A'_n}, \tag{3.1}$$

where every term is one-form valued. It is assumed that the sum here is taken over even spins only, which is easily imposed by requiring the potential to be invariant under $\pi_{A'} \to -\pi_{A'}$.

The "field strength" is defined by

$$F := dA + \frac{1}{2}\{A, A\}, \tag{3.2}$$

where we introduced the Poisson bracket of functions of $\pi_{A'}$

$$\{f(\pi), g(\pi)\} = \partial^{C'} f \partial_{C'} g = \sum_n \frac{1}{n!} \sum_{k+m=n} \frac{n!}{k!m!} f^{A'(k)C'} g^{B'(m)}{}_{C'} \pi_{A'_1} \dots \pi_{A'_k} \pi_{B'_1} \dots \pi_{B'_m}. \tag{3.3}$$

The higher-spin self-dual gravity equations are

$$F \wedge F = 0. \tag{3.4}$$

This, in particular, implies that F is a decomposable 2-form, fact which will be of importance below. Restricting to the first terms in the above sums we recover respectively, an $\mathfrak{su}(2)$-valued connection $A^{A'B'}$, its curvature $F^{A'B'}$ and $\Lambda \neq 0$ self-dual gravity equations $F^{(A'B'} \wedge F^{C'D')} = 0$. For more information about the spin-2 case we refer the reader to [33], see also [34] for a thorough discussion on how the field equations $F^{(A'B'} \wedge F^{C'D')} = 0$ relate to the Mason–Wolf twistor action [35].

3.1.2 Gauge invariance

These equations enjoy several gauge symmetries. First, for any

$$\xi = \sum_{n=2}^{\infty} \frac{1}{n!} \xi^{A'(n)} \pi_{A'_1} \dots \pi_{A'_n}$$

we have the "higher-spin gauge transformations"

$$\delta_\xi A = d\xi + \{A, \xi\} \tag{3.5}$$

and for any

$$\eta^\mu = \sum_{n=0}^{\infty} \frac{1}{n!} \eta^{\mu A'(n)} \pi_{A'_1} \dots \pi_{A'_n}$$

the "higher-spin generalised diffeomorphisms"

$$\delta_\eta A = dx^\nu \, \eta^\mu F_{\mu\nu} \,. \tag{3.6}$$

The first terms in these sums respectively give standard $\mathfrak{su}(2)$ gauge transformations $\delta_\xi A = d_A \xi$ and Lie derivatives (up to a gauge transformation): $\delta_\eta A = \iota_\eta F = \mathcal{L}_\eta A - d_A (\iota_\eta A)$.

3.2 Geometrical realisation in twistor space

3.2.1 Higher-spin fields as a connection on the twistor space

Let $\left(x, \pi^{A'}\right)$ be local coordinates on $\mathcal{T} \to M$. It will here be crucial that the fibers are equipped with a preferred volume form $\epsilon_{A'B'}$ and that the twistor space is therefore equipped with the Poisson structure

$$\epsilon^{B'A'} \frac{\partial}{\partial \pi^{A'}} \frac{\partial}{\partial \pi^{B'}} + \epsilon^{B'A'} \frac{\partial}{\partial \hat{\pi}^{A'}} \frac{\partial}{\partial \hat{\pi}^{B'}}. \tag{3.7}$$

We will start by considering the following 1-form on the twistor space

$$\tau^{A'} \partial_{A'} = \left(d\pi^{A'} + \mathcal{A}(x, \pi, \hat{\pi})^{A'} \right) \frac{\partial}{\partial \pi^{A'}} \,. \tag{3.8}$$

This object receives the following geometric interpretation: at every point $p \in \mathcal{T}$

$$P = \tau^{A'} \frac{\partial}{\partial \pi_{A'}} + \hat{\tau}^{A'} \frac{\partial}{\partial \hat{\pi}_{A'}}$$

defines a projector $P_p \colon T_p\mathcal{T} \to \mathcal{V}_p$ on the vertical tangent bundle. This splits the tangent bundle as

$$T_p\mathcal{T} = \mathcal{V}_p + H_p$$

where H_p is the kernel of P_p. The corresponding "horizontal" distribution H on $TS' \to M$ is a connection in the sense of Ehresmann. A general horizontal vector field is one of the form $\xi = V^\mu(x, \pi, \hat{\pi}) D_\mu$ with

$$D_\mu = \partial_\mu - \mathcal{A}^{A'}_\mu(x, \pi, \hat{\pi}) \partial_{A'} - \hat{\mathcal{A}}^{A'}_\mu(x, \pi, \hat{\pi}) \hat{\partial}_{A'}. \tag{3.9}$$

We have the following proposition:

Proposition 3.1. *A general horizontal vector field ξ is an infinitesimal symmetry of the Poisson structure*

$$\mathcal{L}_\xi \left(\epsilon^{A'B'} \frac{\partial}{\partial \pi^{A'}} \frac{\partial}{\partial \pi^{B'}} + \epsilon^{A'B'} \frac{\partial}{\partial \hat{\pi}^{A'}} \frac{\partial}{\partial \hat{\pi}^{B'}} \right) = 0 \tag{3.10}$$

if and only if $\partial_{A'} V^\mu = \hat{\partial}_{A'} V^\mu = 0$ and

$$\hat{\partial}_{B'} \mathcal{A}^{A'}{}_\mu = 0, \qquad \qquad \partial_{[B'} \mathcal{A}_{A']\mu} = 0. \tag{3.11}$$

These can always be solved locally as

$$\mathcal{A}(x, \pi)^{A'} := -\epsilon^{A'B'} \partial_{B'} A(x, \pi), \tag{3.12}$$

for some $A(x, \pi)$.

Proof is by a computation. We have

$$\mathcal{L}_\xi \left(\epsilon^{A'B'} \frac{\partial}{\partial \pi^{A'}} \frac{\partial}{\partial \pi^{B'}} \right) = \epsilon^{A'B'} [\xi, \frac{\partial}{\partial \pi^{A'}}] \otimes \frac{\partial}{\partial \pi^{B'}} + \epsilon^{A'B'} \frac{\partial}{\partial \pi^{A'}} \otimes [\xi, \frac{\partial}{\partial \pi^{B'}}], \tag{3.13}$$

and

$$[\xi, \frac{\partial}{\partial \pi^{A'}}] = -(\partial_{A'} V^\mu)(\partial_\mu - \mathcal{A}_\mu^{C'} \partial_{C'} - \hat{\mathcal{A}}_\mu^{C'} \hat{\partial}_{C'}) + V^\mu (\partial_{A'} \mathcal{A}_\mu^{C'} \partial_{C'} + \partial_{A'} \hat{\mathcal{A}}_\mu^{C'} \hat{\partial}_{C'}). \tag{3.14}$$

The absence of the $\partial_\mu \otimes \partial_{A'}$ terms in the Lie derivative implies $\partial_{A'} V^\mu = 0$, and the similar reasoning for the $\partial_\mu \otimes \hat{\partial}_{A'}$ terms implies $\hat{\partial}_{A'} V^\mu = 0$. On the other hand, the $\partial_{A'} \otimes \partial_{B'}$ terms in the Lie derivative are

$$\epsilon^{B'A'} V^\mu \partial_{A'} \mathcal{A}_\mu^{C'} \partial_{C'} \otimes \partial_{B'} + \epsilon^{B'A'} \partial_{A'} \otimes V^\mu \partial_{B'} \mathcal{A}_\mu^{C'} \partial_{C'} = V^\mu (\partial^{A'} \mathcal{A}_\mu^{B'})(\partial_{A'} \otimes \partial_{B'} - \partial_{B'} \otimes \partial_{A'}), \tag{3.15}$$

and so indeed the absence of such terms in the Lie derivative implies $\partial_{[A'} \mathcal{A}_{B']} = 0$. The absence of the $\partial_{A'} \otimes \hat{\partial}_{B'}$ terms directly implies $\partial_{A'} \hat{\mathcal{A}}_{B'} = 0$, which is equivalent to $\hat{\partial}_{A'} \mathcal{A}_{B'} = 0$. Thus (3.12) is a necessary and sufficient condition for the horizontal vector field to be a Poisson symmetry.

We are thus led to consider Ehresmann connections $\mathcal{A}(x,\pi)^{A'}$ given by a derivative of the higher-spin field (3.1) (just as for the Yang-Mills case, this is really making use of Hartogs's extension theorem which allows to extend the holomorphicity from $(\mathbb{C}^2)^*$ to the whole of $\mathbb{C}^2$). Explicitly, in terms of the higher-spin fields

$$\mathcal{A}(x,\pi)^{A'} = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} A(x)^{A' A'(n-1)} \pi_{A'_1} \ldots \pi_{A'_{(n-1)}}. \tag{3.16}$$

In particular, restricting oneself to the case of linear connections (and thus to a connection in the more usual sense of principal bundles),

$$\tau^{A'} \partial_{A'} = \left(d\pi^{A'} + A(x)^{A'B'} \pi_{B'} \right) \frac{\partial}{\partial \pi^{A'}}$$

amounts to going from higher spins to the spin-2 field.

At every point $p \in \mathcal{T}$, curvature of this connection is given by the 2-form

$$\mathcal{F}_p \left| \begin{array}{ccc} H_p \times H_p & \to & V_p \\ (X,Y) & \mapsto & P([X,Y]) \end{array} \right.$$

where $[,]$ is the usual Lie Bracket on vector fields, and P is the above projector on the vertical vector fields. This definition makes manifest the fact that curvature of a connection is the obstruction to the integrability of the corresponding horizontal distribution.

It is clear that the required projection on the vertical distribution can be computed as

$$\begin{aligned} -\tau^{0'} \wedge \tau^{1'} \wedge \mathcal{F}^{A'} &= -\tau^{0'} \wedge \tau^{1'} \wedge d\tau^{A'} = \tau^{0'} \wedge \tau^{1'} \wedge \left(\partial^{A'} d_x A + d\pi^{B'} \partial_{B'} \partial^{A'} A \right) \\ &= \tau^{0'} \wedge \tau^{1'} \wedge \left(\partial^{A'} d_x A + (\tau^{B'} + \partial^{B'} A) \partial_{B'} \partial^{A'} A \right) \\ &= \tau^{0'} \wedge \tau^{1'} \left(\partial^{A'} d_x A + \partial^{B'} A \, \partial^{A'} \partial_{B'} A \right) = \tau^{0'} \wedge \tau^{1'} \partial^{A'} \left(d_x A + \frac{1}{2} \partial^{B'} A \, \partial_{B'} A \right) \\ &= \tau^{0'} \wedge \tau^{1'} \partial^{A'} F. \end{aligned} \tag{3.17}$$

Here we denoted by d_x the exterior derivative with respect to the coordinates on the base. We thus see the appearance of the field strength (3.2). Here, to pass to the last line we used the fact that $d\pi^{A'}$ can be replaced by $\partial^{A'} A$ on the kernel of $\tau^{A'}$. Thus the "higher-spin field strength" (3.2) genuinely is the curvature of the "higher-spin potential" (3.1) when the later is properly understood as an Ehresmann connection (3.8).

3.2.2 Field equations

The field equations (3.4) imply that the 2-form F is decomposable, and thus of the form $F = \theta^0 \wedge \theta^1$. We now define the following 4-form

$$\Omega := \tau^{C'} \wedge \tau_{C'} \wedge F. \tag{3.18}$$

This form is factorisable

$$\Omega = 2\tau^{0'} \wedge \tau^{1'} \wedge \theta^0 \wedge \theta^1. \tag{3.19}$$

We can now define an almost complex structure by requiring $\left(\tau^{0'}, \tau^{1'}, \theta^0, \theta^1\right)$ to be the basis of $(1,0)$-forms. Thus, we define $T^*_{(1,0)}\mathcal{T} := Span\left(\tau^{0'}, \tau^{1'}, \theta^0, \theta^1\right)$. The next step is to see whether the HS-SDGR field equations make the almost complex structures defined in this way integrable.

Let J be an almost complex structure, let $(\theta^i)_{i\in\{1...4\}}$ be a basis of $(1,0)$-forms and let Ω be the $(4,0)$-form given by $\Omega = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$. The Nijenhuis tensor is defined as $\mathcal{N}^i := d\theta^i\big|_{(0,2)}$. It can be computed as

$$\theta^i \wedge d\Omega = \mathcal{N}^i \wedge \Omega\,,$$

however, we shall evaluate the components of this tensor directly, in order to better understand the meaning of the equations arising.

Let us start by computing $d\tau^{A'}$. We have

$$\begin{aligned} d\tau^{A'} &= d(d\pi^{A'} - \partial^{A'}A) = d_x\partial^{A'}A + d\pi^{B'} \wedge \partial_{B'}\partial^{A'}A = \\ &= \partial^{A'}d_xA + (\tau^{B'} + \partial^{B'}A) \wedge \partial_{B'}\partial^{A'}A = \partial^{A'}F + \tau^{B'} \wedge \partial_{B'}\partial^{A'}A. \end{aligned} \tag{3.20}$$

A piece of the Nijenhuis tensor of the almost complex structure is obtained by taking the $(0,2)$ component in $d\tau^{A'}$. The last term in (3.20) has a $(1,0)$ $\tau^{B'}$ factor, and so does not survive the projection to $(0,2)$. Therefore, a necessary condition for the almost complex structure to be integrable is

$$\partial^{A'}F\Big|_{(0,2)} = 0. \tag{3.21}$$

This equation indeed follows by differentiating (3.4). Indeed, we get

$$(\partial^{A'}F) \wedge F = 0, \tag{3.22}$$

which is equivalent to (3.21). Thus, there is no obstruction to integrability from $d\tau^{A'}$.

To compute the other components of the Nijenhuis tensor we need to compute the exterior derivative of the decomposable 2-form F. We have

$$dF = d_xF + d\pi^{B'}\partial_{B'}F. \tag{3.23}$$

The first term here can be simplified

$$d_xF = d_x(d_xA + \frac{1}{2}\{A, A\}) = \{d_xA, A\} = \{F, A\}, \tag{3.24}$$

which is effectively the Bianchi identity for F. The second term can be transformed by replacing $d\pi^{B'} = \tau^{B'} + \partial^{B'} A$. We get

$$\begin{aligned} dF &= \{F, A\} + (\tau^{B'} + \partial^{B'} A)\partial_{B'} F \\ &= \{F, A\} + \{A, F\} + \tau^{B'} \partial_{B'} F = \tau^{B'} \partial_{B'} F \end{aligned} \tag{3.25}$$

We want to show that there is no $(1, 2)$ component here. However, given that $\tau^{B'}$ is $(1, 0)$, this is equivalent to showing that $\partial^{B'} F$ does not have the $(0, 2)$ part. As we know from the discussion following (3.21), this follows from the field equations by differentiation with respect to π. Thus, the almost complex structure is integrable, in exact parallel with the spin-2 case.

3.2.3 Gauge transformations

As we will see, the invariance of the HS-SDGR equations under "higher-spin gauge transformations" (3.5) and "generalised diffeomorphisms" (3.6) is essentially equivalent to their invariance under diffeomorphisms of the twistor space.

Higher-spin gauge transformations infinitesimally Let $\xi = \xi^{A'}(x, \pi, \hat{\pi})\partial_{A'} + \hat{\xi}^{A'}(x, \pi, \hat{\pi})\hat{\partial}_{A'}$ be a real vector field along the fibres of $\mathcal{T}$. The action of the Lie derivative on the Ehresmann connection is

$$\mathcal{L}_\xi \left(\tau^{A'} \partial_{A'}\right) = (d\xi^{A'} - \xi^{B'} \partial_{B'} \partial^{A'} A)\partial_{A'} - \tau^{A'}[\xi, \partial_{A'}] \tag{3.26}$$

$$= \left(d_x \xi^{A'} + d\hat{\pi}^{C'} \hat{\partial}_{C'} \xi^{A'} + \partial^{B'} A \partial_{B'} \xi^{A'} - \xi^{B'} \partial_{B'} \partial^{A'} A\right) \partial_{A'} + \left(\tau^{A'} \partial_{A'} \hat{\xi}^{C'}\right) \hat{\partial}_{C'}. \tag{3.27}$$

Now, ξ is a symmetry of the Poisson structure if and only if

$$\xi^{A'}(x, \pi, \hat{\pi}) = \partial^{A'} \xi(x, \pi) \tag{3.28}$$

i.e. if and only if it is hamiltonian and holomorphic. We get in this case

$$\mathcal{L}_\xi \left(\tau^{A'} \partial_{A'}\right) = \left(d_x \partial^{A'} \xi + \partial^{B'} A \partial_{B'} \partial^{A'} \xi - \partial^{B'} \xi \partial_{B'} \partial^{A'} A\right) \partial_{A'} = \partial^{A'} \left(d_x \xi + \{A, \xi\}\right) \partial_{A'}.$$

This last expression corresponds to the "higher-spin gauge transformations" (3.5) which therefore correspond to infinitesimal vertical (Poisson) diffeomorphisms in twistor space.

Non-linear realisation of higher-spin gauge transformations As we just saw, higher-spin gauge transformations can be interpreted as the action of the Lie derivative along the fibres. This suggests to consider the non-linear action of vertical automorphisms of the bundle

$$f \left| \begin{array}{ccc} \mathbb{T} & \to & \mathbb{T} \\ (x^\mu, \pi_{A'}) & \mapsto & (x^\mu, f_{A'}(x, \pi)) \end{array} \right.$$

which are Poisson symmetries

$$f_* \left(\epsilon^{A'B'} \partial_{A'} \partial_{B'} + \epsilon^{A'B'} \hat{\partial}_{A'} \hat{\partial}_{B'}\right) = \epsilon^{A'B'} \partial_{A'} \partial_{B'} + \epsilon^{A'B'} \hat{\partial}_{A'} \hat{\partial}_{B'} \tag{3.29}$$

We now want to show that these vertical Poisson diffeomorphisms act on the space of connections (3.8) satisfying (3.12) and thus provide a non-linear realisation of the "higher-spin gauge transformations".

In order to prove this let us consider the horizontal distribution D_μ given by (3.9) and its push-forward $f_*(D_\mu)$ under a vertical Poisson diffeomorphism. As we know from (3.1), $f_*(D_\mu)$ defines a connection satisfying (3.12) if and only if

$$[f_*(D_\mu), \epsilon^{A'B'}\partial_{A'}\partial_{B'} + \epsilon^{A'B'}\hat{\partial}_{A'}\hat{\partial}_{B'}] = 0\,.$$

We then have

$$\begin{aligned}[f_*(D_\mu), \epsilon^{A'B'}\partial_{A'}\partial_{B'} + \epsilon^{A'B'}\hat{\partial}_{A'}\hat{\partial}_{B'}] &= [f_*(D_\mu), f_*(\epsilon^{A'B'}\partial_{A'}\partial_{B'} + \epsilon^{A'B'}\hat{\partial}_{A'}\hat{\partial}_{B'})] \\ &= f_*([D_\mu, \epsilon^{A'B'}\partial_{A'}\partial_{B'} + \epsilon^{A'B'}\hat{\partial}_{A'}\hat{\partial}_{B'}]) \\ &= 0\,,\end{aligned} \tag{3.30}$$

where the first equality uses that f is a Poisson symmetry, the second follows from the identity $f_*[X,Y] = [f_*X, f_*Y]$ and the third from our assumptions on D_μ. Thus, indeed vertical Poisson diffeomorphisms act on the space of connections satisfying (3.12) and provide the non-linear realisation of the higher-spin gauge symmetry.

In order to see the action of these diffeomorphisms on the field equations we need to see how it acts on the curvature. However $F^{A'} = \partial^{A'}F = \mathcal{L}_{\partial_{A'}}F$ is the curvature of $\tau^{A'}$ and thus a tensorial object. It follows that F is simply a 2-form.

The invariance of the field equations under the higher-spin gauge transformations directly follows from their invariance under the vertical Poisson diffeomorphisms.

Generalised diffeomorphisms The situation with generalised diffeomorphisms is more subtle, as it is not easy to provide them with a clear geometrical interpretation. In fact, we shall see that these transformations are in general not diffeomorphisms, and so it would be better to call the symmetry of the field equations that they generate "the shift symmetry" rather than "generalised diffeomorphisms".

We start by verifying that these transformations are indeed a symmetry of the field equations. Let us consider horizontal vector fields of the form $\eta = \eta\,(x,\pi)^\mu\,\partial_\mu$ and the shift symmetry

$$\delta A = \eta \lrcorner F\,. \tag{3.31}$$

In order to prove the (on-shell) invariance of the field equation

$$\delta_\eta\,(F\wedge F) = 2\delta_\eta F\wedge F = 0 \tag{3.32}$$

we will need to rewrite the variation as

$$\begin{aligned}\delta_\eta F &= d_x(\eta\lrcorner F) + \{\eta\lrcorner F, A\} \\ &= d(\eta\lrcorner F) - d\pi^{A'}\partial_{A'}(\eta\lrcorner F) + \partial^{A'}(\eta\lrcorner F)\partial_{A'}A \\ &= d(\eta\lrcorner F) - \tau^{A'}\mathcal{L}_{\partial_{A'}}(\eta\lrcorner F) \\ &= \mathcal{L}_\eta F - \eta\lrcorner dF - \tau^{A'}\big((\mathcal{L}_{\partial_{A'}}\eta)\lrcorner F) + (\eta\lrcorner(\mathcal{L}_{\partial_{A'}}F)\big) \\ &= \mathcal{L}_\eta F - \tau^{A'}(\mathcal{L}_{\partial_{A'}}\eta)\lrcorner F + \eta\lrcorner\left(-dF + \tau^{A'}\partial_{A'}F\right) \\ &= \mathcal{L}_\eta F - \tau^{A'}(\mathcal{L}_{\partial_{A'}}\eta)\lrcorner F\end{aligned} \tag{3.33}$$

where to get to the line before last we exchanged the insertion of η with $\tau^{A'}$ and thus got an additional minus sign, and to get to the last line we used the identity (3.25). It follows that

$$\begin{aligned} \delta_\eta F \wedge F &= \left(\mathcal{L}_\eta F - \tau^{A'}(\mathcal{L}_{\partial_{A'}}\eta)\lrcorner F\right) \wedge F \\ &= \frac{1}{2}\mathcal{L}_\eta\,(F\wedge F) - \frac{1}{2}\tau^{A'}(\mathcal{L}_{\partial_{A'}}\eta)\lrcorner\,(F\wedge F) \\ &= 0\,. \end{aligned} \tag{3.34}$$

Where in the last step we used the field equation.

However, as noted already in [28], generalised diffeomorphisms do not coincide with the Lie derivative in the direction of the vector field η. The disagreement is by terms containing the derivative of $\eta^\mu(x,\pi)$ with respect to π. Indeed, we consider the vector field $\eta^\mu D_\mu = \eta^\mu(\partial_\mu + \partial^{A'}A_\mu\partial_{A'})$. We have

$$\begin{aligned} \mathcal{L}_{(\eta^\mu D_\mu)}\left(\tau^{A'}\partial_{A'}\right) &= -\eta^\mu(\partial_\mu\partial^{B'}A)\partial_{B'} - \eta^\mu(\partial^{B'}A_\mu)(\partial_{B'}\partial^{A'}A)\partial_{A'} \\ &\quad + d(\eta^\mu\partial^{A'}A_\mu)\partial_{A'} - (d\eta^\mu)(\partial^{A'}A_\mu)\partial_{A'} + \tau^{A'}[\eta^\mu D_\mu, \partial_{A'}] \\ &= -(\eta\lrcorner\partial^{A'}F)\partial_{A'} - \tau^{A'}(\partial_{A'}\eta^\mu)D_\mu \\ &= -\partial^{A'}(\eta\lrcorner F)\partial_{A'} + (\partial^{A'}\eta)\lrcorner F\partial_{A'} - \tau^{A'}(\partial_{A'}\eta^\mu)D_\mu\,, \end{aligned} \tag{3.35}$$

where we have used

$$(\eta\lrcorner\partial^{A'}F)_\nu = \eta^\mu\partial_\mu\partial^{A'}A_\nu - \eta^\mu\partial_\nu\partial^{A'}A_\mu + \eta^\mu\partial^{B'}\partial^{A'}A_\mu\partial_{B'}A_\nu - \eta^\mu\partial^{B'}\partial^{A'}A_\nu\partial_{B'}A_\mu\,. \tag{3.36}$$

The first term in (3.35) is the desired generalised diffeomorphism

$$\delta_\eta(\tau^{A'}\partial_{A'}) = -\partial^{A'}(\eta\lrcorner F)\partial_{A'}\,.$$

The remaining terms all contain $\partial_{A'}\eta^\mu$. So, in general the generalised diffeomorphism does not coincide with the Lie derivative by terms containing $\partial_{A'}\eta^\mu$.

3.3 A higher-spin non-linear graviton correspondence

The discussion above can be summarised as the following theorem.

Let $S' \xrightarrow{\pi} M^4$ be the bundle of spinors on M^4, the twistor space $\mathcal{T} \xrightarrow{\pi} M^4$ is obtained by deleting from S' the zero section. Let U be an open set of M^4 and let $V = \pi^*(U)$ be the corresponding open set of $\mathcal{T}$. We define the horizontal distributions on TV to be those in the kernel of the projection $P\colon T\mathcal{T} \to \mathcal{V}$

$$P = \tau^{A'}\frac{\partial}{\partial\pi^{A'}} + \hat{\tau}^{A'}\frac{\partial}{\partial\hat{\pi}^{A'}}, \qquad \tau^{A'} := d\pi^{A'} + \mathcal{A}^{A'}(x,\pi,\hat{\pi})\,.$$

They are parametrised by Ehresmann connection $\mathcal{A}^{A'}(x,\pi,\hat{\pi})$.

Theorem 3.1. *For horizontal distributions that are infinitesimal symmetries of the Poisson structure* (3.7) *the Ehresmann connection $\mathcal{A}^{A'}$ is potential: $\mathcal{A}^{A'} = -\epsilon^{A'B'}\partial_{B'}A$, where $A = A(x,\pi)$. In particular, the Ehresmann connection $\mathcal{A}^{A'}$ of Poisson horizontal distributions is independent of $\hat{\pi}$. Furthermore, its curvature 2-form is also potential $\mathcal{F}^{A'} = -\epsilon^{A'B'}\partial_{B'}F$,* *where $F = F(x, \pi) = dA + (1/2)\{A, A\}$. There is a one-to-one correspondence (up to a gauge) between solutions of the higher-spin self-dual gravity equations on U, with $A(x, \pi)$ as the generating function* (3.1)*, and Poisson horizontal distributions on TV that have decomposable $F \wedge F = 0$ curvature potential F. Together with $\tau^{A'}$, the two simple factors of F define an almost complex structure on V that is integrable.*

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