Frequently Hypercyclic Random Vectors

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Abstract

Some results concerning the existence of almost surely frequently hypercyclic random vectors have been proved in the literature for certain chaotic weighted shifts. This is of interest for at least two reasons. It is usually difficult to find explicit (frequently) hypercyclic vectors, and random vectors have a probability distribution whose ergodic properties can be studied. The first objective of the thesis is to extend the previously known results. In particular, we prove that every chaotic weighted shift on very general sequence spaces and every operator satisfying the Frequent Hypercyclicity Criterion admits an almost surely frequently hypercyclic random vector.

We also investigate the case of semigroups. The desired random vector is constructed using a stochastic integral. Although our general result requires that this integral is well-defined, we can apply it to the translation semigroups on the space of entire functions.

The second part of the thesis deals with the rate of growth of frequently hypercyclic functions. We present two methods. Recently, a probabilistic approach provided a quasi-optimal rate of growth for the differentiation operator and the Taylor shift. Based on these results and the first part of the thesis, we obtain a general criterion for chaotic weighted shifts. The rate of growth is expressed as a function depending only on the weights, multiplied by some logarithmic factor. We give several examples of shifts defined on the space of entire functions or the space of holomorphic functions on the unit disk, recovering previous results and finding new ones. We also consider the differentiation operators on the space of harmonic functions on the plane and weighted shifts on Köthe sequence spaces. The possible optimality of the growth is also discussed.

On spaces of holomorphic functions, we can also ask whether the growth holds outside some small, but possibly unbounded, set. We give results in this direction, which are stated for general random complex series. This second approach seems to be new in linear dynamics. In particular, we prove that for any chaotic weighted shift, the growth sought by the previous method does hold outside such a set.

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Contents

Introduction

The theory of dynamical systems studies the long-term behaviour of systems that evolves through time. A dynamical system consists of a set, called the space of states, and a map on this set that describes the evolution of the system. One may think of the motion of objects subjected to forces, or the evolution of the number of people in a given population.

Linear dynamics studies a particular case of dynamical systems, where the space of states is a vector space and the evolution map is linear. Hypercyclicity, that is, the existence of a dense orbit, is the main concept of linear dynamics. This term seems to have been introduced by Beauzamy in [13] around 1986 while working on the invariant subspace problem. The first operators known to be hypercyclic were the translation operators by Birkhoff [16] in 1929, the differentiation operator by MacLane [68] in 1952 and the multiples of the backward shift by Rolewicz [84] in 1969.

The foundations of linear dynamics were made in the unpublished thesis of Kitai [58] in 1982 and by Godefroy and Shapiro [42] in 1991. Since then, many other variants of hypercyclicity have been introduced and studied.

In linear dynamics, we thus consider continuous and linear maps T from a vector space E to itself, called *operators*. The vector space E is usually a *Fréchet space* or an *F-space*; these are particular cases of metric spaces generalizing Banach spaces. We are interested in dense orbits i.e., in finding a vector $x \in E$ such that the set of all $T^n(x)$, $n \in \mathbb{N}$, called the *orbit* of x, is dense in E. Such a vector is said to be *hypercyclic* for T, and T is *hypercyclic*. Although it has been proved that once an operator admits a hypercyclic vector, it admits a large supply of such vectors in a topological sense, it is usually not so easy to construct just one hypercyclic vector.

In their work, Bayart and Grivaux [7] introduced the notion of *frequent hypercyclicity*, that is, the existence of a hypercyclic vector visiting any non-empty open set of the space many times in a precise sense. This was the beginning of the connections between linear dynamics and ergodic theory, which studies dynamical systems from a measure-theoretic point of view. This notion is strictly stronger than hypercyclicity. Moothathu [73] proved that the set of frequently hypercyclic vectors is always negligible, in contrast to hypercyclicity.

A second well-studied variant of hypercyclicity is chaos: an operator T is *chaotic* if it is hypercyclic and has a dense set of periodic points. It turns out that the hypercyclic operators mentioned above are also frequently hypercyclic and chaotic.

Random vectors. Bayart and Matheron [10] used ergodic theory to obtain results on the existence of frequently hypercyclic vectors via the existence of an *ergodic* or *strongly mixing measure* for T, two notions from ergodic theory. They obtain the following result as a by-product of their methods.

As usual, $(e_n)_{n\in\mathbb{N}}$ stands for the canonical sequence of ℓ^p, the space of p-summable sequences, $1 \leq p < \infty$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. The notation $\mathbb{N}_0$ will denote the set of all strictly positive integers.

Theorem 0.0.1 ([10, Remark at p. 121])**.** *Let $T : \ell^p \longrightarrow \ell^p$ be a weighted shift with weight sequence $(w_n)_{n\in\mathbb{N}_0}$ i.e., $T(e_0) = 0$ and $T(e_n) = w_n e_{n-1}$ for any $n \in \mathbb{N}_0$. If T is chaotic, then the random vector*

$$\sum_{n=0}^{\infty} \frac{X_n}{w_1 \dots w_n} e_n \tag{0.0.1}$$

is almost surely frequently hypercyclic for T, where $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent standard Gaussian random variables. Furthermore, its distribution is strongly mixing for T.

Here, a *standard Gaussian random variable* means that the random variable follows the normal distribution with mean 0 and variance 1. Note that here and in the sequel, the random variables are real if the vector space is real, and complex if it is complex.

Another well-known and historical example of a frequently hypercyclic operator is the differentiation operator D defined on the space of entire functions $H(\mathbb{C})$, given by $D(f) = f'$, $f \in H(\mathbb{C})$. Nikula [80] proved that the random series

$$\sum_{n=0}^{\infty} \frac{X_n}{n!} z^n \tag{0.0.2}$$

is almost surely frequently hypercyclic for D by directly showing that this random vector has a full probability of being frequently hypercyclic. The sequence of independent random variables $(X_n)_{n\in\mathbb{N}}$ must satisfy some decay condition on their distributions, which is satisfied by Gaussian variables. Mouze and Munnier [74] gave a simpler proof of Nikula's result by using the so-called Birkhoff ergodic theorem under a weaker assumption on the probability distribution. Bayart and Matheron [11] also showed the almost sure frequent hypercyclicity of the random vector (0.0.2) and proved that its distribution is strongly mixing for the operator D as well.

Remark that the operator D is a weighted shift by identifying an entire function with its sequence of Taylor coefficients and noticing that $D(z^n) = nz^{n-1}$ for all integers $n \in \mathbb{N}_0$. Here again, the random vector (0.0.2) is constructed as follows: from the fixed point $\sum_{n\geq 0} z^n/n!$ of D, where $n!$ corresponds to the product of the first n weights of the operator D, each term of the series is multiplied by a random coefficient.

Mouze and Munnier [75] obtained the same result for the Taylor shift: it is the operator T on $H(\mathbb{D})$ such that $T(f) = \sum_{n\geq 0} a_{n+1} z^n$, $f = \sum_{n\geq 0} a_n z^n \in H(\mathbb{D})$. Here,

$H(\mathbb{D})$ denotes the space of holomorphic functions on the unit disk $\mathbb{D}$. They proved that the random vector

$$\sum_{n=0}^{\infty} X_n z^n$$

is almost surely frequently hypercyclic for T, where $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent standard Gaussian random variables. Note that the sequence of weights of T is the constant sequence with constant 1. They also constructed in [74] an almost surely frequently hypercyclic random vector for some polynomials of chaotic weighted shifts on ℓ^p, $1 \leq p < \infty$.

These results were, to our knowledge, the only ones about frequently hypercyclic random vectors. Given these similarities, the following question may be asked.

Question 0.0.2. *Let T be a weighted shift on a vector space E. Under which conditions is the random vector*

$$\sum_{n=0}^{\infty} \frac{X_n}{w_1 \dots w_n} e_n \tag{0.0.3}$$

almost surely convergent in E and frequently hypercyclic for T?

Rate of growth. Let us now discuss another problem in linear dynamics. On spaces of functions, the growth of frequently hypercyclic functions of a given operator can be studied. Let us consider the space $H(\mathbb{C})$ of entire functions, and let T be a frequently hypercyclic operator on the space $H(\mathbb{C})$. An *admissible rate of growth* for the frequently hypercyclic functions of T is a real-valued map g defined on the set of positive numbers such that there is some frequently hypercyclic function for T with the property that $\sup_{|z|=r} |f(z)| \leq g(r)$ for all positive real numbers $r \geq 0$ large enough. A classical method to show the existence of frequently hypercyclic vectors with a given growth is as follows: the operator T is restricted to a Banach space F consisting of entire functions satisfying the growth condition and such that it is continuously embedded in $H(\mathbb{C})$. Then one proves that the sequence of maps $(T_n)_{n\in\mathbb{N}}$ is *frequently universal*, where T_n is the restriction of the operator T^n to the space F. Frequent universality for a sequence of maps is defined as frequent hypercyclicity.

As we have seen above, the differentiation operator D on $H(\mathbb{C})$ has an almost surely frequently hypercyclic random vector (0.0.2). Using its structure, Nikula [80] bounded the sup-norm of the series (0.0.2) and thus obtained an admissible rate of growth for D.

Theorem 0.0.3 ([80, Proposition 2])**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables. Then there exists a constant $C > 0$ such that almost surely,*

$$\sup_{|z|=r} \left| \sum_{n=0}^{\infty} \frac{X_n}{n!} z^n \right| \leq C\sqrt{\log(r)} \frac{e^r}{r^{1/4}} \tag{0.0.4}$$

for all $r > 0$ large enough.

Blasco, Bonilla and Grosse-Erdmann [17] showed that for any function ψ with $\lim_{r\to\infty} \psi(r) = 0$, there is no frequently hypercyclic entire function f for D that satisfies

$$\sup_{|z|=r} |f(z)| \leq \psi(r)\frac{e^r}{r^{1/4}}$$

for every sufficiently large real number $r > 0$. Drasin and Saksman [34] proved that the map $r \longmapsto e^r/r^{1/4}$ is in fact optimal.

Theorem 0.0.4 ([34, Theorem 1.1])**.** *For any $C > 0$, there exists a frequently hypercyclic entire function f for D such that $\sup_{|z|=r} |f(z)| \leq Ce^r/r^{1/4}$ for all $r > 0$.*

Mouze and Munnier [75] also used a probabilistic approach to get an admissible rate of growth for the frequently hypercyclic functions of the Taylor shift and obtained the following result.

Theorem 0.0.5 ([75, Proposition 3.5 and p. 627])**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables. Then there exists a constant $C > 0$ such that almost surely,*

$$\sup_{|z|=r} \left| \sum_{n=0}^{\infty} X_n z^n \right| \leq C\sqrt{|\log(1-r)|}\frac{1}{\sqrt{1-r}} \tag{0.0.5}$$

for all $0 < r < 1$ close enough to 1.

They even proved that the map $r \longmapsto 1/\sqrt{1-r}$ is the optimal rate of growth for the frequently hypercyclic functions of the Taylor shift.

In the proofs of Theorems 0.0.3 and 0.0.5, the idea was to bound the first terms of the random series, which already give the desired growth, and to show that the remaining terms are small. As far as we know, these were the only results about admissible rates of growth with a probabilistic approach.

Let us mention one more fact: the upper bounds in Theorems 0.0.3 and 0.0.5 have the same form. Indeed, one can prove the following inequalities:

$$\frac{e^r}{r^{1/4}} \asymp \sqrt{\sum_{n=0}^{\infty} \frac{r^{2n}}{n!^2}}$$

valid for $r > 0$ large, and

$$\frac{1}{\sqrt{1-r}} \asymp \sqrt{\sum_{n=0}^{\infty} r^{2n}}$$

valid for $0 < r < 1$ close enough to 1, where $a \asymp b$ means $a \leq b$ and $a \geq b$ up to some constants independent of r. In both cases, the upper bound is the ℓ^2-norm of the sequence $(r^n/(w_1 \dots w_n))_{n\in\mathbb{N}}$, where $(w_n)_{n\in\mathbb{N}_0}$ is the weight sequence of the shift. The logarithmic factor that appears in (0.0.4) and (0.0.5) comes from the probabilistic tools used in the proofs.

Let T be a chaotic weighted shift on $H(\mathbb{C})$ with weight sequence $(w_n)_{n\in\mathbb{N}_0}$. We will see in this work that (0.0.3) defines an almost surely frequently hypercyclic random vector for T, where the sequence $(X_n)_{n\in\mathbb{N}}$ of independent random variables is standard Gaussian. Therefore, in order to get an admissible rate of growth, we can try to bound the sup-norm of the random series (0.0.3). It turns out that the formal series $f := \sum_{n\geq 0} z^n/(w_1 \dots w_n)$ determines chaos: T is chaotic if and only if the series converges. Therefore, given a chaotic weighted shift on $H(\mathbb{C})$, there is an associated entire function given by f. From this function, we construct a random vector by multiplying each Taylor coefficient of f by a Gaussian random variable. This leads us to the following more general problem.

Question 0.0.6. *Let $f = \sum_{n\in\mathbb{N}} a_n z^n$ be an entire function. Under which assumptions do we have a constant $C > 0$ such that almost surely,*

$$\sup_{|z|=r} \Big|\sum_{n=0}^{\infty} a_n X_n z^n\Big| \leq C\sqrt{\log(A(r))}\sqrt{\sum_{n=0}^{\infty} |a_n|^2 r^{2n}} \tag{0.0.6}$$

for all $r \geq 0$ large enough, where A is some real-valued function on $[0, \infty[$?

In this approach, the inequality (0.0.6) should be valid for any $r > 0$ large. We can also ask whether, under less restrictive assumptions, this inequality could hold outside some small, but possibly unbounded, set. A set $E \subseteq [1, \infty[$ is of *finite logarithmic measure* if the integral $\int_E x^{-1}\mathrm{d}x$ is finite. Such a set may be unbounded but has to be not very large since $\int_1^\infty x^{-1}\mathrm{d}x$ diverges. It seems that this definition is common in the literature regarding complex analysis.

All we have said also applies to chaotic shifts on $H(\mathbb{D})$, and to holomorphic functions $f \in H(\mathbb{D})$.

Outline. In order to make the text as self-contained and understandable as possible, a preliminary chapter contains the background of this thesis. The first section is intended as a brief introduction to linear dynamics. The second section recalls some definitions and results from probability and measure theory. The last sections are devoted to the Pettis integral, the so-called Itô integral and Gaussian measures.

Chapter 1 contains the results concerning the existence of an almost surely frequently hypercyclic vector. In particular, we prove that every chaotic weighted shift on very general sequence spaces has such a random vector of the form (0.0.3). Incidentally, it has an ergodic probability distribution. The question of whether this distribution was also strongly mixing led us to extend our theorem to more operators, including bilateral weighted shifts, and to prove the strong mixing property. The first main theorem of the chapter states that under some deterministic assumptions on the operator and a decay condition on the distribution of the random variables, there is an almost surely frequently hypercyclic random vector. The second main general result, which will be a consequence of the first one, says that such a sequence of random variables can always be constructed under the deterministic assumptions. The so-called Frequent Hypercyclicity Criterion is often useful to show the frequent hypercyclicity of a given operator. Bayart and Matheron [11] proved that an operator satisfying the criterion admits a strongly mixing Gaussian measure. Murillo-Arcila and Peris [77] also proved that such an operator has a strongly mixing measure by using a constructive method. Our approach will provide another proof of this fact.

The continuous counterpart of a single operator are the C_0-semigroups. Frequent hypercyclicity can also be defined in this framework. Therefore, we can ask whether, for a given semigroup, a random vector that is almost surely frequently hypercyclic can be found. This is investigated in Chapter 2. Unfortunately, we did not manage to find any fully satisfactory result: the random vector is constructed using a stochastic integral, and our main theorem requires that this integral is well-defined. Nevertheless, it can be applied to the translation semigroups on the space of entire functions.

Chapter 3 is devoted to the problem of the rate of growth for general random complex series. The two approaches mentioned above are presented. First, we will answer Question 0.0.6, and prove that for any entire function, the inequality (0.0.6) holds for every positive real number outside a set of finite logarithmic measure. An analogous result is also proved for holomorphic functions on the unit disk. Next, we give general conditions under which an entire function, or a function in the space $H(\mathbb{D})$, satisfies the inequality (0.0.6) for every large enough real number. With an ad hoc assumption, this result is applied to entire functions of finite order.

We will return to linear dynamics and chaotic weighted shifts in Chapter 4. The results regarding the growth valid outside a set of finite logarithmic measure immediately yield an admissible rate of growth for the frequently hypercyclic functions. As far as we know, this approach is new in linear dynamics. The main work then consists of applying the general results from Chapter 3 to get the rate of growth valid everywhere. We did not manage to do this for any chaotic weighted shifts, but we give several examples. In particular, we will recover the results of Nikula [80] and Mouze and Munnier [75]. On the space of entire functions, we find an admissible rate of growth for the frequently hypercyclic functions of the so-called Dunkl and Aron-Markose operators. Weighted shifts on $H(\mathbb{D})$ studied by Mouze and Munnier in [76] are considered next. Differential operators on the space of harmonic functions of the plane and weighted shifts on the Köthe sequence spaces are also studied. The possible optimality of the rate of growth is discussed in the last section of the chapter.

Finally, an appendix contains some results and proofs about Bochner spaces, γ-radonifying operators and stochastic calculus in Fréchet spaces that are used in Chapter 2. They are not included in the main text because the Banach case is already known in the literature, and these results are simply a generalization to the Fréchet case. There is also a list of notations used throughout the thesis after the appendix.

Prerequisites

In this preliminary chapter, we mainly explain the framework of this thesis, the theory of linear dynamical systems. This is the content of the first section. The second section recalls some definitions and tools from probability theory. The last three sections are devoted to the background material needed for Chapter 2. We will review the Pettis integral, the Itô calculus and the Gaussian measures.

Throughout the thesis, let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. The notation $\mathbb{N}$ denotes the set of non-negative integers, and $\mathbb{N}_0$ denotes the set of strictly positive integers.

0.1 Linear Dynamics

The theory of dynamical systems studies the long-term behaviour of systems that evolves through time. More precisely, a *dynamical system* is a pair (E, T) where E is a set and $T : E \longrightarrow E$ is a map on E that describes the evolution of the system. Starting from a point $x_0 \in E$ say, the system will reach the state $T(x_0)$ at the next time. Then the next state will be $T^2(x_0)$, and so on.

In the specific setting of linear dynamical systems, the set of states E is a vector space endowed with some topology and the evolution map T is a continuous and linear map. We give in this section a brief introduction to the field and state the concepts and results that will be used throughout this work. For the proofs and further reading, see the books [10] and [47].

In general, results in linear dynamics are stated in the Fréchet space framework. They are a generalization of the Banach spaces where the norm is replaced by a sequence of seminorms.

Definition 0.1.1. Let E be a $\mathbb{K}$-vector space. A *seminorm* $p : E \longrightarrow [0, \infty[$ is a function satisfying the following two properties: for every $x, y \in E$ and $\lambda \in \mathbb{K}$,

(i) $p(x + y) \leq p(x) + p(y)$,

(ii) $p(\lambda x) = |\lambda| p(x)$.

All that remains for a seminorm to be a norm is the separation condition. The analogous notion is a separating sequence of seminorms.

Definition 0.1.2. A sequence of seminorms $(p_n)_{n\geq 1}$ is *separating* if for every $x \in E$, $p_n(x) = 0$ for all $n \geq 1$ implies $x = 0$.

Definition 0.1.3. A *Fréchet space* E is a vector space endowed with a separating sequence of seminorms $(p_n)_{n\geq 1}$, which is complete in the metric $d : E \times E \longrightarrow [0,\infty[$ defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x-y))$$

for every $x, y \in E$.

By considering $(\max_{1\leq k\leq n} p_k)_{n\geq 1}$, we can always assume that the sequence of seminorms is non-decreasing.

Example 0.1.4. Here are some examples of Fréchet spaces.

(1) Of course, every Banach space is a Fréchet space.

(2) The space of entire functions, denoted by $H(\mathbb{C})$, is a Fréchet space with the sequence of seminorms $p_n(f) := \sup_{|z|=n} |f(z)|$, $f \in H(\mathbb{C})$, $n \geq 1$. This corresponds to the topology of local uniform convergence, that is, uniform convergence on all compact sets. Similarly, the space of holomorphic functions on the unit disk $\mathbb{D}$, denoted by $H(\mathbb{D})$, is also a Fréchet space. In both spaces, the subspace of polynomials with rational coefficients is dense.

(3) The vector space $\mathbb{K}^{\mathbb{N}}$ of all sequences endowed with the seminorms $p_n(x) := \max_{0\leq k\leq n} |x_k|$, $x = (x_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, $n \geq 1$, is a Fréchet space.

A slight generalization of a Fréchet space is the concept of F-space.

Definition 0.1.5. An *F-norm* on a $\mathbb{K}$-vector space E is a function $\|\cdot\| : E \longrightarrow [0,\infty[$ satisfying for all $x, y \in E$ and $\lambda \in \mathbb{K}$,

(i) $\|x+y\| \leq \|x\| + \|y\|$,

(ii) $\|\lambda x\| \leq \|x\|$ if $|\lambda| \leq 1$,

(iii) $\lim_{\lambda\to 0} \|\lambda x\| = 0$,

(iv) if $\|x\| = 0$ then $x = 0$.

An F-norm thus looks like a norm, but there is not, in general, the homogeneity property. It is replaced by the following weaker property: for all $\lambda \in \mathbb{K}$ and $x \in E$,

$$\|\lambda x\| \leq (|\lambda| + 1)\|x\|. \tag{0.1.1}$$

Definition 0.1.6. An *F-space* E is a vector space endowed with an F-norm such that E is complete under the induced metric.

A Fréchet space E is an F-space with the F-norm $\|x\| := d(x,0)$, $x \in E$, where d is the metric defined in Definition 0.1.3, see [47, Proposition 2.8]. It is easy to see that this F-norm is bounded, which is not the case of a norm.

Fréchet spaces are the locally convex F-spaces, see [88, Theorem 1.24 and Remarks 1.38(b)].

The following two lemmas characterize the convergence of sequences, the Cauchy condition, and the continuity of linear maps in terms of seminorms, see [47, Lemma 2.6 and Proposition 2.11].

Lemma 0.1.7. *Let E be a Fréchet space, and let $(p_n)_{n\geq 1}$ be a sequence of seminorms generating its topology. Let $(x_n)_{n\in\mathbb{N}} \subseteq E$ be a sequence of vectors in E and $x \in E$.*

(i) *The sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in E if and only if for all $k \geq 1$, one has $\lim_{n\to\infty} p_k(x_n - x) = 0$.*

(ii) *The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in E if and only if for all $k \geq 1$, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (E, p_k).*

Lemma 0.1.8. *Let E, F be two Fréchet spaces, and let $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ be sequences of seminorms defining the topology of E and F, respectively. Let $T : E \longrightarrow F$ be a linear map. Then T is continuous if and only if for all $n \in \mathbb{N}_0$, there are $m \in \mathbb{N}_0$ and $C > 0$ such that for every $x \in E$, one has*

$$q_n(T(x)) \leq C p_m(x).$$

We will say that a map $T : E \longrightarrow E$ on an F-space E is an *operator* if it is continuous and linear.

We are interested in studying the properties of the orbit of a vector x_0, that is, the different states the system will take, starting from the state x_0 and following the evolution map.

Definition 0.1.9. Let E be an F-space and $T : E \longrightarrow E$ be an operator. For all $x \in E$, the *orbit* of x under T is the set

$$\operatorname{Orb}(x, T) = \big\{T^n(x) \mid n \in \mathbb{N}\big\}.$$

The main notion of linear dynamics is the concept of hypercyclicity.

Definition 0.1.10. Let E be an F-space. An operator $T : E \longrightarrow E$ is *hypercyclic* if there exists $x \in E$ whose orbit is dense in E. Such a vector is called a *hypercyclic vector* for T.

Note that if a space admits a hypercyclic operator, it is necessarily separable since every orbit is at most countable. The first question to ask is whether such an operator exists. Let us first point out that hypercyclicity is an infinite-dimensional phenomenon.

Theorem 0.1.11 ([47, Corollary 2.59])**.** *Let E be a Banach space of finite dimension. If E is not the zero vector space then there are no hypercyclic operators on E.*

Obviously, the identity map is not hypercyclic unless the space is trivial. On a Banach space, any operator with an operator norm less than 1 is not hypercyclic, since then every orbit is bounded.

One way to prove that an operator is hypercyclic is to explicitly construct a hypercyclic vector, see for example [47, Example 2.18]. Another method is to use the Birkhoff Transitivity Theorem below.

Recall that a *G_δ-set* G is a subset of an F-space E of the form $G = \bigcap_{n\in\mathbb{N}} O_n$, where $(O_n)_{n\in\mathbb{N}}$ is a family of open sets of E.

Theorem 0.1.12 (Birkhoff Transitivity Theorem)**.** *Let E be a separable F-space and $T : E \longrightarrow E$ be an operator. Then T is hypercyclic if and only if for any non-empty open sets $U, V \subseteq E$, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. In that case, the set of hypercyclic vectors of T is a dense G_δ-set.*

Proof. Let x be a hypercyclic vector for T. Let $U, V \subseteq E$ be non-empty open sets. There exists $m \in \mathbb{N}$ such that $T^m(x) \in U$. Since an F-space has no isolated points, the vector $T^m(x)$ is hypercyclic for T, and there exists some $n \in \mathbb{N}$ such that $T^n(T^m(x)) \in V$. Therefore $T^n(U) \cap V \neq \emptyset$.

For the converse, let $(U_k)_{k\in\mathbb{N}}$ be a countable base of open subsets of E, which exists since E is separable. Notice that the set of hypercyclic vectors of T is exactly the G_δ-set $\bigcap_{k\geq 0} \bigcup_{n\geq 0} T^{-n}(U_k)$. By assumption, each open set $\bigcup_{n\geq 0} T^{-n}(U_k)$, $k \geq 0$, is dense in E. Therefore, the Baire Category Theorem, see [88, Theorem 2.2], yields that the set of hypercyclic vectors of T is dense, and T is hypercyclic. $\square$

An operator satisfying the necessary and sufficient condition of the theorem is said to be *topologically transitive*.

Example 0.1.13. Let us prove that the differentiation operator

$$D : H(\mathbb{C}) \longrightarrow H(\mathbb{C}), f \longmapsto f'$$

is hypercyclic. Let $U, V \subseteq E$ be non-empty open sets. There exist some polynomials $p = \sum_{k=0}^{N} a_k z^k$ and $q = \sum_{k=0}^{N} b_k z^k$ such that $p \in U$ and $q \in V$. Let $n \geq N+1$ be a positive integer and define

$$h := p + \sum_{k=0}^{N} \frac{k! b_k}{(k+n)!} z^{k+n}.$$

Then $D^n(h) = q$ and for any $R > 0$, one has

$$\sup_{|z|\leq R} |p(z) - h(z)| \leq \sum_{k=0}^{N} \frac{k! |b_k|}{(k+n)!} R^{k+n},$$

Therefore, for n large enough, we get $h \in U$ and $D^n(h) \in V$. Since this holds for any non-empty open sets $U, V \subseteq E$, we have shown that D is hypercyclic by Theorem 0.1.12.

More generally, it has been proved by Bonet and Peris [21, Theorem 1] that every infinite-dimensional separable Fréchet space supports a hypercyclic operator. In contrast, there are infinite-dimensional separable F-spaces whose continuous operators are exactly the multiples of the identity, see [56, Section 7.6], and which therefore do not admit any hypercyclic operator.

A first stronger notion than hypercyclicity is the concept of chaos. There are different versions of chaos, but the one most commonly accepted in linear dynamics is chaos in the sense of Devaney. Recall that a vector $x \in E$ of an operator $T : E \longrightarrow E$ is *periodic* if there exists some integer $n \geq 1$ such that $T^n(x) = x$.

Definition 0.1.14. Let E be an F-space. An operator $T : E \longrightarrow E$ is *chaotic* if it is hypercyclic and has a dense set of periodic points.

A chaotic operator then means that in any non-empty open set of the space, however small, one can find a periodic vector, which has a very regular obit, and also a hypercyclic vector, which has a very irregular orbit. This is related to the so-called butterfly effect, saying that small changes in the initial conditions may lead to very different effects.

We will see below some examples of chaotic operators. In contrast to hypercyclicity, there exists an infinite-dimensional separable Banach space that supports no chaotic operator, as observed by Bonet, Martínez-Giménez and Peris in [20].

An important class of operators is the class of weighted shifts. They are fairly simple operators, and new notions and results of linear dynamics are usually first tested on weighted shifts.

The space $\mathbb{K}^{\mathbb{N}}$ (resp. $\mathbb{K}^{\mathbb{Z}}$) of all sequences is endowed with the topology of coordinatewise convergence.

Definition 0.1.15. A *sequence space over* $\mathbb{N}$ *(resp. over* $\mathbb{Z}$*)* E is a subspace of $\mathbb{K}^{\mathbb{N}}$ (resp. $\mathbb{K}^{\mathbb{Z}}$) such that convergence in E implies convergence in $\mathbb{K}^{\mathbb{N}}$ (resp. $\mathbb{K}^{\mathbb{Z}}$). A Banach (Fréchet, F-) space of this kind is called a *Banach (Fréchet, F-) sequence space*. The vectors $e_n = (\dots, 0, 1, 0, \dots)$ where 1 lies at the n-th coordinate, $n \in \mathbb{N}$ (resp. $n \in \mathbb{Z}$), are called the *canonical unit sequences*.

Let E be a sequence space over $\mathbb{N}$ (resp. over $\mathbb{Z}$) such that the canonical unit sequences span a dense subspace. A *unilateral (resp. bilateral) weighted shift* $T : E \longrightarrow E$ is an operator such that $T(e_0) = 0$ and $T(e_n) = w_n e_{n-1}$ for all $n \in \mathbb{N}_0$ (resp. $T(e_n) = w_n e_{n-1}$ for all $n \in \mathbb{Z}$), where $(w_n)_n$ is a sequence of non-zero scalars called the *weight sequence*.

We will often simply say *weighted shift* for a unilateral weighted shift.

It is pointless to consider a weight sequence with some zero elements since such a shift cannot be hypercyclic. Indeed, suppose for example that $T : E \longrightarrow E$ is a weighted shift with $w_1 = 0$. Then $|e_0^*(T^n(x) - e_0)| = 1$ for all $n \in \mathbb{N}_0$, where e_0^* is the linear map giving the first coordinate, and x is not hypercyclic for T, for every $x \in E$.

Proposition 0.1.16. *Let* $T : E \longrightarrow E$ *be a unilateral or bilateral weighted shift on an F-sequence space* E*. Then* T *is continuous.*

Proof. By the Closed Graph Theorem, see [88, Theorem 2.15], it suffices to prove that T has a closed graph. Let $(x_n)_{n\geq 0} \subseteq E$ be such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} T(x_n) = y$, where $x, y \in E$. Since convergence in E implies coordinatewise convergence, we get that $T(x) = y$, and T has a closed graph. □

The previous proposition means that a weighted shift on E is continuous if and only if it is a linear map on E.

Example 0.1.17. Obvious examples of Banach sequence spaces are the spaces ℓ^p, $1 \leq p < \infty$, and c_0. Note that it is pointless for our purposes to consider ℓ^∞ since this space is not separable. The space of entire functions $H(\mathbb{C})$ can be viewed as a Fréchet sequence space. Indeed, every entire function is identified with its sequence of Taylor coefficients at 0. This also holds true for the space $H(\mathbb{D})$.

There is a criterion to test the hypercyclicity or chaoticity of a weighted shift that is only a condition on the weights.

First recall the definition of unconditional convergence.

Definition 0.1.18. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of vectors in an F-space E. The series $\sum_{n\in\mathbb{N}} x_n$ *converges unconditionally* if for any permutation $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ the series

$$\sum_{n=0}^{\infty} x_{\sigma(n)}$$

converges.

A basis $(e_n)_{n\in\mathbb{N}}$ of an F-space E is *unconditional* if for every $x \in E$, the series $x = \sum_{n\geq 0} x_n e_n$ converges unconditionally.

Theorem 0.1.19 ([45, Theorems 7 and 8])**.** *Let E be an F-sequence space over $\mathbb{N}$ in which $(e_n)_{n\in\mathbb{N}}$ is a basis. Let T be a weighted shift on E with weight sequence $(w_n)_{n\in\mathbb{N}_0}$.*

(i) *T is hypercyclic if and only if there exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that*

$$\lim_{k\to\infty} \Big(\prod_{j=1}^{n_k} w_j\Big)^{-1} e_{n_k} = 0.$$

(ii) *If $(e_n)_{n\in\mathbb{N}}$ is unconditional, then T is chaotic if and only if the series*

$$\sum_{n=0}^{\infty} \Big(\prod_{j=1}^{n} w_j\Big)^{-1} e_n$$

converges in E.

Example 0.1.20. On the Banach spaces $E = \ell^p$, $1 \leq p < \infty$, or $E = c_0$, a weighted shift $T : E \longrightarrow E$ is continuous (and well-defined) if and only if its sequence of weights $(w_n)_{n\geq 1}$ is bounded. By Theorem 0.1.19, T is hypercyclic if and only if $\sup_{n\geq 1} \prod_{j=1}^{n} |w_j| = \infty$.

The backward shift B on E, that is, the weighted shift with constant weights equal to 1, is not hypercyclic. This fact was already known since the operator norm of B is equal to 1.

Example 0.1.21. On $H(\mathbb{C})$, the space of all entire functions, a weighted shift $T : H(\mathbb{C}) \longrightarrow H(\mathbb{C})$ with respect to the basis of monomials is an operator on $H(\mathbb{C})$ if and only if $\sup_{n\geq 1} |w_n|^{1/n} < \infty$. By Theorem 0.1.19, T is hypercyclic if and only if $\sup_{n\geq 1} \big(\prod_{j=1}^{n} |w_j|\big)^{1/n} = \infty$, and chaotic if and only if $\lim_{n\to\infty} \big(\prod_{j=1}^{n} |w_j|\big)^{1/n} = \infty$.

The differentiation operator D from Example 0.1.13 is a weighted shift since $D(\sum_{n\geq 0} a_n z^n) = \sum_{n\geq 0} (n+1) a_{n+1} z^n$. Since $\lim_{n\to\infty} n!^{1/n} = \infty$, we recover the fact that D is hypercyclic. It is even chaotic since the series $\sum_{n\geq 0} z^n/n! = e^z$ converges in $H(\mathbb{C})$.

Example 0.1.22. Similarly, a weighted shift $T : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ is an operator on $H(\mathbb{D})$ if and only if $\limsup_{n\geq 1} |w_n|^{1/n} \leq 1$. By Theorem 0.1.19, T is hypercyclic or chaotic if and only if, respectively,

$$\limsup_{n\to\infty} \Big(\prod_{j=1}^{n} |w_j|\Big)^{1/n} \geq 1, \qquad \liminf_{n\to\infty} \Big(\prod_{j=1}^{n} |w_j|\Big)^{1/n} \geq 1.$$

For bilateral weighted shifts, we have the following characterization. Unconditional bases and convergence are defined in the same way as for the sequence spaces over $\mathbb{N}$.

Theorem 0.1.23 ([45, Theorems 6 and 9])**.** *Let E be an F-sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Let T be a bilateral weighted shift on E with weight sequence $(w_n)_{n\in\mathbb{Z}}$.*

(i) *T is hypercyclic if and only if there exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that, for any $j \in \mathbb{Z}$, one has*

$$\lim_{k\to\infty} \Big(\prod_{m=j-n_k+1}^{j} w_m\Big) e_{j-n_k} = 0 \text{ and } \lim_{k\to\infty} \Big(\prod_{m=j+1}^{j+n_k} w_m\Big)^{-1} e_{j+n_k} = 0.$$

(ii) *If $(e_n)_{n\in\mathbb{Z}}$ is unconditional, then T is chaotic if and only if the series*

$$\sum_{n=-\infty}^{0} \Big(\prod_{j=n+1}^{0} w_j\Big) e_n + \sum_{n=1}^{\infty} \Big(\prod_{j=1}^{n} w_j\Big)^{-1} e_n$$

converges in E.

Many notions of linear dynamics are variants of hypercyclicity. We have already encountered the concept of chaos. A hypercyclic vector x for an operator T means that x visits every non-empty open set via T at least once. In fact, it is easy to see that the orbit of x meets every non-empty open set infinitely often. How often such a vector visits a set can be quantified.

Definition 0.1.24. The *lower density* of a set $A \subseteq \mathbb{N}$ denoted by $\underline{\text{dens}}(A)$ is the quantity

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{|A \cap \{0,\ldots,N\}|}{N+1}.$$

This leads to the definition of frequent hypercyclicity introduced by Bayart and Grivaux in [7].

Definition 0.1.25. Let E be an F-space. An operator $T : E \longrightarrow E$ is *frequently hypercyclic* if there exists $x \in E$ such that, for every non-empty open set U of E, the set $\{n \in \mathbb{N} \mid T^n(x) \in U\}$ has positive lower density. Such a vector is called a *frequently hypercyclic vector* for T.

Of course, frequent hypercyclicity implies hypercyclicity.

As with hypercyclicity, one can try to explicitly construct a frequently hypercyclic vector to prove that a given operator is frequently hypercyclic, see for example [47, Example 9.6]. But for weighted shifts, there is a sufficient condition relating only to the weights, which is much simpler.

Proposition 0.1.26. *Let E be an F-sequence space over $\mathbb{N}$ in which* $\operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ *is dense. Let $T : E \longrightarrow E$ be a weighted shift. If the series*

$$\sum_{n=0}^{\infty} \Big(\prod_{j=1}^{n} w_j \Big)^{-1} e_n$$

converges unconditionally then T is frequently hypercyclic.

Proposition 0.1.27. *Let E be an F-sequence space over $\mathbb{Z}$ in which* $\operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$ *is dense. Let $T : E \longrightarrow E$ be a bilateral weighted shift. If the series*

$$\sum_{n=-\infty}^{0} \Big(\prod_{j=n+1}^{0} w_j \Big) e_n + \sum_{n=1}^{\infty} \Big(\prod_{j=1}^{n} w_j \Big)^{-1} e_n$$

converges unconditionally then T is frequently hypercyclic.

The previous two results can be proved with the Frequent Hypercyclicity Criterion below, see [22, Theorem 4.3].

Example 0.1.28. The differentiation operator D is frequently hypercyclic on $H(\mathbb{C})$. Indeed, recall from Example 0.1.21 that D is a weighted shift with sequence of weights $(n)_{n\geq 1}$. Since $\sum_{n\geq 0} z^n/n! = e^z$ converges unconditionally in $H(\mathbb{C})$, we deduce by Proposition 0.1.26 that D is frequently hypercyclic.

Proposition 0.1.26 combined with Theorem 0.1.19, and Proposition 0.1.27 combined with Theorem 0.1.23, yield the following result.

Corollary 0.1.29 ([22, Corollary 4.4])**.** *Let E be an F-sequence space in which the canonical unit sequences form an unconditional basis. Then every chaotic shift on E is frequently hypercyclic.*

For weighted shifts on ℓ^p, $1 \leq p < \infty$, the converse holds.

Theorem 0.1.30 ([12, Theorem 4])**.** *Let $T : \ell^p \longrightarrow \ell^p$ be a weighted shift with sequence of weights $(w_n)_{n\in\mathbb{N}_0}$, where $1 \leq p < \infty$. Then T is frequently hypercyclic if and only if the series $\sum_{n=0}^{\infty} \prod_{j=1}^{n} |w_j|^{-p}$ converges.*

That is to say, by Theorem 0.1.19, a weighted shift on ℓ^p, $1 \leq p < \infty$, is frequently hypercyclic if and only if it is chaotic.

On the space c_0, there exists a frequently hypercyclic weighted shift that is not chaotic as showed by Bayart and Grivaux [9, Corollary 5.2]. On c_0 or on each space ℓ^p, $1 \leq p < \infty$, Menet [72, Theorem 1.2] proved that there exists a chaotic operator that is not frequently hypercyclic.

As for chaos, there exists an infinite-dimensional separable Banach space over $\mathbb{K}$ that supports no frequently hypercyclic operators. This was proved by Shkarin in [90, Corollaries 1.4 and 1.5].

A useful tool for proving the frequent hypercyclicity of a given operator is the so-called Frequent Hypercyclicity Criterion.

Theorem 0.1.31 (Frequent Hypercyclicity Criterion)**.** *Let T be an operator on a separable F-space E. Assume that there exist a dense subset E_0 of E and a map $S : E_0 \longrightarrow E_0$ such that for any $x \in E_0$, the following conditions hold:*

(i) $\sum_{n=0}^{\infty} T^n(x)$ *is unconditionally convergent,*

(ii) $\sum_{n=0}^{\infty} S^n(x)$ *is unconditionally convergent,*

(iii) $TS(x) = x$.

Then T is frequently hypercyclic.

This version of the criterion for operators defined on an F-space has been proved by Bonilla and Grosse-Erdmann [22, Theorem 2.1]. An operator satisfying the conditions of Theorem 0.1.31 is also chaotic, see [22, Remark 2.2(b)]. Consequently, the example of Bayart and Grivaux [9, Corollary 5.2] of a frequently hypercyclic but not chaotic weighted shift on c_0 shows that not every frequently hypercyclic operator satisfies the criterion.

As we will see in Chapter 1, frequent hypercyclicity is related to ergodic theory.

Definition 0.1.32. Let $(M, \mathcal{B}, \mu)$ be a probability space. A measurable map $T : M \longrightarrow M$ is *measure-preserving*, or μ is *T-invariant*, if $\mu(T^{-1}(A)) = \mu(A)$ for every $A \in \mathcal{B}$.

If T is measure-preserving then it is, or μ is,

(i) *ergodic* if for every $A \in \mathcal{B}$ such that $A = T^{-1}(A)$, we have $\mu(A) \in \{0, 1\}$,

(ii) *strongly mixing* if $\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$ for every $A, B \in \mathcal{B}$,

(iii) *exact* if every $A \in \mathcal{B}$ belonging to $\bigcap_{n\geq 0} T^{-n}(\mathcal{B})$ satisfies $\mu(A) \in \{0, 1\}$.

We remark that exactness implies strong mixing, and strong mixing implies ergodicity, see [32, pp. 50, 87].

The link between frequent hypercyclicity and ergodic theory will be made via the so-called Birkhoff Ergodic Theorem. For a proof of this result, see e.g. [98, Theorem 1.14].

Theorem 0.1.33 (Birkhoff Ergodic Theorem)**.** *Let $(M, \mathcal{B}, \mu)$ be a probability space and $T : M \longrightarrow M$ be a measure-preserving and ergodic map. Let $f \in L^1(M, \mu)$. Then*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f \circ T^n = \int_M f \mathrm{d}\mu \ \ \mu\text{-}a.s.$$

With a single operator as the evolution map, we follow the evolution of a system on discrete times; the concept of C_0-semigroup is the continuous counterpart.

Definition 0.1.34. Let $(T_t)_{t\geq 0}$ be a family of operators on a Fréchet space E. It is called a *C_0-semigroup* if

(i) $T_0 = I$,

(ii) $T_{t+s} = T_t T_s$ for all $s, t \geq 0$,

(iii) $\lim_{s\to t} T_s(x) = T_t(x)$ for all $x \in E$ and $t \geq 0$.

Definition 0.1.35. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup. The map $A : \mathrm{Dom}(A) \longrightarrow E$ defined by

$$A(x) := \lim_{t\to 0} \frac{T_t(x) - x}{t}$$

is called the *generator* of $(T_t)_{t\geq 0}$, where $\mathrm{Dom}(A)$ is the set of vectors where the above limit exists.

The generator is clearly a linear map. A C_0-semigroup is fully determined by its generator, see [60, Theorem 3], and the domain of the generator is a dense subspace of the space, see [60, Proposition 1.3]. Here are some useful properties.

Proposition 0.1.36. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Fréchet space E, and let A be its generator.*

(i) *For every $x \in \mathrm{Dom}(A)$ and $t \geq 0$, we have $T_t(x) \in \mathrm{Dom}(A)$ and $AT_t(x) = T_tA(x)$.*

(ii) *Let $\lambda \in \mathbb{K}$ and $x \in E$. Then $x \in \mathrm{Dom}(A)$ and $A(x) = \lambda x$ if and only if $T_t(x) = e^{\lambda t}x$ for all $t \geq 0$.*

Proof. The first assertion is proved in [60, Proposition 1.2(1)].

For the second one, assume that $T_t(x) = e^{\lambda t}x$ for any $t \geq 0$. We then have

$$\frac{T_t(x) - x}{t} = \frac{e^{\lambda t}x - x}{t} = \frac{e^{\lambda t} - 1}{t}x$$

for any $t > 0$. We conclude that $x \in \mathrm{Dom}(A)$ and $A(x) = \lambda x$.

Assume now that $A(x) = \lambda x$. It is readily check that $(e^{-\lambda t}T_t)_{t\geq 0}$ is a C_0-semigroup with generator $A - \lambda I$ defined on $\mathrm{Dom}(A)$. By applying [60, Proposition 1.2(2)] to this semigroup, we get that

$$e^{-\lambda t}T_t(x) - x = \int_0^t e^{-\lambda s}T_s(A(x) - \lambda x)\mathrm{d}s = 0$$

for any $t \geq 0$. This concludes the proof. □

Example 0.1.37. Let $I \in \{[0,\infty[, \mathbb{R}\}$. An *admissible weight function* ρ is a measurable function $\rho : I \longrightarrow]0,\infty[$ such that there exist $M \geq 1$ and $w \in \mathbb{R}$ such that $\rho(s) \leq Me^{wt}\rho(t+s)$ for every $s \in I$ and $t \geq 0$.

For an admissible weight function ρ and a real $1 \leq p < \infty$, we define the Banach space

$$L^p_\rho(I) := \left\{ f : I \longrightarrow \mathbb{K} \mid \int_I |f(x)|^p \rho(x)\mathrm{d}x < \infty \right\},$$

endowed with the norm $\|f\|_{L^p_\rho(I)} := (\int_I |f(x)|^p\rho(x)\mathrm{d}x)^{1/p}$, $f \in L^p_\rho(I)$. The *translation semigroup* on $L^p_\rho(I)$ is defined by

$$T_t(f)(x) := f(x+t),\ x \in I, t \geq 0,$$

for every $f \in L^p_\rho(I)$. Its generator is the differentiation operator defined on the space of absolutely continuous functions in $L^p_\rho(I)$ with derivative in $L^p_\rho(I)$, see [36, Proposition II.1].

With a given operator on a Banach space, one can construct a semigroup by taking the exponential of that operator.

Definition 0.1.38. Let $A : E \longrightarrow E$ be an operator on a Banach space E. We define the *exponential* of A by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

The series $\sum_{n=0}^{\infty} \frac{A^n}{n!}$ converges since it is absolutely convergent.

Proposition 0.1.39 ([36, Proposition I.2.11]). *Let $A : E \longrightarrow E$ be an operator on a Banach space E. Then $(e^{tA})_{t\geq 0}$ is a C_0-semigroup.*

For further reading on semigroups on Banach spaces, see for example the book by Engel and Nagel [36]. For the case of Fréchet spaces, see the book by Yosida [99, Chapter IX] and the papers by Kōmura [60] and Ouchi [83].

The notions of hypercyclicity and chaos are easily adapted to the continuous case. Here, a vector $x \in E$ is *periodic* for a C_0-semigroup $(T_t)_{t\geq 0}$ if there exists $t > 0$ such that $T_t(x) = x$.

Definition 0.1.40. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Fréchet space E.

(i) For all $x \in E$, the *orbit* of x under $(T_t)_{t\geq 0}$ is the set

$$\mathrm{Orb}\big(x, (T_t)_{t\geq 0}\big) = \big\{T_t(x) \mid t \geq 0\big\}.$$

(ii) $(T_t)_{t\geq 0}$ is *hypercyclic* if there exists $x \in E$ whose orbit is dense in E. Such a vector is called a *hypercyclic vector* for $(T_t)_{t\geq 0}$.

(iii) $(T_t)_{t\geq 0}$ is *chaotic* if it is hypercyclic and has a dense set of periodic points.

In [28, Theorem 2.5], Conejero proved that every separable infinite-dimensional Fréchet space which is not $\mathbb{K}^{\mathbb{N}}$ admits a hypercyclic C_0-semigroup, while $\mathbb{K}^{\mathbb{N}}$ has no hypercyclic C_0-semigroup by a result of Shkarin [91, Corollary 1.7].

As for chaos, Bermúdez, Bonilla and Martinón [14, Theorem 3.3] proved that there exists a separable infinite-dimensional Banach space that supports no chaotic C_0-semigroup.

Conejero, Müller and Peris [29, Theorem 2.3] proved that a C_0-semigroup $(T_t)_{t\geq 0}$ is hypercyclic if and only if each T_t, $t > 0$, is hypercyclic.

For frequent hypercyclicity, we must first redefine the lower density.

Definition 0.1.41. The *lower density* of a measurable set $A \subseteq [0, \infty[$ denoted by $\underline{\mathrm{dens}}(A)$ is the quantity

$$\underline{\mathrm{dens}}(A) = \liminf_{N\to\infty} \frac{\lambda(A \cap [0, N])}{N},$$

where λ is the Lebesgue measure on $[0, \infty[$.

Definition 0.1.42. Let E be a Fréchet space. A C_0-semigroup $(T_t)_{t\geq 0}$ on E is *frequently hypercyclic* if there exists $x \in E$ such that, for every non-empty open set U of E, the set $\{t \geq 0 \mid T_t(x) \in U\}$ has positive lower density. Such a vector is called a *frequently hypercyclic vector* for $(T_t)_{t\geq 0}$.

Note that for any open set $U \subseteq E$, the set $\{t \geq 0 \mid T_t(x) \in U\}$ is indeed measurable since the map $t \longmapsto T_t(x)$ is continuous by (iii) of Definition 0.1.34.

This notion was extended from a single operator to semigroups by Badea and Grivaux [6].

Example 0.1.43. The translation semigroup of Example 0.1.37 is chaotic if and only if it is frequently hypercyclic if and only if $\int_I \rho(x)\mathrm{d}x < \infty$, see [69, Theorems 3.9 and 3.10], where the proof is given for real spaces but is also valid for complex spaces.

Remark 0.1.44. Mangino and Peris [70, Proposition 2.1] proved that a C_0-semigroup $(T_t)_{t\geq 0}$ is frequently hypercyclic if and only if each T_t, $t > 0$, is frequently hypercyclic. Moreover, a careful reading of their proof and the result of Conejero, Müller and Peris [29, Theorem 3.2] show that a vector $x \in E$ is frequently hypercyclic for $(T_t)_{t\geq 0}$ if and only if it is frequently hypercyclic for each $T_t, t > 0$.

There are also continuous versions of the concepts of ergodic theory.

Definition 0.1.45. Let $(E, \mathcal{B}, \mu)$ be a probability space, where E is a Fréchet space. A C_0-semigroup $(T_t)_{t\geq 0}$ on E is *measure-preserving*, or μ is $(T_t)_{t\geq 0}$-*invariant*, if $\mu(T_t^{-1}(A)) = \mu(A)$ for every $t \geq 0$ and $A \in \mathcal{B}$.

If $(T_t)_{t\geq 0}$ is measure-preserving then it is, or μ is,

(i) *ergodic* if for every $A \in \mathcal{B}$ such that $A = T_t^{-1}(A)$ for all $t \geq 0$, we have $\mu(A) \in \{0, 1\}$,

(ii) *strongly mixing* if $\lim_{t\to\infty} \mu(T^{-t}(A) \cap B) = \mu(A)\mu(B)$ for every $A, B \in \mathcal{B}$.

Again, remark that strong mixing implies ergodicity, see [30, p. 25].

There is a version of the Birkhoff Ergodic Theorem for semigroups, see [30, Chapter 1, Theorem 1].

Theorem 0.1.46 (Birkhoff Ergodic Theorem)**.** *Let $(E, \mathcal{B}, \mu)$ be a probability space, where E is a Fréchet space. Let $(T_t)_{t\geq 0}$ be a measure-preserving and ergodic C_0-semigroup, and let $f \in L^1(E, \mu)$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N f(T_t(x))\mathrm{d}t = \int_E f \mathrm{d}\mu \quad \mu\text{-}a.s.$$

0.2 Probability theory

Some definitions and elementary results from probability theory are recalled in this section. For much more information on random vectors taking values in a metric space, see [95, Chapter II] or [51, Appendix E]. See also [50, Chapter 1] for a general theory of measurable functions and integration theory in Banach spaces.

In this section, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The σ-algebra of Borel sets of a topological space E is denoted by $\mathscr{B}(E)$.

Definition 0.2.1. Let E be a metric space and $(S, \mathcal{B})$ be a measurable space. A map $f : S \longrightarrow E$ is *measurable* if $f^{-1}(A) \in \mathcal{B}$ for every $A \in \mathscr{B}(E)$.

Definition 0.2.2. Let E be an F-space. A *random vector* $X : \Omega \longrightarrow E$ is a measurable map $v : (\Omega, \mathcal{A}) \longrightarrow (E, \mathscr{B}(E))$.

The *distribution* of a random vector $X : \Omega \longrightarrow E$ is defined exactly as in the scalar case: it is the probability measure

$$\mathbb{P}_X : \mathscr{B}(E) \longrightarrow [0,1], A \longrightarrow \mathbb{P}(X \in A).$$

Two random vectors X and Y and *identically distributed* if their distributions are equal.

An important notion in probability theory is the concept of independence.

Definition 0.2.3. Let J be a set and $(X_j)_{j\in J}$ be a family of random vectors taking values in an F-space E. The random vectors $X_j, j \in J$, are *independent* if for all $n \geq 1$, $A_1, \ldots, A_n \in \mathscr{B}(E)$ and $j_1, \ldots, j_n \in J$, we have

$$\mathbb{P}(X_{j_1} \in A_1, \ldots, X_{j_n} \in A_n) = \prod_{k=1}^{n} \mathbb{P}(X_{j_k} \in A_k).$$

A family $(X_j)_{j\in J}$ of *independent and identically distributed* random vectors, abbreviated i.i.d., is a family of independent random vectors such that $\mathbb{P}_{X_k} = \mathbb{P}_{X_j}$ for all $k, j \in J$.

Definition 0.2.4. Let E be an F-space. The *support* of a probability measure $\mu : \mathscr{B}(E) \longrightarrow [0,1]$ is the set

$$\operatorname{supp}(\mu) = \bigcap_F F,$$

where the intersection is taken over the closed sets $F \subseteq E$ of full measure.

By [18, Proposition 7.2.9], the set $\operatorname{supp}(\mu)$ has full measure.

Definition 0.2.5. A probability measure $\mu : \mathscr{B}(E) \longrightarrow [0,1]$ on an F-space E has *full support* if $\mu(O) > 0$ for any non-empty open set $O \subseteq E$.

Notice that μ has full support if and only if $\operatorname{supp}(\mu) = E$. Indeed, suppose that $\mu(O) = 0$ for some non-empty open set $O \subseteq E$. Then $F := \complement O$ is closed and has full measure, and $\operatorname{supp}(\mu) \subseteq F \subsetneq E$. For the converse, assume that $\operatorname{supp}(\mu) \neq E$. Then, because $\operatorname{supp}(\mu)$ is closed, there exists an open set $O \subseteq E$ such that $O \subseteq \complement\operatorname{supp}(\mu)$. Since $\operatorname{supp}(\mu)$ has full measure, we deduce that $\mu(O) = 0$.

We recall two elementary and well-known results that will be used in several places in this work, namely the Markov inequality and the Borel-Cantelli lemma.

The expectation of a non-negative or integrable random variable $X : \Omega \longrightarrow \mathbb{K}$ is denoted by $\mathbb{E}(X)$.

Proposition 0.2.6 (Markov inequality). *Let X be a non-negative random variable. For every $t > 0$, we have*

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

Lemma 0.2.7 (Borel-Cantelli lemma)**.** *Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a sequence of measurable events. If $\sum_{n\geq 0} \mathbb{P}(A_n) < \infty$ then*

$$\mathbb{P}\Big(\bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} A_n\Big) = 0.$$

There is also a kind of converse of the lemma but we will not use it in this work.
Let us recall some modes of convergences of measures and random vectors.

Definition 0.2.8. Let E be a metric space. A sequence $(\nu_n)_{n\in\mathbb{N}}$ of positive measures on $(E, \mathscr{B}(E))$ *converges weakly* to a measure ν if

$$\lim_{n\to\infty} \int_E f(x)\mathrm{d}\nu_n = \int_E f(x)\mathrm{d}\nu$$

for every bounded continuous function $f : E \longrightarrow \mathbb{R}$.

Definition 0.2.9. Let $(S, \mathcal{B}, \mu)$ be a measure space, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable maps taking values in a metric space (E, d) and $f : S \longrightarrow E$ be a measurable map.

(i) The sequence $(f_n)_{n\in\mathbb{N}}$ *converges in measure* to f if $\lim_{n\to\infty} \mu(d(f_n, f) \geq \varepsilon) = 0$ for every $\varepsilon > 0$.

(ii) The sequence $(f_n)_{n\in\mathbb{N}}$ *converges almost everywhere* to f if there exists a set $A \in \mathcal{B}$ such that $\mu(A) = 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \notin A$.

Of course, there is also the notion of convergence in the spaces L^p, $1 \leq p < \infty$. See Section A.1 for the definition of the spaces L^p for functions taking values in a Fréchet space.

The proof of the next result can be found in [51, Proposition E.1.5] for a probability space, but it is the same for any measure space.

Lemma 0.2.10. *Let $(S, \mathcal{B}, \mu)$ be a measure space, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable maps on $(S, \mathcal{B}, \mu)$ taking values in a metric space (E, d) and $f : S \longrightarrow E$ be a measurable map. If $(f_n)_{n\in\mathbb{N}}$ converges in measure to f then there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ converging almost everywhere to f.*

Proof. By assumption, construct by induction an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $\mu(d(f_{n_k}, f) \geq 2^{-k}) \leq 2^{-k}$ for all $k \in \mathbb{N}$. Then we have

$$\mu\Big(\bigcap_{k_0\geq 0} \bigcup_{k\geq k_0} \big\{x \in S \mid d(f_{n_k}(x), f(x)) \geq 2^{-k}\big\}\Big) = 0.$$

Therefore, almost everywhere, there exists $k_0 \geq 0$ such that $d(f_{n_k}, f) \leq 2^{-k}$ for every $k \geq k_0$, and thus $(f_{n_k})_{k\in\mathbb{N}}$ converges almost everywhere to f. □

For random vectors, we usually say *almost sure convergence* for almost everywhere convergence, and *convergence in probability* for convergence in measure.

Definition 0.2.11. A sequence of random vectors $(X_n)_{n\in\mathbb{N}}$ *converges in distribution* if $(\mathbb{P}_{X_n})_{n\in\mathbb{N}}$ converges weakly.

It is not difficult to show that almost sure convergence implies convergence in probability, and that convergence in probability implies convergence in distribution, see [51, Proposition E.1.5].

0.3 Pettis integral

We will use the Pettis integral in Chapter 2. We recall here the definition. For more on this topic, see [50, Subsection 1.2.c] in the case of Banach space-valued functions, or [93].

Definition 0.3.1. Let $(S, \mathcal{B}, \mu)$ be a measure space and E be a Fréchet space.

(i) A function $f : S \longrightarrow E$ is *weakly measurable* if x^*f is measurable for every $x^* \in E^*$.

(ii) Let $1 \leq p < \infty$. A weakly measurable function $f : S \longrightarrow E$ is L^p*-weakly integrable*, or *weakly* L^p, if $x^*f \in L^p(S; \mathbb{K})$ for all $x^* \in E^*$.

Definition 0.3.2. Let $(S, \mathcal{B}, \mu)$ be a measure space, E be a Fréchet space and $f : S \longrightarrow E$ be a L^p-weakly integrable function, where $1 \leq p < \infty$. Then f is *Pettis integrable* if for every $A \in \mathcal{B}$, there exists $x_A \in E$ such that for all $x^* \in E^*$, we have $x^*(x_A) = \int_A x^* f \mathrm{d}\mu$.

0.4 Stochastic calculus

We recall in this section the Itô integral. It will be used in Chapter 2. There are many references on this topic, see for example [59] or [81] for more information.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. First of all, let us recall the definition of the Brownian motion.

Definition 0.4.1. A real-valued stochastic process $(B_t)_{t\geq 0}$ is a *Brownian motion* if

(i) $B_0 = 0$ almost surely,

(ii) $(B_t)_{t\geq 0}$ has *independent increments* i.e., for any $n \in \mathbb{N}_0$ and $0 \leq t_0 < \cdots < t_n$, the random variables $B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent,

(iii) $(B_t)_{t\geq 0}$ has *stationary increments* i.e., for any $0 \leq s \leq t$, the random variables $B_t - B_s$ and B_{t-s} have the same distribution,

(iv) for every $t \geq 0$, B_t is a centred Gaussian random variable with variance t,

(v) almost surely, the paths $t \longmapsto B_t$ are continuous.

There are several proofs of the existence of the Brownian motion, and they can be found in many books. See for example [59, Theorem 21.9].

For the remainder of the section, let $(B_t)_{t\geq 0}$ be a Brownian motion.

The Itô integral is defined in the same way as the Riemann integral, by approaching functions to integrate by 'simple' functions. First, recall the definition of a step function, that is, piecewise constant functions.

Definition 0.4.2. A *step function* $\phi : [0, \infty[\longrightarrow \mathbb{R}$ is a function of the form $\phi = \sum_{i=1}^n a_i \mathbf{1}_{]t_{i-1}, t_i[}$, where $n \in \mathbb{N}_0$, $a_i \in \mathbb{R}$ for all $1 \leq i \leq n$ and $0 \leq t_0 \leq \cdots \leq t_n$.

Any square-integrable function $\phi : [0, \infty[\longrightarrow \mathbb{R}$ can be approximated by step functions, see [50, Remark 1.2.20]. If $\phi = \sum_{i=1}^n a_i \mathbf{1}_{]t_{i-1}, t_i[}$ is a step function, define $\int_0^\infty \phi(t) \mathrm{d}B_t := \sum_{i=1}^n a_i (B_{t_i} - B_{t_{i-1}})$. It is easy to check that

$$\Big\| \int_0^\infty \phi(t) \mathrm{d}B_t \Big\|_{L^2(\Omega;\mathbb{R})} = \|\phi\|_{L^2([0,\infty[)}.$$

Therefore, by denoting $\mathcal{E}$ as the space of step functions, we obtain a linear isometry

$$\mathcal{I} : \mathcal{E} \longrightarrow L^2(\Omega, \mathbb{P}), \phi \longmapsto \int_0^\infty \phi(t) \mathrm{d}B_t.$$

Since $\mathcal{E}$ is dense in $L^2([0, \infty[)$, we can extend $\mathcal{I}$ to the whole space $L^2([0, \infty[)$.

Definition 0.4.3. Let $\phi : [0, \infty[\longrightarrow \mathbb{R}$ be a square-integrable function. Let $(\phi_n)_{n\geq 0}$ be a sequence of step functions converging to ϕ in $L^2([0, \infty[)$. The *Itô integral* of ϕ is the Gaussian random variable

$$\int_0^\infty \phi(t) \mathrm{d}B_t := \lim_{n\to\infty} \int_0^\infty \phi_n(t) \mathrm{d}B_t,$$

where the limit is taken in $L^2(\Omega; \mathbb{R})$.

The Itô integral of a square-integrable function is indeed a Gaussian random variable since a converging sequence of centred Gaussian random variables necessarily converges to a Gaussian random variable, see [86, Lemma 2.1]. It is also centred since the Brownian motion is centred.

We get the celebrated Itô isometry.

Theorem 0.4.4 (Itô isometry). *Let $\phi : [0, \infty[\longrightarrow \mathbb{R}$ be a square-integrable function. Then we have*

$$\Big\| \int_0^\infty \phi(t) \mathrm{d}B_t \Big\|_{L^2(\Omega;\mathbb{R})} = \|\phi\|_{L^2([0,\infty[)}.$$

To consider complex spaces as well, we define a stochastic integral with respect to a complex Brownian motion.

Definition 0.4.5. Let $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ be two independent Brownian motions. The stochastic process $(B_t)_{t\geq 0} := (B_t^1 + iB_t^2)_{t\geq 0}$ is called a *complex Brownian motion*.

Definition 0.4.6. Let $(B_t)_{t\geq 0} := (B_t^1 + iB_t^2)_{t\geq 0}$, where $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ are two independent real Brownian motions. Let $\phi \in L^2([0,\infty[;\mathbb{C})$. The *Itô integral* of ϕ is the complex random variable

$$\int_0^\infty \phi(t)\mathrm{d}B_t := \int_0^\infty \phi_1(t)\mathrm{d}B_t^1 - \int_0^\infty \phi_2(t)\mathrm{d}B_t^2 + i\Big(\int_0^\infty \phi_2(t)\mathrm{d}B_t^1 + \int_0^\infty \phi_1(t)\mathrm{d}B_t^2\Big),$$

where ϕ_1 and ϕ_2 are respectively the real and imaginary parts of ϕ.

There is also a version of the Itô isometry for the complex case.

Lemma 0.4.7. *Let $\phi \in L^2([0,\infty[;\mathbb{C})$ and $(B_t)_{t\geq 0} := (B_t^1 + iB_t^2)_{t\geq 0}$, where $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ are two independent real Brownian motions. Then we have*

$$\Big\| \int_0^\infty \phi(t)\mathrm{d}B_t \Big\|_{L^2(\Omega;\mathbb{C})} = \sqrt{2}\|\phi\|_{L^2([0,\infty[;\mathbb{C})}.$$

Proof. Let ϕ_1 and ϕ_2 be respectively the real and imaginary parts of ϕ. The definition of the Itô integral with respect to a complex Brownian motion yields

$$\begin{aligned}
&\Big| \int_0^\infty \phi(t)\mathrm{d}B_t \Big|^2 \\
&= \Big(\int_0^\infty \phi_1(t)\mathrm{d}B_t^1\Big)^2 + \Big(\int_0^\infty \phi_2(t)\mathrm{d}B_t^2\Big)^2 - 2\int_0^\infty \phi_1(t)\mathrm{d}B_t^1 \int_0^\infty \phi_2(t)\mathrm{d}B_t^2 \\
&\quad + \Big(\int_0^\infty \phi_2(t)\mathrm{d}B_t^1\Big)^2 + \Big(\int_0^\infty \phi_1(t)\mathrm{d}B_t^2\Big)^2 + 2\int_0^\infty \phi_2(t)\mathrm{d}B_t^1 \int_0^\infty \phi_1(t)\mathrm{d}B_t^2.
\end{aligned}$$

By taking the expectations on both sides, by Theorem 0.4.4 and since $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ are independent and centred, we get the desired equality. $\square$

The Brownian motion is often indexed by the interval $[0,\infty[$. We may also define it on $\mathbb{R}$ by taking two independent Brownian motions $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$, and setting $B_t := B_t^1$ if $t \geq 0$ and $B_t := B_{-t}^2$ if $t < 0$.

0.5 Gaussian measures

In Chapter 2, we will need the notion of covariance operators for Gaussian measures. This concept is the generalization of the covariance matrix for a finite-dimensional Gaussian distribution. Our main references are [10, Chapter 5], [19] and [95].

Throughout this section, let E be a separable real or complex Fréchet space and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

We first recall the definition of a complex Gaussian random variable.

Definition 0.5.1. Let $X : \Omega \longrightarrow \mathbb{K}$ be a random variable.

(i) If $\mathbb{K} = \mathbb{C}$ then $X : \Omega \longrightarrow \mathbb{C}$ is a *(complex) Gaussian random variable*, or has *Gaussian distribution*, if its real and imaginary parts are independent and have Gaussian distribution with the same mean and variance.

(ii) If $\mathbb{K} = \mathbb{R}$ and X is a Gaussian random variable then it is *standard* if has mean 0 and variance 1.

(iii) If $\mathbb{K} = \mathbb{C}$ and X is a Gaussian random variable then it is *standard* if its real and imaginary parts have mean 0 and variance equal to $\sqrt{2}/2$.

(iv) A sequence $(X_n)_{n\in\mathbb{N}}$ of independent Gaussian random variables is a *standard Gaussian sequence* if each X_n, $n \in \mathbb{N}$, is standard.

Note that we consider the constant variable 0 to be Gaussian, that may be thought as a degenerate Gaussian.

If the space E is real (resp. complex), then the random variables are assumed to be real (resp. complex).

Definition 0.5.2. A probability measure μ on E is a *Gaussian measure* on E if for every $x^* \in E^*$, the random variable x^* defined on the probability space $(E, \mathscr{B}(E), \mu)$ has a centred Gaussian distribution.

A *Gaussian random vector* is a random vector whose distribution is a Gaussian measure.

A well-known result, the so-called Fernique integrability theorem, says that a Gaussian measure has moments of any order. For a proof of this result, see [19, Corollary 2.8.6].

Theorem 0.5.3 (Fernique integrability theorem)**.** *Let μ be a Gaussian measure on E. For every $1 \le p < \infty$ and every continuous seminorm $\|\cdot\|$ on E, the integral*

$$\int_E \|x\|^p \mathrm{d}\mu(x)$$

is finite.

The next result says that the almost sure convergence of a Gaussian series is equivalent to the convergence in $L^p(\Omega; E)$ for any $1 \le p < \infty$. See Section A.1 of the appendix for a definition of the spaces $L^p(\Omega; E)$, $1 \le p < \infty$, and a proof of their completeness.

Theorem 0.5.4. *For any sequence of vectors $(x_n)_{n\in\mathbb{N}} \subseteq E$ and any sequence $(g_n)_{n\in\mathbb{N}}$ of i.i.d. centred Gaussian random variables, the following assertions are equivalent:*

(i) *for all $1 \le p < \infty$, the series $\sum_{n=0}^{\infty} g_n x_n$ converges in $L^p(\Omega; E)$,*

(ii) *there exists $1 \le p < \infty$ such that $\sum_{n=0}^{\infty} g_n x_n$ converges in $L^p(\Omega; E)$,*

(iii) *the series $\sum_{n=0}^{\infty} g_n x_n$ converges in probability,*

(iv) *the series $\sum_{n=0}^{\infty} g_n x_n$ converges almost surely.*

Proof. The implication (i) $\implies$ (ii) is clear and (ii) $\implies$ (iii) follows from Markov's inequality. The equivalence (iii) $\iff$ (iv) is proved in [25, Theorem 1.3.2].

Let us prove (iv) $\implies$ (i). Let $1 \le p < \infty$. Let $\|\cdot\|$ be a continuous seminorm on E. Since the random vector $\sum_{n\ge 0} g_n x_n$ converges almost surely, it has a

Gaussian distribution by [10, Example 5.8]. Note that the proof given there for Banach spaces carries over verbatim to Fréchet spaces. By Theorem 0.5.3, we thus have $\mathbb{E}(\| \sum_{n\geq 0} g_n x_n\|^p) < \infty$. Therefore, by [25, Corollaries 1.7.2 and 1.7.3], we get that

$$\lim_{k\to\infty} \mathbb{E}\Big(\| \sum_{n=k}^{\infty} g_n x_n\|^p\Big) = 0,$$

and the result follows. □

An important tool for studying a Gaussian measure is the covariance operator. A proof of the next theorem is given in [19, Theorem 3.2.3 and Corollary 3.2.5]. The Mackey topology $\tau(E^*, E)$ is defined in Definition A.3.4.

Theorem 0.5.5. *Let μ be a Gaussian measure on E. There exists a continuous conjugate-linear map $Q : (E^*, \tau(E^*, E)) \longrightarrow E$ such that for any $x^*, y^* \in E^*$, we have*

$$y^*Qx^* = \int_E y^*(z)\overline{x^*(z)}\mathrm{d}\mu(z) = \langle y^*, x^*\rangle_{L^2(E,\mu)}.$$

Definition 0.5.6. Let μ be a Gaussian measure. The operator $Q : E^* \longrightarrow E$ of Theorem 0.5.5 is called the *covariance operator* of μ.

A conjugate-linear map $Q : E^* \longrightarrow E$ is a *Gaussian covariance operator* if it is the covariance operator of some Gaussian measure on E.

Recall that the *characteristic functional* of a Borel measure μ on E, denoted by $\widehat{\mu}$, is defined by

$$\widehat{\mu}(x^*) := \int_E e^{i\mathrm{Re}(x^*(x))}\mathrm{d}\mu(x)$$

for every $x^* \in E^*$. The covariance operator fully determines the characteristic functional of a Gaussian measure, as the next result says.

Theorem 0.5.7. *Let μ be a Gaussian measure with covariance operator Q. For all $x^* \in E^*$, one has*

$$\widehat{\mu}(x^*) = e^{-cx^*Qx^*},$$

where $c = 1/2$ if E is real and $c = 1/4$ if E is complex.

A proof for the real case can be found in [19, Theorem 2.2.4 and p. 45]. The proof given there also holds for the complex case, or see [10, Theorem 5.9(b)] for measures on complex Banach spaces. Again, the proof holds for Fréchet spaces.

This result in turn implies that two Gaussian measures with the same covariance operator are equal, see [18, Lemma 7.13.5].

There exists a characterization for a conjugate-linear map to be a Gaussian covariance operator. We will need the notion of γ-radonifying operators.

Definition 0.5.8. Let H be a separable Hilbert space. A continuous and linear map $T : H \longrightarrow E$ is *γ-radonifying* if for some orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H, the series $\sum_{n\geq 0} g_n T(e_n)$ converges almost surely, where $(g_n)_{n\in\mathbb{N}}$ is a standard Gaussian sequence.

If $T : H \longrightarrow E$ is γ-radonifying then $\sum_{n\geq 0} g_n T(e_n)$ converges almost surely for any orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H, see [10, Remark 5.12]. The proof in the reference still holds for Fréchet spaces; use [25, Theorem 1.3.2] for the equivalence between the almost sure convergence and the convergence of the characteristic functionals used in the proof of [10, Remark 5.12].

Again, a proof of Theorem 0.5.9 is given in the cited reference in the case of a Banach space, but the proof remains the same in the case of a Fréchet space.

If H is a separable Hilbert space, the term *canonical conjugate-linear identification operator* stands for the bijective conjugate-linear map $I : H^* \longrightarrow H$ given by the Riesz theorem, which identifies H with its dual.

Theorem 0.5.9 ([10, Theorem 5.13]). *Let $Q : E^* \longrightarrow E$ be a conjugate-linear map. The following assertions are equivalent:*

(i) *Q is a Gaussian covariance operator,*

(ii) *Q has a γ-radonifying square root i.e., there exist a separable Hilbert space E and a γ-radonifying operator $K : H \longrightarrow E$ such that $Q = KIK^*$, where $I : H^* \longrightarrow H$ is the canonical conjugate-linear identification operator.*

More precisely, we can take $H^* = \overline{E^*}$ where the closure is taken in $L^2(\mu)$ and $K^* : E^* \longrightarrow H^*, x^* \longmapsto x^*$.

The following lemma says that every square root of a Gaussian covariance operator is necessarily γ-radonifying.

Lemma 0.5.10. *If $Q : E^* \longrightarrow E$ is a Gaussian covariance operator, and if $Q = KIK^*$, where $K : H \longrightarrow E$ is continuous, H is a separable Hilbert space and $I : H^* \longrightarrow H$ is the canonical conjugate-linear operator, then K is γ-radonifying.*

Proof. Let $x^* \in E^*$. Then

$$\begin{aligned} x^*Qx^* = x^*(KIK^*(x^*)) &= (x^* \circ K)(IK^*(x^*)) \\ &= \langle IK^*(x^*), IK^*(x^*)\rangle_H = \|K^*(x^*)\|_H^2. \end{aligned}$$

By [10, Remark 5.12] and Theorem 0.5.7, we deduce that K is γ-radonifying. $\square$

Definition 0.5.11. A family $\mathcal{R}$ of Borel probability measures on E is *uniformly tight* if for every $\varepsilon > 0$, there exists a compact set $K \subseteq E$ such that $\nu(K) > 1 - \varepsilon$ for every $\nu \in \mathcal{R}$.

Recall that a map $R : E^* \longrightarrow E$ is *positive* if $x^*Rx^* \geq 0$ for all $x^* \in E^*$, and is *symmetric* if $x^*Ry^* = \overline{y^*Rx^*}$ for all $x^*, y^* \in E^*$. It is easy to see that any Gaussian covariance operator is positive symmetric. In addition, note that a symmetric map is necessarily conjugate-linear.

Theorem 0.5.12 ([97, Theorem 8.8]). *Let $Q : E^* \longrightarrow E$ be the covariance operator of a Gaussian measure μ_Q on E. Let $\mathcal{R}$ be the family of positive symmetric operators $R : E^* \longrightarrow E$ such that for every $x^* \in E^*$, one has $x^*Rx^* \leq x^*Qx^*$. Then each*

$R \in \mathcal{R}$ is the covariance operator of a Gaussian measure μ_R on E, and the family $\{\mu_R \mid R \in \mathcal{R}\}$ is uniformly tight. Moreover, for all $R \in \mathcal{R}$, we have

$$\int_E \|x\|^p \mathrm{d}\mu_R(x) \le \int_E \|x\|^p \mathrm{d}\mu_Q(x)$$

for all $1 \le p < \infty$ and all continuous seminorms $\|\cdot\|$ on E.

The cited reference gives the proof for Banach spaces, and it carries over verbatim to Fréchet spaces.

The interest in the covariance operator is that properties involving a Gaussian measure can usually be formulated in terms of its covariance operator. Here are three examples that we will need in Chapter 2. The reference given for the last result is stated for a single operator on a Banach space, but the proof for semigroups on a Fréchet space is the same.

If $M \subseteq E^*$, we define the *orthogonal complement* of M by

$$M^\perp := \Big\{ x \in E \mid x^*(x) = 0 \text{ for all } x^* \in M \Big\}.$$

Proposition 0.5.13 ([10, Proposition 5.18])**.** *Let μ be a Gaussian measure with covariance operator $Q : E^* \longrightarrow E$. Then $\operatorname{supp}(\mu) = \operatorname{Ker}(Q)^\perp$. In particular, μ has full support if and only if Q is one-to-one.*

Theorem 0.5.14 ([10, Proposition 5.22 and Theorem 5.24])**.** *Let $(T_t)_{t\ge 0}$ be a C_0-semigroup on E. Let μ be a Gaussian measure with covariance operator $Q : E^* \longrightarrow E$.*

(i) *The measure μ is $(T_t)_{t\ge0}$-invariant if and only if $T_t Q T_t^* = Q$ for all $t \ge 0$.*

(ii) *If μ is $(T_t)_{t\ge0}$-invariant, then $(T_t)_{t\ge0}$ is strongly mixing with respect to μ if and only if for any $x^*, y^* \in E^*$, one has $\lim_{t\to\infty} y^* Q T_t^*(x^*) = 0$.*

Chapter 1

Random vectors for frequently hypercyclic operators

Let $B_w : \ell^p \longrightarrow \ell^p$ be a weighted shift on ℓ^p, $1 \leq p < \infty$, where $w = (w_n)_{n\geq 1}$ is the sequence of weights. It is known that if B_w is chaotic then the random vector $\sum_{n\geq 0} \frac{X_n}{w_1 \dots w_n} e_n$ is almost surely frequently hypercyclic for B_w, where $(X_n)_{n\geq 0}$ is a sequence of independent and identically distributed non-constant Gaussian random variables, see [10, Section 5.5.2] or [11, Section 7.1]. Furthermore, this random vector also induces a strongly mixing Gaussian measure for B_w.

In [80], Nikula proved that $\sum_{n\geq 0} \frac{X_n}{n!} e_n$ is almost surely frequently hypercyclic for the differentiation operator D on the space $H(\mathbb{C})$ of entire functions, where the distribution of the i.i.d. variables $(X_n)_{n\geq 0}$ satisfies some conditions and $(e_n)_{n\geq 0}$ is the sequence of monomials. In [74], Mouze and Munnier relaxed the condition on the distribution. The result was also proved by Bayart and Matheron in [11, Remark 2 after Proposition 8.1] in the Gaussian case, and the random vector $\sum_{n\geq 0} \frac{X_n}{n!} e_n$ also induces a strongly mixing Gaussian measure for D. As a last example, Mouze and Munnier proved in [75, Theorem 1.3] that $\sum_{n\geq 0} X_n e_n$ is almost surely frequently hypercyclic for the so-called Taylor shift.

The aim of this chapter is to generalize these results to very general chaotic weighted shifts and even to a larger class of operators. However, the sequence $(X_n)_{n\geq 0}$ might not be Gaussian.

Given an operator $T : E \longrightarrow E$ on a locally bounded or locally convex separable F-space E, we will find conditions on T and on the distribution of a random variable X such that X will allow us to define a frequently hypercyclic random vector for T. This is the content of Theorem 1.1.8. Section 1.1 is devoted to its proof. In the second section, we deduce three important special cases of the theorem: we obtain conditions under which the desired random variable X exists (Theorem 1.2.3), or can be chosen to be subgaussian (Theorem 1.2.10) or Gaussian (Theorem 1.2.12).

In Section 1.3, these results will be applied to chaotic weighted shifts on very general sequence spaces. We will also give a new proof of a result of Murillo-Arcila and Peris [77] by showing that every operator satisfying the Frequent Hypercyclicity Criterion admits a strongly mixing invariant measure with full support, where we obtain a rather explicit construction of such a measure. However, our result will hold for operators defined on a locally bounded or locally convex separable F-space whereas Murillo-Arcila and Peris proved the result for operators defined on a separable F-space.

In the last section, other ways to quantify the number of visits of a vector in open sets will be considered, giving variants of frequent hypercyclicity. We will prove that the random vector constructed in Theorem 1.1.8 also exhibits those dynamical properties.

Throughout this chapter, if nothing else is said, let E be a locally bounded or locally convex separable F-space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If the space E is complex (resp. real), a random variable X is assumed to take complex (resp. real) values. Every random variable considered will be defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

1.1 Frequent hypercyclicity

The aim of this section is to prove Theorem 1.1.8. We begin with three lemmas.

Lemma 1.1.1 ([40, Lemma 6.6])**.** *Let $(F, \mathcal{A})$ and $(G, \mathscr{B}(G))$ be two measurable spaces with G a metric space and $\mathscr{B}(G)$ the σ-algebra of Borel sets of G. Let $(f_n)_{n\geq 0}$ be a sequence of measurable maps $f_n : F \longrightarrow G$, $n \geq 0$. Assume that $(f_n)_{n\geq 0}$ converges pointwise to a function $f : F \longrightarrow G$. Then f is measurable.*

Proof. Since $\mathscr{B}(G)$ is the σ-algebra generated by the open subsets of G, it suffices to show that $f^{-1}(C) \in \mathcal{A}$ for every closed subset C of G. So let $C \subseteq G$ be a closed subset of G. It is easily verified that

$$f^{-1}(C) = \bigcap_{k\geq 1} \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n \geq n_0} f_n^{-1}\big(\{x \in X \mid \operatorname{dist}(x, C) < 1/k\}\big).$$

Since $f^{-1}(C)$ can be written as countable unions and intersections of sets of $\mathcal{A}$, we have $f^{-1}(C) \in \mathcal{A}$. $\square$

The proof of Lemma 1.1.2 should already be known. A proof in the case of a Banach space can be given by using [51, Corollary E.1.17].

Lemma 1.1.2. *Let F be a metric space. Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of random variables with values in F such that for every $n \in \mathbb{N}$, X_n and Y_n have the same distribution. If $(X_n)_{n\in\mathbb{N}}$ (resp. $(Y_n)_{n\in\mathbb{N}}$) converges almost surely to X (resp. Y) then the random variables X and Y have the same distribution.*

Proof. By assumption, for every bounded continuous function $h : F \longrightarrow \mathbb{R}$ and every $n \in \mathbb{N}$, we have $\mathbb{E}(h(X_n)) = \mathbb{E}(h(Y_n))$. By taking the limit when n goes to ∞, we get $\mathbb{E}(h(X)) = \mathbb{E}(h(Y))$. (At this point, one can use [51, Corollary E.1.17] to conclude the proof.)

Now, let $A \in \mathscr{B}(F)$ and $\varepsilon > 0$. There exists an open set $U \subseteq F$ containing A such that $\mathbb{P}(X \in U \setminus A) \leq \varepsilon$ and $\mathbb{P}(Y \in U \setminus A) \leq \varepsilon$ by [31, Proposition 18.3]. For all $k \geq 0$, define the bounded and continuous function $f_k : F \longrightarrow \mathbb{R}$ by $f_k(x) :=$

$\min(1, k\text{dist}(x, F \setminus U))$, $x \in F$. By the Dominated Convergence Theorem, there exists $k \geq 0$ large enough such that

$$\Big| \int_\Omega \big(\mathbf{1}_U(X) - f_k(X)\big) \mathrm{d}\mathbb{P} \Big| \leq \varepsilon \quad \text{and} \quad \Big| \int_\Omega \big(\mathbf{1}_U(Y) - f_k(Y)\big) \mathrm{d}\mathbb{P} \Big| \leq \varepsilon.$$

Therefore, $|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \leq 4\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for every $A \in \mathscr{B}(F)$, and X and Y have the same distribution. □

The proof of Lemma 1.1.3 comes from [51, Proposition E.1.12].

Lemma 1.1.3. *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of real random variables such that for every $n \in \mathbb{N}$, X_n and Y_n are independent. If $(X_n)_{n\in\mathbb{N}}$ (resp. $(Y_n)_{n\in\mathbb{N}}$) converges almost surely to X (resp. Y) then the random variables X and Y are independent.*

Proof. By almost sure convergence, for every bounded and continuous functions f and $g : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}(f(X)g(Y)) = \lim_{n\to\infty} \mathbb{E}(f(X_n)g(Y_n)) &= \lim_{n\to\infty} \mathbb{E}(f(X_n))\mathbb{E}(g(Y_n)) \\ &= \mathbb{E}(f(X))\mathbb{E}(g(Y)). \end{aligned}$$

This shows that X and Y are independent by [51, Proposition E.1.10]. □

The next result gives conditions under which the random vector $\sum_{n=-\infty}^{\infty} X_n u_n$ is almost surely frequently hypercyclic.

Proposition 1.1.4. *Let $T : E \longrightarrow E$ be an operator and let $(u_n)_{n\in\mathbb{Z}}$ be a sequence in E such that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$. Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that the random vector*

$$v := \sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well-defined and $\mathbb{P}(v \in O) > 0$ for every non-empty open subset O of E. Then v is almost surely frequently hypercyclic for the operator T and induces a strongly mixing measure with full support for T.

Proof. We can assume that the series defining v is convergent everywhere. Indeed, restrict the random variables X_n, $n \in \mathbb{Z}$, to a subset of Ω of full measure on which the series defining v converges. Hence, we assume that the convergence is everywhere, and v is measurable by Lemma 1.1.1.

Define the probability measure

$$\mu : \mathscr{B}(E) \longrightarrow [0, 1], A \longmapsto \mathbb{P}(v \in A).$$

In fact, the measure μ is the probability distribution of the random vector v.

First, we show that μ is T-invariant. Let $A \in \mathscr{B}(E)$. By the definitions of μ and v and continuity of T we have

$$\mu(T^{-1}(A)) = \mathbb{P}(T(v) \in A) = \mathbb{P}\left(\sum_{n=-\infty}^{\infty} X_n u_{n-1} \in A\right) = \mathbb{P}\left(\sum_{n=-\infty}^{\infty} X_{n+1} u_n \in A\right).$$

Since $(X_n)_{n\in\mathbb{Z}}$ is a sequence of i.i.d. random variables, we have

$$\mathbb{P}\left(\sum_{n=-\infty}^{\infty} X_{n+1} u_n \in A\right) = \mathbb{P}\left(\sum_{n=-\infty}^{\infty} X_n u_n \in A\right)$$

by Lemma 1.1.2. We conclude by definition of μ that $\mu(T^{-1}(A)) = \mathbb{P}(v \in A) = \mu(A)$. The measure μ is thus T-invariant.

Now we claim that μ is T-strongly mixing. Let f and g be two bounded and continuous real-valued functions defined on E. We aim to show that $\lim_{n\to\infty} \int_E (f \circ T^n) g \mathrm{d}\mu = \int_E f \mathrm{d}\mu \int_E g \mathrm{d}\mu$. Since the set of bounded continuous functions on E is dense in $L^2(E,\mu)$ by [31, Theorem 18.1], this will imply the claim by [31, Criterion at p. 26]. First, by definition of μ, this is equivalent to showing that

$$\lim_{n\to\infty} \int_\Omega f(T^n(v)) g(v) \mathrm{d}\mathbb{P} = \int_\Omega f(v) \mathrm{d}\mathbb{P} \int_\Omega g(v) \mathrm{d}\mathbb{P}.$$

Let $\varepsilon > 0$. By the Dominated Convergence Theorem and since f and g are continuous and bounded, there exists $N \geq 1$ such that

$$\left\| g\left(\sum_{k=-\infty}^{N} X_k u_k\right) - g(v) \right\|_{L^1(\Omega,\mathbb{P})} < \varepsilon \tag{1.1.1}$$

and

$$\left\| f\left(\sum_{k=-N}^{\infty} X_k u_k\right) - f(v) \right\|_{L^1(\Omega,\mathbb{P})} < \varepsilon. \tag{1.1.2}$$

Let $n > 2N$. We have

$$\begin{aligned} f(T^n(v))g(v) = f(T^n(v))g(v) - f(T^n(v))g\left(\sum_{k=-\infty}^{N} X_k u_k\right) \\ + f(T^n(v))g\left(\sum_{k=-\infty}^{N} X_k u_k\right) - f\left(\sum_{k=-N}^{\infty} X_{k+n} u_k\right) g\left(\sum_{k=-\infty}^{N} X_k u_k\right) \\ + f\left(\sum_{k=-N}^{\infty} X_{k+n} u_k\right) g\left(\sum_{k=-\infty}^{N} X_k u_k\right). \end{aligned} \tag{1.1.3}$$

For the first two terms, using the assumption that f is bounded and the inequality

(1.1.1) yield

$$\begin{aligned}
&\left| \int_\Omega f(T^n(v))g(v)\mathrm{d}\mathbb{P} - \int_\Omega f(T^n(v))g\left(\sum_{k=-\infty}^{N} X_k u_k\right)\mathrm{d}\mathbb{P}\right| \\
&\qquad\qquad \leq \|f\|_\infty \left\| g\left(\sum_{k=-\infty}^{N} X_k u_k\right) - g(v)\right\|_{L^1(\Omega,\mathbb{P})} \leq \|f\|_\infty \varepsilon.
\end{aligned}$$

Now, for the third and fourth terms, using the linearity and continuity of T,

$$\begin{aligned}
&\left| \int_\Omega \left(f(T^n(v))g\Big(\sum_{k=-\infty}^{N} X_k u_k\Big) - f\Big(\sum_{k=-N}^{\infty} X_{k+n} u_k\Big) g\Big(\sum_{k=-\infty}^{N} X_k u_k\Big)\right)\mathrm{d}\mathbb{P}\right| \\
&\qquad \leq \|g\|_\infty \left\| f\Big(\sum_{k=-\infty}^{\infty} X_{k+n} u_k\Big) - f\Big(\sum_{k=-N}^{\infty} X_{k+n} u_k\Big)\right\|_{L^1(\Omega,\mathbb{P})} \\
&\qquad = \|g\|_\infty \left\| f\Big(\sum_{k=-\infty}^{\infty} X_k u_k\Big) - f\Big(\sum_{k=-N}^{\infty} X_k u_k\Big)\right\|_{L^1(\Omega,\mathbb{P})} \\
&\qquad \leq \|g\|_\infty \varepsilon,
\end{aligned}$$

where we have used Lemma 1.1.2 for the equality and (1.1.2) for the last inequality.

For the last term of (1.1.3), since the random variables X_n, $n \in \mathbb{Z}$, are i.i.d. and $n > 2N$, we have, by Lemma 1.1.3 applied to $(f(\sum_{k=-N}^{M} X_{k+n}u_k))_{M\geq 1}$ and $(g(\sum_{k=-M}^{N} X_k u_k))_{M\geq 1}$ and then Lemma 1.1.2 applied to $(f(\sum_{k=-N}^{M} X_{k+n}u_k))_{M\geq 1}$ and $(f(\sum_{k=-N}^{M} X_k u_k))_{M\geq 1}$,

$$\begin{aligned}
&\int_\Omega f\Big(\sum_{k=-N}^{\infty} X_{k+n} u_k\Big) g\Big(\sum_{k=-\infty}^{N} X_k u_k\Big)\mathrm{d}\mathbb{P} \\
&= \int_\Omega f\Big(\sum_{k=-N}^{\infty} X_{k+n} u_k\Big)\mathrm{d}\mathbb{P} \int_\Omega g\Big(\sum_{k=-\infty}^{N} X_k u_k\Big)\mathrm{d}\mathbb{P} \\
&= \int_\Omega f\Big(\sum_{k=-N}^{\infty} X_k u_k\Big)\mathrm{d}\mathbb{P} \int_\Omega g\Big(\sum_{k=-\infty}^{N} X_k u_k\Big)\mathrm{d}\mathbb{P}.
\end{aligned}$$

Therefore, using again (1.1.1) and (1.1.2) gives

$$\begin{aligned}
&\left| \int_\Omega f\left(\sum_{k=-N}^{\infty} X_{k+n} u_k\right) g\left(\sum_{k=-\infty}^{N} X_k u_k\right)\mathrm{d}\mathbb{P} - \int_\Omega f(v)\mathrm{d}\mathbb{P}\int_\Omega g(v)\mathrm{d}\mathbb{P}\right| \\
&\quad \leq \|f\|_\infty \left\| g\left(\sum_{k=-\infty}^{N} X_k u_k\right) - g(v)\right\|_{L^1(\Omega,\mathbb{P})} + \|g\|_\infty \left\| f\left(\sum_{k=-N}^{\infty} X_k u_k\right) - f(v)\right\|_{L^1(\Omega,\mathbb{P})} \\
&\quad \leq \|f\|_\infty \varepsilon + \|g\|_\infty \varepsilon.
\end{aligned}$$

We can finally conclude that

$$\left|\int_\Omega f(T^n(v))g(v)\mathrm{d}\mathbb{P} - \int_\Omega f(v)\mathrm{d}\mathbb{P}\int_\Omega g(v)\mathrm{d}\mathbb{P}\right| \leq 2\|f\|_\infty\varepsilon + 2\|g\|_\infty\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, we eventually get that $\lim_{n\to\infty}\int_\Omega f(T^n(v))g(v)\mathrm{d}\mathbb{P} = \int_\Omega f(v)\mathrm{d}\mathbb{P}\int_\Omega g(v)\mathrm{d}\mathbb{P}$. The measure μ is thus T-strongly mixing.

Let O be a non-empty open subset of E. The Birkhoff Ergodic Theorem can be applied to T and μ and gives

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}\mathbb{I}_O \circ T^n = \mu(O)\ \mu\text{-a.s.}$$

Let A be a Borel subset of E such that $\mu(A) = 1$ and the previous equality holds everywhere on A. Then, if $B := v^{-1}(A) \subseteq \Omega$, we have $\mathbb{P}(B) = \mathbb{P}(v^{-1}(A)) = \mu(A) = 1$ and

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}\mathbb{I}_O \circ T^n(v) = \mathbb{P}(v \in O) > 0$$

on B. Since E is a separable F-space, we can take a countable base of open subsets of E and get that almost surely, $\{n \geq 0 \mid T^n(v) \in O\}$ has positive lower density for every non-empty open subset O of E. The random vector v is therefore almost surely frequently hypercyclic for the operator T. □

Remark 1.1.5. If T admits an invariant and ergodic probability measure μ of full support then T is frequently hypercyclic on E. This result is well-known, see e.g. [8, Proposition 3.12] for complex Hilbert spaces; the proof given there also holds for F-spaces without any modification.

If $u_n = 0$ for every $n \leq -1$ in Proposition 1.1.4, Kolmogorov's zero-one law can be used to prove that μ is ergodic, as it is done in [74]. In fact, the same argument even shows that the measure induced by v is exact for T.

Proposition 1.1.6. *Let $T : E \longrightarrow E$ be an operator and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in E such that $T(u_n) = u_{n-1}$ for every $n \geq 1$ and $T(u_0) = 0$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables. Assume that the random vector*

$$v := \sum_{n=0}^{\infty} X_n u_n$$

is almost surely well-defined and $\mathbb{P}(v \in O) > 0$ for every non-empty open subset O of E. Then v is almost surely frequently hypercyclic for the operator T and induces an exact measure with full support for T.

Proof. Let $A \in \bigcap_{n\geq 0} T^{-n}(\mathscr{B}(E))$. We claim that $\mathbb{P}(v \in A) \in \{0, 1\}$.

Let $n \geq 0$, there exists $B \in \mathscr{B}(E)$ such that $A = T^{-n}(B)$. We then have, using Lemma 1.1.1,

$$\begin{aligned}\{v \in A\} = \{T^n(v) \in B\} &= \Big\{T^n\Big(\sum_{k=0}^{\infty} X_k u_k\Big) \in B\Big\} \\ &= \Big\{\sum_{k=n}^{\infty} X_k u_{k-n} \in B\Big\} = \Big\{\sum_{k=0}^{\infty} X_{n+k} u_k \in B\Big\} \in \sigma(X_n, X_{n+1}, \dots).\end{aligned}$$

We conclude by Kolmogorov's zero-one law, see [66, Chapitre 0, Proposition II.2]. □

By Propositions 1.1.4 and 1.1.6, in order to prove Theorem 1.1.8, it remains to show that the series $v = \sum_{n\in\mathbb{Z}} X_n u_n$ converges almost surely and the probability on E induced by v has full support. We first need a lemma.

Lemma 1.1.7 ([87], Theorem 15.5)**.** *Let $(x_n)_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n\geq 1} x_n$ converges and $x_n < 1$ for all $n \geq 1$. Then $\prod_{n\geq 1}(1 - x_n) > 0$.*

Proof. Let $n_0 \geq 1$ be large enough such that $x_n \leq 1/2$ for every $n \geq n_0$ and let $N > n_0$. Then

$$\begin{aligned}-\log\Big(\prod_{n=n_0}^{N} (1 - x_n)\Big) &= \sum_{n=n_0}^{N} \log(1/(1 - x_n)) = \sum_{n=n_0}^{N} \log(1 + x_n/(1 - x_n)) \\ &\leq \sum_{n=n_0}^{N} x_n/(1 - x_n) \leq \sum_{n=n_0}^{N} 2x_n.\end{aligned}$$

The result follows since the series $\sum_{n\geq 1} x_n$ converges. □

The proof of Theorem 1.1.8 uses some ideas from the proof of Mouze and Munnier [74, Theorem 2.3]. In particular, the idea of the condition on the distribution of the random variable X comes from that theorem.

Theorem 1.1.8. *Let $T : E \longrightarrow E$ be an operator and let $(u_n)_{n\in\mathbb{Z}}$ be a sequence in E such that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$ and* span$\{u_n \mid n \in \mathbb{Z}\}$ *is dense in E. Let X be a random variable of full support and let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. copies of X. Assume that there exists a sequence of positive numbers $(\delta_n)_{n\in\mathbb{Z}}$ such that*

$$\sum_{n\in\mathbb{Z}} \mathbb{P}\left(|X| \geq \delta_n\right) < \infty$$

and the series $\sum_{n\in\mathbb{Z}} \delta_n u_n$ is unconditionally convergent in E. Then the random vector

$$v := \sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well-defined and frequently hypercyclic for the operator T, and it induces a strongly mixing measure with full support for T. If $u_n = 0$ for all $n \leq -1$ then the measure is even exact for T.

Proof. Let $(\delta_n)_{n\geq 1}$ be given by the assumption. Because $\sum_{n\in\mathbb{Z}}\mathbb{P}\left(|X|\geq\delta_n\right)$ converges, it follows from the Borel-Cantelli lemma that

$$\mathbb{P}\left(\bigcup_{n_0\geq 1}\bigcap_{|n|\geq n_0}\left\{|X_n|<\delta_n\right\}\right)=1$$

and hence, almost surely, $|X_n|<\delta_n$ for every $|n|$ large enough. Therefore, by the unconditional convergence of $\sum_{n\in\mathbb{Z}}\delta_n u_n$, the random vector v is almost surely well-defined, see [57, Theorems 3.3.8 and 3.3.9].

By Propositions 1.1.4 and 1.1.6, it remains to show that $\mathbb{P}(v\in O)>0$ for every non-empty open subset O of E. It is enough to show this on a base of open subsets of E.

Let $\|\cdot\|$ be an F-norm defining the topology of E. Let $\eta>0$ and $y=\sum_{n=-d}^{d}y_n u_n\in E$. We will prove that $\mathbb{P}(v\in B_{\|\cdot\|}(y,\eta))>0$, where $B_{\|\cdot\|}(y,\eta)$ is the open ball for $\|\cdot\|$ centred at y and of radius η. Let $(\delta_n)_{n\in\mathbb{Z}}$ be the sequence given by assumption. Since $\sum_{n\in\mathbb{Z}}\delta_n u_n$ converges unconditionally, there exists an integer $N\geq d$ such that $\|\sum_{|n|\geq N+1}\alpha_n u_n\|<\eta/2$ whenever $|\alpha_n|\leq\delta_n$ for all $n\in\mathbb{Z}$. Define

$$B:=\left\{\left\|\sum_{n=-N}^{N}(X_n-y_n)u_n\right\|<\frac{\eta}{2}\right\}\subseteq\Omega$$

and

$$A:=B\cap\Big\{|X_n|<\delta_n \text{ for all } |n|\geq N+1\Big\},$$

where $y_n=0$ if $d+1\leq|n|\leq N$. By the triangle inequality we get on A

$$\|v-y\|\leq\left\|\sum_{n=-N}^{N}(X_n-y_n)u_n\right\|+\left\|\sum_{|n|\geq N+1}X_n u_n\right\|<\frac{\eta}{2}+\frac{\eta}{2}=\eta.$$

This shows that $A\subseteq\{v\in B_{\|\cdot\|}(y,\eta)\}$. Thus it suffices to prove that $\mathbb{P}(A)>0$. Since $(X_n)_{n\in\mathbb{Z}}$ is i.i.d., we have

$$\mathbb{P}(A)=\mathbb{P}(B)\prod_{|n|\geq N+1}\left(1-\mathbb{P}\left(|X|\geq\delta_n\right)\right).$$

Since X has full support and $(X_n)_{n\in\mathbb{Z}}$ is i.i.d., we get $\mathbb{P}(B)>0$. By Lemma 1.1.7, the product is positive since the series $\sum_{n\in\mathbb{Z}}\mathbb{P}(|X|\geq\delta_n)$ converges and X has full support. $\square$

1.2 Existence of a distribution

There still remains a question in Theorem 1.1.8: does there exist a random variable X satisfying the condition on the distribution? We begin with a simple proposition.

Proposition 1.2.1. *Let $(\delta_n)_{n\geq 0}$ be a sequence of positive numbers. Then there exist a probability space $(\Omega,\mathcal{A},\mathbb{P})$ and a random variable $X:\Omega\longrightarrow\mathbb{K}$ with full support and $\sum_{n\geq 0}\mathbb{P}(|X|\geq\delta_n)<\infty$ if and only if $\lim_{n\to\infty}\delta_n=\infty$.*

Proof. It is easy to prove that if such a variable X exists then $(\delta_n)_{n\in\mathbb{N}}$ must converge to ∞. Indeed, assume that $(\delta_{n_k})_{k\geq 1}$ is bounded by some $M > 0$ where $(n_k)_{k\geq 0}$ is increasing. Then $\sum_{k\geq 0}\mathbb{P}(|X| \geq \delta_{n_k}) \geq \sum_{k\geq 0}\mathbb{P}(|X| \geq M) = \infty$ since X has full support.

Now suppose that $\lim_{n\to\infty}\delta_n = \infty$. By considering $\inf_{k\geq n}\delta_k$, $n \geq 0$, we can assume without loss of generality that $(\delta_n)_{n\geq 0}$ is non-decreasing. By replacing δ_n with $\delta_n - 1/n$ and dropping some δ_n, if necessary, we may also assume that $(\delta_n)_{n\geq 0}$ is a (strictly) increasing sequence of positive numbers. Define $U_0 = B(0,\delta_0)$ and for each $k \geq 1$, $U_k := B(0,\delta_k) \setminus B(0,\delta_{k-1})$ and set $m_k := \lambda(U_k)$, $k \geq 0$, where $B(0,r)$ is the open ball in $\mathbb{K}$ of center 0 and radius r and λ is the Lebesgue measure on $\mathbb{K}$. Note that $(U_k)_{k\in\mathbb{N}}$ is a partition of $\mathbb{K}$. Define

$$\rho := 2^{-1}\sum_{k\geq 0}\frac{1}{2^k m_k}\mathbf{1}_{U_k}.$$

Since

$$\int_{\mathbb{K}}\rho \mathrm{d}\lambda = 2^{-1}\sum_{k\geq 0}\frac{1}{2^k} = 1,$$

ρ is a density on $\mathbb{K}$ and we consider the probability space $(\mathbb{K},\mathscr{B}(\mathbb{K}),\rho\mathrm{d}\lambda)$ and the random variable $X = \mathrm{Id}_{\mathbb{K}}$. It is then enough to show that $\int_O \rho\mathrm{d}\lambda > 0$ for every non-empty open set O of $\mathbb{K}$ and $\sum_{n\geq 0}\int_{\mathbb{K}\setminus B(0,\delta_n)}\rho\mathrm{d}\lambda < \infty$.

By using the definition of ρ, we get

$$\begin{aligned}\sum_{n\geq 0}\int_{\mathbb{K}\setminus B(0,\delta_n)}\rho\mathrm{d}\lambda &= \sum_{n\geq 0}\sum_{j\geq n+1}\int_{U_j}\rho\mathrm{d}\lambda = 2^{-1}\sum_{n\geq 0}\sum_{j\geq n+1}\frac{1}{2^j}\\ &= 2^{-1}\sum_{n\geq 0}\frac{1}{2^n} = 1.\end{aligned}$$

This shows that $\sum_{n\geq 0}\mathbb{P}(|X| \geq \delta_n)$ converges.

Let O be a non-empty open subset of $\mathbb{K}$. Let $u \in \mathbb{K}$ and $\varepsilon > 0$ be such that $B(u,\varepsilon) \subseteq O$. There is some $k \in \mathbb{N}$ such that $u \in U_k$. Then we have

$$\int_{B(u,\varepsilon)}\rho\mathrm{d}\lambda \geq \int_{B(u,\varepsilon)\cap U_k}\rho\mathrm{d}\lambda = \frac{2^{-1}}{2^k m_k}\lambda(B(u,\varepsilon)\cap U_k).$$

Since $\lambda(B(u,\varepsilon)\cap U_k) > 0$, we can conclude that $\int_{B(u,\varepsilon)}\rho\mathrm{d}\lambda > 0$ and hence $\int_O \rho\mathrm{d}\lambda > 0$. □

Lemma 1.2.2. *Let $(e_n)_{n\geq 0}$ be a sequence in E. For every sequence of scalars $(\varepsilon_n)_{n\geq 0}$ such that the series $\sum_{n\geq 0}\varepsilon_n e_n$ is unconditionally convergent, there exists a sequence of positive numbers $(\delta_n)_{n\geq 0}$ such that $\sum_{n\geq 0}\delta_n e_n$ is unconditionally convergent and $|\varepsilon_n| = o(\delta_n)$.*

Proof. Let $\|\cdot\|$ be an F-norm defining the topology of E. Since $\sum_{n\geq 0}\varepsilon_n e_n$ is unconditionally convergent and by using [57, Theorems 3.3.8 and 3.3.9], we can construct inductively an increasing sequence of positive integers $(N_k)_{k\geq 1}$ such that for every

$k \geq 1$, every sequence $(\alpha_n)_{n\geq 0}$ of scalars with $\sup_{n\geq 0} |\alpha_n| \leq 1$ and every finite set $F \subseteq \mathbb{N}$ with $\min F > N_k$, one has $\| \sum_{n\in F} \alpha_n \varepsilon_n e_n \| \leq 1/k^2$. For each $n > N_1$, there exists a unique $k \geq 1$ such that $N_k < n \leq N_{k+1}$, and we set $\delta_n = k^{1/2}|\varepsilon_n|$. We then have for any $1 \leq k < k'$ and every finite set $F \subseteq \mathbb{N}$ with $N_k < \min F \leq \max F \leq N_{k'}$,

$$\Big\| \sum_{n\in F} \delta_n e_n \Big\| = \Big\| \sum_{s=k}^{k'-1} \sum_{n=N_s+1,\, n\in F}^{N_{s+1}} \delta_n e_n \Big\| \leq \sum_{s=k}^{k'-1} (1+s^{1/2})s^{-2},$$

where we have used the property (0.1.1) of an F-norm. Since $\sum_{s\geq 1}(1+s^{1/2})s^{-2}$ is convergent, we conclude that the series $\sum_{n\geq 0} \delta_n e_n$ is unconditionally convergent too. In addition, we have that $|\varepsilon_n| = o(\delta_n)$ as n goes to ∞. □

We immediately deduce the main result of this section, which gives conditions for an operator to have a frequently hypercyclic random vector.

Theorem 1.2.3. *Let T be an operator on E and let $(u_n)_{n\in\mathbb{Z}}$ be a sequence in E. Assume that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$, the series $\sum_{n\in\mathbb{Z}} u_n$ is unconditionally convergent and* span$\{u_n \mid n \in \mathbb{Z}\}$ *is dense in E. Then there exists a random variable X with full support such that the random vector*

$$\sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well-defined and frequently hypercyclic for the operator T, and it induces a strongly mixing measure with full support for T, where $(X_n)_{n\in\mathbb{Z}}$ is a sequence of i.i.d. copies of X. If $u_n = 0$ for all $n \leq -1$ then the measure is even exact for T.

Proof. Let $(\delta_n)_{n\in\mathbb{Z}}$ be the sequence of positive numbers obtained by applying Lemma 1.2.2 to $\sum_{n\geq 0} u_n$ and $\sum_{n\leq -1} u_n$. Then $\lim_{n\to\infty} \delta_n = \infty$ and $\lim_{n\to-\infty} \delta_n = \infty$. The result follows by applying Proposition 1.2.1 to $(\min(\delta_n, \delta_{-n}))_{n\geq 0}$ in order to obtain the existence of a random variable X of full support such that $\sum_{n\in\mathbb{Z}} \mathbb{P}(|X| \geq \delta_n) < \infty$, and then by using Theorem 1.1.8. □

Remark 1.2.4. The random variable X in Theorem 1.2.3 can be assumed to be centred. Indeed, the random vector $\sum_{n=-\infty}^{\infty}(X_n - \mathbb{E}(X))u_n$ is still frequently hypercyclic for the operator T since $\sum_{n=-\infty}^{\infty} u_n$ is a fixed point of T.

Remark 1.2.5. In this chapter, we are mostly only interested in the existence of a random variable X as given in Theorem 1.2.3. For a more precise information on which random variable can be employed, one has to go back to Theorem 1.1.8.

In view of later applications in Chapters 3 and 4, we would like the random variable X to be subgaussian. We present two ways to achieve this.

Definition 1.2.6. A real random variable X is *subgaussian* if there exist some $\sigma > 0$ and $M > 0$ such that $\mathbb{E}(e^{\lambda X}) \leq Me^{\lambda^2\sigma^2}$ for every $\lambda \in \mathbb{R}$. A complex random variable X is *subgaussian* if its real and imaginary parts are subgaussian.

A sequence of random variables $(X_n)_{n\geq 0}$ is *subgaussian* if each X_n, $n \geq 0$, is subgaussian with the same constants σ and M.

One could call $(X_n)_{n\geq 0}$ a ***uniformly subgaussian*** sequence to stress the fact that the constant σ is the same for each random variable of the sequence. A Gaussian variable is of course subgaussian, see [66, Chapitre 8, Proposition I.1].

Being a subgaussian random variable X means that the expectation of $e^{\lambda X}$ is no greater, up to a factor, than the one if X was Gaussian. Another way to define a subgaussian variable is by bounding the tail of the probability distribution.

Lemma 1.2.7. *Let X be a random variable. Then X is subgaussian if and only if there exists $K > 0$ and $\tau > 0$ such that $\mathbb{P}(|X| > t) \leq Ke^{-t^2/\tau^2}$ for every $t \geq 0$.*

A sequence of random variables $(X_n)_{n\geq 0}$ is subgaussian if and only if each X_n, $n \geq 0$, satisfies this property with the same constants τ and K.

Proof. Without loss of generality, we can assume that X is a real variable. First, assume that X is subgaussian. Let $t \geq 0$ and $\lambda > 0$. Markov's inequality yields

$$\mathbb{P}(|X| > t) = \mathbb{P}(X > t \text{ or } X < -t) \leq \mathbb{P}(e^{\lambda X} > e^{\lambda t}) + \mathbb{P}(e^{-\lambda X} > e^{\lambda t}) \leq 2Me^{\lambda^2\sigma^2 - t\lambda},$$

where $M > 0$ and $\sigma > 0$ are the constants in Definition 1.2.6. Take $\lambda := t/(2\sigma^2)$ to conclude the first part of the proof.

Now, assume that for every $t \geq 0$, we have $\mathbb{P}(|X| > t) \leq Ke^{-t^2/\tau^2}$, and let $\lambda \in \mathbb{R}$. Without loss of generality, we can assume that $\lambda > 0$. Use the formula $\mathbb{E}(Y) = \int_0^\infty \mathbb{P}(Y > t)\mathrm{d}t$ (see [66, Chapitre 0, Proposition IV.2]) for a positive random variable Y to get

$$\begin{aligned}
\mathbb{E}(e^{\lambda X}) = \int_0^\infty \mathbb{P}(e^{\lambda X} > t)\mathrm{d}t &= \int_0^1 \mathbb{P}(e^{\lambda X} > t)\mathrm{d}t + \int_1^\infty \mathbb{P}(X > \log(t)/\lambda)\mathrm{d}t \\
&\leq 1 + K\int_1^\infty e^{-\log(t)^2/(\tau^2\lambda^2)}\mathrm{d}t.
\end{aligned}$$

By the change of variables $u = \log(t)$, we have

$$\begin{aligned}
\int_1^\infty e^{-\log(t)^2/(\tau^2\lambda^2)}\mathrm{d}t &= \int_0^\infty e^u e^{-u^2/(\tau^2\lambda^2)}\mathrm{d}u \\
&= \int_0^\infty \exp\left(-\left(\frac{u}{\tau\lambda} - \frac{\tau\lambda}{2}\right)^2 + \frac{\lambda^2\tau^2}{4}\right)\mathrm{d}u \\
&= e^{\frac{\lambda^2\tau^2}{4}}\int_0^\infty \exp\left(-\left(\frac{u}{\tau\lambda} - \frac{\tau\lambda}{2}\right)^2\right)\mathrm{d}u
\end{aligned}$$

A last change of variables $y = u/(\tau\lambda) - (\tau\lambda)/2$ finally yields

$$\mathbb{E}(e^{\lambda X}) \leq 1 + K\tau\lambda e^{\frac{\lambda^2\tau^2}{4}}\int_{-\frac{\tau\lambda}{2}}^\infty e^{-y^2}\mathrm{d}y,$$

hence, since the last integral is bounded above by $\int_{-\infty}^\infty e^{-y^2}\mathrm{d}y$, one can find some positive constants M and σ such that $\mathbb{E}(e^{\lambda X}) \leq Me^{\lambda^2\sigma^2}$ for all $\lambda \in \mathbb{R}$. $\square$

The definition of a subgaussian variable and a version of Lemma 1.2.7 can be found in [54, pp. 4-5]. In [54], a subgaussian variable is in fact subgaussian with constant $M = 1$ in our setting. We will need this restriction in Chapter 3, but that definition from [54] is in fact equivalent to be centred and subgaussian.

Lemma 1.2.8. *A real random variable X is subgaussian with constant $M = 1$ if and only if X is subgaussian and centred.*

Proof. Suppose that X is subgaussian with constants $M = 1$ and $\sigma > 0$. For all $\lambda \in \mathbb{R}$, we have $\mathbb{E}(e^{\lambda X}) \leq e^{\lambda^2\sigma^2}$. By using the Dominated Convergence Theorem, this is equivalent to

$$\sum_{n=0}^{\infty} \frac{\lambda^n \mathbb{E}(X^n)}{n!} \leq \sum_{n=0}^{\infty} \frac{\lambda^{2n}\sigma^{2n}}{n!},$$

hence $1 + \lambda\mathbb{E}(X) + o(\lambda) \leq 1 + o(\lambda)$. Now, if $\lambda > 0$, then $\mathbb{E}(X) \leq o(\lambda)/\lambda$, implying that $\mathbb{E}(X) \leq 0$ by letting λ converge to 0. The same argument with $\lambda < 0$ shows that $\mathbb{E}(X) \geq 0$, and $\mathbb{E}(X) = 0$.

Assume now that X is centred and subgaussian with some positive constants σ and M given by Definition 1.2.6. Pick $\tau > \sigma$ and define $K := [-\sqrt{\frac{\log(M)}{\tau^2-\sigma^2}}, \sqrt{\frac{\log(M)}{\tau^2-\sigma^2}}]$. For every $\lambda \notin K$, we have

$$\mathbb{E}(e^{\lambda X}) \leq M e^{\lambda^2\sigma^2} \leq e^{\lambda^2\tau^2}.$$

Now let $\nu \geq \tau$ that will be taken large. Let $\lambda \in K$, $\lambda \neq 0$. We aim to show that $\mathbb{E}(e^{\lambda X}) \leq e^{\lambda^2\nu^2}$, which by the Dominated Convergence Theorem is equivalent to

$$\sum_{n=0}^{\infty} \frac{\lambda^n \mathbb{E}(X^n)}{n!} \leq \sum_{n=0}^{\infty} \frac{\lambda^{2n}\nu^{2n}}{n!},$$

or

$$\sum_{n=0}^{\infty} \frac{\lambda^n \mathbb{E}(X^{n+2})}{(n+2)!} \leq \sum_{n=1}^{\infty} \frac{\lambda^{2(n-1)}\nu^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}\nu^{2(n+1)}}{(n+1)!}.$$

Thus we want to prove that $\mathbb{E}(X^2)/2 + f(\lambda) \leq \nu^2 + g(\lambda)$ for any $\lambda \in K$, where $f, g : K \longrightarrow \mathbb{R}$ are some continuous functions. This is equivalent to show that $\sup_{\lambda\in K}(f(\lambda) - g(\lambda)) \leq \nu^2 - \mathbb{E}(X^2)/2$. Since f and g are continuous and K is compact, this supremum is finite, and such a ν exists. □

The first approach to allow X to be subgaussian is by assuming the unconditional convergence of the series $\sum_{n\in\mathbb{Z}^*} \sqrt{\log(|n|)}u_n$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. This assumption guarantees the almost sure convergence of the random series $\sum_{n=-\infty}^{\infty} X_n u_n$, where $(u_n)_{n\in\mathbb{Z}}$ is a sequence in E. This fact will be used in Chapter 3, so we highlight it in the following lemma.

Lemma 1.2.9. *Let $(u_n)_{n\in\mathbb{Z}}$ be a sequence of vectors of E. Assume that the series $\sum_{n\in\mathbb{Z}^*} \sqrt{\log(|n|)}u_n$ is unconditionally convergent. Then, for every subgaussian sequence $(X_n)_{n\in\mathbb{Z}}$, the random vector*

$$\sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well-defined. In particular, the result holds for every non-constant Gaussian variable.

Proof. Let $c > 0$. Let $K > 0$ and $\tau > 0$ be the constants associated with $(X_n)_{n\in\mathbb{Z}}$ in Lemma 1.2.7. We then have

$$\sum_{n\in\mathbb{Z}^*} \mathbb{P}\left(|X_n| \geq c\sqrt{\log(|n|)}\right) \leq K \sum_{n\in\mathbb{Z}^*} e^{-c^2\log(|n|)/\tau^2} = \sum_{n\in\mathbb{Z}^*} \frac{K}{|n|^{c^2/\tau^2}}.$$

If $c^2 > \tau^2$ then the last series converges and $\sum_{n\in\mathbb{Z}^*} \mathbb{P}(|X| \geq c\sqrt{\log(|n|)})$ converges. It follows from the Borel-Cantelli lemma that

$$\mathbb{P}\bigg(\bigcup_{n_0\geq 1}\bigcap_{|n|\geq n_0}\left\{|X_n| < c\sqrt{\log(|n|)}\right\}\bigg) = 1$$

and hence, almost surely, $|X_n| < c\sqrt{\log(|n|)}$ for every $|n|$ large enough. Therefore, by the unconditional convergence of $\sum_{n\in\mathbb{Z}^*}\sqrt{\log(|n|)}u_n$, the series $\sum_{n\in\mathbb{Z}} X_n u_n$ is almost surely convergent. Furthermore, it is also measurable by Lemma 1.1.1. □

Theorem 1.2.10. *Let T be an operator on E and let $(u_n)_{n\in\mathbb{Z}}$ be a sequence in E. Assume that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$, $\operatorname{span}\{u_n \mid n \in \mathbb{Z}\}$ is dense in E and assume that the series $\sum_{n\in\mathbb{Z}^*}\sqrt{\log(|n|)}u_n$ is unconditionally convergent. Then for every subgaussian random variable X with full support, the random vector*

$$\sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well-defined and frequently hypercyclic for the operator T, and it induces a strongly mixing measure with full support for T, where $(X_n)_{n\in\mathbb{Z}}$ is a sequence of i.i.d. copies of X. If $u_n = 0$ for all $n \leq -1$, then the measure is even exact for T. In particular, the result holds for every non-constant Gaussian variable.

Proof. As in the proof of Lemma 1.2.9, we have that there is some $c > 0$ such that

$$\sum_{n\in\mathbb{Z}^*} \mathbb{P}\left(|X| \geq c\sqrt{\log(|n|)}\right) < \infty.$$

The result then follows by Theorem 1.1.8. □

The next result uses a different assumption on $(u_n)_{n\in\mathbb{Z}}$ than Theorem 1.2.10, in the case where E is a Banach space. Recall the definition of type.

Definition 1.2.11. Let E be a Banach space and $1 \leq p \leq 2$. Then E has *type p* if there exists $C > 0$ such that for every $x_1, \dots, x_n \in E$, $n \geq 1$,

$$\bigg\|\sum_{k=1}^{n} g_k x_k\bigg\|_{L^1(\Omega,\mathbb{P};E)} \leq C\bigg(\sum_{k=1}^{n}\|x_k\|^p\bigg)^{1/p},$$

where $(g_k)_{k=1}^n$ is a sequence of independent standard Gaussian variables.

The definition is usually expressed with a Rademacher sequence and sometimes in terms of the $L^2(\Omega,\mathbb{P};E)$-norm. But by [51, Proposition 7.1.18] and the Kahane-Khintchine inequalities [51, Theorem 6.2.6], this leads to the same definition.

Theorem 1.2.12. *Assume that E is a Banach space of type $1 \leq p \leq 2$. Let T be an operator on E and let $(u_n)_{n\in\mathbb{Z}}$ be a sequence in E. Assume that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$ and $\operatorname{span}\{u_n \mid n \in \mathbb{Z}\}$ is dense in E. Assume that the series $\sum_{n=-\infty}^{\infty} \|u_n\|^p$ converges. Let X be a standard Gaussian random variable of full support and let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. copies of X. Then the random vector*

$$v := \sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well-defined and frequently hypercyclic for the operator T, and it induces a strongly mixing measure with full support for T. If $u_n = 0$ for all $n \leq -1$, then the measure is even exact for T.

Proof. Since E has type p, we have for every $M \geq N$ that

$$\mathbb{E}\left(\left\| \sum_{n=N}^{M} X_n u_n \right\|\right) \leq C_p \left(\sum_{n=N}^{M} \|u_n\|^p \right)^{1/p}, \tag{1.2.1}$$

where $C_p > 0$ is some constant depending only on p. Therefore, the random series $\sum_{n=-\infty}^{\infty} X_n u_n$ converges in $L^1(\Omega; E)$, and since $(X_n)_{n\in\mathbb{Z}}$ is a standard Gaussian sequence of independent random variables, v is almost surely well-defined by Theorem 0.5.4.

By Proposition 1.1.4, it remains to show that $\mathbb{P}(v \in O) > 0$ for every non-empty open subset O of E. It is enough to show this on a base of open subsets of E.

So let $\eta > 0$ and $y = \sum_{n=-d}^{d} y_n u_n \in E$. We will prove that $\mathbb{P}(v \in B_{\|\cdot\|}(y, \eta)) > 0$ where $B_{\|\cdot\|}(y, \eta)$ is the open ball centred at y and of radius η. Let $N \geq d$ be an integer. Define

$$B := \left\{ \left\| \sum_{n=-N}^{N} (X_n - y_n) u_n \right\| < \frac{\eta}{2} \right\}, \; C := \left\{ \left\| \sum_{|n|\geq N+1} X_n u_n \right\| < \frac{\eta}{2} \right\},$$

where $y_n := 0$ if $d < |n| \leq N$, and let $A := B \cap C$. By the triangle inequality we get on A

$$\|v - y\| \leq \left\| \sum_{n=-N}^{N} (X_n - y_n) u_n \right\| + \left\| \sum_{|n|\geq N+1} X_n u_n \right\| < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

This shows that $A \subseteq \{v \in B_{\|\cdot\|}(y, \eta)\}$. Thus it suffices to prove that $\mathbb{P}(A) > 0$. Since $(X_n)_{n\in\mathbb{Z}}$ is i.i.d., we have by Lemma 1.1.3 that

$$\mathbb{P}(A) = \mathbb{P}(B)\mathbb{P}(C).$$

Since X has full support and $(X_n)_{n\in\mathbb{Z}}$ is i.i.d., we get $\mathbb{P}(B) > 0$. The last step is to show that $\mathbb{P}(C) > 0$. The Markov inequality yields

$$1 - \mathbb{P}(C) = \mathbb{P}\left(\left\| \sum_{|n|\geq N+1} X_n u_n \right\| \geq \eta/2\right) \leq (\eta/2)^{-1} \mathbb{E}\left(\left\| \sum_{|n|\geq N+1} X_n u_n \right\|\right).$$

It follows from (1.2.1) that if we take $N \geq d$ large enough then $1 - \mathbb{P}(C) < 1$, i.e. $\mathbb{P}(C) > 0$. □

Remark 1.2.13. We end this section by noticing that all of our results can be extended, with the same proofs, to an operator T such that there exist finitely many sequences of vectors $(u_n^{(k)})_{n\in\mathbb{Z}}$, $1 \leq k \leq N$, such that $\operatorname{span}\{u_n^{(k)} \mid n \in \mathbb{Z}, 1 \leq k \leq N\}$ is dense in E and $T(u_n^{(k)}) = u_{n-1}^{(k)}$ for all $n \in \mathbb{Z}$ and $1 \leq k \leq N$. In that case, the sequence of random variables is replaced by a family of i.i.d. random variables $(X_{n,k})_{n\in\mathbb{Z},1\leq k\leq N}$.

1.3 Applications

1.3.1 Weighted shifts

We list the applications of Theorem 1.2.3 and Theorem 1.2.10 to unilateral and bilateral weighted shifts.

If T is a (unilateral) weighted shift with sequence of weights $(w_n)_{n\geq 1}$, define $\beta_n := w_1 \dots w_n$ if $n \geq 1$, and $\beta_0 := 1$. If T is a bilateral weighted shift with sequence of weights $(w_n)_{n\in\mathbb{Z}}$, define $\beta_n := w_1 \dots w_n$ if $n \geq 1$, $\beta_n := (\prod_{k=-n+1}^{0} w_k)^{-1}$ if $n \leq -1$, and $\beta_0 := 1$.

In the first result, let E be a locally bounded or locally convex F-sequence space over $\mathbb{N}$ in which $\operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ is dense. We then apply the results of Section 1.2 to $u_n = \frac{e_n}{\beta_n}$ for $n \geq 0$ and $u_n = 0$ for $n \leq -1$.

Theorem 1.3.1. *Let $T : E \longrightarrow E$ be a weighted shift with sequence of weights $(w_n)_{n\geq 1}$.*

(i) *Assume that the series $\sum_{n\in\mathbb{N}} \frac{e_n}{\beta_n}$ is unconditionally convergent. Then there exists a random variable X with full support such that the random vector*

$$\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n$$

is almost surely well-defined and frequently hypercyclic for the operator T, and it induces an exact measure with full support for T, where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. copies of X.

(ii) *If the series $\sum_{n\geq 1} \frac{\sqrt{\log(n)}}{\beta_n} e_n$ is unconditionally convergent then X can be any subgaussian random variable with full support. In particular, the result holds for every non-constant Gaussian variable.*

This generalizes the qualitative parts of [74, Theorem 2.3] and [80, Theorem 1]; their quantitative parts are contained in Theorem 1.1.8.

We next consider a locally bounded or locally convex F-sequence space E over $\mathbb{Z}$ in which $\operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$ is dense. We then apply the results of Section 1.2 to $u_n = \frac{e_n}{\beta_n}$, $n \in \mathbb{Z}$.

Theorem 1.3.2. *Let $T : E \longrightarrow E$ be a bilateral weighted shift with sequence of weights $(w_n)_{n\in\mathbb{Z}}$.*

(i) *Assume that the series* $\sum_{n\in\mathbb{Z}} \frac{e_n}{\beta_n}$ *is unconditionally convergent. Then there exists a random variable* X *with full support such that the random vector*

$$\sum_{n=-\infty}^{\infty} \frac{X_n}{\beta_n} e_n$$

is almost surely well-defined and frequently hypercyclic for the operator T, *and it induces a strongly mixing measure with full support for* T, *where* $(X_n)_{n\in\mathbb{Z}}$ *is a sequence of i.i.d. copies of* X.

(ii) *If the series* $\sum_{n\in\mathbb{Z}^*} \frac{\sqrt{\log(|n|)}}{\beta_n} e_n$ *is unconditionally convergent then* X *can be any subgaussian random variable with full support. In particular, the result holds for every non-constant Gaussian variable.*

Theorems 1.3.1 and 1.3.2 apply, in particular, to any chaotic unilateral or bilateral weighted shift on a F-sequence space in which $(e_n)_n$ is an unconditional basis, see Theorems 0.1.19 and 0.1.23. The existence of an exact or strongly mixing measure with full support has already been proved in [77, Corollary 2 and Remark 3]. A different approach has also led to the existence of a strongly mixing measure in [67, Theorem 1] for a class of weighted shifts on $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $1 \leq p < \infty$.

It was already known that, in this setting, the unconditional convergence of $\sum_n \frac{e_n}{\beta_n}$ implies the frequent hypercyclicity of the weighted shift, see Propositions 0.1.26 and 0.1.27. By modifying the coefficients of this series, one can construct periodic points for the shift. This was used to prove that the convergence of $\sum_n \frac{e_n}{\beta_n}$ implies that the shift is chaotic in Theorems 0.1.19 and 0.1.23, see [45, Theorem 8]. By multiplying the coefficients of this series with random variables, we now get an almost surely frequently hypercyclic random vector that induces an exact or strongly mixing measure. This phenomenon was already known for chaotic weighted shifts on ℓ^p, $1 \leq p < \infty$, see [11, Section 7.1], for the so-called Taylor shift, see [75, Theorem 1.3], or for the differentiation operator on $H(\mathbb{C})$, see [80, Theorem 1]. This now holds for very general chaotic weighted shifts.

Remark 1.3.3. The bilateral weighted shift on $\ell^2(\mathbb{Z})$ with weights $w_n = 2$, $n \geq 1$, and $w_n = 1/2$, $n \leq 0$, is invertible and satisfies the assumptions of Theorem 1.3.2. On the other hand, no invertible measure preserving transformation can be exact, see [32, p. 86]. Thus the measure induced by the vector v in Theorem 1.1.8 cannot be exact for all operators T.

It might be an interesting fact that on the space $H(\mathbb{C})$ of entire functions or the space $H(D(0,R))$ of holomorphic functions on $D(0,R) := \{z \in \mathbb{C} \mid |z| < R\}$, every chaotic weighted shift satisfies the assumption of the second assertion of Theorem 1.3.1.

Theorem 1.3.4. *On the space* $E = H(\mathbb{C})$ *or* $H(D(0,R))$ *with* $R > 0$, *let* $T : E \longrightarrow E$ *be a chaotic weighted shift with sequence of weights* $(w_n)_{n\geq 1}$. *Then for every subgaussian random variable* X *with full support the random series*

$$\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} z^n$$

is almost surely holomorphic and frequently hypercyclic for the operator T, and it induces an exact measure with full support for T, where $(X_n)_{n\in\mathbb{Z}}$ is a sequence of i.i.d. copies of X. In particular, the result holds for every non-constant Gaussian variable.

Proof. By Theorem 1.3.1, it suffices to show that if T is chaotic on E then the series $\sum_{n\geq 1}\frac{\sqrt{\log(n)}}{\beta_n}e_n$ is unconditionally convergent in E.

On $H(\mathbb{C})$, T is chaotic if and only if $\lim_{n\to\infty}|\beta_n|^{1/n}=\infty$, see Example 0.1.21. Therefore, for any $r\geq 1$ and $0<\rho<1$, there exists $n_0\geq 1$ such that for every $n\geq n_0$, we have $r^n\sqrt{\log(n)}/|\beta_n|\leq\rho^n$.

On $H(D(0,R))$, T is chaotic if and only if $\limsup_{n\to\infty}|\beta_n|^{-1/n}\leq 1/R$, see Example 0.1.22. Let $0<r<R$ and $0<\rho<1$ be such that $r<\rho R$, there exists $n_0\geq 1$ such that for every $n\geq n_0$, we have $\log(n)^{1/(2n)}|\beta_n|^{-1/n}\leq\rho/r$ and hence $r^n\sqrt{\log(n)}/|\beta_n|\leq\rho^n$. □

For the differentiation operator on $H(\mathbb{C})$, this result was proved in the Gaussian case in [11, Remark 2 after Proposition 8.1]. For the Taylor shift on $H(\mathbb{D})=H(D(0,1))$, which is given by the weights $w_n=1$, $n\geq 1$, the frequent hypercyclicity of the random function was proved in the Gaussian case in [75, Theorem 1.3].

One can ask the same question about the spaces ℓ^p, $1\leq p<\infty$. In fact, it is already known that $\sum_{n=0}^{\infty}\frac{X_n}{\beta_n}e_n$ is almost surely well-defined and frequently hypercyclic on those spaces if the random variables X_n, $n\geq 0$, are Gaussian and the weighted shift is chaotic, see [10, Section 5.5.2] or [11, Section 7.1]. However, the second assertion of Theorem 1.3.1 cannot be applied to every chaotic weighted shift defined on ℓ^p, $1\leq p<\infty$. Indeed, consider the sequence $(\beta_n)_{n\geq 1}:=(\log(n)^{1/2+1/p}n^{1/p})_{n\geq 1}$. Then $\sum_{n\geq 0}\sqrt{\log(n)}/\beta_n e_n$ is not in ℓ^p but the weighted shift associated with $(\beta_n)_{n\geq 1}$ is chaotic. Note that Theorem 1.2.12 can be applied to any chaotic weighted shift on ℓ^p, for every $1\leq p\leq 2$, since ℓ^p has type $\min(p,2)$ by [51, Proposition 7.1.4], $1\leq p<\infty$.

However, on the space of all sequences $\omega:=\mathbb{K}^{\mathbb{N}}$, every shift T is chaotic since $\sum_{n\geq 0}e_n/\beta_n$ converges, where $(\beta_n)_{n\geq 0}$ is the sequence associated with T. If X is a random variable then $\lim_{t\to\infty}\mathbb{P}(|X|>t)=0$, and there exists a sequence $(\delta_n)_{n\geq 0}$ of positive numbers such that $\sum_{n\geq 0}\mathbb{P}(|X|>\delta_n)$ is finite. Furthermore, $\sum_{n\geq 0}\delta_n e_n/\beta_n$ is unconditionally convergent; recall that ω is endowed with the coordinatewise convergence, see Example 0.1.4. Therefore, Theorem 1.1.8 implies the next result.

Theorem 1.3.5. *Let $T:\omega\longrightarrow\omega$ be a weighted shift with sequence of weights $(w_n)_{n\geq 1}$. Let X be a random variable with full support. Then the random vector $\sum_{n=0}^{\infty}\frac{X_n}{\beta_n}e_n$ is almost surely well-defined and frequently hypercyclic for the operator T, and it induces an exact measure with full support for T, where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. copies of X.*

In their article [74], Mouze and Munnier have also studied some polynomials of a frequently hypercyclic weighted shift on ℓ^p, $1\leq p<\infty$. But their Lemma 4.1 says that certain polynomials of a weighted shift can be seen as a shift with respect to another basis. The proof of this lemma shows the following.

Lemma 1.3.6 ([74, Lemma 4.1])**.** *Let $T:\mathbb{K}^{\mathbb{N}}\longrightarrow\mathbb{K}^{\mathbb{N}}$ be a weighted shift. Let $P(z)=\sum_{k=1}^{d}a_kz^k$ be a polynomial with $a_1\neq 0$. Then there exist vectors $u_n=\sum_{j=0}^{n}\beta_{j,n}e_j$,*

$n \geq 0$, such that $(u_n)_{n\geq 0}$ is an algebraic basis of $\mathbb{K}^{\mathbb{N}}$ and $P(T)(u_n) = u_{n-1}$ for every $n \geq 1$.

In fact, this result implies that $\operatorname{span}\{e_n \mid n \in \mathbb{N}\} = \operatorname{span}\{u_n \mid n \in \mathbb{N}\}$. Note also that $P(T)(u_0) = 0$. Therefore, together with Theorem 1.2.3, we deduce the following result.

Theorem 1.3.7. *Let E be a locally bounded or locally convex F-sequence space in which $\operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ is dense. Let $T : E \longrightarrow E$ be a weighted shift and $P(z) = \sum_{k=1}^{d} a_k z^k$ be a polynomial with $a_1 \neq 0$. Assume that the series $\sum_{n\geq 0} u_n$ is unconditionally convergent in E, where $(u_n)_{n\geq 0}$ is given by Lemma 1.3.6. Then there exists a random variable X with full support such that the random vector*

$$v := \sum_{n=0}^{\infty} X_n u_n$$

is almost surely well-defined and frequently hypercyclic for the operator $P(T)$, and it induces an exact measure with full support for $P(T)$, where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. copies of X.

This result improves and generalizes the qualitative part of [74, Theorem 4.3]; its quantitative part is contained in Theorem 1.1.8.

1.3.2 Operators satisfying the Frequent Hypercyclicity Criterion

In [77, Theorem 1], Murillo-Arcila and Peris proved that every operator satisfying the Frequent Hypercyclicity Criterion, see Theorem 0.1.31, has a strongly mixing invariant measure with full support. They used the Bernoulli shift on a subset of $\mathbb{N}^{\mathbb{Z}}$ to construct such a measure. In [11, Proposition 8.1], it is even shown that such operators admit a strongly mixing Gaussian measure. We will show here the existence of a strongly mixing measure of full support as the distribution of some random vector $\sum_{n\in\mathbb{Z}} X_n u_n$. We will need Lemma 1.3.9. Its proof is contained in the proof of [44, Lemma 3.2]. In this subsection, E will be again a locally bounded or locally convex separable F-space.

For this subsection, we need a weaker notion than hypercyclicity.

Definition 1.3.8. Let T be an operator on E. A vector $x \in E$ is called *supercyclic* for T if the set

$$\{\lambda T^n(x) \mid n \in \mathbb{N}, \lambda \in \mathbb{K}\}$$

is dense in E.

Lemma 1.3.9. *Let T be an operator on E satisfying the Frequent Hypercyclicity Criterion, and let S and E_0 be respectively the map and dense set given by that criterion. Let $(a_k)_{k\geq 1}$ be a sequence of non-zero scalars. If $(x_k)_{k\geq 1}$ is a dense sequence in E_0 then there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that the vector $x := \sum_{k\geq 1} a_k S^{n_k}(x_k)$ is well-defined and supercyclic for T.*

Proof. By conditions (i) and (ii) of the Frequent Hypercyclicity Criterion, we know that $(T^n(x))_{n\geq 0}$ and $(S^n(x))_{n\geq 0}$ converge to 0 for every $x \in E_0$. Therefore, together with (iii), one can construct by induction an increasing sequence of positive integers $(n_k)_{k\geq 1}$ such that $\|a_k S^{n_k}(x_k)\| \leq \frac{1}{2^k}$ for every $k \geq 1$, and

$$\left\|\frac{1}{a_l}T^{n_l}\Big(\sum_{j=1}^{k} a_j S^{n_j}(x_j)\Big) - x_l\right\| < \frac{1}{2^l} \text{ for every } 1 \leq l \leq k,$$

where $\|\cdot\|$ is an F-norm defining the topology of E. The first condition tells us that $(\sum_{k=1}^{s} a_k S^{n_k}(x_k))_{s\geq 1}$ is Cauchy in E, hence converges. The second condition tells us that the vector $x := \sum_{k\geq 1} a_k S^{n_k}(x_k)$ is supercyclic for T. □

Theorem 1.3.10. *Let T be an operator on E satisfying the Frequent Hypercyclicity Criterion. Then there exists a supercyclic vector x for T, a sequence $(u_n)_{n\geq 0}$ in E with $u_0 = x$ and $T(u_n) = u_{n-1}$ for every $n \geq 1$, and a random variable X with full support such that the random vector*

$$v := \sum_{n=0}^{\infty} X_n T^n(x) + \sum_{n=1}^{\infty} X_{-n} u_n$$

is almost surely well-defined and frequently hypercyclic for the operator T, and it induces a strongly mixing measure with full support for T, where $(X_n)_{n\in\mathbb{Z}}$ is a sequence of i.i.d. copies of X.

Proof. Let S be the map and E_0 be the dense set given by the Frequent Hypercyclicity Criterion and let $\|\cdot\|$ be an F-norm defining the topology of E. Let $(x_k)_{k\geq 1}$ be a dense sequence in E_0.

For each $k \geq 1$, choose a real number $0 < a_k < 1$ such that

$$\sup_{F\subseteq\mathbb{N},\ F \text{ finite}} \left\|\sum_{n\in F} a_k S^n(x_k)\right\| \leq \frac{1}{2^k} \tag{1.3.1}$$

and

$$\sup_{F\subseteq\mathbb{N},\ F \text{ finite}} \left\|\sum_{n\in F} a_k T^n(x_k)\right\| \leq \frac{1}{2^k}. \tag{1.3.2}$$

This is possible by (i) and (ii) of the Frequent Hypercyclicity Criterion. Indeed, by unconditional convergence and [57, Theorems 3.3.8 and 3.3.9], there exists $N \geq 1$ such that $\|\sum_{n\in F} a_k S^n(x_k)\| \leq 2^{-k-1}$ whenever $\min F \geq N$ and $|a_k| \leq 1$, and by continuity one can choose $a_k > 0$ small enough to get $\|\sum_{n\in F} a_k S^n(x_k)\| \leq 2^{-k-1}$ whenever $\max F \leq N$. The same arguments hold for the second inequality.

Now let $(n_k)_{k\geq 1}$ be the sequence given by Lemma 1.3.9 and define the vector $x := \sum_{k\geq 1} a_k S^{n_k}(x_k)$. If $n \geq 0$, by the triangle inequality and (1.3.1), we have for every $M \geq N \geq 1$ that

$$\left\|\sum_{k=N}^{M} a_k S^{n_k+n}(x_k)\right\| \leq \sum_{k=N}^{M} \|a_k S^{n_k+n}(x_k)\| \leq \sum_{k=N}^{M} \frac{1}{2^k}$$

and hence

$$u_n := \sum_{k\geq 1} a_k S^{n_k+n} x_k,$$

$n \geq 0$, is well-defined, where $u_0 = x$. We also set $u_n = T^{-n}(x)$, $n \leq -1$. It is then easy to check that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$. We will apply Theorem 1.2.3 to $(u_n)_{n\in\mathbb{Z}}$. For the statement of the theorem, we then replace $(X_n)_{n\in\mathbb{Z}}$ by $(X_{-n})_{n\in\mathbb{Z}}$. Note that $\operatorname{span}\{u_n \mid n \in \mathbb{Z}\}$ is dense in E since x is supercyclic for T.

Thus it remains to show that $\sum_{n\in\mathbb{Z}} u_n$ is unconditionally convergent. Let $\varepsilon > 0$ and let $k_0 \geq 1$ be such that $\sum_{k\geq k_0+1} 2^{1-k} \leq \varepsilon$. For each $k \geq 1$, by (i) and (ii) of the Frequent Hypercyclicity Criterion, there exists $N_k \geq 1$ such that

$$\left\| \sum_{n\in F} a_k T^n(x_k) \right\| < \frac{\varepsilon}{k_0} \quad \text{and} \quad \left\| \sum_{n\in F} a_k S^n(x_k) \right\| < \frac{\varepsilon}{k_0}$$

for every finite set $F \subseteq \mathbb{N}$ with $\min F \geq N_k$. Let $F \subseteq \mathbb{N}$ be a finite subset with $\min F \geq \max_{1\leq k\leq k_0}(N_k + n_k)$. We have

$$\begin{aligned}\sum_{n\in F} u_{-n} &= \sum_{n\in F}\sum_{k\geq 1} a_k T^n S^{n_k}(x_k) = \sum_{k\geq 1}\sum_{n\in F} a_k T^n S^{n_k}(x_k) \\ &= \sum_{k=1}^{k_0}\sum_{n\in F} a_k T^n S^{n_k}(x_k) + \sum_{k\geq k_0+1}\sum_{n\in F} a_k T^n S^{n_k}(x_k).\end{aligned}$$

The first term is smaller than ε with respect to $\|\cdot\|$ since $\min F \geq N_k + n_k$ for each $1 \leq k \leq k_0$. The triangle inequality, condition (iii) of the Frequent Hypercyclicity Criterion and inequalities (1.3.1) and (1.3.2) yield

$$\begin{aligned}\left\| \sum_{k\geq k_0+1}\sum_{n\in F} a_k T^n S^{n_k}(x_k) \right\| &\leq \sum_{k\geq k_0+1} \left\| \sum_{n\in F} a_k T^n S^{n_k}(x_k) \right\| \\ &\leq \sum_{k\geq k_0+1} \left(\left\| \sum_{n\in F, n<n_k} a_k S^{n_k-n}(x_k) \right\| + \left\| \sum_{n\in F, n\geq n_k} a_k T^{n-n_k}(x_k) \right\| \right) \\ &\leq \sum_{k\geq k_0+1} \frac{2}{2^k}.\end{aligned}$$

By definition of k_0, we finally get $\|\sum_{n\in F} u_{-n}\| \leq 2\varepsilon$. This shows the unconditional convergence of $\sum_{n\leq 0} u_n$.

Again by the triangle inequality and (1.3.1), we also have

$$\begin{aligned}\left\| \sum_{n\in F} u_n \right\| &= \left\| \sum_{n\in F}\sum_{k\geq 1} a_k S^{n_k+n}(x_k) \right\| \\ &\leq \sum_{k=1}^{k_0} \left\| \sum_{n\in F} a_k S^{n_k+n}(x_k) \right\| + \sum_{k\geq k_0+1} \left\| \sum_{n\in F} a_k S^{n_k+n}(x_k) \right\| \\ &\leq \sum_{k=1}^{k_0} \left\| \sum_{n\in F} a_k S^{n_k+n}(x_k) \right\| + \sum_{k\geq k_0+1} \frac{1}{2^k}.\end{aligned}$$

As before, the first term is smaller than ε since $\min F \geq N_k$ for each $1 \leq k \leq k_0$, and the second term is smaller than ε by definition of k_0. This shows the unconditional convergence of $\sum_{n\geq 0} u_n$. □

Example 1.3.11. Bonilla and Grosse-Erdmann [22, Theorem 4.2] used the Frequent Hypercyclicity Criterion to prove the frequent hypercyclicity of the translation operators $T_a : H(\mathbb{C}) \longrightarrow H(\mathbb{C}), f \longrightarrow f(\cdot + a)$, $a \in \mathbb{C} \setminus \{0\}$. Therefore, Theorem 1.3.10 says that each operator T_a, $a \in \mathbb{C} \setminus \{0\}$, has an almost surely frequently hypercyclic random vector.

1.4 A-frequent hypercyclicity

A frequently hypercyclic vector for a given operator T means that this vector visits via T each non-empty open set of the space plenty of times. The number of visits is quantified by the lower density. Quite recently in the literature, other ways of quantifying how often the orbit of a vector visits every region of the space have been studied, see [38] and [39].

The notions of regular and strongly regular matrices can be found in [23].

Definition 1.4.1. Let $A = (a_{i,j})_{i,j\geq 1}$ be an infinite matrix of complex numbers. Then A is

(i) *regular* if $\sup_{n\geq 1} \sum_{j\geq 1} |a_{n,j}| < \infty$, $\lim_{n\to\infty} a_{n,j} = 0$ for every $j \geq 1$, and $\lim_{n\to\infty} \sum_{j\geq 1} a_{n,j} = 1$,

(ii) *strongly regular* if A is regular and $\lim_{n\to\infty} \sum_{j\geq 1} |a_{n,j} - a_{n,j+1}| = 0$,

(iii) *stochastic* if for all $n, j \geq 1$, one has $a_{n,j} \geq 0$ and $\sum_{j\geq 1} a_{n,j} = 1$.

Definition 1.4.2. Let A be a regular matrix with non-negative real entries. The *lower A-density* of a set $F \subseteq \mathbb{N}$ is the quantity

$$\underline{d}_A(F) = \liminf_{n\to\infty} \sum_{j=1}^{\infty} a_{n,j} \mathbf{1}_F(j).$$

Definition 1.4.3. Let A be a regular matrix with non-negative real entries. Let E be an F-space. An operator $T : E \longrightarrow E$ is *A-frequently hypercyclic* if there exists $x \in E$ such that, for every non-empty open set U of E, the set $\{n \in \mathbb{N} \mid T^n(x) \in U\}$ has positive lower A-density. Such a vector is called a *A-frequently hypercyclic vector* for T.

Example 1.4.4. When the matrix A is given by $a_{n,j} = 1/n$ for $1 \leq j \leq n$ and $a_{n,j} = 0$ for $j > n$, A-frequent hypercyclicity is the classic frequent hypercyclicity. When $a_{n,j} = 1/(j \log(n+1))$ for $1 \leq j \leq n$ and $a_{n,j} = 0$ for $j > n$, A-frequent hypercyclicity is the so-called log-frequent hypercyclicity.

The random vector constructed in Theorem 1.1.8 is frequently hypercyclic. In order to prove that this vector is also A-frequently hypercyclic for certain regular matrices A, the following generalized Birkhoff ergodic theorem will be needed.

Theorem 1.4.5 ([53, Theorem 8]). *Let $(M, \mathcal{B}, \mu)$ be a probability space and $A = (a_{i,j})_{i,j\geq 1}$ be a stochastic strongly regular matrix. Let $T : M \longrightarrow M$ be a measure-preserving and ergodic map, and let $f \in L^1(M, \mu)$. Then*

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} a_{n,j}(f \circ T^{j-1}) = \int_M f \mathrm{d}\mu \ \mu\text{-a.s.}$$

Theorem 1.4.6. *Let E be a locally bounded or locally convex F-space. Let $T : E \longrightarrow E$ be an operator and let $(u_n)_{n\in\mathbb{Z}}$ be a sequence in E such that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$ and $\operatorname{span}\{u_n \mid n \in \mathbb{Z}\}$ is dense in E. Let X be a random variable of full support and let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. copies of X. Assume that there exists a sequence of positive numbers $(\delta_n)_{n\in\mathbb{Z}}$ such that*

$$\sum_{n\in\mathbb{Z}} \mathbb{P}\left(|X| \geq \delta_n\right) < \infty$$

and the series $\sum_{n\in\mathbb{Z}} \delta_n u_n$ is unconditionally convergent in E. Then the random vector

$$v := \sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well-defined. Furthermore, let A be a stochastic strongly regular matrix. Then v is almost surely A-frequently hypercyclic for T.

Proof. The proof is exactly the same as the proof of Theorem 1.1.8 by using Theorem 1.4.5 instead of the Birkhoff Ergodic Theorem. □

Theorem 1.4.5 allows us to recover the result of Ernst and Mouze [39, Theorem 3.9] in the case of iterates of a single operator. They considered the matrix $\widetilde{D}_s = (a_{n,j})_{n,j\geq 1}$ given by $a_{n,j} := \alpha_j / \sum_{k=k_0}^{n} \alpha_k$ for $1 \leq j \leq n$ and $a_{n,j} = 0$ for $j > n$, where $\alpha_k := e^{k/\log_s(k)}$ if $k \geq k_0$ and $\alpha_k = 0$ otherwise. Here, $s \geq 1$ is a positive integer, $\log_s$ denotes the logarithm iterates s times and k_0 is a sufficiently large positive integer such that the sequence $(k/\log_s(k))_{k\geq k_0}$ is increasing. Let us show that the matrix $\widetilde{D}_s$ is strongly regular and stochastic.

Lemma 1.4.7. *The matrix $\widetilde{D}_s$ defined above is strongly regular and stochastic.*

Proof. It is trivially checked that $\widetilde{D}_s$ is stochastic and regular. Let us show that $\widetilde{D}_s$ is strongly regular.

Let $n \geq 1$ be large. Define $S_n := \sum_{k=k_0}^{n} \alpha_k$. By definition of $\widetilde{D}_s$, we have

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} |a_{n,j} - a_{n,j+1}| = \sum_{k=k_0}^{n-1} \frac{\alpha_{k+1} - \alpha_k}{S_n} + \frac{\alpha_n}{S_n} = \frac{2\alpha_n}{S_n} - \frac{\alpha_{k_0}}{S_n}.$$

We will show that $S_n \asymp e^{n/\log_s(n)} \log_s(n)$, where $a \asymp b$ means $a \leq b$ and $b \leq a$ up to some constants independent of $n \in \mathbb{N}$ or $x > 0$, and this will imply that $\lim_{n\to\infty} \alpha_n / S_n = 0$, thus will conclude the proof.

Define the function

$$f : [k_0, \infty[\longrightarrow \mathbb{R}, x \longmapsto e^{x/\log_s(x)} \log_s(x).$$

For every $x \geq k_0$, the derivative of f is given by

$$\begin{aligned}\partial_x f(x) &= e^{x/\log_s(x)} \log_s(x)\Big(\frac{1}{\log_s(x)} - \frac{x}{\log_s(x)^2}\frac{1}{x\prod_{j=1}^{s-1}\log_j(x)}\Big) + \frac{e^{x/\log_s(x)}}{x\prod_{j=1}^{s-1}\log_j(x)} \\ &= e^{x/\log_s(x)}\left(1 + \frac{1}{\prod_{j=1}^{s-1}\log_j(x)}\Big(\frac{1}{x} - \frac{1}{\log_s(x)}\Big)\right). \qquad (1.4.1)\end{aligned}$$

From this, it follows that

$$\partial_x f(x) \asymp e^{x/\log_s(x)}. \qquad (1.4.2)$$

The map $x \longmapsto \frac{1}{x} - \frac{1}{\log_s(x)}$ is increasing for $x > 0$ large. Then the second factor in the right-hand term of (1.4.1) is positive and increasing, and since the first factor is also positive and increasing for x large, we deduce that the derivative of f is increasing for x large. Therefore, for all $n \geq 1$ large enough, we have

$$\begin{aligned}f(n) &= \sum_{k=k_0}^{n-1}(f(k+1) - f(k)) = \sum_{k=k_0}^{n-1}\int_k^{k+1}\partial_x f(x)\mathrm{d}x \\ &\lesssim \sum_{k=k_0}^{n-1}\partial_x f(k+1) \asymp \sum_{k=k_0}^{n}\partial_x f(k), \qquad (1.4.3)\end{aligned}$$

and similarly

$$f(n) \gtrsim \sum_{k=k_0}^{n}\partial_x f(k) - \partial_x f(n). \qquad (1.4.4)$$

By using (1.4.3) and (1.4.2), we get

$$\frac{\partial_x f(n)}{\sum_{k=k_0}^{n}\partial_x f(k)} \lesssim \frac{\partial_x f(n)}{f(n)} \asymp \frac{e^{n/\log_s(n)}}{e^{n/\log_s(n)}\log_s(n)},$$

which converges to 0 when n goes to ∞. Then, by (1.4.3), (1.4.4) and (1.4.2), we finally get that $f(n) \asymp S_n$. □

Corollary 1.4.8. *Let E be a locally bounded or locally convex F-space. Let $T : E \longrightarrow E$ be an operator satisfying the Frequent Hypercyclicity Criterion. Then T has a random vector that is almost surely $\widetilde{D}_s$-frequently hypercyclic for T for any $s \geq 1$.*

Remark 1.4.9. It was already pointed out in [39, Remark 3.10(2)] that [39, Theorem 3.9] can also be proved thanks to ergodic theory arguments via Theorem 1.4.5. Theorem 1.4.6 originates from that remark.

Chapter 2

Random vectors and C_0-semigroups

We investigate in this chapter the case of C_0-semigroups. Two approaches to finding a frequently hypercyclic vector are presented.

The idea of the first approach is to try to mimic what has been done in Section 1.1. However, only a result analogous to Proposition 1.1.4 for semigroups will be proved, namely Proposition 2.2.2. The sequence of random variables $(X_n)_{n\in\mathbb{Z}}$ in Proposition 1.1.4 is replaced by the Brownian motion. This means that for semigroups, we are *a priori* quite restrictive since only the normal distribution is considered. For a C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Fréchet space E, we will assume that there exists a family $(u_t)_{t\in\mathbb{R}}$ of vectors of E such that $T_s(u_t) = u_{t-s}$ for every $s \geq 0$ and $t \in \mathbb{R}$. Instead of a series as in Proposition 1.1.4, we will integrate this family of vectors with respect to the Brownian motion. It is thus a stochastic integral for functions with values in a Fréchet space. It should be noted that we have not been able to find any examples of this method.

The second approach is inspired by the results of Chakir and El Mourchid [26]. Again, we will construct a frequently hypercyclic random vector via a stochastic integral, see Theorem 2.3.1. The family $(u_t)_t$ will consist of eigenvectors of the generator of the semigroup with respect to purely imaginary eigenvalues, hence only complex Fréchet spaces will be considered. This time, we will have three examples of this approach.

The stochastic integral for scalar-valued functions is the well-known Itô integral, see Section 0.4. The case of Fréchet space-valued integrands is more delicate and seems to be quite recent in the literature. We will use the definition of van Neerven and Weis [96]. Their stochastic integral is defined in a Pettis manner for Banach space-valued functions defined on a bounded interval. Nevertheless, it can be extended for Fréchet space-valued functions defined on an arbitrary interval, and their results still hold with the same proofs.

Recall that by Remark 0.1.44, a C_0-semigroup $(T_t)_{t\geq 0}$ has the same set of frequently hypercyclic vectors as each T_t, $t > 0$. Therefore, we could apply Theorem 1.2.3 to T_1, but this would not guarantee that the resulting measure is strongly mixing for the semigroup.

This chapter is divided into three parts. Section 2.1 defines and states some properties of the stochastic integral for Fréchet space-valued functions. Some proofs that differ from [96] are postponed to Section A.3 of the appendix. Sections 2.2 and 2.3 present the first and second approaches, respectively.

For the remainder of the chapter, let E be a separable Fréchet space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $(B_t)_{t\in\mathbb{R}}$ be a Brownian motion over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. This stochastic process is real (resp. complex) if E is real (resp. complex).

2.1 Stochastic integral in Fréchet spaces

This section is devoted to the definition and some properties of the stochastic integral in vector spaces developed by van Neerven and Weis [96]. They defined the integral for functions on a bounded interval and with values in a real Banach space. Nevertheless, it can be defined for functions on an arbitrary interval and with values in a real or complex Fréchet space, and the relevant properties still hold with essentially the same proofs. For measurable functions taking values in a separable Banach space, the definition of the stochastic integral from [96] coincides with that of Rosiński and Suchanecki [86], see Theorem 2.1.6.

Proofs of the results for which modifications in the Fréchet case might not be obvious are given in Section A.3. In the sequel, we will only prove some results that are not in [96].

For the remainder of this section, let $I \subseteq \mathbb{R}$ be an interval, which can be unbounded. See Section 0.4 for a reminder of the Itô integral.

Definition 2.1.1. Let $\phi : I \longrightarrow E$ be a weakly L^2 function. Then ϕ is called *stochastically integrable* if for all measurable sets $A \in \mathscr{B}(I)$, there exists a random vector $Y_A : \Omega \longrightarrow E$ such that for all $x^* \in E^*$, one has

$$x^*(Y_A) = \int_I \mathbf{1}_A(t) x^*\big(\phi(t)\big) \mathrm{d}B_t$$

almost surely. In that case, we write $Y_A = \int_A \phi(t) \mathrm{d}B_t$.

The random vectors Y_A are Gaussian since $x^*(Y_A)$ is a Gaussian random variable for any $x^* \in E^*$. Furthermore, they are uniquely determined almost surely. Indeed, assume that Y is a random vector such that $x^*(Y) = 0$ almost surely, for any $x^* \in E^*$. Since E is separable, the topological space $(E^*, \sigma(E^*, E))$ is also separable by [89, Subsection IV.1.7]. Let $(x_n^*)_{n\in\mathbb{N}} \subseteq E^*$ be a dense sequence in $(E^*, \sigma(E^*, E))$, which thus separates points of E. Then almost surely, $x_n^*(Y) = 0$ for all $n \in \mathbb{N}$, which is equivalent to $Y = 0$ almost surely.

By the Fernique theorem, see Theorem 0.5.3, we have $Y_A \in L^p(\Omega; E)$ for every $1 \le p < \infty$. See Section A.1 for the definition of the spaces $L^p(\Omega; E)$, $1 \le p < \infty$.

Here are two easy consequences of Definition 2.1.1.

Lemma 2.1.2 ([96, pp. 138 and 139])**.** *Let $\phi, \psi : I \longrightarrow E$ be two stochastically integrable functions, let $a, b \in \mathbb{K}$ and $T : E \longrightarrow F$ be a continuous linear map,* *where F is another separable Fréchet space. Then the functions $a\phi + b\psi$ and $T\phi$ are stochastically integrable, and we have*

$$\int_I \big(a\phi(t) + b\psi(t)\big)\mathrm{d}B_t = a\int_I \phi(t)\mathrm{d}B_t + b\int_I \psi(t)\mathrm{d}B_t$$

and

$$\int_I T\phi(t)\mathrm{d}B_t = T\int_I \phi(t)\mathrm{d}B_t.$$

In the case of scalar-valued functions, any square-integrable function is stochastically integrable. In a Fréchet space, this is no longer true: for every $1 \leq p < 2$, there exists a bounded measurable function defined on $[0,1]$ with values in ℓ^p that is not stochastically integrable, see [86, Example 3.1].

Definition 2.1.3. Let $\phi : I \longrightarrow E$ be a weakly L^2 function. We define the conjugate-linear map $I_\phi : L^2(I) \longrightarrow (E^*)'$ by

$$I_\phi(f) : E^* \longrightarrow \mathbb{K}, x^* \longmapsto \int_I x^*(\phi(t))\overline{f(t)}\mathrm{d}t$$

for every $f \in L^2(I)$.

Here, $(E^*)'$ denotes the algebraic dual of E^*.

We will say that I_ϕ is γ-*radonifying* if I_ϕ takes values in E and the linear map $I_\phi S$ is γ-radonifying, where $S : L^2(I)^* \longrightarrow L^2(I)$ is the canonical isometry. This is equivalent to saying that $\sum_{n\geq 0} g_n I_\phi(f_n)$ converges almost surely in E, where $(g_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. standard Gaussian random variables and $(f_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(I)$. Note that $L^2(I)$ is separable by [50, Proposition 1.2.29].

The map associated with a weakly L^2 function characterizes its stochastic integrability, as the next result says. Its proof is postponed to Section A.3.

Theorem 2.1.4 ([96, Theorem 2.3])**.** *Let $\phi : I \longrightarrow E$ be a weakly L^2 function. The following assertions are equivalent, where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$:*

(i) *ϕ is stochastically integrable,*

(ii) *there exists a random vector $Y : \Omega \longrightarrow E$ such that for every $x^* \in E^*$, we have*

$$x^*(Y) = \int_I x^*(\phi(t))\mathrm{d}B_t$$

almost surely,

(iii) *there exists a Gaussian measure μ on E with covariance operator $Q : E^* \longrightarrow E$ such that for every $x^* \in E^*$,*

$$c\int_I |x^*(\phi(t))|^2\mathrm{d}t = x^*Qx^*,$$

(iv) *there exist a separable Hilbert space H and a γ-radonifying operator $T : H \longrightarrow E$ such that for every $x^* \in E^*$,*

$$c \int_I |x^*(\phi(t))|^2 \mathrm{d}t \leq \|T^*(x^*)\|_H^2,$$

(v) *the map I_ϕ takes values in E, and is continuous and γ-radonifying.*

If one of these assertions holds, then ϕ is Pettis integrable on every bounded interval included in I, and μ is the distribution of the random vector $\int_I \phi(t)\mathrm{d}B_t$.

This theorem also shows that if a function is stochastically integrable with respect to some Brownian motion, then it is stochastically integrable with respect to any Brownian motion.

Although the stochastic integral is defined in a Pettis manner, any stochastically integrable function can be approximated by step functions.

Definition 2.1.5. A *step function* $\phi : I \longrightarrow E$ is a function of the form $\phi = \sum_{i=1}^n a_i \mathbf{1}_{]t_{i-1},t_i[}$ where $a_i \in E$ for all $1 \leq i \leq n$, $n \in \mathbb{N}_0$ and $t_0 \leq \cdots \leq t_n \in I$.

Remark that every step function is stochastically integrable.

Theorem 2.1.6 ([96, Theorem 2.5])**.** *Let $\phi : I \longrightarrow E$ be a weakly L^2 function. Then ϕ is stochastically integrable if and only if there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of step functions such that*

(i) *for all $x^* \in E^*$, $\lim_{n\to\infty} x^*\phi_n = x^*\phi$ in measure,*

(ii) *there exists a random vector $Y : \Omega \longrightarrow E$ such that $Y = \lim_{n\to\infty} \int_I \phi_n(t)\mathrm{d}B_t$ in probability.*

In that case, we have $Y = \int_I \phi(t)\mathrm{d}B_t$, the convergence in (i) *is in $L^2(I)$, and the convergence in* (ii) *is in $L^p(\Omega; E)$ for every $1 \leq p < \infty$.*

See Section A.3 for the proof of Theorem 2.1.6. This characterization of the stochastic integral has been taken as its definition in [86] for Banach space-valued measurable functions.

The proof of the next result follows the same lines as [96, Corollary 2.8].

Proposition 2.1.7. *Let $\phi : [0, \infty[\longrightarrow E$ be stochastically integrable. Then*

$$\lim_{t\to\infty} \int_0^t \phi(s)\mathrm{d}B_s = \int_0^\infty \phi(s)\mathrm{d}B_s$$

in $L^p(\Omega; E)$, for every $1 \leq p < \infty$.

Proof. Let $\|\cdot\|$ be a continuous seminorm, and let $1 \leq p < \infty$. Let $(t_n)_{n\geq 0}$ be a sequence of positive numbers converging to ∞. Let $n \geq 0$. Denote by R_n the covariance operator of the distribution ν_n of $\int_{t_n}^\infty \phi(s)\mathrm{d}B_s$, and let Q be the covariance operator of the distribution μ of $\int_0^\infty \phi(s)\mathrm{d}B_s$. By Theorem 2.1.4, we get

$$x^*R_nx^* = c\int_{t_n}^\infty |x^*\phi|^2 \mathrm{d}s \leq c\int_0^\infty |x^*\phi|^2 \mathrm{d}s = x^*Qx^*,$$

where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$. By Theorem 0.5.12, this implies that the sequence $(\nu_n)_{n\geq 0}$ is uniformly tight. On the other hand, we also have $\lim_{n\to\infty} x^* R_n x^* = 0$. By Theorem 0.5.7, we deduce that $\lim_{n\to\infty} \widehat{\nu_n}(x^*) = 1$ for all $x^* \in E^*$. Therefore, by [95, Theorems I.3.6 and IV.3.1], we get that $\lim_{n\to\infty} \nu_n = \delta_0$ weakly, where δ_0 is the Dirac measure at 0 (that is, $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$, for all $A \in \mathscr{B}(E)$).

We conclude the proof by using [19, Corollary 3.8.8]. We just need to check that

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{\{x\in E, \|x\|^p > R\}} \|x\|^p \mathrm{d}\nu_n(x) = 0.$$

Let $\varepsilon > 0$. Since $(\nu_n)_{n\geq 0}$ is uniformly tight, there exists a compact set $K \subseteq E$, thus bounded, such that $\nu_n(K) > 1-\varepsilon$. Let $R > 0$ be so large that $K \subseteq \{x \in E, \|x\|^p \leq R\}$. For all $n \geq 0$, by the Cauchy-Schwarz inequality and Theorem 0.5.12, we get that

$$\begin{aligned}\int_{\{x\in E, \|x\|^p > R\}} \|x\|^p \mathrm{d}\nu_n(x) &\leq \Big(\int_E \|x\|^p \mathrm{d}\nu_n(x) \Big)^{1/2} \Big(\int_E \mathbf{1}^2_{\{x\in E, \|x\|^p > R\}} \mathrm{d}\nu_n(x) \Big)^{1/2} \\ &\leq \Big(\int_E \|x\|^p \mathrm{d}\mu(x) \Big)^{1/2} \varepsilon^{1/2}.\end{aligned}$$

We conclude that $\sup_{n\geq 0} \int_{\{x\in E, \|x\|^p > R\}} \|x\|^p \mathrm{d}\nu_n(x) \leq M\varepsilon^{1/2}$ for some constant $M \geq 0$. Since ε was arbitrary, we are done. □

Lemma 2.1.8. *Let $\phi : I \longrightarrow E$ be stochastically integrable. Then for any $s \in \mathbb{R}$, $\phi(\cdot - s)$ is stochastically integrable on $I + s$ and the random vectors $\int_{I+s} \phi(t-s)\mathrm{d}B_t$ and $\int_I \phi(t)\mathrm{d}B_t$ are identically distributed.*

Proof. For all $x^* \in E^*$, we have by a change of variables and Theorem 2.1.4 that

$$c \int_{I+s} |x^*(\phi(t-s))|^2 \mathrm{d}t = c \int_I |x^*(\phi(t))|^2 \mathrm{d}t = x^* Q x^*,$$

where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$, and Q is the covariance operator of the distribution μ of $\int_I \phi(t)\mathrm{d}B_t$. By Theorem 2.1.4, we deduce that $\phi(\cdot - s)$ is stochastically integrable and μ is the distribution of $\int_{I+s} \phi(t-s)\mathrm{d}B_t$. □

Lemma 2.1.9. *Let $A, B \in \mathscr{B}(I)$ and $\phi, \varphi : I \longrightarrow E$ be two stochastically integrable functions. If A and B are disjoint then the random vectors $\int_A \phi(t)\mathrm{d}B_t$ and $\int_B \varphi(t)\mathrm{d}B_t$ are independent.*

Proof. The random vectors $X := \int_A \phi(t)\mathrm{d}B_t$ and $Y := \int_B \varphi(t)\mathrm{d}B_t$ are independent if and only if $x^*(X)$ and $y^*(Y)$ are independent for all $x^*, y^* \in E^*$, see [25, p. 23]. By Lemma 2.1.2 and Itô isometry, we have

$$\begin{aligned}\big\langle x^*(X), y^*(Y) \big\rangle_{L^2(\Omega)} &= \Big\langle \int_A x^*(\phi(t))\mathrm{d}B_t, \int_B y^*(\varphi(t))\mathrm{d}B_t \Big\rangle_{L^2(\Omega)} \\ &= c\big\langle x^*\phi, y^*\varphi \big\rangle_{L^2(I)} = 0,\end{aligned}$$

where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$. Since $(x^*(X), y^*(Y))$ is a Gaussian random vector by linearity of the stochastic integral, this implies that $x^*(X)$ and $y^*(Y)$ are independent by [51, Proposition E.2.12]. □

We end this section with a result characterizing the stochastically integrable functions with values in a space of functions. This will be useful for concrete examples.

Recall the definition of cotype.

Definition 2.1.10. Let E be a Banach space and $2 \leq q \leq \infty$. Then E has *cotype q* if there exists $C > 0$ such that for all $n \in \mathbb{N}_0$ and every $x_1, \ldots, x_n \in E$, we have

$$\|(x_k)_{k=1}^n\|_q \leq C \Big\| \sum_{k=1}^n g_k x_k \Big\|_{L^1(\Omega,\mathbb{P};E)},$$

where $(g_k)_{k=1}^n$ is a sequence of independent standard Gaussian variables.

Every Banach space has cotype ∞. The space c_0 has no finite cotype, see [51, Corollary 7.1.10].

Theorem 2.1.11. *Let $(S, \mathcal{B}, \mu)$ be a σ-finite measure space, let $1 \leq p < \infty$, and set $E := L^p(S, \mathcal{B}, \mu)$. Then a measurable function $\phi : I \longrightarrow E$ is stochastically integrable if and only if*

$$\Big\| \Big(\int_I |\phi(t, \cdot)|^2 \mathrm{d}t \Big)^{1/2} \Big\|_E < \infty.$$

Theorem 2.1.11 is a direct consequence of [96, Corollary 2.10] by noticing that $L^p(S, \mathcal{B}, \mu)$ has finite cotype for any $1 \leq p < \infty$, see [51, Proposition 7.1.4].

2.2 Random vector: First method

The idea of the first method to get a random vector for semigroups is to reuse the arguments of Section 1.1 for a single operator by replacing series with stochastic integrals. However, we have not been able to prove results analogous to Theorems 1.1.8 or 1.2.3 for C_0-semigroups. Furthermore, we did not find any example of C_0-semigroups that satisfies the assumptions of the only result we obtained, Proposition 2.2.2. Nevertheless, this part was left as it might still be interesting.

The final parts of the proofs of the main results of Sections 2.2 and 2.3 are the same, so we put it in a lemma.

Lemma 2.2.1. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on E. Let $v : \Omega \longrightarrow E$ be a random vector such that its probability distribution is ergodic for $(T_t)_{t\geq 0}$ and $\mathbb{P}(v \in O) > 0$ for every non-empty open subset O of E. Then v is almost surely frequently hypercyclic for $(T_t)_{t\geq 0}$.*

Proof. Let μ be the probability distribution of v. Let O be a non-empty open subset of E. The Birkhoff Ergodic Theorem for semigroups, Theorem 0.1.46, can be applied to T and μ and gives

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N (\mathbf{1}_O \circ T_t)(x) \mathrm{d}t = \mu(O) \ \mu\text{-a.s.}$$

Let A be a Borel subset of E such that $\mu(A) = 1$ and the previous equality holds everywhere on A. Then, if we set $B := v^{-1}(A) \subseteq \Omega$, we have $\mathbb{P}(B) = \mathbb{P}(v^{-1}(A)) = \mu(A) = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N (\mathbf{1}_O \circ T_t)(v)\mathrm{d}t = \mathbb{P}(v \in O) > 0$$

on B. Since E is separable, we can take a countable base of open subsets of E and get that almost surely, the set $\{t \geq 0 \mid T_t(v) \in O\}$ has positive lower density for every non-empty open subset O of E. The random vector v is therefore almost surely frequently hypercyclic for $(T_t)_{t\geq 0}$. □

The next proposition is the analogous result to Proposition 1.1.4. Their proofs are the same, but we need to use the Birkhoff Ergodic Theorem for semigroups through Lemma 2.2.1, and replace all series with stochastic integrals.

Proposition 2.2.2. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on E, and let $(u_t)_{t\in\mathbb{R}}$ be a family of vectors in E such that $T_s(u_t) = u_{t-s}$ for every $s \geq 0$ and $t \in \mathbb{R}$. Let $(B_t)_{t\in\mathbb{R}}$ be a Brownian motion. Assume that $t \longmapsto u_t$ is stochastically integrable, and set*

$$v := \int_{\mathbb{R}} u_t \mathrm{d}B_t.$$

If $\mathbb{P}(v \in O) > 0$ for every non-empty open subset O of E, then v is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t\geq 0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. Define the probability measure

$$\mu : \mathscr{B}(E) \longrightarrow [0, 1], A \longmapsto \mathbb{P}(v \in A).$$

Let $s \geq 0$, we first show that μ is T_s-invariant. Let $A \in \mathscr{B}(E)$. By the definitions of μ and v and by Lemmas 2.1.2 and 2.1.8, we have

$$\mu(T_s^{-1}(A)) = \mathbb{P}(T_s(v) \in A) = \mathbb{P}\left(\int_{\mathbb{R}} u_{t-s}\mathrm{d}B_t \in A\right) = \mathbb{P}\left(\int_{\mathbb{R}} u_t \mathrm{d}B_t \in A\right).$$

We conclude by definitions of v and μ that $\mu(T_s^{-1}(A)) = \mathbb{P}(v \in A) = \mu(A)$. The measure μ is thus T_s-invariant.

Now we claim that μ is $(T_t)_{t\geq 0}$-strongly mixing. Let f and g be two bounded and continuous real-valued functions defined on E. We aim to show that

$$\lim_{t\to\infty} \int_E (f \circ T_t) g \mathrm{d}\mu = \int_E f \mathrm{d}\mu \int_E g \mathrm{d}\mu.$$

Since the set of bounded continuous functions on E is dense in $L^2(E, \mu)$ by [31, Theorem 18.1], this will imply the claim by [31, p. 26]. First, by definition of μ, this is equivalent to showing that

$$\lim_{t\to\infty} \int_\Omega f(T_t(v))g(v)\mathrm{d}\mathbb{P} = \int_\Omega f(v)\mathrm{d}\mathbb{P} \int_\Omega g(v)\mathrm{d}\mathbb{P}.$$

Let $\varepsilon > 0$. Since f and g are continuous and bounded, by Proposition 2.1.7, Lemma A.1.3 and the Dominated Convergence Theorem, there exists $N \geq 0$ such that

$$\left\| g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big) - g(v) \right\|_{L^1(\Omega,\mathbb{P})} < \varepsilon \tag{2.2.1}$$

and

$$\left\| f\Big(\int_{-N}^{\infty} u_t \mathrm{d}B_t\Big) - f(v) \right\|_{L^1(\Omega,\mathbb{P})} < \varepsilon. \tag{2.2.2}$$

Let $s > 2N$, write

$$\begin{aligned} f(T_s(v))g(v) &= f(T_s(v))g(v) - f(T_s(v))g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big) \\ &\quad + f(T_s(v))g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big) - f\Big(\int_{-N+s}^{\infty} u_{t-s} \mathrm{d}B_t\Big)g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big) \\ &\quad + f\Big(\int_{-N+s}^{\infty} u_{t-s} \mathrm{d}B_t\Big)g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big). \end{aligned} \tag{2.2.3}$$

For the first two terms, using the assumption that f is bounded and the inequality (2.2.1) yield

$$\begin{aligned} &\left| \int_\Omega f(T_s(v))g(v)\mathrm{d}\mathbb{P} - \int_\Omega f(T_s(v))g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big)\mathrm{d}\mathbb{P} \right| \\ &\qquad \leq \|f\|_\infty \left\| g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big) - g(v) \right\|_{L^1(\Omega,\mathbb{P})} \leq \|f\|_\infty \varepsilon. \end{aligned}$$

Now, for the third and fourth terms, define the random vectors

$$X := \int_{-\infty}^{-N+s} u_{t-s} \mathrm{d}B_t, Y := \int_{-N+s}^{\infty} u_{t-s} \mathrm{d}B_t, U := \int_{-\infty}^{-N} u_t \mathrm{d}B_t, V := \int_{-N}^{\infty} u_t \mathrm{d}B_t.$$

Then Lemma 2.1.8 implies that X and U are identically distributed, and so are Y and V. By Lemma 2.1.9, X and Y are independent, and so are U and V. Therefore, the random variables $f(X+Y) - f(Y)$ and $f(U+V) - f(V)$ are identically distributed. Now, by using Lemma 2.1.2, we get that

$$\begin{aligned} &\left| \int_\Omega \Big(f(T_s(v))g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big) - f\Big(\int_{-N+s}^{\infty} u_{t-s} \mathrm{d}B_t\Big)g\Big(\int_{-\infty}^{N} u_t \mathrm{d}B_t\Big)\Big)\mathrm{d}\mathbb{P} \right| \\ &\qquad \leq \|g\|_\infty \left\| f\Big(\int_{-\infty}^{\infty} u_{t-s} \mathrm{d}B_t\Big) - f\Big(\int_{-N+s}^{\infty} u_{t-s} \mathrm{d}B_t\Big) \right\|_{L^1(\Omega,\mathbb{P})} \\ &\qquad = \|g\|_\infty \left\| f\Big(\int_{-\infty}^{\infty} u_t \mathrm{d}B_t\Big) - f\Big(\int_{-N}^{\infty} u_t \mathrm{d}B_t\Big) \right\|_{L^1(\Omega,\mathbb{P})} \\ &\qquad \leq \|g\|_\infty \varepsilon, \end{aligned}$$

where we have used (2.2.2) for the last inequality. For the last term of (2.2.3), since $s > 2N$ and by using Lemma 2.1.9 and then Lemma 2.1.8, one has

$$\begin{aligned}
&\int_\Omega f\Big(\int_{-N+s}^{\infty} u_{t-s}\mathrm{d}B_t\Big)g\Big(\int_{-\infty}^{N} u_t\mathrm{d}B_t\Big)\mathrm{d}\mathbb{P}\\
&= \int_\Omega f\Big(\int_{-N+s}^{\infty} u_{t-s}\mathrm{d}B_t\Big)\mathrm{d}\mathbb{P}\int_\Omega g\Big(\int_{-\infty}^{N} u_t\mathrm{d}B_t\Big)\mathrm{d}\mathbb{P}\\
&= \int_\Omega f\Big(\int_{-N}^{\infty} u_t\mathrm{d}B_t\Big)\mathrm{d}\mathbb{P}\int_\Omega g\Big(\int_{-\infty}^{N} u_t\mathrm{d}B_t\Big)\mathrm{d}\mathbb{P}.
\end{aligned}$$

Therefore, using again (2.2.1) and (2.2.2) gives

$$\begin{aligned}
&\left|\int_\Omega f\left(\int_{-N+s}^{\infty} u_{t-s}\mathrm{d}B_t\right) g\left(\int_{-\infty}^{N} u_t\mathrm{d}B_t\right)\mathrm{d}\mathbb{P} - \int_\Omega f(v)\mathrm{d}\mathbb{P}\int_\Omega g(v)\mathrm{d}\mathbb{P}\right|\\
&\leq \|f\|_\infty\left\|g\left(\int_{-\infty}^{N} u_t\mathrm{d}B_t\right) - g(v)\right\|_{L^1(\Omega,\mathbb{P})} + \|g\|_\infty\left\|f\left(\int_{-N}^{\infty} u_t\mathrm{d}B_t\right) - f(v)\right\|_{L^1(\Omega,\mathbb{P})}\\
&\leq \|f\|_\infty\varepsilon + \|g\|_\infty\varepsilon.
\end{aligned}$$

We can finally conclude that

$$\left|\int_\Omega f(T_s(v))g(v)\mathrm{d}\mathbb{P} - \int_\Omega f(v)\mathrm{d}\mathbb{P}\int_\Omega g(v)\mathrm{d}\mathbb{P}\right| \leq 2\|f\|_\infty\varepsilon + 2\|g\|_\infty\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, we eventually get that $\lim_{t\to\infty}\int_\Omega f(T_t(v))g(v)\mathrm{d}\mathbb{P} = \int_\Omega f(v)\mathrm{d}\mathbb{P}\int_\Omega g(v)\mathrm{d}\mathbb{P}$. The measure μ is thus $(T_t)_{t\geq 0}$-strongly mixing.

The result now follows by Lemma 2.2.1. □

In the same spirit as Theorem 1.2.12, we have the following result.

Theorem 2.2.3. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space E of type 2. Let $(u_t)_{t\in\mathbb{R}}$ be a family of vectors in E such that $T_s(u_t) = u_{t-s}$ for every $s \geq 0$ and $t \in \mathbb{R}$, and $\operatorname{span}\{u_t \mid t \notin A\}$ is dense in E for every $A \in \mathscr{B}(\mathbb{R})$ with zero Lebesgue measure. Let $(B_t)_{t\in\mathbb{R}}$ be a Brownian motion. Assume that $t \longmapsto u_t \in L^2(\mathbb{R}; E)$. Then the random vector*

$$v := \int_{\mathbb{R}} u_t\mathrm{d}B_t.$$

is almost surely well-defined and frequently hypercyclic for $(T_t)_{t\geq 0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. We first prove that v is almost surely well-defined. Define the linear map $J : \phi \longmapsto \int_{\mathbb{R}} \phi(t)\mathrm{d}B_t$ defined on the space of E-valued step functions. Let $\phi = \sum_{i=1}^n a_i\mathbf{1}_{A_i}$ be a step function. Since E has type 2 and by using properties (ii) to (iv) of Definition 0.4.1, we have

$$\|J(\phi)\|^2_{L^2(\Omega;E)} \leq C\sum_{i=1}^{n}\lambda(A_i)\|a_i\|_E^2 = C\|\phi\|^2_{L^2(\mathbb{R};E)},$$

where $C > 0$ is some constant depending only on E and λ is the Lebesgue measure on $\mathbb{R}$. Therefore, J is a continuous embedding, and the density of the space of step functions in $L^2(\mathbb{R}; E)$, see [50, Remark 1.2.20], allows us to extend J on $L^2(\mathbb{R}; E)$. Theorem 2.1.6 then implies that $J(\phi) = \int_{\mathbb{R}} \phi \mathrm{d}B_t$ for every $\phi \in L^2(\mathbb{R}; E)$. This shows that v is almost surely well-defined since the map $t \longmapsto u_t$ belongs to $L^2(\mathbb{R}; E)$.

Let Q be the covariance operator of the probability distribution $\mathbb{P}_v$ of v. By Proposition 0.5.13, it suffices to show that Q is one-to-one to prove that $\mathbb{P}_v$ has full support. Let $x^* \in \operatorname{Ker}(Q)$. By definition of Q, we have

$$0 = x^*(Q(x^*)) = \int_E |x^*|^2 \mathrm{d}\mathbb{P}_v = \int_\Omega |x^*(v)|^2 \mathrm{d}\mathbb{P},$$

implying that $x^*(v) = 0$ almost surely. By definition of v and Lemma 2.1.2, we get that

$$0 = x^*(v) = \int_{\mathbb{R}} x^*(u_t) \mathrm{d}B_t$$

almost surely. Therefore, Itô's isometry yields that $x^*(u_t) = 0$ for every $t \in \mathbb{R}$ outside some set of zero Lebesgue measure. By assumption, we get $x^* = 0$.

The result now follows by Proposition 2.2.2. □

Remark 2.2.4. In the proof of Theorem 2.2.3, in order to show that v is almost surely well-defined, we actually proved that $L^2(\mathbb{R}; E)$ is continuously embedded in the space of stochastically integrable functions. This was already known in [86, Proposition 5.2].

As mentioned above, we have not been able to prove analogous results to Theorems 1.1.8 or 1.2.3 for C_0-semigroups. In the proof of Proposition 2.2.2, the independent and stationary increments of the Brownian motion play a crucial role. The idea was then to prove the existence of a Lévy process with respect to which $t \longmapsto u_t$ is stochastically integrable, under some deterministic conditions on $(u_t)_{t\in\mathbb{R}}$. A sought condition was the Pettis integrability of $t \longmapsto u_t$. If this plan could work, one would then hope to find another proof of the frequent hypercyclicity criterion for semigroups [78, Theorem 3], as we did for operators.

2.3 Random vector: Second method

In our second approach, we will make assumptions close to those of the main result of Chakir and El Mourchid [26, Theorem 3.2]. The semigroup will be defined on a complex Fréchet space since we will consider purely imaginary eigenvalues of the generator.

Theorem 2.3.1. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable complex Fréchet space E. Let A be the generator of $(T_t)_{t\geq 0}$, and let $(u_t)_{t\in I}$ be a family of vectors in* $\operatorname{Dom}(A)$ *such that $A(u_t) = itu_t$ for every $t \in I$, where I is a non-empty interval of $\mathbb{R}$, and* $\operatorname{span}\{u_t \mid t \notin B\}$ *is dense in E for every $B \in \mathscr{B}(I)$ with zero Lebesgue measure. Let $(B_t)_{t\in I}$ be a complex Brownian motion. Assume that $t \longmapsto u_t$ is stochastically integrable, and set*

$$v := \int_I u_t \mathrm{d}B_t.$$

Then v is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t\geq 0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. Let Q be the covariance operator of the probability distribution $\mathbb{P}_v$ of v.

Let $x^*, y^* \in E^*$. By using the definition of Q and Lemmas 2.1.2 and 0.4.7, we have

$$\begin{aligned} y^*Qx^* &= \int_E y^*(x)\overline{x^*(x)}\mathrm{d}\mathbb{P}_v(x) = \int_\Omega y^*\Big(\int_I u_t \mathrm{d}B_t\Big)\overline{x^*\Big(\int_I u_t \mathrm{d}B_t\Big)}\mathrm{d}\mathbb{P} \\ &= \mathbb{E}\Big(\int_I y^*(u_t)\mathrm{d}B_t \overline{\int_I x^*(u_t)\mathrm{d}B_t}\Big) = 2\int_I y^*(u_t)\overline{x^*(u_t)}\mathrm{d}t; \end{aligned} \tag{2.3.1}$$

note that we have used the linearisation of the Itô isometry.

We first prove that $\mathbb{P}_v$ is $(T_t)_{t\geq 0}$-invariant. Let $x^*, y^* \in E^*$ and $t \geq 0$. By using (2.3.1) twice, (ii) of Proposition 0.1.36 and the assumption on $(u_s)_{s\in I}$, we have

$$\begin{aligned} y^*T_tQT_t^*x^* &= T_t^*y^*QT_t^*x^* = 2\int_I T_t^*y^*(u_s)\overline{T_t^*x^*(u_s)}\mathrm{d}s \\ &= 2\int_I y^*T_t(u_s)\overline{x^*T_t(u_s)}\mathrm{d}t = 2\int_I e^{its}y^*(u_s)e^{-its}\overline{x^*(u_s)}\mathrm{d}s \\ &= y^*Qx^*. \end{aligned}$$

By (i) of Theorem 0.5.14, this shows that $\mathbb{P}_v$ is $(T_t)_{t\geq 0}$-invariant.

Now, let us show that $\mathbb{P}_v$ is strongly mixing for $(T_t)_{t\geq 0}$. Let $x^*, y^* \in E^*$ and $t \geq 0$. By using (2.3.1), (ii) of Proposition 0.1.36 and the assumption on $(u_t)_{t\in I}$, we have

$$\begin{aligned} y^*QT_t^*(x^*) &= 2\int_I y^*(u_s)\overline{T_t^*x^*(u_s)}\mathrm{d}s = 2\int_I y^*(u_s)\overline{x^*(e^{its}u_s)}\mathrm{d}s \\ &= 2\int_I e^{-its}y^*(u_s)\overline{x^*(u_s)}\mathrm{d}s. \end{aligned}$$

Since $s \longmapsto y^*(u_s)\overline{x^*(u_s)}$ is integrable, by the Riemann-Lebesgue lemma, see [50, Lemma 2.4.3], we get that $\lim_{t\to\infty} y^*QT_t^*(x^*) = 0$. We conclude that $\mathbb{P}_v$ is strongly mixing for $(T_t)_{t\geq 0}$ by (ii) of Theorem 0.5.14.

Finally, we show that $\mathbb{P}_v$ has full support. It suffices to show that Q is one-to-one by Proposition 0.5.13. Let $x^* \in \mathrm{Ker}(Q)$. Again by (2.3.1), we have

$$0 = x^*Qx^* = 2\int_I |x^*(u_s)|^2\mathrm{d}s,$$

hence $x^*(u_s) = 0$ almost everywhere on I. Therefore, $x^* = 0$ by hypothesis and linearity and continuity of x^*.

We conclude the proof by using Lemma 2.2.1. □

As a first application, let us apply Theorem 2.3.1 to the chaotic translation semigroups. See Example 0.1.37 for the definition. It is shown in [26, Example 4.1] that these semigroups admit a strongly mixing Gaussian measure, by constructing a stochastic process. We reuse their arguments to apply Theorem 2.3.1.

Theorem 2.3.2. *Let $1 \leq p < \infty$, $I \in \{[0,\infty[, \mathbb{R}\}$ and an admissible weight $\rho : I \longrightarrow]0,\infty[$. If the integral $\int_I \rho(x)\mathrm{d}x$ is finite, then the random vector*

$$\int_{\mathbb{R}} \frac{e_t}{\sqrt{1+t^2}} \mathrm{d}B_t,$$

where $e_t(x) = e^{itx}$, $x \in I, t \in \mathbb{R}$, is almost surely frequently hypercyclic for the translation semigroup $(T_t)_{t\geq 0}$ on $L^p_\rho(I)$, and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. For every $s \in \mathbb{R}$, define the function $u_s : I \longrightarrow \mathbb{C}$ by

$$u_s(x) := \frac{e^{isx}}{\sqrt{1+s^2}}$$

for all $x \in I$. Since $\int_I \rho(x)\mathrm{d}x < \infty$, we get that $u_s \in L^p_\rho(I)$ for all $s \in \mathbb{R}$.

It is clear that $T_t(u_s) = e^{ist} u_s$ for every $s \in \mathbb{R}$ and $t \geq 0$. By (ii) of Proposition 0.1.36, this implies that $u_s \in \mathrm{Dom}(A)$ and $A(u_s) = isu_s$ for every $s \in \mathbb{R}$, where A is the generator of the semigroup.

Now, let $\phi \in (L^p_\rho(I))^* = L^q_{\rho^{-q/p}}(I)$, where $1/p + 1/q = 1$, be such that $\phi(u_s) = 0$ for all $s \notin B$, where $B \in \mathscr{B}(\mathbb{R})$ has zero Lebesgue measure. Then

$$0 = \phi(u_s) = \int_I \frac{e^{isx}}{\sqrt{1+s^2}} \phi(x)\mathrm{d}x,$$

hence $\int_I e^{isx}\phi(x)\mathrm{d}x = 0$ for all $s \notin B$. Since $\phi \in L^1(I)$ by the fact that $\int_I \rho(x)\mathrm{d}x < \infty$ and Hölder's inequality, we get that $\phi = 0$ almost everywhere by [87, Theorem 9.11], and thus $\phi = 0$. This shows that $\mathrm{span}\{u_s \mid s \notin B\}$ is dense in $L^p_\rho(I)$ for any $B \in \mathscr{B}(\mathbb{R})$ with zero Lebesgue measure.

We now prove that

$$\Big\| \Big(\int_{\mathbb{R}} |u_s(\cdot)|^2 \mathrm{d}s \Big)^{1/2} \Big\|_{L^p_\rho(I)} < \infty.$$

For any $x \in I$, we have

$$\int_{\mathbb{R}} |u_s(x)|^2 \mathrm{d}s = \int_{\mathbb{R}} \frac{1}{1+s^2} \mathrm{d}s,$$

and thus

$$\Big\| \Big(\int_{\mathbb{R}} |u_s(\cdot)|^2 \mathrm{d}s \Big)^{1/2} \Big\|^p_{L^p_\rho(I)} = \int_I \Big(\int_{\mathbb{R}} \frac{1}{1+s^2} \mathrm{d}s \Big)^{p/2} \rho(x)\mathrm{d}x,$$

which is finite since $\int_I \rho(x)\mathrm{d}x < \infty$.

By Theorem 2.1.11, the map $s \longmapsto u_s$ is stochastically integrable. We conclude the proof by applying Theorem 2.3.1. ☐

Remark 2.3.3. By Example 0.1.43, the translation semigroup on $L^p_\rho(I)$ is chaotic if and only if $\int_I \rho(x)\mathrm{d}x$ is finite. Therefore, by Theorem 2.3.2, we have found a frequently hypercyclic random vector for any chaotic translation semigroup.

The second example was studied by El Mourchid and Latrach in [35, Proposition 3.4], who proved the existence of a strongly mixing Gaussian measure for the semigroup under consideration. It turns out that their arguments are the same to check the assumptions of Theorem 2.3.1.

Theorem 2.3.4. *Let $1 \leq p < \infty$ and A be the weighted shift on ℓ^p with sequence of weights $(w_n)_{n\geq 1}$. Set $\beta_n := w_1 \dots w_n$ for all $n \geq 1$, and $\beta_0 := 1$. If the quantity $\limsup_{n\to\infty}(1/|\beta_n|)^{1/n}$ is finite, then there exists $\eta > 0$ such that the random vector*

$$\int_{-\eta}^{\eta} \Big(\frac{i^n t^n}{\beta_n}\Big)_{n\geq 0} \mathrm{d}B_t$$

is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t\geq 0} := (e^{tA})_{t\geq 0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. For all $\lambda \in \mathbb{C}$, define the sequence $v_\lambda := (\lambda^n/\beta_n)_{n\geq 0}$. Then $v_\lambda \in \ell^p$ if and only if $\sum_{n\geq 0} |\lambda|^{np}/|\beta_n|^p < \infty$, and in that case we have $A(v_\lambda) = (\lambda^{n+1}/\beta_n)_{n\geq 0} = \lambda v_\lambda$. The power series $\sum_{n\geq 0} \lambda^n/\beta_n$ has a radius of convergence $R := 1/\rho$, where $\rho := \limsup_{n\to\infty}(1/|\beta_n|)^{1/n} < \infty$.

Let $\eta \in \,]0, R[$, and define $u_s := v_{is}$ for every $-\eta < s < \eta$. By our previous calculations, we have $u_s \in \ell^p$ and $A(u_s) = isu_s$ for all $-\eta < s < \eta$.

Let $\phi = (\phi_n)_{n\geq 0} \in (\ell^p)^* = \ell^q$, where $1/p + 1/q = 1$, be such that $\phi(u_s) = 0$ for almost all $-\eta < s < \eta$. Define the analytic function $F : D(0, R) \longrightarrow \mathbb{C}$ by

$$F(\lambda) := \sum_{n=0}^{\infty} \frac{\lambda^n \phi_n}{\beta_n} = \phi(v_\lambda)$$

for all $\lambda \in D(0, R)$. Since $F = 0$ almost everywhere on $i]-\eta, \eta[$, we conclude that $F = 0$ on $D(0, R)$, and $\phi = 0$. This means that $\mathrm{span}\{u_t \mid t \notin B\}$ is dense in ℓ^p for any $B \in \mathscr{B}(]-\eta, \eta[)$ with zero Lebesgue measure.

For all $n \geq 0$, we have

$$\int_{-\eta}^{\eta} |u_s(n)|^2 \mathrm{d}s = \int_{-\eta}^{\eta} \frac{|is|^{2n}}{|\beta_n|^2} \mathrm{d}s \leq \frac{2\eta\eta^{2n}}{|\beta_n|^2},$$

implying that

$$\Big\| \Big(\int_{-\eta}^{\eta} |u_s(\cdot)|^2 \mathrm{d}s \Big)^{1/2} \Big\|_{\ell^p}^p \leq \sum_{n=0}^{\infty} \frac{2^{p/2}\eta^{p/2}\eta^{np}}{|\beta_n|^p},$$

which is finite since $\eta < R$. Therefore, by Theorem 2.1.11, the map $s \longmapsto u_s$ is stochastically integrable.

We conclude the proof by applying Theorem 2.3.1. □

We now study the translation semigroups on $H(\mathbb{C})$. We will need the notion of *Hilbert seminorm*: it is a seminorm $p : H(\mathbb{C}) \longrightarrow [0, \infty[$ for which there exists a semi-scalar product $\langle \cdot, \cdot \rangle$ on $H(\mathbb{C})$ such that $p(x) = \langle x, x \rangle$ for all $x \in H(\mathbb{C})$.

Theorem 2.3.5. *For each $t \geq 0$, define the translation operator*

$$T_t : H(\mathbb{C}) \longrightarrow H(\mathbb{C}), f \longmapsto f(\cdot + t).$$

Then the random vector $\int_0^{2\pi} e^{itz} \mathrm{d}B_t$ is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t\geq 0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. It is easy to check that $(T_t)_{t\geq 0}$ is indeed a C_0-semigroup. Furthermore, its generator is the differentiation operator $D : H(\mathbb{C}) \longrightarrow H(\mathbb{C})$.

For every $s \in [0, 2\pi]$, set $u_s(z) := e^{isz}$, $z \in \mathbb{C}$. Clearly, $D(u_s) = isu_s$ for any $s \in [0, 2\pi]$. The space $\operatorname{span}\{u_s \mid s \notin B\}$ is dense in $H(\mathbb{C})$ by [47, Lemma 2.34], for any $B \in \mathscr{B}([0, 2\pi])$ with zero Lebesgue measure. Furthermore, the map $s \longmapsto u_s$ is weakly L^2 since $\{u_s \mid s \in [0, 2\pi]\}$ is bounded in $H(\mathbb{C})$. We now show that the map I_ϕ of Definition 2.1.3 takes values in $H(\mathbb{C})$ and is γ-radonifying, where $\phi(s) := u_s$, $s \in [0, 2\pi]$.

Let $f \in L^2([0, 2\pi])$. Define the function $F : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$F(z) := \int_0^{2\pi} u_s(z)\overline{f(s)}\mathrm{d}s$$

for every $z \in \mathbb{C}$. It is well-defined since the maps $s \longmapsto u_s(z)$ and f are in $L^2([0, 2\pi])$, for any $z \in \mathbb{C}$. Let $z, h \in \mathbb{C}$, we have

$$\frac{F(z+h) - F(z)}{h} = \int_0^{2\pi} \frac{e^{is(z+h)} - e^{isz}}{h}\overline{f(s)}\mathrm{d}s.$$

An application of the Dominated Convergence Theorem then shows that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = \int_0^{2\pi} ise^{isz}\overline{f(s)}\mathrm{d}s,$$

and F is holomorphic on $\mathbb{C}$.

Let $x^* \in H(\mathbb{C})^*$. We show that $x^*(F) = \int_0^{2\pi} x^*(u_s)\overline{f(s)}\mathrm{d}s = I_\phi(f)(x^*)$. Let $N \geq 1$. Then by linearity of x^* and the integral, one has

$$\begin{aligned}
x^*\Big(\sum_{k=0}^N \frac{i^k z^k}{k!} \int_0^{2\pi} s^k\overline{f(s)}\mathrm{d}s\Big) &= \sum_{k=0}^N x^*\Big(\frac{i^k z^k}{k!}\Big) \int_0^{2\pi} s^k\overline{f(s)}\mathrm{d}s \\
&= \int_0^{2\pi} \sum_{k=0}^N x^*\Big(\frac{i^k z^k}{k!}\Big) s^k\overline{f(s)}\mathrm{d}s \\
&= \int_0^{2\pi} x^*\Big(\sum_{k=0}^N \frac{i^k s^k z^k}{k!}\Big)\overline{f(s)}\mathrm{d}s. \qquad (2.3.2)
\end{aligned}$$

The right-hand side converges to $\int_0^{2\pi} x^*(u_s)\overline{f(s)}\mathrm{d}s$ by the Dominated Convergence Theorem. As for the left-hand side, let us prove that it converges to $x^*(F)$ as N goes to ∞. Let $r > 0$ be fixed. Then

$$\begin{aligned}\Big\| \sum_{k=0}^{N} \frac{i^k z^k}{k!} \int_0^{2\pi} s^k \overline{f(s)} \mathrm{d}s - F(z) \Big\|_r &= \Big\| \int_0^{2\pi} \sum_{k=N+1}^{\infty} \frac{i^k s^k z^k}{k!} \overline{f(s)} \mathrm{d}s \Big\|_r \\ &\leq \int_0^{2\pi} \sum_{k=N+1}^{\infty} \frac{s^k r^k}{k!} |f(s)| \mathrm{d}s \leq \sum_{k=N+1}^{\infty} \frac{r^k}{k!} \|s^k\|_{L^2([0,2\pi])} \|f\|_{L^2([0,2\pi])},\end{aligned}$$

where we have used the Cauchy-Schwarz for the last inequality. The right-hand side converges to 0 when N goes to ∞. By continuity of x^*, we conclude that the left-hand side of (2.3.2) converges to $x^*(F)$. Therefore, $x^*(F) = I_\phi(f)(x^*)$, so that $I_\phi(f) = F$, and hence I_ϕ takes values in $H(\mathbb{C})$. The map I_ϕ is also continuous: for every $r \geq 0$, we have $\|I_\phi(f)\|_r \leq \sqrt{2\pi} e^{2\pi r} \|f\|_{L^2([0,2\pi])}$.

By Theorem 2.1.4, in order to prove that $t \longmapsto u_t$ is stochastically integrable, it remains to show that I_ϕ is γ-radonifying. Let $(f_n)_{n\in\mathbb{Z}} = (e^{ins}/\sqrt{2\pi})_{n\in\mathbb{Z}}$ be the canonical orthonormal basis of $L^2([0,2\pi])$. By [71, Examples 28.9(4)] and [71, Lemma 28.1], the space $H(\mathbb{C})$ has a system of Hilbert seminorms generating its topology. Let $\|\cdot\|$ be such a seminorm. Since it is continuous, there exist some $r > 0$ and $C > 0$ such that $\|\cdot\| \leq C \|\cdot\|_r$ Let $(g_n)_{n\in\mathbb{Z}}$ be an i.i.d. Gaussian sequence. Let $0 \leq N \leq M$ be two positive integers; we have, by using the facts that $\|\cdot\|$ is a Hilbert seminorm for the first equality and g_n, $n \geq 0$, are independent for the second one,

$$\begin{aligned}\mathbb{E}\Big(\Big\| \sum_{n=N}^{M} g_n I_\phi(f_n) \Big\|^2 \Big) &= \sum_{n,m=N}^{M} \mathbb{E}\big(g_n \overline{g_m}\big) \langle I_\phi(f_n), I_\phi(f_m) \rangle \\ &= \sum_{n=N}^{M} \|I_\phi(f_n)\|^2 \\ &\leq C \sum_{n=N}^{M} \|I_\phi(f_n)\|_r^2 \\ &= C \sum_{n=N}^{M} \Big\| \int_0^{2\pi} e^{isz} \overline{f_n(s)} \mathrm{d}s \Big\|_r^2,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the semi-scalar product associated to $\|\cdot\|$. For each $n \in \mathbb{N}$ such that $n > r$ and all $z \in \mathbb{C}$ such that $|z| = r$, we have

$$\int_0^{2\pi} e^{isz} \overline{f_n(s)} \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{isz} e^{-isn} \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \frac{e^{i2\pi(z-n)} - 1}{i(z-n)},$$

and then

$$\mathbb{E}\Big(\Big\| \sum_{n=N}^{M} g_n I_\phi(f_n) \Big\|^2 \Big) \leq \frac{C}{2\pi} \sum_{n=N}^{M} \Big\| \frac{e^{i2\pi(z-n)} - 1}{i(z-n)} \Big\|_r^2 \leq \frac{C}{2\pi} \sum_{n=N}^{M} \frac{(e^{2\pi r} + 1)^2}{(n-r)^2}$$

provided that $N > r$. We can conclude that the series

$$\sum_{n=0}^{\infty} g_n I_\phi(f_n)$$

converges in $L^2(\Omega; E)$. The same arguments show that $\sum_{n\leq 0} g_n I_\phi(f_n)$ also converges in $L^2(\Omega; E)$, hence I_ϕ is γ-radonifying by Theorem 0.5.4. By Theorem 2.1.4, the map $s \longmapsto u_s$ is thus stochastically integrable.

The result now follows by Theorem 2.3.1. $\square$

Remark 2.3.6. By Remark 0.1.44, any frequently hypercyclic vector for the semigroup $(T_t)_{t\geq 0}$ is also a frequently hypercyclic for each T_t, $t > 0$. Therefore, the random vector $\int_0^{2\pi} e^{itz} \mathrm{d}B_t$ is almost surely frequently hypercyclic for the translation operator $T_1 : H(\mathbb{C}) \longrightarrow H(\mathbb{C}), f \longmapsto f(\cdot + 1)$; see also Example 1.3.11.

We repeat our remark made at the end of Section 2.2: in Theorem 2.3.1, we still assume that the map $t \longmapsto u_t$ is stochastically integrable. Then one may ask if we could find a deterministic assumption to get the stochastic integrability of $t \longmapsto u_t$, possibly replacing the Brownian motion with another stochastic process.

Chapter 3

Rate of growth of random power series

For a given frequently hypercyclic weighted shift defined on the space of entire functions $H(\mathbb{C})$, the rate of growth of the frequently hypercyclic vectors of this shift can be studied. More precisely, an *admissible rate of growth* for the frequently hypercyclic functions of an operator T on $H(\mathbb{C})$ is a map $g : [0, \infty[\longrightarrow [0, \infty[$ for which there exists a frequently hypercyclic function f for T such that

$$\sup_{|z|=r} |f(z)| \leq g(r)$$

for all $r \geq 0$ large enough.

Recall from Chapter 1 that for a given chaotic weighted shift T on the space $H(\mathbb{C})$ with weights $(w_n)_{n\geq 1}$, we have proved in Theorem 1.3.4 that the random vector

$$\sum_{n=0}^{\infty} \frac{X_n}{w_1 \dots w_n} z^n$$

is almost surely an entire function and frequently hypercyclic for T, where the complex random variables X_n, $n \geq 0$, are i.i.d. and subgaussian with full support.

In the present chapter, we are interested in the rate of growth of general random series $\sum_{n\geq 0} a_n X_n e_n$ where $(e_n)_{n\in\mathbb{N}}$ is a sequence of polynomials and $f = \sum_{n\geq 0} a_n e_n$ is entire. We will prove that almost surely, the inequality

$$\max_{|z|=r} \left| \sum_{n\geq 0} a_n X_n e_n(z) \right| \leq c\sqrt{\log(A(r))} \sqrt{\sum_{n\geq 0} |a_n|^2 \max_{|z|=r} |e_n(z)|^2} \tag{3.0.1}$$

holds for a large amount of r's, where A is some function and $c > 0$ is some constant. Two approaches to the problem are presented. In both approaches, random series on the unit disk $\mathbb{D}$ will also be considered.

The first approach leads to a general result without restrictions on f. The rate of growth will be valid for any r outside some set of finite logarithmic measure. This extends results of Erdős and Rényi [37] and Kuryliak, Skaskiv and Skaskiv [63], and improves a result of Kuryliak [62, Theorem 3]; see Theorems 3.1.17 and 3.2.10.

The second approach provides an inequality valid for any r large enough, but only for some functions f. However, unlike the first method, there will be a single general result, namely Theorem 3.3.4, for functions defined on $\mathbb{C}$ or $\mathbb{D}$. We obtain, in particular, a generalization of a result of Nikula [80, Proposition 2].

Throughout this chapter, we will use the following notations. Every random variable considered will be defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If a and b are two positive real numbers, the notation $a \lesssim b$ means that there exists some $C > 0$ such that $a \leq Cb$ and C does not depend on any current variable such as $n \in \mathbb{N}$, $r > 0$ or $\omega \in \Omega$. For example, $r^2 - 2r + 1 \lesssim r^2$ for $r > 0$ large enough means that there exist $C > 0$ and $r_0 > 0$ such that for every $r \geq r_0$, one has $r^2 - 2r + 1 \leq Cr^2$. The notation $a \asymp b$ means $a \lesssim b$ and $b \lesssim a$. To make the reading easier, $\log_m$ means the logarithm iterated m times, and $\log_0$ is the identity map. Lastly, if a complex-valued function f is defined on a closed disk centred at the origin and of radius $r > 0$ then we define

$$\|f\|_r := \sup_{|z|=r} |f(z)|.$$

3.1 On $\mathbb{C}$ with an exceptional set

We first study the rate of growth for random power series on $\mathbb{C}$ with subgaussian coefficients, where we accept a certain exceptional set of radii r. This is the situation typically encoutered in the Wiman-Valiron theory, see [48] and [49]. The main ideas in this section come from Erdős and Rényi [37], Kuryliak [62] and Steele [92].

In this section, $(e_n)_{n\geq 0}$ will always denote the sequence of monomials i.e., $e_n(z) = z^n$ for every $z \in \mathbb{C}$ and $n \geq 0$.

First of all, we must make sure that the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely convergent in $H(\mathbb{C})$. This is a corollary of Lemma 1.2.9.

Lemma 3.1.1. *Let $f = \sum_{n\geq 0} a_n e_n$ be an entire function and $(X_n)_{n\geq 0}$ be a subgaussian sequence. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely an entire function.*

Proof. Since f is entire, we have $\lim_{n\to\infty} |a_n|^{1/n} = 0$. Therefore, for every $r > 0$, there exist $0 < \rho < 1$ and $n_0 \geq 1$ such that for every $n \geq n_0$, $r^n \sqrt{\log(n)}|a_n| \leq \rho^n$. This implies that $\sum_{n\geq 1} \sqrt{\log(n)} a_n e_n$ converges unconditionally in $H(\mathbb{C})$. Lemma 1.2.9 then assures us that $\sum_{n\geq 0} a_n X_n e_n$ is almost surely an entire function. □

Definition 3.1.2. Let $f = \sum_{n\geq 0} a_n e_n$ be an entire function. We define the functions μ_f, S_f and $G_f : [0, \infty[\longrightarrow \mathbb{R}$ for any $r \geq 0$ by

$$\mu_f(r) = \sup_{n\geq 0} |a_n| r^n,$$

$$S_f(r) = \sqrt{\sum_{n=0}^{\infty} |a_n|^2 r^{2n}},$$

$$G_f(r) = \sum_{n=0}^{\infty} |a_n| r^n.$$

The function μ_f is called the *maximum term* of f.

The function μ_f is standard in the theory of entire functions. The function S_f is less common, but appears in Erdős and Rényi [37] and Steele [92]. Note that $S_f(r) = \sqrt{\sum_{n\geq 0} |a_n|^2 \max_{|z|=r} |z^n|^2}$, so that it coincides with the term on the right-hand side of (3.0.1).

Since f is an entire function, the maps μ_f, S_f and G_f are well-defined. If f is not a constant then μ_f converges to ∞ as the next result implies.

Lemma 3.1.3. *For every entire function f, the maximum term μ_f is continuous. If f is not constant then* $\lim_{r\to\infty} \mu_f(r) = \infty$.

Proof. For the continuity of μ_f, see [52, Satz 4.2]. If f is not constant then there exists $n \geq 1$ such that $a_n \neq 0$, and since $\mu_f(r) \geq |a_n| r^n$ for every $r \geq 0$, the result follows. □

Example 3.1.4. Let $f(z) := e^z = \sum_{n\geq 0} r^n/n!$, $z \in \mathbb{C}$. By noticing that $r^n/n! = r \dots r/(1 \dots n)$, it is clear that $(r^n/n)_{n\geq 0}$ reaches its maximum at $\lfloor r \rfloor$. Then Stirling's formula yields

$$\mu_f(r) \asymp \frac{r^{\lfloor r \rfloor} e^{\lfloor r \rfloor}}{\lfloor r \rfloor^{\lfloor r \rfloor} \sqrt{2\pi \lfloor r \rfloor}},$$

which implies that $\mu_f(r) \asymp e^r/\sqrt{r}$.

Let $(X_n)_{n\geq 0}$ be a subgaussian sequence of centred independent random variables and $f = \sum_{n\geq 0} a_n e_n$ be an entire function. The aim of this section is to prove that there exists a measurable set E of finite logarithmic measure and some constant $c > 0$ such that, almost surely, the inequality

$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \leq c \sqrt{\log_2(\mu_f(r))} S_f(r) \tag{3.1.1}$$

holds for all $r \notin E$ large enough.

Definition 3.1.5. A measurable set $E \subseteq [0, \infty[$ is of *finite logarithmic measure* if $\int_{E\cap[1,\infty[} t^{-1} \mathrm{d}t$ is finite.

Example 3.1.6. If $E = \bigcup_{n\geq 1} [a_n, b_n] \subseteq [0, \infty[$ with $a_n < b_n < a_{n+1}$ for all $n \geq 1$, then E is of finite logarithmic measure if and only if $\sum_{n\geq 2} (b_n - a_n)/a_n < \infty$. Thus, the sets $\bigcup_{n\geq 1} [n^\beta, n^\beta + 1/n^\alpha]$, $\alpha, \beta > 0$ with $\alpha + \beta > 1$, and $\bigcup_{n\geq 1} [n^\alpha, n^\alpha e^{1/n^\beta}]$, $\alpha > 0$, $\beta > 1$, are of finite logarithmic measure.

Remark 3.1.7. In order to show that some property holds outside a set of finite logarithmic measure, it suffices to prove that there exists a set of finite logarithmic measure such that the property holds outside this set and for r sufficiently large.

The next two lemmas are inspired by [63, Lemma 3.2 and p. 143].

The first lemma is an application of [49, Lemma 6.15]. For the sake of completeness, we provide the proof.

Lemma 3.1.8. *Let $f = \sum_{n\geq 0} a_n e_n$ be a non-constant entire function. Then for every $\delta > 0$ there exists an open set $E \subseteq [0, \infty[$ of finite logarithmic measure such that for every $r \notin E$, one has*

$$\partial_r \log(G_f(r)) \leq \frac{1}{r} \log^{1+\delta}(G_f(r)).$$

Proof. Let $r_0 > 0$ be such that $G_f(r) > 1$ for all $r \geq r_0$, which is possible since f is non-constant. Let $E \subseteq]r_0, \infty[$ be the set where the inequality of the lemma is false. Since both sides of the inequality are continuous, the set E is open. Using the change of variables $x = \log(G_f(r))$ yields

$$\int_{E\cap[1,\infty[} \frac{1}{r}\mathrm{d}r \leq \int_{E\cap[1,\infty[} \frac{\partial_r \log(G_f(r))}{\log^{1+\delta}(G_f(r))}\mathrm{d}r \leq \int_1^\infty \frac{1}{x^{1+\delta}}\mathrm{d}x < \infty,$$

so that E is of finite logarithmic measure. □

Lemma 3.1.9. *Let $f = \sum_{n\geq 0} a_n e_n$ be a non-constant entire function. Then, for every $\delta > 0$, there exists an open set E of finite logarithmic measure such that for any $r \notin E$, one has*

$$\sum_{n=0}^{\infty} n|a_n|r^n \leq G_f(r) \log^{1+\delta}(G_f(r))$$

Proof. First notice that for every $r > 0$, one has $\partial_r G_f(r) = r^{-1} \sum_{n\geq 0} n|a_n|r^n$ and thus

$$\sum_{n=0}^{\infty} n|a_n|r^n = r\partial_r G_f(r) = rG_f(r)\partial_r \log(G_f(r)).$$

Let E be the open set given by Lemma 3.1.8. Then we get for every $r \notin E$

$$\sum_{n=0}^{\infty} n|a_n|r^n \leq rG_f(r)\frac{1}{r} \log^{1+\delta}(G_f(r)).$$

□

Lemma 3.1.13 is proved in Kahane [55, Chapter 6, Theorem 2] and will be crucial for our purposes. It will also be used in the next section. For the sake of completeness, we provide its proof. We will need the next three results.

Lemma 3.1.10 ([55, Chapter 6, Theorem 1])**.** *Let (T, μ) be a measurable space, where T is a separable topological space and μ is a finite Borel measure on T, and let B be a complex vector space of bounded continuous complex-valued functions defined on T and closed under complex conjugation. Let $(X_n)_{n=1}^K$ be a finite subgaussian sequence of centred independent real random variables with constant $\sigma > 0$, and $(f_n)_{n=1}^K$ be a finite sequence of elements of B. Assume that there exists some $\rho > 0$ such that, for* *every real function $f \in B$, there exists a measurable set $I \subseteq T$ such that $\mu(I) \geq \mu(T)/\rho$ and $|f(t)| \geq \|f\|_\infty/2$ for every $t \in I$. Then for every $s \geq 1/(2\rho)$, the random vector $P = \sum_{n=1}^{K} X_n f_n$ satisfies*

$$\mathbb{P}\left(\|P\|_\infty > 8\sigma\sqrt{\log(2\rho s)}\sqrt{\sum_{n=1}^{K}\|f_n\|_\infty^2}\right) \leq \frac{2}{s},$$

where $\|f\|_\infty := \sup_{t\in T}|f(t)|$ for each $f \in B$.

Proof. First assume that for every $1 \leq n \leq K$, the function f_n is real and set $r := \sum_{n=1}^{K}\|f_n\|_\infty^2$ and $C := \|P\|_\infty$. Assume that $f_n \neq 0$ for some $1 \leq n \leq K$. By assumptions on the measurable space (T,μ), C is measurable. Indeed, since T is separable, there exists a dense countable subset $D \subseteq T$, hence $C = \sup_{t\in D}|P(t)|$ by continuity of P. Since each $P(t)$, $t \in D$, is measurable, the random variable C is also measurable.

Let $\lambda > 0$. Since the random variables X_n, $1 \leq n \leq K$, are independent, centred and subgaussian, by Lemma 1.2.8, we have for every $t \in T$,

$$\mathbb{E}(e^{\lambda P(t)}) = \mathbb{E}\left(\prod_{n=1}^{K} e^{\lambda X_n f_n(t)}\right) = \prod_{n=1}^{K}\mathbb{E}(e^{\lambda X_n f_n(t)}) \leq \prod_{n=1}^{K} e^{\lambda^2 f_n(t)^2\sigma^2} \leq e^{\lambda^2 r\sigma^2}, \tag{3.1.2}$$

where $\sigma > 0$ is the constant associated with $(X_n)_{n=1}^{K}$ in Definition 1.2.6. Without loss of generality, we can assume that $\mu(T) = 1$. By assumption, for every $\omega \in \Omega$, there exists a measurable set $I \subseteq T$ depending on ω such that $\mu(I) \geq 1/\rho$ and $|P(t)| \geq C/2$ for every $t \in I$, hence $C/2 \leq P(t)$ or $C/2 \leq -P(t)$. This implies that

$$e^{\lambda C/2} \leq \rho\int_I e^{\lambda C/2}\mathrm{d}\mu \leq \rho\int_T\left(e^{\lambda P(t)} + e^{-\lambda P(t)}\right)\mathrm{d}\mu$$

and therefore

$$\mathbb{E}(e^{\lambda C/2}) \leq \rho\mathbb{E}\left(\int_T\left(e^{\lambda P(t)} + e^{-\lambda P(t)}\right)\mathrm{d}\mu\right).$$

By the Fubini theorem and (3.1.2), we deduce that $\mathbb{E}(e^{\lambda C/2}) \leq 2\rho e^{\lambda^2 r\sigma^2}$. Therefore, Markov's inequality yields

$$\mathbb{P}\left(\frac{C}{2} \geq \lambda r\sigma^2 + \frac{\log(2\rho s)}{\lambda}\right) \leq \mathbb{E}\left(\exp\left(\frac{\lambda C}{2} - \log(2\rho s) - \lambda^2 r\sigma^2\right)\right) \leq \frac{1}{s}.$$

By taking $\lambda = \sigma^{-1}\sqrt{\log(2\rho s)}/\sqrt{r}$ and recalling that $C = \|P\|_\infty$, we get

$$\mathbb{P}\left(\|P\|_\infty \geq 4\sqrt{r}\sigma\sqrt{\log(2\rho s)}\right) \leq \frac{1}{s}.$$

Now, if P is complex, then by taking the real and imaginary parts of P and applying to them the previous inequality, we get that

$$\begin{aligned}\mathbb{P}\left(\|P\|_\infty \geq 8\sqrt{r}\sigma\sqrt{\log(2\rho s)}\right) &\leq \mathbb{P}\left(\|\mathrm{Re}(P)\|_\infty \geq 4\sqrt{r}\sigma\sqrt{\log(2\rho s)}\right)\\ &\quad + \mathbb{P}\left(\|\mathrm{Im}(P)\|_\infty \geq 4\sqrt{r}\sigma\sqrt{\log(2\rho s)}\right) \leq \frac{2}{s}.\end{aligned}$$

Note that $\mathrm{Re}(P)$ and $\mathrm{Im}(P)$ are in B since B is closed under complex conjugation. □

For the proof of Bernstein's inequality, see [66, Chapitre 5, Lemme IV.12]. A map $t \longmapsto \sum_{n=-N}^{N} a_n e^{int}$ where $a_n \in \mathbb{C}$ for all $-N \leq n \leq N$, and $N \geq 1$, is called a *complex trigonometric polynomial of degree less than or equal to* N.

Lemma 3.1.11 (Bernstein inequality). *For every complex trigonometric polynomial* $p : [0, 2\pi] \longrightarrow \mathbb{C}$ *of degree less than or equal to* $N \geq 1$*, one has* $\|p'\|_\infty \leq N\|p\|_\infty$.

Lemma 3.1.12 ([55, Chapter 5, Proposition 5]). *For every complex trigonometric polynomial* $p : [0, 2\pi] \longrightarrow \mathbb{C}$ *of degree less than or equal to* $N \geq 1$*, there exists a closed set* $I \subseteq [0, 2\pi]$ *of Lebesgue measure* $1/N$ *such that* $|p(t)| \geq \|p\|_\infty/2$ *for every* $t \in I$.

Proof. Since p is continuous, there exists some $t_0 \in [0, 2\pi]$ such that $|p(t_0)| = \|p\|_\infty$. Extend p periodically to $\mathbb{R}$, and let $t \in \mathbb{R}$. By the Mean Value Theorem, $|p(t)-p(t_0)| \leq \sup_{s\in[0,2\pi]} |p'(s)||t - t_0|$. Then, by Bernstein's inequality, we get $|p(t) - p(t_0)| \leq N\|p\|_\infty|t - t_0|$, hence

$$|p(t)| \geq |p(t_0)| - |p(t) - p(t_0)| \geq (1 - N|t - t_0|)\|p\|_\infty.$$

If $t \in I := \big\{s \mod 2\pi \mid s \in [t_0 - 1/(2N), t_0 + 1/(2N)]\big\}$ then $|p(t)| \geq \|p\|_\infty/2$. This concludes the proof since I has indeed Lebesgue measure $1/N$. □

Combining the two previous results yields Lemma 3.1.13, with $T = [0, 2\pi]$ endowed with the Lebesgue measure, B the space of complex trigonometric polynomials, $\rho = 2\pi N$ and $s = 2N^p$.

In the following results, the random variables X_n may be complex: one needs to apply Lemma 3.1.10 to $\mathrm{Re}(X_n)$ and $\mathrm{Im}(X_n)$.

Lemma 3.1.13 ([55, Chapter 6, Theorem 2]). *Let* $(X_n)_{n=1}^K$ *be a subgaussian sequence of centred independent random variables. Let* $(a_n)_{n=1}^K$ *be a finite sequence of complex numbers,* $(q_n)_{n=1}^K$ *be a finite sequence of complex trigonometric polynomials of degree less than or equal to* $N \geq 1$ *and* $p > 0$ *a real number. Then there exists a constant* $c > 0$ *that depends only on the distribution of* $(X_n)_{n=1}^K$ *and* p *such that*

$$\mathbb{P}\left(\sup_{\theta\in[0,2\pi]} \left|\sum_{n=1}^{K} a_n X_n q_n(\theta)\right| > c\sqrt{\log(N)}\sqrt{\sum_{n=1}^{K} |a_n|^2\|q_n\|_\infty^2}\right) \leq \frac{c}{N^p}.$$

Lemma 3.1.13 says that the first terms of the random series $\sum_{n\geq 0} a_n X_n e_n$ already give the kind of rate of growth (3.1.1) we are seeking. One may then hope that the rest of the terms are small in some sense.

To get a rate of growth for a random entire function, we will use a deterministic result which gives a link between the maximum modulus of an entire function f i.e., $\|f\|_r$, $r > 0$, and its maximum term μ_f. This theorem can be proved with the Wiman-Valiron theory, see Hayman [48, Theorem 6] and also [37, p. 48] and [92, p. 550]. This result is due to Rosenbloom [85].

Theorem 3.1.14. *Let* f *be a non-constant entire function. For every* $\delta > 0$*, there exists a set* $E \subseteq [0, \infty[$ *of finite logarithmic measure such that for every* $r \notin E$*, one has*

$$\|f\|_r \leq \mu_f(r)\sqrt{\log(\mu_f(r))}\Big(\log_2(\mu_f(r))\Big)^{1+\delta}.$$

Both sides of the inequality given by Theorem 3.1.14 are continuous where they are defined, see Lemma 3.1.3 and [52, Satz 1.1]. Consequently, the set E can be chosen to be open.

Lemma 3.1.15. *Let $f = \sum_{n\geq 0} a_n e_n$ be a non-constant entire function and let $(X_n)_{n\geq 0}$ be a subgaussian sequence of centred independent random variables. Let $0 < \alpha < 1$ and $\delta > 0$. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely an entire function, and there exist a constant $c > 0$ and a set $E \subseteq [0, \infty[$ of finite logarithmic measure such that for any $r \notin E$,*

$$\mathbb{P}\left(\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \geq c\sqrt{\log(N)} S_f(r) \right) \lesssim \frac{1}{N^{2/(1-\alpha)}}$$

for every $N \geq \big(\log(\mu_f(r))\big)^{\frac{3}{2}+\delta}$, and E can be chosen to be open.

Proof. The series $\sum_{n\geq 0} a_n X_n e_n$ is almost surely an entire function by Lemma 3.1.1. Take the open set $E \subseteq [0, \infty[$ as the finite union of the open sets of finite logarithmic measure given by Lemma 3.1.9 and Theorem 3.1.14 applied to $z \longmapsto \sum_{n\geq 0} |a_n| z^n$ and $\delta/2 > 0$, and set $\beta := 1/(1-\alpha)$.

Define $B_n := \{|X_n| \geq n^\alpha\} \subseteq \Omega$ for each $n \geq 1$. We get that, since $(X_n)_{n\geq 0}$ is a subgaussian sequence,

$$\mathbb{P}(B_n) \lesssim e^{-n^{2\alpha}/\tau^2} \lesssim \frac{1}{n^3}$$

for every $n \geq 1$ and some $\tau > 0$ given by Lemma 1.2.7.

Let $r \notin E$ be large enough and $N \geq \big(\log(\mu_f(r))\big)^{\frac{3}{2}+\delta}$. Define $B(r) := \bigcup_{n>N^\beta} B_n$. Then

$$\mathbb{P}(B(r)) \leq \sum_{n>N^\beta} \mathbb{P}(B_n) \lesssim \sum_{n>N^\beta} \frac{1}{n^3} \lesssim \frac{1}{N^{2\beta}} \tag{3.1.3}$$

for $r > 0$ large enough. On the complement of $B(r)$, we get that

$$\begin{aligned} \left\| \sum_{n>N^\beta} a_n X_n e_n \right\|_r \leq \sum_{n>N^\beta} |X_n||a_n| r^n &\leq \sum_{n>N^\beta} n^\alpha |a_n| r^n \\ &\leq N^{-1} \sum_{n>N^\beta} n |a_n| r^n. \end{aligned}$$

The last inequality holds because if $n \geq N^\beta$ then $n^\alpha \leq n/N$. By Lemma 3.1.9 and Theorem 3.1.14, we finally get that

$$\begin{aligned} \left\| \sum_{n>N^\beta} a_n X_n e_n \right\|_r &\leq N^{-1} G_f(r) \log^{1+\frac{\delta}{2}}(G_f(r)) \\ &\lesssim N^{-1} \mu_f(r) \sqrt{\log(\mu_f(r))} \log_2^{1+\frac{\delta}{2}}(\mu_f(r)) \log^{1+\frac{\delta}{2}}(\mu_f(r)) \\ &\lesssim N^{-1} \mu_f(r) \log^{\frac{3}{2}+\delta}(\mu_f(r)) \leq \mu_f(r) \leq S_f(r). \end{aligned} \tag{3.1.4}$$

Therefore, there is a constant $\widetilde{c} > 0$ such that if $r \notin E$ is large enough then

$$\mathbb{P}\left(\left\| \sum_{n>N^\beta} a_n X_n e_n \right\|_r > \widetilde{c} S_f(r)\right) \leq \mathbb{P}(B(r)) \lesssim \frac{1}{N^{2\beta}},$$

where the first inequality holds by (3.1.4), and the second one holds by (3.1.3).

By Lemma 3.1.13 applied to $q_n = r^n e^{int}$, $n \geq 0$, $p = 2$ and $K = \lfloor N^\beta \rfloor$, we have on the other hand

$$\mathbb{P}\left(\left\| \sum_{0 \leq n \leq N^\beta} a_n X_n e_n \right\|_r \geq c\sqrt{\log(N)} S_f(r)\right) \lesssim \frac{1}{N^{2\beta}}$$

for some constant $c \geq \widetilde{c}$, and hence

$$\mathbb{P}\left(\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \geq 2c\sqrt{\log(N)} S_f(r)\right) \lesssim \frac{1}{N^{2\beta}} + \frac{1}{N^{2\beta}}$$

for r large enough, $r \notin E$. □

The next lemma is the last result we will need. Versions of it can be found in [37, p. 49], [62, Lemma 8] or [92, p. 555].

Lemma 3.1.16. *Let $h : [r_0, \infty[\longrightarrow [0, \infty[$ be a continuous non-decreasing function such that $\lim_{r\to\infty} h(r) = \infty$ and $h(r_0) \geq 1$, where $r_0 \geq 0$. Let $E \subseteq [r_0, \infty[$ be an open set of finite logarithmic measure. Then there exists a non-decreasing sequence $(r_k)_{k\in J}$, where $J \subseteq \mathbb{N}_0$, such that for every $k \in J$,*

(i) $r_k \notin E$,

(ii) $h(r_k) \geq k$,

(iii) *for any $r \notin E$, there exists $k \in J$ such that $r \leq r_k$ and $h(r_k) \leq h(r) + 1$.*

Proof. Define for each $k \geq 1$ the closed, possibly empty, set

$$U_k := \{r \geq r_0 \mid k \leq h(r) \leq k+1\}.$$

These sets are indeed closed since h is continuous. They are also bounded since $\lim_{r\to\infty} h(r) = \infty$, and thus they are compact. Define $J := \{k \in \mathbb{N}_0 \mid U_k \setminus E \neq \emptyset\}$. For each $k \in J$, there exists $r_k \in U_k \setminus E$ such that $r_k = \sup(U_k \setminus E)$. Since $\lim_{r\to\infty} h(r) = \infty$ and E is of finite logarithmic measure, the set J is infinite.

Let $r \geq r_0$. Since $h(r_0) \geq 1$, there exists $k \in \mathbb{N}_0$ such that $k \leq h(r) \leq k+1$. If $r \notin E$ then $k \in J$, and $r \leq r_k$ by definition of r_k. By definition of U_k, we also have $h(r_k) \leq h(r) + 1$. □

Theorem 3.1.17 is the main result of this section. First, thanks to the Borel-Cantelli lemma, we will prove the desired inequality for a suitable sequence $(r_n)_{n\geq 1}$ chosen with Lemma 3.1.16. The properties of this sequence and the Maximum Principle will conclude the proof of Theorem 3.1.17.

Theorem 3.1.17. *Let $f = \sum_{n\geq 0} a_n e_n$ be a non-constant entire function and $(X_n)_{n\geq 0}$ be a subgaussian sequence of centred independent random variables. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely an entire function, and there exist a constant $c > 0$ and an open set $E \subseteq [0,\infty[$ of finite logarithmic measure such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \leq c\sqrt{\log_2(\mu_f(r))}S_f(r)$$

for every $r \geq r_0$, $r \notin E$.

Proof. Take the open set $E \subseteq [0,\infty[$ of finite logarithmic measure as the finite union of the sets given by Lemma 3.1.15 applied to f, $\alpha \in]0,1[$ and a fixed $\delta > 0$, and Theorem 3.1.14 applied to $z \longmapsto \sum_{n\geq 0} |a_n| z^n$ and $\delta > 0$. By Lemma 3.1.16 applied to $h = \log(S_f)$ and $r_0 \geq 0$ so large that $\log(S_f(r_0)) \geq 1$, we get a non-decreasing sequence $(r_k)_{k\in J}$ converging to ∞, where $J \subseteq \mathbb{N}_0$, and satisfying assertions (i), (ii) and (iii) of the lemma.

Set $\beta := 1/(1-\alpha)$. Define for each $k \in J$ the real number

$$N_k := \log^{\frac{3}{2}+\delta}(\mu_f(r_k))$$

and the measurable set

$$A_k := \left\{ \left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_{r_k} \geq c\sqrt{\log(N_k)}S_f(r_k) \right\} \subseteq \Omega,$$

where $c > 0$ is the constant of Lemma 3.1.15. We can assume that $N_k \geq 1$ for all $k \in J$. Then (i) of Lemma 3.1.16, Lemma 3.1.15 and the definition of N_k imply that

$$\sum_{k\in J} \mathbb{P}(A_k) \lesssim \sum_{k\in J} \frac{1}{N_k^{2\beta}} = \sum_{k\in J} \frac{1}{\log^{\beta(3+2\delta)}(\mu_f(r_k))}.$$

By Theorem 3.1.14, for every $r \notin E$, we have

$$\mu_f(r) \leq S_f(r) \leq \sum_{n\geq 0} |a_n| r^n \leq \mu_f(r)\sqrt{\log(\mu_f(r))}\Big(\log_2(\mu_f(r))\Big)^{1+\delta}.$$

This implies that $\log(S_f(r)) \asymp \log(\mu_f(r))$ for $r \notin E$. Therefore

$$\sum_{k\in J} \mathbb{P}(A_k) \lesssim \sum_{k\in J} \frac{1}{\log^{\beta(3+2\delta)}(S_f(r_k))} \lesssim \sum_{k=1}^{\infty} \frac{1}{k^{\beta(3+2\delta)}} < \infty$$

by (i) and (ii) of Lemma 3.1.16. This in turn implies by the Borel-Cantelli lemma that for almost surely every $\omega \in \Omega$, there exists $k_0(\omega) \in J$ such that for every $k \in J$ with $k \geq k_0(\omega)$,

$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_{r_k} \leq c\sqrt{\log(N_k)}S_f(r_k). \tag{3.1.5}$$

Let $r \notin E$ be large, and let $k \geq k_0(\omega)$ be such that $k \in J$ given by (iii) of Lemma 3.1.16. The Maximum Principle and inequality (3.1.5) yield

$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \leq \left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_{r_k} \leq c\sqrt{\log(N_k)} S_f(r_k)$$
$$\lesssim \sqrt{\log_2(\mu_f(r_k))} S_f(r_k).$$

Since $\mu_f \leq S_f$ and $S_f(r_k) \leq e S_f(r)$ by (iii) of Lemma 3.1.16, we get that

$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \lesssim \sqrt{\log_2(S_f(r_k))} S_f(r_k) \lesssim \sqrt{\log_2(S_f(r))} S_f(r).$$

We conclude the proof by using the fact that $\log(S_f(r)) \lesssim \log(\mu_f(r))$ for $r \notin E$. Note that the constant c in (3.1.5) is possibly replaced by a larger constant $\widetilde{c}$, but independently of $\omega \in \Omega$. □

In Kuryliak [62], the sequence $(r_k)_{k\geq 1}$ was constructed from the maximum term μ_f. The idea of constructing this sequence from S_f instead comes from [37] and [92].

The previous theorem includes the result of Erdős and Rényi [37, Theorem 2] who used Rademacher random variables. Indeed, every bounded random variable is subgaussian. Recall that a *Rademacher variable* is a random variable $X : \Omega \longrightarrow \mathbb{R}$ such that $\mathbb{P}(X = 1) = 1/2 = \mathbb{P}(X = -1)$. In their main result, [37, Theorem 1], Erdős and Rényi obtained a rate of growth written in terms of the maximum term. The following theorem extends this result to arbitrary centred subgaussian sequences.

Theorem 3.1.18. *Let $f = \sum_{n\geq 0} a_n e_n$ be a non-constant entire function and $(X_n)_{n\geq 0}$ be a subgaussian sequence of centred independent random variables. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely an entire function and for every $\delta > 0$, there exist a constant $c > 0$ and an open set $E \subseteq [0, \infty[$ of finite logarithmic measure such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \leq c\mu_f(r)\big(\log(\mu_f(r))\big)^{1/4}\big(\log_2(\mu_f(r))\big)^{1+\delta}$$

for every $r \geq r_0$, $r \notin E$.

Proof. This result is a direct consequence of Theorem 3.1.17. Let $\delta > 0$. Define the open set E as the finite union of the sets given by Theorem 3.1.14 applied to $z \longmapsto \sum_{n\geq 0} |a_n| z^n$ and Theorem 3.1.17 applied to f. Let $r \notin E$, Theorem 3.1.14 yields

$$S_f^2(r) \leq \mu_f(r) \sum_{n\geq 0} |a_n| r^n \leq \mu_f(r)\mu_f(r)\sqrt{\log(\mu_f(r))}\Big(\log_2(\mu_f(r))\Big)^{1+\delta}.$$

It remains to apply Theorem 3.1.17 to conclude the proof. □

This corollary also improves the special case of $\rho = 1$ in the result of Kuryliak [62, Theorem 3] since in the article, the exponent of the iterated logarithm is $3/2 + \delta$.

3.2 On the disk with an exceptional set

We now study the rate of growth for random power series on $\mathbb{D}$ with subgaussian coefficients, where we accept a certain exceptional set of radii r. Kuryliak, Skaskiv and Skaskiv [63] obtained a result on $\mathbb{D}$ and more generally on a polydisk for uniformly bounded random variables. Theorem 3.2.10 generalizes [63, Theorem 2.3] to subgaussian random variables, in the case of holomorphic functions on $\mathbb{D}$. The main ideas in this section come from [63], combined with ideas of the previous section.

Again in this section, $(e_n)_{n\geq 0}$ will always denote the sequence of monomials i.e., $e_n(z) = z^n$ for every $z \in \mathbb{D}$ and $n \geq 0$.

First of all, we must make sure that the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely convergent in $H(\mathbb{D})$. Like Lemma 3.1.1, this is a corollary of Lemma 1.2.9.

Lemma 3.2.1. *Let $f = \sum_{n\geq 0} a_n e_n \in H(\mathbb{D})$ and let $(X_n)_{n\geq 0}$ be a subgaussian sequence. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely a holomorphic function on $\mathbb{D}$.*

Proof. Since f is holomorphic on $\mathbb{D}$, we have $\limsup_{n\to\infty} |a_n|^{1/n} \leq 1$. Therefore, for every $0 < r < 1$, there exist $0 < \rho < 1$ and $n_0 \geq 1$ such that for every $n \geq n_0$, $r^n \sqrt{\log(n)}|a_n| \leq \rho^n$. This implies that $\sum_{n\geq 1} \sqrt{\log(n)} a_n e_n$ converges unconditionally in $H(\mathbb{D})$. Lemma 1.2.9 then ensures that $\sum_{n\geq 0} a_n X_n e_n$ is almost surely a holomorphic function on $\mathbb{D}$. □

The aim of this section is to bound the sup-norm of the random power series $\sum_{n\geq 0} a_n X_n e_n$ outside some subset of finite logarithmic measure of $[0, 1[$.

Definition 3.2.2. A measurable set $E \subseteq [0, 1[$ is of *finite logarithmic measure* if $\int_E \frac{1}{1-t}\mathrm{d}t$ is finite.

Example 3.2.3. If $E = \bigcup_{n\geq 1}[a_n, b_n] \subseteq [0, \infty[$ with $0 < a_n < b_n < a_{n+1} < 1$ for all $n \geq 1$, then E is of finite logarithmic measure if and only if $\sum_{n\geq 1}(b_n - a_n)/(1 - b_n) < \infty$. Thus, the sets $\bigcup_{n\geq 2}[1 - 1/n^\alpha, 1 - 1/n^\alpha + 1/n^\beta]$, $\alpha, \beta > 0$, $\beta > 1 + \alpha$, are of finite logarithmic measure.

Remark 3.2.4. In order to show that some property holds outside a set of finite logarithmic measure, it suffices to prove that there exists a set of finite logarithmic measure such that the property holds outside this set and for r close enough to 1.

The proof of the main theorem of this section, Theorem 3.2.10, is similar to the the proof of Theorem 3.1.17. The first term of the series will give the growth we are seeking, and the remaining terms will be smaller than the first ones. Lemma 3.2.8 will be the probabilistic tool needed, and it is again a corollary of Lemma 3.1.13.

The maximum term μ_f of a function $f \in H(\mathbb{D})$ and the functions S_f and G_f are defined exactly in the same way as for entire functions, see Definition 3.1.2.

For the sake of completeness, we provide the proof of the next lemma.

Lemma 3.2.5 ([61, Lemma 1])**.** *Let $f = \sum_{n\geq 0} a_n e_n \in H(\mathbb{D})$ be non-constant with $\sum_{n\geq 0} |a_n| > 1$. Then for every $\delta > 0$ there exists an open set $E \subseteq [0, 1[$ of finite logarithmic measure such that for every $r \notin E$, one has*

$$\partial_r \log(G_f(r)) \leq \frac{1}{1-r} \log^{1+\delta}(G_f(r)).$$

Proof. Let $0 < r_0 < 1$ and $\varepsilon > 0$ be such that $\log(G_f(r)) \geq \varepsilon$ for all $r_0 < r < 1$. Let $E \subseteq]r_0, 1[$ be the set where the inequality is false. It is an open set since both sides of the inequality are continuous. Using the change of variables $x = \log(G_f(r))$ yields

$$\int_E \frac{1}{1-r}\mathrm{d}r \leq \int_E \frac{\partial_r \log(G_f(r))}{\log^{1+\delta}(G_f(r))}\mathrm{d}r \leq \int_\varepsilon^1 \frac{1}{x^{1+\delta}}\mathrm{d}x < \infty.$$

$\square$

Theorem 3.2.6 will be the analogous deterministic theorem to Theorem 3.1.14 we will need.

Theorem 3.2.6 ([61, Theorem 2])**.** *Let $f = \sum_{n\geq 0} a_n e_n \in H(\mathbb{D})$ be non-constant. Then for every $\delta > 0$ there exists an open set $E \subseteq [0, 1[$ of finite logarithmic measure such that*

$$\|f\|_r \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \log^{\frac{1+\delta}{2}}\Big(\frac{\mu_f(r)}{1-r}\Big)$$

for every $r \notin E$.

Lemma 3.2.7. *Let $f = \sum_{n\geq 0} a_n e_n \in H(\mathbb{D})$ be non-constant, and let $\delta > 0$. There exists an open set E of finite logarithmic measure such that for any $r \notin E$, one has*

$$\sum_{n=0}^{\infty} n|a_n|r^n \lesssim \frac{\mu_f(r)}{(1-r)^{2+\delta}} \log^{3(1+\delta)/2}\Big(\frac{\mu_f(r)}{1-r}\Big).$$

Proof. By multiplying f with a constant we can assume that $\mu_f(r) > 1$ if r is sufficiently big, and then also $\sum_{n\geq 0}|a_n| > 1$.

First notice that $\sum_{n\geq 0} n|a_n|r^n \asymp \partial_r G_f(r) = G_f(r)\partial_r \log(G_f(r))$. Take the open set $E \subseteq [0, 1[$ as the finite union of the open sets of finite logarithmic measure given by Lemma 3.2.5 and Theorem 3.2.6 applied to $z \longmapsto \sum_{n\geq 0}|a_n|z^n$ and $\delta > 0$. Then we get for every $r \notin E$ close enough to 1,

$$\begin{aligned}\sum_{n=0}^{\infty} n|a_n|r^n \lesssim{}& \frac{\mu_f(r)}{(1-r)^{1+\delta}} \log^{\frac{1+\delta}{2}}\big(\frac{\mu_f(r)}{1-r}\big)\frac{1}{1-r}\Bigg(\log(\mu_f(r)) \\ &+ (1+\delta)\Big(\log\big(\frac{1}{1-r}\big) + \frac{1}{2}\log_2\big(\frac{\mu_f(r)}{1-r}\big)\Big)\Bigg)^{1+\delta}.\end{aligned}$$

Thus we get that

$$\begin{aligned}\sum_{n=0}^{\infty} n|a_n|r^n &\lesssim \frac{\mu_f(r)}{(1-r)^{2+\delta}} \log^{\frac{1+\delta}{2}}\big(\frac{\mu_f(r)}{1-r}\big)(1+\delta)^{1+\delta}\Big(\frac{3}{2}\log\big(\frac{\mu_f(r)}{1-r}\big)\Big)^{1+\delta} \\ &= \big(\frac{3}{2}\big)^{1+\delta}(1+\delta)^{1+\delta}\frac{\mu_f(r)}{(1-r)^{2+\delta}}\log^{3(1+\delta)/2}\big(\frac{\mu_f(r)}{1-r}\big).\end{aligned}$$

$\square$

The next lemma is analogous to Lemma 3.1.15.

Lemma 3.2.8. *Let $f = \sum_{n\geq 0} a_n e_n \in H(\mathbb{D})$ be non-constant, and let $(X_n)_{n\geq 0}$ be a subgaussian sequence of centred independent random variables. Let $0 < \alpha < 1$ and $\delta > 0$. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely a holomorphic function on $\mathbb{D}$, and there exist a constant $c > 0$ and an open set $E \subseteq [0,1[$ of finite logarithmic measure such that for any $r \notin E$,*

$$\mathbb{P}\left(\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_r \geq c\sqrt{\log(N)}\frac{\mu_f(r)}{(1-r)^{\frac{1+\delta}{2}}}\log^{(1+\delta)/4}\left(\frac{\mu_f(r)}{1-r}\right)\right) \lesssim \frac{1}{N^{2/(1-\alpha)}}$$

for every $N \geq \frac{1}{(1-r)^{2+\delta}}\Big(\log\big(\frac{\mu_f(r)}{1-r}\big)\Big)^{3(1+\delta)/2}$.

Proof. The series $\sum_{n\geq 0} a_n X_n e_n$ is almost surely holomorphic on $\mathbb{D}$ by Lemma 3.2.1. Take the open set $E \subseteq [0,1[$ as the finite union of the open sets of finite logarithmic measure given by Theorem 3.2.6 and Lemma 3.2.7 applied to $\delta > 0$, and set $\beta := 1/(1-\alpha)$.

Define $B_n := \{|X_n| \geq n^\alpha\} \subseteq \Omega$ for each $n \geq 1$. We get that, since $(X_n)_{n\geq 0}$ is a subgaussian sequence,

$$\mathbb{P}(B_n) \lesssim e^{-n^{2\alpha}/\tau^2} \lesssim \frac{1}{n^3}$$

for every $n \geq 1$ and some $\tau > 0$ given by Lemma 1.2.7.

Let $r \notin E$ be close enough to 1 and $N \geq (1-r)^{-(2+\delta)}\Big(\log\big(\mu_f(r)/(1-r)\big)\Big)^{3(1+\delta)/2}$. Define now $B(r) := \bigcup_{n>N^\beta} B_n$. Then

$$\mathbb{P}(B(r)) \leq \sum_{n>N^\beta} \mathbb{P}(B_n) \lesssim \sum_{n>N^\beta} \frac{1}{n^3} \lesssim \frac{1}{N^{2\beta}}. \tag{3.2.1}$$

On the complement of $B(r)$, we get that

$$\begin{aligned}\left\|\sum_{n>N^\beta} a_n X_n e_n\right\|_r \leq \sum_{n>N^\beta} |X_n||a_n|r^n &\leq \sum_{n>N^\beta} n^\alpha |a_n| r^n \\ &\leq \frac{1}{N}\sum_{n>N^\beta} n|a_n|r^n.\end{aligned}$$

The last inequality holds because if $n \geq N^\beta$ then $n^\alpha \leq n/N$. By Lemma 3.2.7, we finally get that

$$\left\|\sum_{n>N^\beta} a_n X_n e_n\right\|_r \lesssim \frac{1}{N}\frac{\mu_f(r)}{(1-r)^{2+\delta}}\log^{3(1+\delta)/2}\left(\frac{\mu_f(r)}{1-r}\right) \leq \mu_f(r) \tag{3.2.2}$$

for $r \notin E$. Therefore, there is a constant $\widetilde{c} > 0$ such that if $r \notin E$ is close enough to 1 then

$$\begin{aligned}\mathbb{P}\left(\left\|\sum_{n>N^\beta} a_n X_n e_n\right\|_r > \widetilde{c}\sqrt{\log(N)}\frac{\mu_f(r)}{(1-r)^{\frac{1+\delta}{2}}}\log^{(1+\delta)/4}\left(\frac{\mu_f(r)}{1-r}\right)\right) &\leq \mathbb{P}(B(r)) \\ &\lesssim \frac{1}{N^{2\beta}},\end{aligned}$$

where the first inequality holds by (3.2.2), and the second one holds by (3.2.1).

By Lemma 3.1.13 applied to $q_n = r^n e^{int}$, $n \geq 0$, $p = 2$ and $K = \lfloor N^\beta \rfloor$, we have on the other hand

$$\mathbb{P}\left(\Big\| \sum_{0 \leq n \leq N^\beta} a_n X_n e_n \Big\|_r \geq c\sqrt{\log(N)} S_f(r) \right) \lesssim \frac{1}{N^{2\beta}}$$

for some constant $c \geq \widetilde{c}$. Furthermore, we get by Theorem 3.2.6

$$S_f^2(r) \leq \mu_f(r) \sum_{n=0}^{\infty} |a_n| r^n \leq \frac{\mu_f(r)^2}{(1-r)^{1+\delta}} \log^{(1+\delta)/2}\left(\frac{\mu_f(r)}{1-r}\right),$$

and hence

$$\mathbb{P}\left(\Big\| \sum_{n=0}^{\infty} a_n X_n e_n \Big\|_r \geq 2c\sqrt{\log(N)} \frac{\mu_f(r)}{(1-r)^{\frac{1+\delta}{2}}} \log^{(1+\delta)/4}\left(\frac{\mu_f(r)}{1-r}\right) \right) \lesssim \frac{1}{N^{2\beta}} + \frac{1}{N^{2\beta}}$$

for r close enough to 1, $r \notin E$. □

Lemma 3.2.9. *Let $h, g : [r_0, 1[\longrightarrow [0, \infty[$ be two continuous non-decreasing functions such that $\lim_{r\to\infty} h(r) = \infty$, $h(r_0) \geq 1$ and $g(r_0) \geq 1$, where $0 \leq r_0 < 1$. Let $E \subseteq [r_0, 1[$ be an open set of finite logarithmic measure. Then there exists a family $(r_{l,k})_{(l,k)\in J} \subseteq [r_0, 1[$, where $J \subseteq \mathbb{N}_0^2$, such that for every $(l,k) \in J$,*

(i) *$r_{l,k} \notin E$,*

(ii) *$h(r_{l,k}) \geq k$ and $g(r_{l,k}) \geq l$,*

(iii) *for any $r \notin E$, there exists $(l,k) \in J$ such that $r \leq r_{l,k}$, $h(r_{l,k}) \leq h(r) + 1$ and $g(r_{l,k}) \leq g(r) + 1$.*

Proof. Define for each $k, l \geq 1$ the closed, possibly empty, set

$$U_{l,k} := \big\{ r_0 \leq r < 1 \mid k \leq h(r) \leq k+1 \text{ and } l \leq g(r) \leq l+1 \big\}.$$

These sets are indeed closed since h and g are continuous. They are also bounded since $\lim_{r\to 1} h(r) = \infty$, and thus they are compact. Define $J := \{(l,k) \in \mathbb{N}_0^2 \mid U_{l,k} \setminus E \neq \emptyset\}$. For each $(l,k) \in J$, there exists $r_{l,k} \in U_{l,k} \setminus E$ such that $r_{l,k} = \sup(U_{l,k} \setminus E)$.

Let $r_0 \leq r < 1$. Since $h(r_0) \geq 1$ and $g(r_0) \geq 1$, there exists $(l,k) \in \mathbb{N}_0^2$ such that $k \leq h(r) \leq k+1$ and $l \leq g(r) \leq l+1$. If $r \notin E$ then $(l,k) \in J$, and $r \leq r_{l,k}$ by definition of $r_{l,k}$. By definition of $U_{l,k}$, we also have $h(r_{l,k}) \leq h(r) + 1$ and $g(r_{l,k}) \leq g(r) + 1$. □

Theorem 3.2.10 is the main result of this section.

Theorem 3.2.10. *Let $f = \sum_{n\geq 0} a_n e_n \in H(\mathbb{D})$ be non-constant, and let $(X_n)_{n\geq 0}$ be a subgaussian sequence of centred independent random variables. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely a holomorphic function on $\mathbb{D}$, and for every*

$\delta > 0$, there exist a constant $c > 0$ and an open set $E \subseteq [0,1[$ of finite logarithmic measure such that almost surely, there exists $r_0 > 0$ such that

$$\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_r \leq c \frac{\mu_f(r)}{(1-r)^{\frac{1+\delta}{2}}} \log^{\frac{1+\delta}{4}}\left(\frac{\mu_f(r)}{1-r}\right)$$

for every $r_0 \leq r < 1$, $r \notin E$.

Proof. Without loss of generality, we can assume that $\mu_f(r) > e$ for every $0 < r < 1$ close enough to 1 since f is not a constant. Pick $0 < \alpha < 1$ and set $\beta := 1/(1-\alpha)$. Take the open set $E \subseteq [0,1[$ of finite logarithmic measure given by Lemma 3.2.8 applied to α and $\delta > 0$.

Let $(r_{l,k})_{l,k \in J}$, where $J \subseteq \mathbb{N}_0^2$, be the family given by Lemma 3.2.9 applied to $h(r) = \log(1/(1-r))$, $g = \log(\mu_f)$ and $0 \leq r_0 < 1$ so large that $h(r_0) \geq 1$ and $g(r_0) \geq 1$. Define for each $(l,k) \in J$ the real number

$$N_{l,k} := \frac{1}{(1-r_{l,k})^{2+\delta}} \log^{3(1+\delta)/2}\left(\frac{\mu_f(r_{l,k})}{1-r_{l,k}}\right) \geq 1$$

and the measurable set

$$A_{l,k} := \left\{\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r_{l,k}} \geq c\sqrt{\log(N_{l,k})} \frac{\mu_f(r_{l,k})}{(1-r_{l,k})^{\frac{1+\delta}{2}}} \log^{\frac{1+\delta}{4}}\left(\frac{\mu_f(r_{l,k})}{1-r_{l,k}}\right)\right\},$$

where $c > 0$ is the constant of Lemma 3.2.8. We can assume that $N_{l,k} \geq 1$ for all $(l,k) \in J$. Then Lemma 3.2.8, the definition of $N_{l,k}$ and (i) and (ii) of Lemma 3.2.9 imply that

$$\begin{aligned}\sum_{(l,k)\in J} \mathbb{P}(A_{l,k}) \lesssim \sum_{(l,k)\in J} \frac{1}{N_{l,k}^{2\beta}} &= \sum_{(l,k)\in J} \frac{(1-r_{l,k})^{2\beta(2+\delta)}}{\log^{3\beta(1+\delta)}\left(\frac{\mu_f(r_{l,k})}{1-r_{l,k}}\right)} \\ &\leq \sum_{(l,k)\in J} \frac{1}{e^{k2\beta(2+\delta)}(l+k)^{3\beta(1+\delta)}}.\end{aligned}$$

Therefore $\sum_{(l,k)\in J} \mathbb{P}(A_{l,k}) < \infty$ and by the Borel-Cantelli lemma, we have that for almost surely every $\omega \in \Omega$, there exist $l_0(\omega), k_0(\omega) \geq 1$ such that for every $l \geq l_0(\omega)$ and $k \geq k_0(\omega)$ such that $(l,k) \in J$, one has

$$\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r_{l,k}} \leq c\sqrt{\log(N_{l,k})} \frac{\mu_f(r_{l,k})}{(1-r_{l,k})^{\frac{1+\delta}{2}}} \log^{\frac{1+\delta}{4}}\left(\frac{\mu_f(r_{l,k})}{1-r_{l,k}}\right). \tag{3.2.3}$$

Let $r \notin E$ be such that $r \geq r_{l_0(\omega),k(\omega)}$. By (iii) of Lemma 3.2.9, there exist $l \geq l_0(\omega)$ and $k \geq k_0(\omega)$ such that $(l,k) \in J$ and $r \leq r_{l,k}$. Let $\varepsilon > 0$. The Maximum

Principle and inequality (3.2.3) yield

$$\begin{aligned}\Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_r &\le \Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_{r_{l,k}} \\ &\le c\sqrt{\log(N_{l,k})}\frac{\mu_f(r_{l,k})}{(1-r_{l,k})^{\frac{1+\delta}{2}}}\log^{\frac{1+\delta}{4}}\left(\frac{\mu_f(r_{l,k})}{1-r_{l,k}}\right) \\ &\lesssim \frac{\mu_f(r_{l,k})}{(1-r_{l,k})^{\frac{1+\delta}{2}+\varepsilon}}\log^{\frac{1+\delta}{4}+\varepsilon}\left(\frac{\mu_f(r_{l,k})}{1-r_{l,k}}\right),\end{aligned}$$

the last inequality holds if r is close enough to 1, and then also $r_{l,k}$. By (iii) of Lemma 3.2.9, we finally get that

$$\Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_r \lesssim \frac{\mu_f(r)}{(1-r)^{\frac{1+\delta}{2}+\varepsilon}}\log^{\frac{1+\delta}{4}+\varepsilon}\left(\frac{\mu_f(r)}{1-r}\right).$$

Note that the constant c in (3.2.3) is possibly replaced by a larger constant $\widetilde{c}$, but independently of $\omega \in \Omega$. □

3.3 Without an exceptional set

We now present the second approach. This method yields a rate of growth valid without an exceptional set of finite logarithmic measure. In addition, it can be applied to random entire functions or to random functions defined on the unit disk $\mathbb{D}$.

For the rest of this section, let E be the space $H(\mathbb{C})$ or $H(\mathbb{D})$ and let $w = \infty$ if $E = H(\mathbb{C})$, or $w = 1$ if $E = H(\mathbb{D})$. Assume that the random vector $\sum_{n=0}^{\infty} a_n X_n e_n$ is almost surely well-defined on the Fréchet space E, where $(X_n)_{n\ge 0}$ is a subgaussian sequence of centred independent random variables and $f = \sum_{n\ge 0} a_n e_n \in E$. We want to prove that

$$\Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_r \lesssim \sqrt{\log(A(r))}\sqrt{\sum_{n\ge 0}|a_n|^2\|e_n\|_r^2}$$

holds for $r > 0$ large enough almost surely, under some conditions on the function A. Note that here, $(e_n)_{n\ge 0}$ is no longer necessarily the sequence of monomials.

Definition 3.3.1. Let $(e_n)_{n\ge 0}$ be a sequence in E and $f = \sum_{n\ge 0} a_n e_n \in E$. We define the function $S_f : [0,\infty[\longrightarrow [0,\infty]$ for any $r \ge 0$ by

$$S_f(r) = \sqrt{\sum_{n=0}^{\infty}|a_n|^2\|e_n\|_r^2}. \tag{3.3.1}$$

Proposition 3.3.2. *Let $f = \sum_{n\ge 0} a_n e_n \in E$ where $(e_n)_{n\ge 0}$ is a sequence of polynomials in E such that for every $n \ge 0$, the degree of e_n is at most n. Let $(X_n)_{n\ge 0}$ be a subgaussian sequence of centred independent random variables and $(r_k)_{k\ge 1}$ be a* *sequence of positive numbers converging to w. Assume that there exists a sequence of positive integers $(A(r_k))_{k\geq 1}$ such that $(A(r_k)^{-1})_{k\geq 1}$ is p-summable for some $p>0$. Then there exists $c>0$ such that almost surely, there exists $k_0\geq 1$ such that for every $k\geq k_0$,*

$$\left\|\sum_{n=0}^{A(r_k)} a_n X_n e_n\right\|_{r_k} \leq c\sqrt{\log(A(r_k))}S_f(r_k).$$

Proof. Lemma 3.1.13 gives, for every $k\geq 1$,

$$\mathbb{P}\left(\left\|\sum_{n=0}^{A(r_k)} a_n X_n e_n\right\|_{r_k} > c\sqrt{\log(A(r_k))}S_f(r_k)\right) \leq \frac{c}{A(r_k)^p}.$$

for some constant $c>0$. The result follows by the Borel-Cantelli lemma. □

To show that $\sqrt{\log(A)}S_f$ bounds the sup-norm of $v:=\sum_{n\geq 0} a_n X_n e_n$ along the sequence $(r_k)_{k\geq 1}$, that is, $\|v\|_{r_k}\lesssim \sqrt{\log(A(r_k))}S_f(r_k)$ holds for every $k\geq 1$ large enough almost surely, it remains to estimate $\|\sum_{n\geq A(r_k)+1} a_n X_n e_n\|_{r_k}$ for each $k\geq 1$. Note that $\liminf_{k\to\infty} S_f(r_k)>0$ as soon as there exists $n\geq 0$ such that $a_n e_n\neq 0$.

Proposition 3.3.3. *Let $f=\sum_{n\geq 0} a_n e_n\in E$ where $(e_n)_{n\geq 0}$ is a sequence of polynomials such that for every $n\geq 0$, the degree of e_n is at most n. Let $(X_n)_{n\geq 0}$ be a sub-gaussian sequence of centred independent random variables such that $\sum_{n=0}^{\infty} a_n X_n e_n$ is almost surely convergent and let $(r_k)_{k\geq 1}$ be a sequence of positive numbers converging to w. Assume that there exists a sequence $(A(r_k))_{k\geq 1}$ of positive integers such that $(A(r_k)^{-1})_{k\geq 1}$ is p-summable for some $p>0$ and that, almost surely, the sequence $(\|\sum_{n\geq A(r_k)+1} a_n X_n e_n\|_{r_k})_{k\geq 1}$ is bounded. Then there exists $c>0$ such that almost surely, there exists $k_0\geq 1$ such that for every $k\geq k_0$,*

$$\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r_k} \leq c\sqrt{\log(A(r_k))}S_f(r_k).$$

Proof. By Proposition 3.3.2, there is some $c>0$ such that, almost surely, there exist $M>0$ and $k_1\geq 1$ such that for every $k\geq k_1$,

$$\begin{aligned}\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r_k} &\leq \left\|\sum_{n=0}^{A(r_k)} a_n X_n e_n\right\|_{r_k} + \left\|\sum_{n\geq A(r_k)+1} a_n X_n e_n\right\|_{r_k}\\ &\leq c\sqrt{\log(A(r_k))}S_f(r_k)+M,\end{aligned}$$

where $M>0$ is some constant that depends on $\omega\in\Omega$. If $f=0$ there is nothing to prove. Assume that $f\neq 0$. Since $\liminf_{k\to\infty} S_f(r_k)>0$ and $\lim_{k\to\infty} A(r_k)=\infty$, we deduce that there exists $k_0\geq k_1$ such that for every $k\geq k_0$,

$$\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r_k} \leq 2c\sqrt{\log(A(r_k))}S_f(r_k).$$

□

Now we give conditions when the function $\sqrt{\log(A)}S_f$ is actually a rate of growth for v. This is the main result of this section. The proof uses ideas of the work of Nikula [80, Proposition 2].

Theorem 3.3.4. *Let $f = \sum_{n\geq 0} a_n e_n \in E$ where $(e_n)_{n\geq 0}$ is a sequence of polynomials such that for every $n \geq 0$, the degree of e_n is at most n. Let $(X_n)_{n\geq 0}$ be a subgaussian sequence of centred independent random variables such that $\sum_{n=0}^{\infty} a_n X_n e_n$ is almost surely convergent, and let $(A_j)_{j\geq 1}$ be a non-decreasing sequence of positive functions defined on $]0, w[$ such that A_1 is non-decreasing. Assume that the following conditions hold:*

(i) *the series*
$$\sum_{j\geq 1} \sqrt{\log(A_{j+1}(r))} \sqrt{\sum_{n\geq A_j(r)+1} |a_n|^2 \|e_n\|_r^2}$$
is bounded in $\delta < r < w$ for some $0 < \delta < w$,

(ii) *there exists an increasing sequence of positive numbers $(r_k)_{k\geq 1}$ converging to w such that the family $(A_j(r_k)^{-1})_{j,k\geq 1}$ is well-defined and p-summable for some $p > 0$.*

Then there exists $c > 0$ such that almost surely, there exists $k_0 \geq 1$ such that for every $k \geq k_0$,
$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_{r_k} \leq c\sqrt{\log(A_1(r_k))}S_f(r_k).$$
Furthermore, if the condition

(iii) *the sequences $\big(\log(A_1(r_{k+1}))/\log(A_1(r_k))\big)_{k\geq 1}$ and $\big(S_f(r_{k+1})/S_f(r_k)\big)_{k\geq 1}$ are bounded*

holds then there exists $c > 0$ such that almost surely, there exists $r_0 > 0$ such that
$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_{r} \leq c\sqrt{\log(A_1(r))}S_f(r)$$
for every $r_0 \leq r < w$.

Proof. We can assume that the functions A_j, $j \geq 1$, take integer values greater than or equal to 2. Indeed, the assumptions (i) to (iii) still hold for the sequence $(\lceil A_j \rceil + 2)_{j\geq 1}$.

Lemma 3.1.13 gives, for every $k, j \geq 1$,
$$\mathbb{P}\left(\left\| \sum_{n=A_j(r_k)+1}^{A_{j+1}(r_k)} a_n X_n e_n \right\|_{r_k} > c\sqrt{\log(A_{j+1}(r_k))} \sqrt{\sum_{n\geq A_j(r_k)+1} |a_n|^2 \|e_n\|_{r_k}^2} \right) \leq \frac{c}{A_{j+1}(r_k)^p}$$
for some constant $c > 0$, and hence almost surely for every $k \geq 1$ and $j \geq 1$ with k large enough, we get that
$$\left\| \sum_{n=A_j(r_k)+1}^{A_{j+1}(r_k)} a_n X_n e_n \right\|_{r_k} \leq c\sqrt{\log(A_{j+1}(r_k))} \sqrt{\sum_{n\geq A_j(r_k)+1} |a_n|^2 \|e_n\|_{r_k}^2}$$

by the Borel-Cantelli lemma since the family $(A_j(r_k)^{-1})_{j,k\geq 1}$ is p-summable by (ii). Then the sequence $\left(\left\|\sum_{n\geq A_1(r_k)+1} a_n X_n e_n\right\|_{r_k}\right)_{k\geq 1}$ is almost surely bounded by (i), hence Proposition 3.3.3 gives almost surely the inequality

$$\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r_k} \leq c\sqrt{\log(A_1(r_k))}S_f(r_k)$$

for every $k \geq 1$ large enough, where $c > 0$ is a constant. This proves the first part of the theorem.

Assume that (iii) holds. Let $k \geq 1$ and let $r_k \leq r < r_{k+1}$. By assumption (iii) and the Maximum Principle, we get for almost surely every $\omega \in \Omega$ and if k is large enough that

$$\begin{aligned}\left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r} \leq \left\|\sum_{n=0}^{\infty} a_n X_n e_n\right\|_{r_{k+1}} &\leq c\sqrt{\log(A_1(r_{k+1}))}S_f(r_{k+1}) \\ &= c\sqrt{\frac{\log(A_1(r_{k+1}))}{\log(A_1(r_k))}}\frac{S_f(r_{k+1})}{S_f(r_k)}\sqrt{\log(A_1(r_k))}S_f(r_k) \\ &\leq C\sqrt{\log(A_1(r_k))}S_f(r_k) \leq C\sqrt{\log(A_1(r))}S_f(r),\end{aligned}$$

where $C > 0$ is a constant. This concludes the proof. □

We point out the following fact.

Lemma 3.3.5. *Let $f = \sum_{n\geq 0} a_n e_n$ and $g = \sum_{n\geq 0} b_n e_n$ be elements of E where $(e_n)_{n\geq 0}$ is a sequence of polynomials such that for every $n \geq 0$, the degree of e_n is at most n. Suppose that there exist some $C_1, C_2 > 0$ such that $C_1 a_n \leq b_n \leq C_2 a_n$ for all $n \geq 0$. If f satisfies the assumptions of Theorem 3.3.4 for some sequences $(A_j)_{j\geq 1}$ and $(r_k)_{k\geq 1}$ then g satisfies the assumptions of Theorem 3.3.4 with the same sequences $(A_j)_{j\geq 1}$ and $(r_k)_{k\geq 1}$.*

Proof. If the series in (i) of Theorem 3.3.4 is bounded for f, it also bounded for g by assumption. Condition (ii) depends only on the sequences $(A_j)_{j\geq 1}$ and $(r_k)_{k\geq 1}$, and condition (iii) is again satisfied by assumption on the coefficients of the series defining f and g. □

Without the assumptions (ii) and (iii) of Theorem 3.3.4, we can prove that the function $\sqrt{\log(A_1)}S_f$ bounds the expectation of the sup-norm of the random series $\sum_{n\geq 0} a_n X_n e_n$. The main argument comes from [10, Section 5.5.4].

We first prove two lemmas, the second one being a consequence of the Orlicz-Jensen inequality.

Lemma 3.3.6. *Let $X_1, \ldots, X_N$ be independent real subgaussian random variables with constants $M = 1$ and $\sigma > 0$ in Definition 1.2.6, and let $a_1, \ldots, a_n$ be real numbers. Then $\sum_{n=1}^{N} a_n X_n$ is subgaussian with constants $M = 1$ and $\sigma\sqrt{\sum_{n=1}^{N} |a_n|^2}$.*

Proof. Let $\lambda \in \mathbb{R}$. By independence, we have

$$\mathbb{E}(e^{\lambda \sum_{n=1}^N a_n X_n}) = \prod_{n=1}^N \mathbb{E}(e^{\lambda a_n X_n}) \leq \prod_{n=1}^N e^{\lambda^2\sigma^2|a_n|^2} = e^{\lambda^2\sigma^2 \sum_{n=1}^N |a_n|^2},$$

hence $\sum_{n=1}^N a_n X_n$ is subgaussian with constants $M = 1$ and $\sigma\sqrt{\sum_{n=1}^N |a_n|^2}$ □

Remark 3.3.7. Let $X_1, \dots, X_N$ be complex subgaussian random variables such that the real variables $\mathrm{Re}(X_1), \dots, \mathrm{Re}(X_N), \mathrm{Im}(X_1), \dots, \mathrm{Im}(X_N)$ are independent. Then Lemma 3.3.6 still holds with the same proof and a little more calculations, with $X_1, \dots, X_N$ and complex numbers $a_1, \dots, a_N$. This means that the real and imaginary parts of $\sum_{n=1}^N a_n X_n$ are subgaussian with constants $M = 1$ and $\sigma\sqrt{\sum_{n=1}^N |a_n|^2}$.

Lemma 3.3.8. *Let $X_1, \dots, X_N$ be real subgaussian random variables with constants $K > 0$ and $\tau > 0$ as in Lemma 1.2.7. Then*

$$\mathbb{E}(\max(|X_1|, \dots, |X_N|)) \leq \sqrt{\log(N+1)}\sqrt{K+1}\tau.$$

Proof. Define the function $\psi_2 : [0,\infty[\longrightarrow [0,\infty[, x \longmapsto e^{x^2} - 1$, and

$$\|X\|_{\psi_2} := \inf\left\{a > 0 \mid \mathbb{E}\Big(\psi_2\big(\frac{|X|}{a}\big)\Big) \leq 1\right\} \in [0,\infty]$$

for any random variable X. For any subgaussian random variable X with constants $K > 0$ and $\tau > 0$ in Lemma 1.2.7, we have that for any $a > \sqrt{K+1}\tau$, using the formula $\mathbb{E}(f(Y)) = f(0) + \int_0^\infty f'(t)\mathbb{P}(Y > t)\mathrm{d}t$ for any positive random variable Y and any continuously differentiable function $f : [0,\infty[\longrightarrow [0,\infty[$, see [66, Chapitre 0, Proposition IV.2],

$$\begin{aligned}\mathbb{E}\Big(\psi_2\big(\frac{|X|}{a}\big)\Big) &= \int_0^\infty \frac{2t}{a^2} e^{t^2/a^2}\mathbb{P}(|X| > t)\mathrm{d}t \leq \frac{K}{a^2}\int_0^\infty 2te^{t^2/a^2}e^{-t^2/\tau^2}\mathrm{d}t \\ &= \frac{K}{a^2}\int_0^\infty e^{(1/a^2 - 1/\tau^2)t}\mathrm{d}t = K\frac{\tau^2}{a^2-\tau^2} \leq 1,\end{aligned}$$

thus $\|X\|_{\psi_2} \leq \sqrt{K+1}\tau$. Therefore, the Orlicz-Jensen inequality (see [66, Chapitre 0, Proposition IV.3]) yields

$$\mathbb{E}\big(\max(|X_1|, \dots, |X_N|)\big) \leq \sqrt{\log(N+1)} \max_{1\leq n\leq N} \|X_n\|_{\psi_2} \leq \sqrt{\log(N+1)}\sqrt{K+1}\tau.$$

□

Theorem 3.3.9. *If the hypotheses and the assumption* (i) *of Theorem 3.3.4 hold and if $\liminf_{r\to w} A_1(r) > 1$, then there exists $c > 0$ such that*

$$\mathbb{E}\left(\left\|\sum_{n=0}^\infty a_n X_n e_n\right\|_r\right) \leq c\sqrt{\log(A_1(r))}S_f(r)$$

for every $\delta < r < w$, where $\delta > 0$ is given by assumption (i) *of Theorem 3.3.4.*

Proof. By the triangle inequality, we can assume that the random variables X_n, $n \geq 0$, are real and a_n, $n \geq 0$, are real numbers.

We may assume again that the functions A_j, $j \geq 1$, take integer values greater than or equal to 2.

Let $\delta < r < w$. Let $M \geq N \geq 0$. For a fixed $\omega \in \Omega$, let P_ω be the polynomial $P_\omega(t) := \sum_{n=N}^{M} a_n X_n(\omega) e_n(re^{it})$, $0 \leq t \leq 2\pi$. By the Mean Value Theorem and then by Bernstein's inequality, see Lemma 3.1.11, we have

$$|P_\omega(t) - P_\omega(s)| \leq \|P_\omega'\|_\infty |t-s| \leq M \|P_\omega\|_\infty |t-s|$$

for every $0 \leq t, s \leq 2\pi$. Let $c > 1$, define $\varepsilon := (c-1)/(cM)$ and let Γ be a finite ε-net of the compact interval $[0, 2\pi]$ whose size is of order M. This implies that $\|P_\omega\|_\infty \leq c \sup_{s \in \Gamma} |P_\omega(s)|$.

Now, Lemma 3.3.6 ensures that for any $s \in \Gamma$, $\sum_{n=N}^{M} a_n X_n e_n(re^{is})$ is subgaussian with constants $M = 1$ and $\sigma \sqrt{\sum_{n=N}^{M} |a_n|^2 \|e_n\|_r^2}$. Notice that in the proof of Lemma 1.2.7, one can easily see that $K = 2$ and $\tau = 2\sigma \sqrt{\sum_{n=N}^{M} |a_n|^2 \|e_n\|_r^2}$. Therefore, Lemma 3.3.8 yields that

$$\begin{aligned} \mathbb{E}\left(\left\| \sum_{n=N}^{M} a_n X_n e_n \right\|_r \right) &\leq c \mathbb{E}\left(\sup_{s \in \Gamma} \left| \sum_{n=N}^{M} a_n X_n e_n(re^{is}) \right| \right) \\ &\lesssim \sqrt{\log(M)} \sqrt{\sum_{n=N}^{M} |a_n|^2 \|e_n\|_r^2}. \end{aligned}$$

Applying the previous inequality gives

$$\begin{aligned} \mathbb{E}\left(\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \right) &\leq \mathbb{E}\left(\left\| \sum_{n=0}^{A_1(r)} a_n X_n e_n \right\|_r \right) + \sum_{j \geq 1} \mathbb{E}\left(\left\| \sum_{n=A_j(r)+1}^{A_{j+1}(r)} a_n X_n e_n \right\|_r \right) \\ &\lesssim \sqrt{\log(A_1(r))} S_f(r) + \sum_{j \geq 1} \sqrt{\log(A_{j+1}(r))} \sqrt{\sum_{n \geq A_j(r)+1} |a_n|^2 \|e_n\|_r^2}, \end{aligned}$$

and this yields the result since, by assumption (i) of Theorem 3.3.4, the second term is bounded. □

The proofs of Theorem 3.1.17 and 3.3.4 share the same ideas. The first terms of the random series are more relevant than the tail, and the desired rate of growth is proved to hold along some sequence $(r_k)_{k \geq 1}$. Then, by using the Maximum Principle, the inequality still holds for large $r > 0$.

Functions of finite order. As an application of Theorem 3.3.4 to end this section, let $f = \sum_{n \geq 0} a_n e_n$ be an entire function, where $(e_n)_{n \geq 0}$ is the sequence of monomials. If f is of finite order and satisfies the Assumption 3.3.14 below then we can obtain an admissible rate of growth for the random entire function $\sum_{n \geq 0} a_n X_n e_n$ valid for any large $r > 0$. Note that here, this random series is almost surely convergent by Lemma 3.1.1.

Definition 3.3.10. Let f be a non-constant entire function. The *order* of f, written ρ_f, is the quantity

$$\rho_f = \limsup_{r\to\infty} \frac{\log_2(\|f\|_r)}{\log(r)}.$$

In other words, ρ_f is the least constant such that, for every $\varepsilon > 0$, there is some $c > 0$ such that, for all $r > 0$, one has $\|f\|_r \le ce^{r^{\rho_f+\varepsilon}}$.

The maximum term can also be used to compute the order as the next result says.

Theorem 3.3.11 ([52, Satz 4.5])**.** *Let f be a non-constant entire function. Then we have the equality*

$$\rho_f = \limsup_{r\to\infty} \frac{\log_2(\mu_f(r))}{\log(r)}.$$

Note that $\lim_{r\to\infty} \mu_f(r) = \infty$ by Lemma 3.1.3, and the limit in the theorem is well-defined.

When the order of an entire function is finite, its growth is related to its maximum term. For a proof of the next theorem, see [52, Satz 4.6].

Theorem 3.3.12. *Let f be a non-constant entire function of finite order. Then* $\log(\|f\|_r) \asymp \log(\mu_f(r))$.

Definition 3.3.13. Let $f = \sum_{n\ge 0} a_n e_n$ be a non-constant entire function. Define the function $N_f : [0,\infty[\longrightarrow \mathbb{R}$ by

$$N_f(r) := \inf\left\{n_0 \ge 1 \mid \forall n \ge n_0,\ |a_n| r^n < 1\right\}$$

for all $r \ge 0$.

We will assume the following on the function N_f.

Assumption 3.3.14. *There exist $\varepsilon > 0$, an integer $m \in \{0,1\}$ and a real number $b > 0$ such that for every $r > 0$ large enough,*

$$N_f(re^{\varepsilon}) \lesssim \left(\log_m(S_f(r))\right)^b.$$

The function N_f will be needed to check assumption (i) of Theorem 3.3.4. Then, Assumption 3.3.14 will serve to get the growth written only with S_f, and to ensure that (iii) of Theorem 3.3.4 is satisfied.

The next result is an application of Theorem 3.3.4.

Theorem 3.3.15. *Let $f = \sum_{n\ge 0} a_n e_n$ be a non-constant entire function satisfying Assumption 3.3.14, where $(e_n)_{n\ge 0}$ is the sequence of monomials, and let $(X_n)_{n\ge 0}$ be a sequence of independent centred subgaussian random variables. Then the random vector $\sum_{n\ge 0} a_n X_n e_n$ is almost surely an entire function, and there exists a constant $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} a_n X_n e_n \right\|_r \le c\sqrt{\log_{m+1}(S_f(r))}\, S_f(r)$$

for every $r \ge r_0$.

Proof. Let $\varepsilon > 0$ and $m \in \{0,1\}$ be given by Assumption 3.3.14, and define the function $\mathcal{M}_f :]0,\infty[\longrightarrow \mathbb{R}$ by

$$\mathcal{M}_f(r) = \max\big(N_f(re^{\varepsilon}), \log_m(S_f(r))\big)$$

for every $r > 0$. Since the function S_f is continuous and $\lim_{r\to\infty} S_f(r) = \infty$, there exists an increasing sequence $(r_k)_{k\geq 1}$ of positive real numbers converging to ∞ such that $\log_m(S_f(r_{k+1})) = \log_m(S_f(r_k)) + 1$ and $\log_m(S_f(r_k)) \geq k$ for each $k \geq 1$. Take a real number $p > 1$. We prove that f satisfies the assumptions of Theorem 3.3.4 with $A_j := j\mathcal{M}_f$, $j \geq 1$.

Let us check condition (i). Let $j \geq 1$ and $r > 0$ be large enough. By definitions of $\mathcal{M}_f$ and N_f, we have $|a_n|r^n \leq e^{-\varepsilon n}$ for every $n \geq A_1(r)$, and thus

$$\sum_{n\geq A_j(r)+1} |a_n|^2 r^{2n} \leq \sum_{n\geq A_j(r)+1} e^{-2\varepsilon n} \leq \int_{A_j(r)}^{\infty} e^{-2\varepsilon x}\mathrm{d}x = 2^{-1}\varepsilon^{-1}e^{-2\varepsilon A_j(r)}.$$

In order to verify (i), we will show that

$$\sum_{j\geq 1} \sqrt{\log(A_{j+1}(r))}e^{-\varepsilon A_j(r)} = \sum_{j\geq 1} \sqrt{\log((j+1)A_1(r))}e^{-\varepsilon A_1(r)j} \tag{3.3.2}$$

converges to 0 when r goes to ∞; note that the series converges for all r sufficiently large. The idea is to use the Dominated Convergence Theorem. It is enough to show that for every $j \geq 1$, the function

$$f_j :]1,\infty[\longrightarrow \mathbb{R}, x \longmapsto \log((j+1)x)e^{-2\varepsilon jx}$$

is non-increasing for x sufficiently large, uniformly in j, and that it converges to 0 when x goes to ∞. Indeed, denoting by μ the counting measure on $\mathbb{N}_0$, we have

$$\sum_{j\geq 1} \sqrt{\log((j+1)A_1(r))}e^{-\varepsilon A_1(r)j} = \int_{\mathbb{N}_0} \sqrt{f_j(A_1(r))}\mathrm{d}\mu(j).$$

Since $\lim_{r\to\infty} A_1(r) = \infty$ and A_1 is increasing, let $r_0 \geq 1$ be such that $f_j \circ A_1$ is non-increasing on $[r_0,\infty[$ for all $j \geq 1$. Then $(f_j \circ A_1)(r) \leq (f_j \circ A_1)(r_0)$ for every $j \geq 1$ and $r \geq r_0$, and the series (3.3.2) converges to 0 when x goes to ∞ by the Dominated Convergence Theorem.

First, it is clear that $\lim_{x\to\infty} f_j(x) = 0$. For every $x > 1$, the derivative of f_j is given by

$$\partial_x f_j(x) = -2\varepsilon j \log((j+1)x)e^{-2\varepsilon jx} + \frac{e^{-2\varepsilon jx}}{x},$$

which is negative if and only if $1 < 2\varepsilon jx\log((j+1)x)$. But this holds for every $x > 0$ large enough, uniformly in j, since the right-hand side of the inequality converges to ∞ when x goes to ∞. Thus f_j is decreasing for x large enough, uniformly in $j \geq 1$.

By definition of $\mathcal{M}_f$ and construction of the sequence $(r_k)_{k\geq 1}$, we have

$$\sum_{k=1}^{\infty} \frac{1}{\mathcal{M}_f(r_k)^p} \leq \sum_{k=1}^{\infty} \frac{1}{\log_m(S_f(r_k))^p} \leq \sum_{k=1}^{\infty} \frac{1}{k^p} < \infty.$$

Therefore, assumption (ii) of Theorem 3.3.4 is satisfied.

By construction of $(r_k)_{k\geq 1}$, the sequence $(S_f(r_{k+1})/S_f(r_k))_{k\geq 1}$ is bounded. This implies in turn that the sequence

$$\Big(\frac{\log(\mathcal{M}_f(r_{k+1}))}{\log(\mathcal{M}_f(r_k))}\Big)_{k\geq 1}$$

is bounded. Indeed, by Assumption 3.3.14 and definition of $\mathcal{M}_f$, we have

$$\frac{\log(\mathcal{M}_f(r_{k+1}))}{\log(\mathcal{M}_f(r_k))} \lesssim \frac{\log_{m+1}(S_f(r_{k+1}))}{\log_{m+1}(S_f(r_k))}$$

if k is sufficiently large. Therefore, assumption (iii) of Theorem 3.3.4 is satisfied, and we get that there exists $c > 0$ such that, almost surely, there exists $r_0 > 0$ such that for every $r \geq r_0$,

$$\Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_r \leq c\sqrt{\log(\mathcal{M}_f(r))}S_f(r).$$

Using again Assumption 3.3.14 concludes the proof. $\square$

Remark 3.3.16. Let $f = \sum_{n\geq 0} a_n e_n$ be a non-constant entire function satisfying Assumption 3.3.14, where $(e_n)_{n\geq 0}$ is the sequence of monomials. Let $g = \sum_{n\geq 0} b_n e_n$ be another entire function and assume that there are some $C_1, C_2 > 0$ such that $C_1 a_n \leq b_n \leq C_2 a_n$ for all $n \geq 0$. Then the conclusion of Theorem 3.3.15 also applies to g by Lemma 3.3.5.

We now apply Theorem 3.3.15 to functions of finite order.

Theorem 3.3.17. *Let $f = \sum_{n\geq 0} a_n e_n$ be a non-constant entire function of finite order satisfying Assumption 3.3.14, where $(e_n)_{n\geq 0}$ is the sequence of monomials, and let $(X_n)_{n\geq 0}$ be a sequence of independent centred subgaussian random variables. Then the random vector $\sum_{n\geq 0} a_n X_n e_n$ is almost surely an entire function, and there exists a constant $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_r \leq c\sqrt{\log_{m+1}(\mu_f(r))}S_f(r)$$

for every $r \geq r_0$.

Proof. Since f is of finite order, the entire function $z \longmapsto \sum_{n\geq 0} |a_n| z^n$ is also of finite order by Theorem 3.3.11. By Theorem 3.3.12 applied to $z \longmapsto \sum_{n\geq 0} |a_n| z^n$, we have

$$\log(S_f(r)) \leq \log\left(\sum_{n=0}^{\infty} |a_n| r^n\right) \lesssim \log(\mu_f(r)).$$

Since $\mu_f(r) \leq S_f(r)$ for all $r \geq 0$, this also implies that $\log(S_f(r)) \asymp \log(\mu_f(r))$, and in turn $\log_{m+1}(S_f(r)) \asymp \log_{m+1}(\mu_f(r))$. The result follows from Theorem 3.3.15. $\square$

Remark 3.3.18. Remark 3.3.16 still holds for a function f of finite order satisfying Assumption 3.3.14.

Theorems 3.3.4 and 3.3.17 give a rate of growth valid without an exceptional set of finite logarithmic measure, in contrast to Theorems 3.1.17 and 3.2.10, but only for some functions. Nevertheless, these results will be sufficient to obtain a generalization of the works of Nikula [80] and Mouze and Munnier [75] in Chapter 4 and to consider some other operators.

Example 3.3.19. If $f(z) = e^z = \sum_{n\geq 0} z^n/n!$, then it is not difficult to show that $S_f(r) \asymp e^r/r^{1/4}$, $\log(\mu_f(r)) \asymp r$, and $N_f(r) \lesssim r$, see Example 3.1.4 and Lemmas 4.1.6 and 4.1.8. Thus Theorem 3.3.17 gives the upper bound

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{n!} z^n \right\|_r \leq c\sqrt{\log(r)}\frac{e^r}{r^{1/4}},$$

which confirms the result obtained by Nikula [80, Proposition 2].

Chapter 4

Rate of growth for operators

Theorem 1.3.4 from Chapter 1 says that the random series

$$\sum_{n=0}^{\infty} \frac{X_n}{w_1 \dots w_n} z^n$$

is almost surely holomorphic and frequently hypercyclic for a given chaotic weighted shift T on $H(\mathbb{C})$ (resp. $H(\mathbb{D})$) with sequence of weights $(w_n)_{n\in\mathbb{N}_0}$, and the complex random variables X_n, $n \in \mathbb{N}$, are i.i.d. and subgaussian with full support. As discussed in the introduction of Chapter 3, the present chapter aims to find an upper bound for the maximum modulus of this random vector, which then gives an admissible rate of growth for the frequently hypercyclic functions of T.

When applied to chaotic weighted shifts on $H(\mathbb{C})$ or $H(\mathbb{D})$, Theorems 3.1.17 and 3.3.4 will extend the works of Nikula [80], Bernal-González and Bonilla [15] and Mouze and Munnier [75], see Theorems 4.1.9, 4.1.14 and 4.2.7, respectively.

We will also consider the differential operators on the space of harmonic functions on the plane and chaotic weighted shifts on Köthe sequence spaces in Sections 4.3 and 4.4, respectively.

In Section 4.5, we will discuss the possible optimality of the rate of growth found in the previous sections.

Throughout this chapter, we will use the same notations as in Chapter 3 that we recall here. Every random variable considered will be defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If a and b are two positive real numbers, the notation $a \lesssim b$ means that there exists some $C > 0$ such that $a \leq Cb$ and C does not depend on any current variable such as $n \in \mathbb{N}$, $r > 0$ or $\omega \in \Omega$. The notation $a \asymp b$ means $a \lesssim b$ and $b \lesssim a$. To make the reading easier, $\log_m$ means the logarithm iterated m times. Lastly, if a complex-valued function f is defined on a disk centred at the origin and of radius $r > 0$, then we define

$$\|f\|_r := \sup_{|z|=r} |f(z)|.$$

4.1 Entire functions

In this section, we consider weighted shifts defined on the Fréchet space $H(\mathbb{C})$. Theorem 0.1.19 combined with Theorem 3.1.17 applied to $f(z) = \sum_{n\geq 0} z^n/\beta_n$ immediately yields an admissible rate of growth for each chaotic weighted shift, where the sequence $(\beta_n)_{n\geq 0}$ is defined in Subsection 1.3.1. However, it is only valid outside a set of finite logarithmic measure.

Theorem 4.1.1. *Let T be a chaotic weighted shift on $H(\mathbb{C})$ with respect to the basis of monomials $(e_n)_{n\geq 0}$ and with sequence of weights $(w_n)_{n\geq 1}$. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n$ is almost surely an entire function, is frequently hypercyclic for T and there exist a constant $c > 0$ and an open set $E \subseteq [0, \infty[$ of finite logarithmic measure such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n \right\|_r \leq c\sqrt{\log_2(\mu_f(r))} \sqrt{\sum_{n=0}^{\infty} \frac{r^{2n}}{|\beta_n|^2}}$$

for every $r \geq r_0$, $r \notin E$, where $f := \sum_{n\geq 0} e_n/\beta_n$.

The frequent hypercyclicity of the random vector is obtained by Theorem 1.3.4.

In Section 3.3, another approach for finding a rate of growth for random sums valid for any r large enough was presented. We will see that the first assumptions (i) and (ii) of Theorem 3.3.4 are satisfied for every chaotic weighted with respect to the basis of monomials. Unfortunately, we are in general not able to construct a suitable sequence $(r_k)_{k\geq 1}$ in order to obtain the rate of growth valid for any r large enough.

Proposition 4.1.2. *Let T be a chaotic weighted shift on $H(\mathbb{C})$ with respect to the basis of monomials $(e_n)_{n\geq 0}$ and with sequence of weights $(w_n)_{n\geq 1}$. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n$ is almost surely an entire function, is frequently hypercyclic for T and satisfies, for every $\alpha > 3/2$, the assumption* (i) *of Theorem 3.3.4 with $A_j(r) := \max\left\{n \geq 0 \mid |\beta_n| \leq r^n n^\alpha\right\}.j$, $j \geq 1$, $r > 0$.*

Proof. By Theorem 1.3.4, the random vector $\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n$ is almost surely entire and frequently hypercyclic for T.

First, since T is chaotic on $H(\mathbb{C})$, which is equivalent to $\lim_{n\to\infty} |\beta_n|^{1/n} = \infty$, see Example 0.1.21, the function A_1 is well-defined. Furthermore, we have that $\lim_{r\to\infty} A_1(r) = \infty$. Indeed, let $n \geq 1$ be an integer and $r > |\beta_n|^{1/n}/n^{\alpha/n}$. Then $A_1(r) \geq n$ and since n was arbitrary, $\lim_{x\to\infty} A_1(x) = \infty$.

Let us check assumption (i) of Theorem 3.3.4. Let $j \geq 1$ and $r > 0$. By definition of A_j, we have $r^n/|\beta_n| \leq 1/n^\alpha$ for every $n \geq A_j(r) + 1$ and thus

$$\sum_{n\geq A_j(r)+1} \frac{r^{2n}}{|\beta_n|^2} \leq \sum_{n\geq A_j(r)+1} \frac{1}{n^{2\alpha}} \leq \int_{A_j(r)}^{\infty} \frac{1}{x^{2\alpha}} \mathrm{d}x = (2\alpha - 1)^{-1} A_j(r)^{1-2\alpha}.$$

In order to verify assertion (i) of Theorem 3.3.4, we will show that

$$\sum_{j\geq 1}\sqrt{\log(A_{j+1}(r))}A_j(r)^{1/2-\alpha}=\sum_{j\geq 1}\sqrt{\log\left(A_1(r)(j+1)\right)}(A_1(r)j)^{1/2-\alpha} \tag{4.1.1}$$

converges to 0 when r goes to ∞. The idea is to use the Dominated Convergence Theorem, as in the proof of Theorem 3.3.15. It is enough to show for every $j\geq 1$ that the function

$$f_j:\,]0,\infty[\longrightarrow\mathbb{R}, x\longmapsto\log(xj)(xj)^{1-2\alpha}$$

is non-increasing and converges to 0 when x goes to ∞. First, it is clear that $\lim_{x\to\infty}f_j(x)=0$. For every $x>0$, the derivative of f_j is given by

$$\partial_x f_j(x)=\log(xj)(1-2\alpha)\frac{j}{(xj)^{2\alpha}}+\frac{j}{(xj)^{2\alpha}}$$

which is negative if and only if $1<(2\alpha-1)\log(xj)$. But this holds for every $x>0$ large enough, uniformly in $j\geq 1$, since the right-hand side of the inequality converges to ∞ when x goes to ∞. Thus f_j is decreasing away from 0, uniformly in $j\geq 1$. Note that $j\longmapsto\sqrt{f_j(r)}$ is integrable on $\mathbb{N}$ with respect to the counting measure for every $r>0$ since $\alpha>3/2$, which allows us to use the Dominated Convergence Theorem and conclude that (4.1.1) converges to 0 when r goes to ∞. This concludes the proof. □

Observe that, once it is proved that a weighted shift satisfies the first two assumptions of Theorem 3.3.4, then other weighted shifts still satisfy those assumptions.

Lemma 4.1.3. *Let $(w_n)_{n\geq 1}$ be a weight sequence such that the assumptions of Theorem 3.3.4 are satisfied for $a_n=w_1^{-1}\dots w_n^{-1}$, $n\geq 0$, with some sequence $(A_j)_{j\geq 1}$. Let $(\widetilde{w}_n)_{n\geq 1}$ be a weight sequence such that there exists $c>0$ such that $|\widetilde{w}_1\dots\widetilde{w}_n|\geq c|w_1\dots w_n|$ for every $n\geq 1$. Then the assumptions* (i) *and* (ii) *of Theorem 3.3.4 still hold for $a_n=\widetilde{w}_1^{-1}\dots\widetilde{w}_n^{-1}$, $n\geq 0$, with the same sequence $(A_j)_{j\geq 1}$.*

Condition (ii) of Theorem 3.3.4 can always be satisfied with the functions A_j in Proposition 4.1.2: since $\lim_{r\to\infty}A_1(r)=\infty$, such a sequence $(r_k)_{k\geq 1}$ must exist. This allows us to get the first two assumptions of the theorem.

To fully apply Theorem 3.3.4, we have to choose a sequence $(r_k)_{k\geq 0}$ satisfying both assumptions (ii) and (iii) of the theorem. We do not know whether it is possible to achieve this for any weighted shift on $H(\mathbb{C})$ with the choice of the functions A_j, $j\geq 1$, of Proposition 4.1.2. However, by choosing a slightly different function A_1, and for some weighted shifts, we can fully apply Theorem 3.3.4 through Theorem 3.3.17. This will be the content of the next subsections. It turns out that for those examples, one can apply Proposition 4.1.2. We will only compute the function A_1 of Proposition 4.1.2 for the differentiation operator below. Since we still need to estimate the associated series in (3.3.1) in order to check assumption (iii) of Theorem 3.3.4, we will instead apply Theorem 3.3.17 to the examples in the following subsections. The advantage of this method is that we do not need to estimate the series (3.3.1), although we will be able to do so for some of our examples.

Lemma 4.1.4. *Let T be the differentiation operator on $H(\mathbb{C})$. Then $A_1(r)\lesssim r$ and $\log(r)\lesssim\log(A_1(r))$ where A_1 is defined in Proposition 4.1.2.*

Proof. Let $\alpha > 3/2$. Let $r > 0$ be large and set $n := A_1(r)$. Then $n! \le r^n n^\alpha$, which implies

$$n\log(n) - (n-1) = \int_1^n \log(x)\mathrm{d}x \le \sum_{k=1}^n \log(k) = \log(n!) \le n\log(r) + \alpha\log(n).$$

Thus

$$\log(n) \le \log(r) + \alpha\frac{\log(n)}{n} + \frac{n-1}{n},$$

which yields $A_1(r) \lesssim r$. The other inequality is similarly proved. □

The following result will be used to estimate the series in the formula for the rate of growth of frequently hypercyclic functions.

Theorem 4.1.5 ([33, Theorem IV.2.5])**.** *Let $-\infty \le a < b \le \infty$ and $g, h :]a, b[\longrightarrow \mathbb{R}$ be two twice continuously differentiable functions. Assume that*

(i) *the integral $\int_a^b |g(t)|e^{h(t)}\mathrm{d}t$ is finite,*

(ii) *there exists a unique $a < c < b$ such that $\partial_t^2 h(c) < 0$, h' changes sign only at c, h reaches a maximum at c and $g(c) \neq 0$.*

Then for every $x > 0$ large enough, one has

$$\int_a^b g(t)e^{xh(t)}\mathrm{d}t \asymp \frac{e^{xh(c)}}{\sqrt{x}}.$$

4.1.1 Operators with weights n^α

We consider on the space $H(\mathbb{C})$ the weighted shift with respect to the basis of monomials $(e_n)_{n\ge0}$ of $H(\mathbb{C})$ and with weights $w_n = n^\alpha$, $n \ge 1$, where $\alpha > 0$ is a parameter. We have $\beta_n = n!^\alpha$ for all $n \ge 1$. It is easy to check that $\lim_{n\to\infty} w_n^{1/n} = 1$, hence this operator is well-defined on $H(\mathbb{C})$, and $\lim_{n\to\infty} \beta_n^{1/n} = \infty$, hence it is also chaotic on $H(\mathbb{C})$; see Example 0.1.21.

We will apply Theorem 3.3.17 to the entire function $f := \sum_{n\ge0} e_n/n!^\alpha$. To check that f satisfies Assumption 3.3.14, we will estimate its maximum term and the function N_f in Definition 3.3.13.

Lemma 4.1.6. *For any $r > 0$ large enough, we have*

$$\log(r)\alpha^{-1} \lesssim \log(N_f(r)) \text{ and } N_f(r) \lesssim r^{1/\alpha}.$$

Proof. By definition of N_f, if $n = N_f(r) - 1$ then $r^n/n!^\alpha \ge 1$. This implies

$$\int_1^n \log(x)\mathrm{d}x = \sum_{k=2}^n \int_{k-1}^k \log(x)\mathrm{d}x \le \sum_{k=1}^n \log(k) = \log(n!) \le \alpha^{-1} n\log(r)$$

and then

$$\log(n) \le \frac{\log(r)}{\alpha} + \frac{n-1}{n} \le \log(r^{1/\alpha}) + 1.$$

We conclude that $N_f(r) \lesssim r^{1/\alpha}$.

If $n = N_f(r)$ then $r^n \leq n!^\alpha$, and

$$\int_1^{n+1} \log(x)\mathrm{d}x = \sum_{k=1}^{n} \int_k^{k+1} \log(x)\mathrm{d}x \geq \sum_{k=1}^{n} \log(k) \geq \alpha^{-1} n \log(r),$$

which implies

$$\frac{n\log(r)}{(n+1)\alpha} + \frac{n}{n+1} \leq \log(n+1),$$

hence $\log(r)\alpha^{-1} \lesssim \log(N_f(r))$. □

Lemma 4.1.7. *For any $r > 0$, we have*

$$\mu_f(r) \asymp \frac{e^{\alpha r^{1/\alpha}}}{\sqrt{r}}.$$

Proof. By noticing that

$$\frac{r^n}{n!^\alpha} = \frac{r}{1^\alpha} \cdots \frac{r}{n^\alpha},$$

we easily get by using Stirling's formula

$$\mu_f(r) = \frac{r^{\lfloor r^{1/\alpha} \rfloor}}{\lfloor r^{1/\alpha} \rfloor !^\alpha} \asymp \frac{r^{\lfloor r^{1/\alpha} \rfloor} e^{\lfloor r^{1/\alpha} \rfloor \alpha}}{\lfloor r^{1/\alpha} \rfloor^{\alpha \lfloor r^{1/\alpha} \rfloor} \lfloor r^{1/\alpha} \rfloor^{\alpha/2}}.$$

Thanks to the estimates

$$\frac{e^{\lfloor r^{1/\alpha} \rfloor \alpha}}{e^{r^{1/\alpha}\alpha}} \asymp 1, \quad \frac{\lfloor r^{1/\alpha} \rfloor}{r^{1/\alpha}} \asymp 1 \quad \text{and} \quad \frac{r^{\lfloor r^{1/\alpha} \rfloor}}{\lfloor r^{1/\alpha} \rfloor^{\alpha \lfloor r^{1/\alpha} \rfloor}} \asymp 1,$$

we get the estimate of the lemma. □

Before concluding, we will use Theorem 4.1.5 to estimate the series S_f^2 of Theorem 3.3.17 in the following lemma, whose proof is somewhat technical.

Lemma 4.1.8. *Let $\beta > 0$ be a positive real number. Then for every $r > 0$ large enough, one has*

$$\sum_{n=0}^{\infty} \frac{r^n}{n!^\beta} \asymp r^{\frac{1}{2\beta} - \frac{1}{2}} e^{\beta r^{1/\beta}}.$$

Proof. The proof is divided into two steps. First, we use a comparison series-integral, and then apply Theorem 4.1.5 to conclude.

Let $r > 0$ be large. By Stirling's formula, we have

$$\sum_{n=1}^{\infty} \frac{r^n}{n!^\beta} \asymp \sum_{n=1}^{\infty} \frac{r^n e^{n\beta}}{n^{\beta(n+1/2)}}.$$

Define the function

$$G_r : [1, \infty[\longrightarrow \mathbb{R}, t \longmapsto t\log(re^\beta) - \beta\big(t + \frac{1}{2}\big)\log(t).$$

Its derivative is given by

$$\partial_t G_r(t) = \log(re^{\beta}) - \beta\big(1 + \frac{1}{2t} + \log(t)\big)$$

for every $t > 1$. Then $\partial_t G_r(t) > 0$ if and only if $\frac{\log(r)}{\beta} > \frac{1}{2t} + \log(t)$. The function $t \longmapsto 1/(2t) + \log(t)$ has a unique minimum at $1/2$ and converges to ∞ when t goes to ∞. We deduce that for r large enough, there exists a unique $x = x(r) \geq 1/2$ that maximises G_r and

$$\frac{\log(r)}{\beta} = \frac{1}{2x} + \log(x). \tag{4.1.2}$$

Notice that $\lim_{r\to\infty} x(r) = \infty$. We can now write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{r^n}{n!^{\beta}} &\gtrsim \sum_{2\leq n\leq \lfloor x\rfloor} \int_{n-1}^{n} e^{G_r(t)}\mathrm{d}t + \sum_{n>\lfloor x\rfloor} \int_{n}^{n+1} e^{G_r(t)} \\ &= \int_1^{\infty} e^{G_r(t)}\mathrm{d}t - \int_{\lfloor x\rfloor}^{\lfloor x\rfloor+1} e^{G_r(t)}\mathrm{d}t \\ &\geq \int_1^{\infty} e^{G_r(t)}\mathrm{d}t - e^{G_r(x)}.\end{aligned}$$

Similarly, one shows that

$$\sum_{n=1}^{\infty} \frac{r^n}{n!^{\beta}} \lesssim \int_1^{\infty} e^{G_r(t)}\mathrm{d}t + e^{G_r(x)}. \tag{4.1.3}$$

We begin by estimating the integral in (4.1.3). For all $y \geq 1/x$, one has

$$\begin{aligned}\frac{G_r(xy)}{\beta} &= xy\big(1 + \frac{1}{2x} + \log(x)\big) - \big(xy + \frac{1}{2}\big)\log(xy) \\ &= xy + \frac{y}{2} + xy\log(x) - xy\log(xy) - \frac{\log(xy)}{2} \\ &= x(y - y\log(y)) + \frac{y}{2} - \frac{\log(y)}{2} - \frac{\log(x)}{2}.\end{aligned}$$

The change of variables $t = xy$ then yields

$$\int_1^{\infty} e^{G_r(t)}\mathrm{d}t = \int_{1/x}^{\infty} x\frac{e^{\beta x(y-y\log(y))}}{x^{\beta/2}}\frac{e^{\beta y/2}}{y^{\beta/2}}\mathrm{d}y.$$

Pick $0 < \delta < \beta$. Let $0 < \varepsilon < 1$ be such that $\beta y(1 - \log(y)) < \delta$ for all $0 < y < \varepsilon$. We can now apply Theorem 4.1.5 to $h : y \longmapsto \beta y(1 - \log(y))$ and $g : y \longmapsto e^{\beta y/2}/y^{\beta/2}$ with $c = 1$, and we get

$$\int_{\varepsilon}^{\infty} xe^{xh(y)}g(y)\mathrm{d}y \asymp x^{\frac{1}{2}-\frac{\beta}{2}}e^{\beta x}.$$

By (4.1.2), we have

$$r^{1/\beta} = e^{1/(2x)}x = x\Big(1 + \frac{1}{2x} + o\big(\frac{1}{x}\big)\Big), \tag{4.1.4}$$

and we get

$$\int_{\varepsilon}^{\infty} xe^{xh(y)}g(y)\mathrm{d}y \asymp r^{\frac{1}{2\beta}-\frac{1}{2}}e^{\beta r^{1/\beta}}.$$

Recalling the definition of ε, we also have

$$\int_{1/x}^{\varepsilon} xe^{G_r(xy)}\mathrm{d}y \leq e^{\beta\varepsilon/2}x\int_{1/x}^{\varepsilon} e^{x\delta}\mathrm{d}y \leq \varepsilon e^{\beta\varepsilon/2}xe^{x\delta},$$

which is negligible compared to $x^{1/2-\beta/2}e^{\beta x}$ since $\delta < \beta$. In conclusion, the integral in (4.1.3) is estimated by

$$\int_1^{\infty} e^{G_r(t)}\mathrm{d}t \asymp r^{\frac{1}{2\beta}-\frac{1}{2}}e^{\beta r^{1/\beta}}.$$

We now show that the second term of the right-hand side of (4.1.3) is bounded by $e^{\beta r^{1/\beta}}/\sqrt{r}$, which will finish the proof. By (4.1.4), we can write

$$r^{1/\beta} = x\big(1 + g(x) + G(x)\big), \tag{4.1.5}$$

where $g :]0,\infty[\longrightarrow \mathbb{R}$ and $G :]0,\infty[\longrightarrow \mathbb{R}$ are such that

$$\lim_{y\to\infty} yg(y) = \frac{1}{2} \quad \text{and} \quad \lim_{y\to\infty} \frac{G(y)}{g(y)} = 0. \tag{4.1.6}$$

(a) Let us show that $\lim_{r\to\infty} r^x/x^{\beta x} = e^{-1/2}$. By passing to the logarithm and using (4.1.5), we have

$$\begin{aligned}
x\log(r) - x\beta\log(x) &= \log(r)\big(r^{1/\beta} - xg(x) - xG(x)\big) \\
&\quad - \beta\big(r^{1/\beta} - xg(x) - xG(x)\big)\log\big(r^{1/\beta} - xg(x) - xG(x)\big) \\
&= \log(r)r^{1/\beta} - \log(r)(xg(x) + xG(x)) \\
&\quad - \beta\big(r^{1/\beta} - xg(x) - xG(x)\big)\Big(\log(r^{1/\beta}) + \log\Big(1 - \frac{xg(x)+xG(x)}{r^{1/\beta}}\Big)\Big) \\
&= \log(r)r^{1/\beta} - \log(r)(xg(x) + xG(x)) - \beta r^{1/\beta}\log(r^{1/\beta}) \\
&\quad + \beta(xg(x)+xG(x))\log(r^{1/\beta}) + \beta(xg(x)+xG(x))\log\Big(1 - \frac{xg(x)+xG(x)}{r^{1/\beta}}\Big) \\
&\quad - \beta r^{1/\beta}\log\Big(1 - \frac{xg(x)+xG(x)}{r^{1/\beta}}\Big) \\
&= \beta(xg(x)+xG(x))\log\Big(1 - \frac{xg(x)+xG(x)}{r^{1/\beta}}\Big) - \beta r^{1/\beta}\log\Big(1 - \frac{xg(x)+xG(x)}{r^{1/\beta}}\Big).
\end{aligned}$$

The first term of the right-hand side converges to 0 when r goes to ∞ by (4.1.6). As for the second term, we have

$$r^{1/\beta}\log\Big(1 - \frac{xg(x)+xG(x)}{r^{1/\beta}}\Big) = r^{1/\beta}\Big(-\frac{xg(x)+xG(x)}{r^{1/\beta}} + o\Big(\frac{xg(x)+xG(x)}{r^{1/\beta}}\Big)\Big),$$

which converges to $-1/2$ when r goes to ∞. We conclude that $\lim_{r\to\infty}(x\log(r) - x\beta\log(x)) = -1/2$.

(b) By (4.1.5), we get

$$e^{x\beta} = e^{\beta r^{1/\beta}} e^{-\beta x g(x)} e^{-\beta x G(x)} \asymp e^{\beta r^{1/\beta}}$$

and

$$x^{\beta/2} = (r^{1/\beta} - xg(x) - xG(x))^{\beta/2} \asymp r^{1/2}.$$

These three assertions imply that the second term in (4.1.3) is bounded by the quantity $e^{\beta r^{1/\beta}}/\sqrt{r}$, concluding the proof. □

Theorem 3.3.17 then yields the following result.

Theorem 4.1.9. *Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support, $\alpha > 0$ and let $(e_n)_{n\geq 0}$ be the sequence of monomials. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{n!^\alpha} e_n$ is almost surely an entire function, is frequently hypercyclic for the weighted shift associated with the sequence of weights $(n^\alpha)_{n\geq 1}$ and there exists $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{n!^\alpha} e_n \right\|_r \leq c\sqrt{\log(r)}\, r^{\frac{1}{4\alpha}-\frac{1}{2}} e^{\alpha r^{1/\alpha}}$$

for every $r \geq r_0$.

Proof. First, the random vector $\sum_{n\geq 0} X_n/n!^\alpha e_n$ is almost surely entire and frequently hypercyclic for the weighted shift by Theorem 1.3.4. By using Lemma 4.1.7 and Theorem 3.3.11, we get that $f = \sum_{n\geq 0} e_n/n!^\alpha$ is of finite order. Noticing that $\log(\mu_f) \leq \log(S_f)$, by Lemmas 4.1.6 and 4.1.7, we see that Assumption 3.3.14 is satisfied with $b = 1$, $m = 1$ and any $\varepsilon > 0$. We conclude by applying Theorem 3.3.17. The series S_f^2 is estimated by Lemma 4.1.8 applied to $\beta = 2\alpha$. □

The case $\alpha = 1$ corresponds to the differentiation operator.

Theorem 4.1.10. *Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables and let $(e_n)_{n\geq 0}$ be the sequence of monomials. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{n!} e_n$ is almost surely an entire function, is frequently hypercyclic for the differentiation operator and there exists $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{n!} e_n \right\|_r \leq c\sqrt{\log(r)} \frac{e^r}{r^{1/4}}$$

for every $r \geq r_0$.

This result have already been obtained by Nikula [80, Proposition 2]. By Drasin and Saksman [34, Theorem 1.1], it is already known that $r \longmapsto e^r/r^{1/4}$ is the optimal growth for the differentiation operator.

4.1.2 Dunkl operator

Another example of a weighted shift on $H(\mathbb{C})$ is the Dunkl operator. The product of its first n weights is given by

$$\beta_n := 2^n \left(\left\lfloor \frac{n}{2} \right\rfloor !\right) \Gamma\left(\left\lfloor \frac{n+1}{2} \right\rfloor + \alpha + 1\right) \Gamma(\alpha+1)^{-1},$$

$n \geq 1$, where $\alpha > -1/2$ and Γ is the gamma function, see [33, Exemple III.9.9].

We have the estimate

$$\beta_n \asymp (n+\alpha+1)^{n+\alpha+1} e^{-(n+\alpha+1)}$$

by [15, Lemma 1]. Therefore, we have $\lim_{n\to\infty} \beta_n^{1/n} = \infty$, and the operator is chaotic, see Example 0.1.21. We will apply Theorem 3.3.17 to the entire function $f := \sum_{n\geq 0} e_n/\alpha_n$ where $\alpha_n := (n+\alpha+1)^{n+\alpha+1} e^{-(n+\alpha+1)}$, $n \geq 1$. Since $\alpha_n \asymp \beta_n$, f satisfies the assumptions of Theorem 3.3.4 if and only if $\sum_{n\geq 0} e_n/\beta_n$ does so by Remark 3.3.18. Therefore, we just need to show that Theorem 3.3.17 can be applied to f.

As in the previous section, we must check that Assumption 3.3.14 is satisfied for f. We begin by estimating the function N_f.

Lemma 4.1.11. *For every $0 < \varepsilon < 1$, there exists $r_0 > 0$ such that for every $r \geq r_0$, we have*

$$(1-\varepsilon)\log(r) \leq \log(N_f(r)) \leq (1+\varepsilon)\log(r).$$

Proof. If $n = N_f(r) - 1$ then

$$\log(n+\alpha+1) \leq \frac{n\log(r)}{n+\alpha+1} + 1,$$

which proves the second inequality.

If $n = N_f(r)$ then $\log(r)\frac{n}{n+\alpha+1} + 1 < \log(n+\alpha+1)$. Let $\delta > 0$, there exists $r_0 > 0$ such that for every $r \geq r_0$,

$$(1-\delta)\log(r) \leq \log(r)\frac{n}{n+\alpha+1} + 1 < \log(n+\alpha+1) \leq (1+\delta)\log(n).$$

Now choose $\delta > 0$ such that $(1-\delta)(1+\delta)^{-1} = 1-\varepsilon$, we then have $(1-\varepsilon)\log(r) \leq \log(N_f(r))$. □

Lemma 4.1.12. *For any $r > 0$ large enough, we have*

$$\log(\mu_f(r)) \asymp r.$$

Proof. By definition of μ_f,

$$\mu_f(r) = \max_{n\geq 0} \frac{r^n e^{n+\alpha+1}}{(n+\alpha+1)^{n+\alpha+1}} = \max_{n\geq 0} \left(\frac{re}{n+\alpha+1}\right)^{n+\alpha+1} r^{-(\alpha+1)}.$$

Define the function $g :]0,\infty[\longrightarrow \mathbb{R}$ by $g(x) = (x+\alpha+1)(\log(re) - \log(x+\alpha+1))$ for every $x > 0$. Its derivative is given by

$$\partial_x g(x) = \log(re) - \log(x+\alpha+1) + (x+\alpha+1)(-(x+\alpha+1)^{-1})$$

and is positive if and only if $x < r-\alpha-1$. Therefore, $\mu_f(r)$ is attained at either $\nu_f(r) = \lceil r-\alpha-1 \rceil$ or $\nu_f(r) = \lfloor r-\alpha-1 \rfloor$.

We then have

$$\frac{\log(\mu_f(r))}{r} = \frac{\nu_f(r)+\alpha+1}{r} \log\left(\frac{re}{\nu_f(r)+\alpha+1}\right) - (\alpha+1)\frac{\log(r)}{r}$$

which converges to 1 when r goes to infinity since $\lim_{r\to\infty}(\nu_f(r)+\alpha+1)/r = 1$. □

Before making the conclusion, we will use Theorem 4.1.5 to estimate the series ${S_f}^2$ of Theorem 3.3.17 in the following lemma, whose proof is a little bit technical.

Lemma 4.1.13. *Let $\alpha > -1/2$ be a real number. Then for every $r > 0$ large enough, one has*

$$\sum_{n=0}^{\infty} \frac{r^{2n} e^{2(n+\alpha+1)}}{(n+\alpha+1)^{2(n+\alpha+1)}} \asymp \frac{e^{2r}}{r^{2(\alpha+3/4)}}.$$

Proof. The result is equivalent to

$$\sum_{n=0}^{\infty} \left(\frac{re}{n+\alpha+1}\right)^{2(n+\alpha+1)} \asymp \sqrt{r}e^{2r}.$$

Let $r > 0$ be large, and define the function

$$f_r :]0,\infty[\longrightarrow \mathbb{R}, x \longmapsto 2\big(x+\alpha+1\big)\Big(\log(re) - \log\big(x+\alpha+1\big)\Big).$$

For each $x > 0$, we have $\partial_x f_r(x) > 0$ if and only if $\log(re) - \log(x+\alpha+1) - 1 > 0$ if and only if $r-\alpha-1 > x$. Therefore, the derivative of f_r only vanishes at $x := r-\alpha-1$. We can then write

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{re}{n+\alpha+1}\right)^{2(n+\alpha+1)} &\geq \left(\frac{re}{\alpha+1}\right)^{2(\alpha+1)} \\ &\quad + \sum_{1\leq n\leq \lfloor x\rfloor} \int_{n-1}^{n} \left(\frac{re}{t+\alpha+1}\right)^{2(t+\alpha+1)} \mathrm{d}t + \sum_{n>\lfloor x\rfloor} \int_{n}^{n+1} \left(\frac{re}{t+\alpha+1}\right)^{2(t+\alpha+1)} \mathrm{d}t \\ &\geq \int_0^{\infty} \left(\frac{re}{t+\alpha+1}\right)^{2(t+\alpha+1)} \mathrm{d}t - e^{f_r(x)}.\end{aligned}$$

Similarly, we have

$$\sum_{n=0}^{\infty}\left(\frac{re}{n+\alpha+1}\right)^{2(n+\alpha+1)} \leq \int_0^{\infty} \left(\frac{re}{t+\alpha+1}\right)^{2(t+\alpha+1)} \mathrm{d}t + 2e^{f_r(x)}. \tag{4.1.7}$$

Recalling that $x = r - \alpha - 1$, the second term of the right-hand side of (4.1.7) is

$$e^{f_r(x)} = \Big(\frac{re}{x+\alpha+1}\Big)^{2(x+\alpha+1)} = e^{2r}.$$

We now estimate the integral of the right-hand side of (4.1.7). First, we have for r large such that $re/(\alpha+1) > 1$,

$$\int_0^1 \Big(\frac{re}{t+\alpha+1}\Big)^{2(t+\alpha+1)} \mathrm{d}t \leq \frac{(re)^{2(2+\alpha)}}{(\alpha+1)^{2(2+\alpha)}} \lesssim \sqrt{r}e^{2r}.$$

Therefore, we just need to estimate the integral on $[1,\infty[$ instead on $[0,\infty[$. By the change of variables $t = s+\alpha+1$ and setting $a := 2+\alpha$, we get that

$$\int_1^\infty \Big(\frac{re}{s+\alpha+1}\Big)^{2(s+\alpha+1)} \mathrm{d}s = \int_a^\infty \frac{(re)^{2t}}{t^{2t}} \mathrm{d}t = \int_a^\infty e^{2t\log(re)-2t\log(t)} \mathrm{d}t.$$

Define the function

$$G_r :]0,\infty[\longrightarrow \mathbb{R}, t \longmapsto 2t\log(re) - 2t\log(t).$$

Its derivative is given by

$$\partial_t G_r(t) = 2\log(r) - 2\log(t)$$

for every $t > 0$. The derivative of G_r only vanishes at r. Now, we make the change of variables $t = ry$ to get that

$$\begin{aligned}\int_1^\infty \Big(\frac{re}{s+\alpha+1}\Big)^{2(s+\alpha+1)} \mathrm{d}s &= \int_{a/r}^\infty re^{2ry\log(re)-2ry\log(ry)} \mathrm{d}y \\ &= \int_{a/r}^\infty re^{2ry(1-\log(y))} \mathrm{d}y.\end{aligned}$$

Pick $0 < \delta < 1$. Let $0 < \varepsilon < 1$ be such that $y(1-\log(y)) < \delta$ for all $0 < y < \varepsilon$. We can now apply Theorem 4.1.5 to $h : y \longmapsto 2y(1-\log(y))$ and $g : y \longmapsto 1$ with $c = 1$ to get

$$\int_\varepsilon^\infty re^{2ry(1-\log(y))} \mathrm{d}y \asymp \sqrt{r}e^{2r}.$$

By definition of $\varepsilon > 0$, we also have

$$\int_{a/r}^\varepsilon re^{2ry(1-\log(y))} \leq \varepsilon re^{2r\delta},$$

the right-hand term being negligible compared to $\sqrt{r}e^{2r}$ since $\delta < 1$. In conclusion, the integral of the right-hand side of (4.1.7) is estimated by $\sqrt{r}e^{2r}$, concluding the proof. $\square$

Theorem 3.3.17 then yields the following result.

Theorem 4.1.14. *Let $(\beta_n)_{n\geq 0}$ be the product of the weights associated with the Dunkl operator. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables and let $(e_n)_{n\geq 0}$ be the sequence of monomials. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n$ is almost surely an entire function, is frequently hypercyclic for the Dunkl operator and there exists $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n \right\|_r \leq c\sqrt{\log(r)} \frac{e^r}{r^{\alpha+3/4}}. \tag{4.1.8}$$

for every $r \geq r_0$.

Proof. First, the random vector $\sum_{n\geq 0} X_n/\beta_n e_n$ is almost surely entire and frequently hypercyclic for the Dunkl operator by Theorem 1.3.4. By Lemmas 4.1.11 and 4.1.12, if $\delta > 0$ is fixed and for every $r > 0$ large enough and $0 < \varepsilon < 1$, we have

$$\log(N_f(re^{\delta})) \leq (1+\varepsilon)\log(re^{\delta}) \lesssim (1+\varepsilon)\log_2(\mu_f(r)),$$

hence $N_f(re^{\delta}) \leq \log(\mu_f(r))^C$ for some $C > 0$. Furthermore, Lemma 4.1.12 and Theorem 3.3.11 tell us that $\sum_{n\geq 0} e_n/\alpha_n$ is of finite order, hence Assumption 3.3.14 is satisfied with $b = C$ and $m = 1$; notice that $\log(\mu_f) \leq \log(S_f)$. We conclude that Theorem 3.3.17 can be applied to f, and hence to $\sum_{n\geq 0} e_n/\beta_n$ by Remark 3.3.18. Finally, the series S_f^2 is estimated by Lemma 4.1.13. □

Therefore, (4.1.8) gives a better rate of growth than the one found by Bernal-González and Bonilla [15, Theorem 5]. By their Theorem 6, this could even be the optimal growth. However, their bound works for any function φ tending to infinity instead of $\sqrt{\log(\cdot)}$, but see the discussion in Section 4.6.

4.1.3 Aron-Markose operators

As a last example, we consider the operators $T_{\lambda,b}$ introduced by Aron and Markose [4]. They are defined by $T_{\lambda,b}(f) = f'(\lambda z + b)$, $f \in H(\mathbb{C})$, where the parameters λ and b are complex. If $\lambda \in \mathbb{C} \setminus \{0,1\}$, we know that these operators are weighted shifts with respect to the basis $((z-a)^n)_{n\geq 0}$.

Lemma 4.1.15 ([65, Proposition 2.1])**.** *Let $\lambda \in \mathbb{C} \setminus \{0,1\}$ and $b \in \mathbb{C}$. Then the operator $T_{\lambda,b}(f)$ is a weighted shift with respect to the basis $((z-a)^n)_{n\geq 0}$ with $a := b/(1-\lambda)$, and its weight sequence is $(w_n)_{n\geq 1} = (n\lambda^{n-1})_{n\geq 1}$.*

Proof. Define for each $n \in \mathbb{N}$ the map $e_n(z) = (z-a)^n$, $z \in \mathbb{C}$. Then for every $n \geq 1$ and $z \in \mathbb{C}$, we have

$$T_{\lambda,b}(e_n)(z) = n(\lambda z + b - a)^{n-1} = n\lambda^{n-1}\Big(z + \frac{b-a}{\lambda}\Big)^{n-1} = n\lambda^{n-1} e_{n-1}(z),$$

and $T_{\lambda,b}(e_0) = 0$. □

It is easy to see that $T_{\lambda,b}$ is chaotic on $H(\mathbb{C})$ if and only if $|\lambda| \geq 1$. Indeed, we only need to show that $\lim_{n\to\infty} |\beta_n|^{1/n} = \infty$, where $\beta_n := n!\lambda^{n(n-1)/2}$, $n \geq 1$, see Theorem 0.1.19 and Example 0.1.21. A comparison series-integral allows us to write

$$\begin{aligned}\log(|\beta_n|^{1/n}) &\asymp n^{-1}\int_1^n \log(x)\mathrm{d}x + (n-1)\frac{\log(|\lambda|)}{2} \\ &= \log(n) - \frac{n-1}{n} + (n-1)\frac{\log(|\lambda|)}{2}.\end{aligned}$$

The right-hand side converges to ∞ if $|\lambda| \geq 1$ and to $-\infty$ otherwise, hence the claim.

We will show that the entire function $f(z) := \sum_{n\geq 0} z^n/\beta_n$, $z \in \mathbb{C}$, is of finite order and satisfies Assumption 3.3.14. We can assume without loss of generality that $\lambda > 1$. Indeed, only the modulus of λ matters and if $|\lambda| = 1$, then N_f and μ_f are the same as for the differentiation operator, see Lemmas 4.1.6 and 4.1.7 .

Lemma 4.1.16. *For every $0 < \varepsilon < 1$, there exists $r_0 > 0$ such that for every $r \geq r_0$, we have*

$$(1-\varepsilon)\log(r) \leq \frac{\log(\lambda)}{2}N_f(r) \leq (1+\varepsilon)\log(r).$$

Proof. If $n = N_f(r) - 1$ then

$$\begin{aligned}\log(r)n \geq \log(n!) + n(n-1)\frac{\log(\lambda)}{2} &\geq \sum_{k=2}^{n}\int_{k-1}^{k}\log(x)\mathrm{d}x + n(n-1)\frac{\log(\lambda)}{2} \\ &= n\log(n) - (n-1) + n(n-1)\frac{\log(\lambda)}{2}.\end{aligned}$$

Therefore

$$\log(r) + \frac{n-1}{n} \geq (n+1)\frac{\log(\lambda)}{2}\left(\frac{2\log(n)}{(n+1)\log(\lambda)} + \frac{n-1}{n+1}\right).$$

Let $0 < \delta < 1$, if r is large enough then we have

$$(1+\delta)\log(r) \geq \frac{\log(\lambda)}{2}N_f(r)(1-\delta).$$

Now choose $\delta > 0$ such that $(1+\delta)(1-\delta)^{-1} = 1+\varepsilon$, we then have

$$(1+\varepsilon)\log(r) \geq \frac{\log(\lambda)}{2}N_f(r).$$

A similar argument shows the other inequality of the lemma. $\square$

Lemma 4.1.17. *For any $r > 0$ large enough, we have*

$$\log(\mu_f(r)) \asymp \log(r)^2.$$

Proof. By definition of μ_f and Stirling's formula,

$$\mu_f(r) \asymp (re)^{-1/2}\max_{n\geq 0}\left(\left(\frac{re}{n+1/2}\right)^{n+1/2}\frac{1}{\lambda^{n(n-1)/2}}\right).$$

Let $r > 0$. Define the function $g_r :]0, \infty[\longrightarrow \mathbb{R}$ by

$$g_r(x) = (x + 1/2)(\log(re) - \log(x + 1/2)) - \frac{x(x-1)}{2} \log(\lambda)$$

for every $x > 0$. Its derivative is given by

$$\partial_x g_r(x) = \log(re) - \log(x + 1/2) - 1 - \frac{2x-1}{2} \log(\lambda)$$

and is positive if and only if

$$x \left(1 + \frac{\log(x + 1/2)}{x \log(\lambda)} \right) < \frac{\log(r)}{\log(\lambda)} + \frac{1}{2}.$$

Let $x_{\max}(r)$ be the value of x where g_r reaches its maximum. Therefore, $\mu_f(r)$ is attained at either $\nu_f(r) = \lceil x_{\max}(r) \rceil$ or $\nu_f(r) = \lfloor x_{\max}(r) \rfloor$. Furthermore, $x_{\max}(r)$ is such that, for every $0 < \varepsilon < 1$, there exists $r_0 \geq 1$ such that for every $r \geq r_0$, one has

$$(1 - \varepsilon) \left(\frac{\log(r)}{\log(\lambda)} + \frac{1}{2} \right) \leq x_{\max}(r) \leq \frac{\log(r)}{\log(\lambda)} + \frac{1}{2}.$$

If $\nu_f(r) = \lceil x_{\max}(r) \rceil$, then

$$\begin{aligned}
\log(\mu_f(r)) &\gtrsim -\frac{\log(re)}{2} - \frac{\log(\lambda)}{2} \left(\frac{\log(r)}{\log(\lambda)} + \frac{1}{2} + 1 \right) \left(\frac{\log(r)}{\log(\lambda)} + \frac{1}{2} + 1 - 1 \right) \\
&\quad + \left(\left(\frac{\log(r)}{\log(\lambda)} + \frac{1}{2} \right) (1 - \varepsilon) + \frac{1}{2} \right) \left(\log(re) - \log \left(\frac{\log(r)}{\log(\lambda)} + \frac{1}{2} + 1 + \frac{1}{2} \right) \right) \\
&= \log(r)^2 \Bigg(-\frac{\log(re)}{2\log(r)^2} - \frac{\log(\lambda)}{2} \left(\frac{1}{\log(\lambda)} + \frac{3}{2\log(r)} \right) \left(\frac{1}{\log(\lambda)} + \frac{1}{2\log(r)} \right) \\
&\quad + \left(\left(\frac{1}{\log(\lambda)} + \frac{1}{2\log(r)} \right) (1 - \varepsilon) + \frac{1}{2\log(r)} \right) \\
&\quad . \left(\frac{\log(re)}{\log(r)} - \log(r)^{-1} \log \left(\frac{\log(r)}{\log(\lambda)} + 2 \right) \right) \Bigg).
\end{aligned}$$

The terms in the largest brackets of the right-hand side of the equality converge to

$$-2^{-1} \log(\lambda)^{-1} + \log(\lambda)^{-1}(1 - \varepsilon)$$

when r goes to ∞, which is positive if ε is small enough. Therefore, $\log(\mu_f(r)) \gtrsim \log(r)^2$. Similarly, one can prove that $\log(\mu_f(r)) \lesssim \log(r)^2$, and that $\log(\mu_f(r)) \asymp \log(r)^2$ if $\nu_f(r) = \lfloor x_{\max}(r) \rfloor$. $\square$

Theorem 4.1.18. *Let $(X_n)_{n \geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support, $(e_n)_{n \geq 0} = ((z-a)^n)_{n \geq 0}$, and let $b \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{1\}$ such that $|\lambda| \geq 1$. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{n! \lambda^{n(n-1)/2}} e_n$ is almost surely an* *entire function, is frequently hypercyclic for $T_{\lambda,b}$ and there exists $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{n!\lambda^{n(n-1)/2}} e_n \right\|_r \leq c\sqrt{\log(\mu(r))} \sqrt{\sum_{n=0}^{\infty} \frac{(r+|a|)^{2n}}{n!^2|\lambda|^{n(n-1)}}}$$

for every $r \geq r_0$, where $a := b/(1-\lambda)$ and for each $r \geq 1$, $\mu(r) := \log(r)$ if $|\lambda| > 1$ and $\mu(r) := r$ otherwise.

Proof. The random vector $\sum_{n\geq 0} X_n/(n!\lambda^{n(n-1)/2})e_n$ is almost surely an entire function and frequently hypercyclic for the weighted shift by Theorem 1.3.4; note that its proof carries over verbatim to weighted shifts on $H(\mathbb{C})$ with respect to the basis $(e_n)_{n\geq 0}$.

Assume that $|\lambda| > 1$. Let $\varepsilon > 0$ and $\delta > 0$. By Lemmas 4.1.16 and 4.1.17, for every $r > 0$ large enough, we have

$$2^{-1}\log(|\lambda|)N_f(re^{\delta}) \leq (1+\varepsilon)\log(re^{\delta}) \lesssim (1+\varepsilon)\sqrt{\log(\mu_f(r))},$$

hence $N_f(re^{\delta}) \lesssim \log(\mu_f(r))^{1/2}$ for $r > 0$ large enough. Furthermore, Lemma 4.1.17 and Theorem 3.3.11 tell us that f is of finite order, hence Assumption 3.3.14 is satisfied with $b = 1/2$ and $m = 1$; notice that $\log(\mu_f) \leq \log(S_f)$. If $|\lambda| = 1$ then Lemmas 4.1.6 and 4.1.7 yield the same conclusion with $b = 1$. By applying Theorem 3.3.17, we can conclude that almost surely, for r large enough,

$$\|v\|_r \lesssim \sqrt{\log(\mu(r))} \sqrt{\sum_{n\geq 0} \frac{r^{2n}}{n!^2|\lambda|^{n(n-1)}}},$$

where $v(z) := \sum_{n\geq 0} X_n/\beta_n z^n$, $z \in \mathbb{C}$. Noticing that

$$\|u\|_r = \sup_{|z|=r} |v(z-a)| \leq \sup_{|z|\leq r+|a|} |v(z)| = \|v\|_{r+|a|},$$

where $u := \sum_{n\geq 0} X_n/\beta_n e_n$, we get almost surely the inequality

$$\|u\|_r \lesssim \sqrt{\log(\mu_f(r+|a|))} \sqrt{\sum_{n\geq 0} \frac{(r+|a|)^{2n}}{n!^2|\lambda|^{n(n-1)}}}$$

for any $r > 0$ large enough. This concludes the proof. □

Note that the series in the rate of growth obtained in Theorem 4.1.18 is the series $\sum_{n\geq 0} \|e_n\|_r^2/|\beta_n|^2$ associated with the fixed point $\sum_{n\geq 0} e_n/\beta_n$ of $T_{\lambda,b}$ that appears in Theorem 3.3.4. Indeed, we have $\|e_n\|_r = (r+|a|)^n$ for every $r > 0$ and integer $n \geq 0$. It is enough to show this for $n = 1$. Let $r > 0$. By definition of $\|\cdot\|_r$ and assuming $a = \rho e^{i\varphi} \neq 0$, we get

$$\sup_{0\leq\theta\leq 2\pi} |e^{i\theta}r - a| = \sup_{0\leq\theta\leq 2\pi} \rho\left|e^{i(\theta-\varphi)}\frac{r}{\rho} - 1\right| = \rho \sup_{0\leq\theta\leq 2\pi} \left|e^{i\theta}\frac{r}{\rho} - 1\right|.$$

Define the function $f : [0, 2\pi] \longrightarrow [0, \infty[$ by $f(\theta) := |e^{i\theta}\frac{r}{\rho} - 1|^2$, $\theta \in [0, 2\pi]$. A simple study of function shows that f reaches its maximum at $\theta = \pi$ and $f(\pi) = (r/\rho + 1)^2$. Therefore, $\|e_1\|_r = (r + |a|)$ for every $r > 0$.

Remark 4.1.19. Recall that by Theorem 1.3.1, the probability distribution of the frequently hypercyclic random vector in Theorem 4.1.18 is strongly mixing. It was already known that the Aron-Markose operators are strongly mixing with respect to some Gaussian measure of full support, see [79, Proposition 2.3] and its proof.

We point out that more generally, one can consider the weights $w_n = n^\alpha \lambda^{n-1}$, $n \geq 1$, where $\lambda \in \mathbb{C}$ is such that $|\lambda| \geq 1$, and $\alpha > 0$. The computations for these weights are similar. In the same vein, Theorem 3.3.17 can be applied to the weights defined by $\beta_n = (n + \alpha + 1)^{n+\alpha+1} e^{-(n+\alpha+1)} \lambda^{n(n-1)/2}$, $n \geq 1$, where $\alpha > -1/2$ and $\lambda \in \mathbb{C}$ is such that $|\lambda| \geq 1$.

4.2 Functions on an open disk

In this section, we consider weighted shifts defined on the Fréchet space $H(\mathbb{D})$. Theorem 0.1.19 combined with Theorem 3.2.10 applied to $f(z) = \sum_{n\geq 0} z^n/\beta_n$ immediately yields an admissible rate of growth for each chaotic weighted shift, where the sequence $(\beta_n)_{n\geq 0}$ is defined in Subsection 1.3.1. However, it is only valid outside a set of finite logarithmic measure.

Theorem 4.2.1. *Let T be a chaotic weighted shift on $H(\mathbb{D})$ with respect to the basis of monomials $(e_n)_{n\geq 0}$ and with sequence of weights $(w_n)_{n\geq 1}$. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n$ is almost surely holomorphic on $\mathbb{D}$, is frequently hypercyclic for T and there exist a constant $c > 0$ and an open set $E \subseteq [0, \infty[$ of finite logarithmic measure such that almost surely, there exists $0 < r_0 < 1$ such that*

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n \right\|_r \leq c \frac{\mu_f(r)}{(1-r)^{\frac{1+\delta}{2}}} \log^{\frac{1+\delta}{4}} \left(\frac{\mu_f(r)}{1-r} \right)$$

for every $r_0 \leq r < 1$, $r \notin E$.

The frequent hypercyclicity of the random vector is obtained by Theorem 1.3.4.

As in the previous section, we will show that Theorem 3.3.4 can always be partially applied to chaotic weighted shifts on $H(\mathbb{D})$. Unlike the corresponding result for the space $H(\mathbb{C})$, we distinguish two cases. The only difference lies in the choice of the function A_1 in order to ensure that $\lim_{r\to 1} A_1(r) = \infty$, but the proof is actually the same as that of Proposition 4.1.2.

Proposition 4.2.2. *Let T be a chaotic weighted shift on $H(\mathbb{D})$ with respect to the basis of monomials $(e_n)_{n\geq 0}$ and with sequence of weights $(w_n)_{n\geq 1}$. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support. Assume that there exists $\alpha > 3/2$ such that for every $n_0 \geq 1$, $\inf_{n\geq n_0} |\beta_n|/n^\alpha < 1$. Then the random vector $\sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n$ is almost surely holomorphic on $\mathbb{D}$, is frequently hypercyclic* *for T and satisfies the assumption* (i) *of Theorem 3.3.4 with*

$$A_j(r) := \max\left\{n \geq 0 \mid |\beta_n| \leq r^n n^\alpha\right\}.j,$$

for all $0 \leq r < 1$ and $j \geq 1$.

Proof. The proof is exactly the same as that of Proposition 4.1.2. First note that A_1 is well-defined since T is chaotic, which is equivalent to $\liminf_{n\to\infty} |\beta_n|^{1/n} \geq 1$ by 0.1.22. We just need to show that $\lim_{r\to 1} A_1(r) = \infty$. Let $n_0 \geq 1$; by assumption, there exists $n \geq n_0$ such that $|\beta_n| < n^\alpha$. This is equivalent to $|\beta_n|^{1/n}/n^{\alpha/n} < 1$, thus for every $|\beta_n|^{1/n}/n^{\alpha/n} < r < 1$, we have $A_1(r) \geq n \geq n_0$. Since n_0 was arbitrary, $\lim_{r\to 1} A_1(r) = \infty$. □

The proof of the next result is also exactly the same as that of Proposition 4.1.2.

Proposition 4.2.3. *Let T be a chaotic weighted shift on $H(\mathbb{D})$ with respect to the basis of monomials $(e_n)_{n\geq 0}$ and with sequence of weights $(w_n)_{n\geq 1}$. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support. Assume that there exist $\alpha > 3/2$ and an integer $n_0 \geq 1$ such that for every $n \geq n_0$, $|\beta_n| \geq n^\alpha$. Then the random vector $\sum_{n=0}^\infty \frac{X_n}{\beta_n} e_n$ is almost surely holomorphic on $\mathbb{D}$, is frequently hypercyclic for T and satisfies the assumption* (i) *of Theorem 3.3.4 with $A_j = A.j$, $j \geq 1$, where $A : [0, 1[\longrightarrow [0, \infty[$ is any function such that $\lim_{r\to 1} A(r) = \infty$.*

Remark 4.2.4. Notice that Lemma 4.1.3 still holds for operators on $H(\mathbb{D})$.

Condition (ii) of Theorem 3.3.4 can always be satisfied with the function A_1 in Propositions 4.2.2 or 4.2.3: since $\lim_{r\to\infty} A_1(r) = \infty$, such a sequence $(r_k)_{k\geq 1}$ must exist. This allows us to get assumptions (i) and (ii) of the theorem.

As in the previous section, in order to fully apply Theorem 3.3.4, we have to choose a sequence $(r_k)_{k\geq 0}$ satisfying both assumptions (ii) and (iii) of the theorem. Again, we do not know if it is possible to achieve this for any weighted shift on $H(\mathbb{D})$ with the choice of the function A_1 in Propositions 4.2.2 or 4.2.3. However, by choosing a slightly different function A_1, and for some weighted shifts, we can fully apply Theorem 3.3.4. This will be the content of the next subsections. In Subsection 4.2.1, we will first prove an analogous result to [46, Theorem 1].

Let us compute the function A_1 of Proposition 4.2.2 only for the so-called Taylor shift.

Lemma 4.2.5. *Let $T : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ be the Taylor shift, that is, $T(z^n) = z^{n-1}$ for any $n \geq 1$, and $T(1) = 0$. Then $\log(A_1(r)) \asymp -\log(1-r)$ where A_1 is defined in Proposition 4.2.2.*

Proof. Let $\alpha > 3/2$. Let $0 < r < 1$ be close enough to 1 and set $n := A_1(r)$. Then $1 \leq r^n n^\alpha$, which is equivalent to

$$0 \leq n \log(r) + \alpha \log(n).$$

Thus

$$\frac{n}{\log(n)} \leq \frac{\alpha}{-\log(r)},$$

which yields $\log(A_1(r)) \lesssim -\log(-\log(r))$. A Taylor development at 1 of the logarithm yields the inequality $A_1(r) \lesssim -\log(1-r)$. The other inequality is similarly proved. □

4.2.1 Rate of growth of hypercyclic functions

We begin by proving in this short subsection an analogous result of [46, Theorem 1] on the growth of hypercyclic functions for chaotic weighted shifts on $H(\mathbb{D})$. Its proof is the same as the one of [46, Theorem 1]. As usual, the quantity β_n, $n \geq 0$, stands for the product of the first n weights of the shift, see Subsection 1.3.1.

Theorem 4.2.6. *Let $T : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ be a chaotic weighted shift. Let $\phi : [0,1[\longrightarrow [0,\infty[$ be such that $\lim_{r\to 1} \phi(r) = \infty$. Then there exists a hypercyclic function $f \in H(\mathbb{D})$ for T such that $\|f\|_r \lesssim \phi(r)\mu(r)$ for all $0 < r < 1$ close enough to 1, where $\mu(r) := \max_{n\geq 0} r^n/|\beta_n|$.*

Proof. Without loss of generality, we can assume that $\inf_{0\leq r<1} \phi(r) > 0$. For $f = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ and all $n \geq 0$, define

$$p_n(f) := \sup_{0<r<1} \frac{\| \sum_{j=n}^{\infty} a_j z^j \|_r}{\phi(r)\mu(r)}.$$

Define the space

$$X := \Big\{ f = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \mid \sup_{n\in\mathbb{N}} p_n(f) < \infty \wedge \lim_{n\to\infty} p_n(f) = 0 \Big\}$$

and $\|f\|_X := \sup_{n\geq 0} p_n(f)$, $f \in X$. The space X endowed with the norm $\| \cdot \|_X$ is a Banach space and the set of polynomials is dense in X. Furthermore, X is continuously embedded in $H(\mathbb{D})$.

We will check that the sequence of operators $T_n : X \longrightarrow H(\mathbb{D}), f \longmapsto T^n(f)$ satisfies the hypotheses of the Universality Criterion, see [47, Theorem 3.24]. As in the proof of [46, Theorem 1], all we need is to prove that $\lim_{n\to\infty} \|e_n/\beta_n\|_X = 0$, where $(e_n)_{n\geq 0}$ is the sequence of monomials.

Let $\varepsilon > 0$. There exists $0 < r_0 < 1$ such that for every $r_0 \leq r < 1$, one has $\phi(r) \geq \varepsilon^{-1}$. Then for all $n \geq 0$, we have

$$\sup_{r_0\leq r<1} \frac{r^n}{|\beta_n|\phi(r)\mu(r)} \leq \varepsilon$$

and

$$\sup_{0\leq r\leq r_0} \frac{r^n}{|\beta_n|\phi(r)\mu(r)} \leq \frac{r_0^n}{|\beta_n|} \sup_{0\leq r\leq r_0} \frac{1}{\phi(r)\mu(r)} \leq \frac{r_0^n}{|\beta_n|} \frac{1}{\inf_{0\leq r<1} \phi(r)}.$$

The last right-hand term converges to 0 when n goes to infinity since T is chaotic; recall that T is chaotic if and only if $\sum_{n=0}^{\infty} z^n/\beta_n \in H(\mathbb{D})$, see Theorem 0.1.19. We conclude that $\lim_{n\to\infty} \|e_n/\beta_n\|_X = 0$, as desired. This implies the claim of the theorem like in the proof of [46, Theorem 1]. □

4.2.2 Weighted Taylor shifts

We consider the weighted shift $T_\alpha : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ with respect to the basis of monomials of $H(\mathbb{D})$, with weights $w_n = n^\alpha/(n-1)^\alpha$, $n \geq 2$, and $w_1 = 1$, where

$\alpha \in \mathbb{R}$. This operator is well-defined on $H(\mathbb{D})$. We have $\beta_n = n^\alpha$ for all $n \geq 1$. Since $\lim_{n\to\infty} |\beta_n|^{1/n} = 1$, the operator T_α is also chaotic on $H(\mathbb{D})$, see Example 0.1.22.

These operators have been studied in [76]. The case $\alpha = 0$ corresponds to the so-called Taylor shift.

In order to avoid a too long proof, we will distinguish two cases depending on whether $\alpha \leq 0$ or $\alpha > 0$. We begin with the case $\alpha \leq 0$.

Theorem 4.2.7. *Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support, let $(e_n)_{n\geq 0}$ be the sequence of monomials and let $\alpha \leq 0$. Then the random vector $X_0e_0 + \sum_{n=1}^{\infty} X_n/n^\alpha e_n$ is almost surely holomorphic on $\mathbb{D}$, is frequently hypercyclic for the weighted shift T_α and there exists $c > 0$ such that almost surely, there exists $0 < r_0 < 1$ such that*

$$\left\| X_0e_0 + \sum_{n=1}^{\infty} \frac{X_n}{n^\alpha} e_n \right\|_r \leq c\sqrt{|\log(1-r)|}(1-r)^{\alpha-1/2}$$

for every $r_0 \leq r < 1$.

Proof. The random vector $X_0e_0 + \sum_{n\geq 1} X_n/n^\alpha e_n$ is almost surely holomorphic on $\mathbb{D}$ and frequently hypercyclic for T_α by Theorem 1.3.4. Set $A_j(r) = d^j|\log(1-r)|/(1-r)$ with $d > 1$ to be determined later.

Set $\beta := -2\alpha$ and let $0 < r < 1$. Define the function

$$g_r : [0, \infty[\longrightarrow [0, \infty[, x \longmapsto r^{2x}x^\beta. \tag{4.2.1}$$

It is elementary to show that g_r has a single maximum at $x_{\max}(r) := -\beta/(2\log(r))$. We have $|\log(1-r)|/(1-r) \geq x_{\max}(r)$ for r close to 1 and hence, for every $j \geq 1$, one has $A_j(r) \geq x_{\max}(r)$ and thus

$$\sum_{n\geq A_j(r)+1} g_r(n) \leq \sum_{n\geq A_j(r)+1} \int_{n-1}^{n} g_r(x)\mathrm{d}x = \int_{A_j(r)}^{\infty} g_r(x)\mathrm{d}x.$$

Let $j \geq 1$, two consecutive changes of variables $y = 2|\log(r)|x$ and then $u = y - 2|\log(r)|A_j(r)$ yield

$$\begin{aligned}
\int_{A_j(r)}^{\infty} g_r(x)\mathrm{d}x &= \int_{2|\log(r)|A_j(r)}^{\infty} \frac{y^\beta e^{-y}}{2^{\beta+1}|\log(r)|^{\beta+1}}\mathrm{d}y \\
&= \frac{1}{2^{\beta+1}|\log(r)|^{\beta+1}} \int_0^\infty \Big(u + 2|\log(r)|A_j(r)\Big)^\beta e^{-(u+2|\log(r)|A_j(r))}\mathrm{d}u \\
&= \frac{A_j(r)^\beta e^{-2|\log(r)|A_j(r)}}{2|\log(r)|} \int_0^\infty \Big(u\big(2|\log(r)|A_j(r)\big)^{-1} + 1\Big)^\beta e^{-u}\mathrm{d}u.
\end{aligned}$$

Since $|\log(r)|A_j(r) \geq |\log(1-r)|$ for $0 < r < 1$, the last integral is bounded in $\delta < r < 1$ for every $0 < \delta < 1$.

In order to apply Theorem 3.3.4 to $f := \sum_{n\geq 0} e_n/n^\alpha$, we will show that

$$\sum_{j\geq 1} \sqrt{\log(A_{j+1}(r))}\sqrt{\frac{A_j(r)^\beta e^{-2|\log(r)|A_j(r)}}{|\log(r)|}}$$

converges to 0 when r goes to 1. The idea is to use the Dominated Convergence Theorem. First note that the series converges for any $0 < r < 1$ since $d > 1$. Let $j \geq 1$ and $0 < r < 1$. We make the change of variables $r = 1 - e^{-x}$, where $x > 0$, hence $A_j(r) = d^j x e^x$ for every $j \geq 1$. A simple calculation gives

$$\begin{aligned}\log(A_{j+1}(r))&\frac{A_j(r)^\beta e^{-2|\log(r)|A_j(r)}}{|\log(r)|}\\&= ((j+1)\log(d) + \log(x) + x)\frac{(d^j x e^x)^\beta e^{2\log(1-e^{-x})d^j x e^x}}{-\log(1-e^{-x})}.\end{aligned}$$

Therefore it is enough to show for every $j \geq 1$ that the function

$$f_j :]0,\infty[\longrightarrow]0,\infty[, x \longmapsto x\frac{(xe^x)^\beta e^{2\log(1-e^{-x})Mxe^x}}{-\log(1-e^{-x})}$$

is non-increasing and converges to 0 when x goes to ∞, where $M := d^j$.

By taking the logarithm of f_j, we get for every $x > 0$ that

$$\begin{aligned}&\log(f_j(x)) = \log(x) + \beta\log(xe^x) + 2\log(1-e^{-x})Mxe^x - \log\big(-\log(1-e^{-x})\big)\\&= \log(x) + \beta\log(xe^x) + 2Mx(-1 + o(e^{-x})e^x) - \log\big(e^{-x} + o(e^{-x})\big)\\&= x\bigg(\frac{\log(x)}{x} + \beta + \frac{\beta\log(x)}{x} + 2M(-1 + o(e^{-x})e^x) - \frac{\log(1 + o(e^{-x})e^x)}{x} + 1\bigg).\end{aligned}$$

Thus $\lim_{x\to\infty}\log(f_j(x)) = -\infty$ if $\beta + 1 < 2M$, and $\lim_{x\to\infty} f_j(x) = 0$.

Now we prove that f_j is non-increasing. The derivative of the function $\log(f_j)$ is given by

$$\begin{aligned}\partial_x\log(f_j)(x) &= \frac{1}{x} + \beta\left(\frac{1}{x} + 1\right) + 2M\left(\frac{x}{1-e^{-x}} + \log(1-e^{-x})e^x(x+1)\right)\\&\quad - \frac{e^{-x}}{\log(1-e^{-x})(1-e^{-x})}\\&= \frac{1}{x} + \beta\left(\frac{1}{x} + 1\right) + 2M\left(\frac{x}{1-e^{-x}} + e^x(x+1)\Big(-e^{-x} - \frac{e^{-2x}}{2} + o(e^{-2x})\Big)\right)\\&\quad - \frac{e^{-x}}{(-e^{-x} + o(e^{-x}))(1-e^{-x})},\end{aligned}$$

where in the second equality we have used the Taylor expansions of order 1 and 2 of the logarithm at 1. We conclude that $\lim_{x\to\infty}\partial_x\log(f_j)(x) = \beta - 2M + 1$ for every $x > 0$, hence $\partial_x\log(f_j)(x) < 0$, provided $2M > \beta + 1$. Thus f_j is non-increasing away from 0, uniformly in $j \geq 1$, if $2d > \beta + 1$.

This shows that assumption (i) of Theorem 3.3.4 is satisfied. By taking $r_k = 1 - e^{-k}$, it is clear that assumption (ii) is also satisfied, for any $1 < p < \infty$. It remains to check assumption (iii). It is also clear that $(\log(A_1(r_{k+1}))/\log(A_1(r_k)))_{k\geq 1}$ is bounded. Let $0 < r < 1$. Set $I(r) := \sum_{n\geq 1} n^\beta r^{2n}$, thus $S_f^2(r) = 1 + I(r)$. It suffices to show that $(I(r_{k+1})/I(r_k))_{k\geq 1}$ is bounded. By comparing again series and integrals, where we set $N := \lfloor x_{\max}(r) \rfloor$, we have

$$\begin{aligned}\sum_{n\geq 1} g_r(n) &\leq \sum_{n=1}^{N-1} \int_n^{n+1} g_r(x)\mathrm{d}x + g_r(N) + g_r(N+1) + \sum_{n\geq N+2} \int_{n-1}^{n} g_r(x)\mathrm{d}x \\ &\leq \int_1^\infty g_r(x)\mathrm{d}x + 2g_r(x_{\max}(r)) \\ &\leq \int_0^\infty g_r(x)\mathrm{d}x + 2g_r(x_{\max}(r))\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 1} g_r(n) &\geq \sum_{n=1}^{N} \int_{n-1}^{n} g_r(x)\mathrm{d}x + \sum_{n\geq N+1} \int_n^{n+1} g_r(x)\mathrm{d}x \\ &= \int_0^\infty g_r(x)\mathrm{d}x - \int_N^{N+1} g_r(x)\mathrm{d}x \geq \int_0^\infty g_r(x)\mathrm{d}x - g_r(x_{\max}(r)).\end{aligned}$$

The change of variables $y = 2|\log(r)|x$ and the Taylor expansion of order 1 at 1 of the logarithm yield that, for every $0 < r < 1$ close enough to 1,

$$\int_0^\infty g_r(x)\mathrm{d}x = \int_0^\infty e^{2\log(r)x} x^\beta \mathrm{d}x = \frac{1}{|\log(r)|^{\beta+1}2^{\beta+1}} \int_0^\infty e^{-y} y^\beta \mathrm{d}y \asymp \frac{1}{(1-r)^{\beta+1}}.$$

Recalling the definition (4.2.1) of g_r above, notice that $g_r(x_{\max}(r)) \asymp 1/(1-r)^\beta$. We can conclude that

$$\frac{1}{(1-r)^{\beta+1}} - \frac{1}{(1-r)^\beta} \lesssim I(r) \lesssim \frac{1}{(1-r)^{\beta+1}} + \frac{1}{(1-r)^\beta}, \tag{4.2.2}$$

for every $0 < r < 1$ close enough to 1. Then, applying this inequality for $r = r_k$ and $r = r_{k+1}$ finally gives

$$\frac{I(r_{k+1})}{I(r_k)} \lesssim \left(\frac{1}{(1-r_{k+1})^{\beta+1}} + \frac{1}{(1-r_{k+1})^\beta}\right)\left(\frac{1}{(1-r_k)^{\beta+1}} - \frac{1}{(1-r_k)^\beta}\right)^{-1}$$

for every $k \geq 1$ large enough, and the right-hand side is bounded. This shows that also (iii) of Theorem 3.3.4 holds. Thus we can apply Theorem 3.3.4. Since $\log(A_1(r)) \lesssim |\log(1-r)|$ and, by (4.2.2),

$$S_f(r) = \sqrt{1 + I(r)} \asymp (1-r)^{\alpha - 1/2},$$

the result follows. $\square$

Now we deal with the case $\alpha > 0$.

Theorem 4.2.8. *Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support, let $(e_n)_{n\geq 0}$ be the sequence of monomials and let $\alpha > 0$. Then the random vector $X_0 e_0 + \sum_{n=1}^\infty X_n/n^\alpha e_n$ is almost surely holomorphic on $\mathbb{D}$, is* *frequently hypercyclic for the weighted shift T_α and there exists $c > 0$ such that almost surely, there exists $0 < r_0 < 1$ such that*

$$\left\| X_0 e_0 + \sum_{n=1}^{\infty} \frac{X_n}{n^\alpha} e_n \right\|_r \leq c \begin{cases} \sqrt{|\log(1-r)|}(1-r)^{\alpha-1/2} & \text{if } \alpha < 1/2 \\ \sqrt{|\log(1-r)|} & \text{if } \alpha > 1/2 \\ |\log(1-r)| & \text{if } \alpha = 1/2 \end{cases}$$

for every $r_0 \leq r < 1$.

Proof. The random vector $X_0 e_0 + \sum_{n\geq 1} X_n/n^\alpha e_n$ is almost surely holomorphic on $\mathbb{D}$ and frequently hypercyclic for T_α by Theorem 1.3.4. Since $\beta_n = n^\alpha \geq n^0$, the case $\alpha = 0$ of the previous theorem already gives that the conditions (i) and (ii) of Theorem 3.3.4 are satisfied with the same functions A_j, $j \geq 1$, thanks to Lemma 4.1.3 and Remark 4.2.4. It only remains to check that the second sequence in (iii) is also bounded with $(r_k)_{k\geq 1} = (1 - e^{-k})_{k\geq 1}$.

Let $0 < r < 1$ and define the function

$$g_r :]0, \infty[\longrightarrow [0, \infty[, x \longmapsto r^{2x} x^{-2\alpha}. \tag{4.2.3}$$

This function is decreasing on $]0, \infty[$, which implies

$$\sum_{n\geq 2} g_r(n) \leq \sum_{n\geq 2} \int_{n-1}^{n} g_r(x)\mathrm{d}x = \int_1^\infty g_r(x)\mathrm{d}x$$

and

$$\sum_{n\geq 1} g_r(n) \geq \sum_{n\geq 1} \int_{n}^{n+1} g_r(x)\mathrm{d}x = \int_1^\infty g_r(x)\mathrm{d}x.$$

Define $I(r) := \sum_{n\geq 1} g_r(n)$ and $J(r) := \int_1^\infty g_r(x)\mathrm{d}x$. Assume that the sequence $(J(r_{k+1})/J(r_k))_{k\geq 1}$ is bounded, this would imply that

$$\frac{I(r_{k+1})}{I(r_k)} \leq \frac{J(r_{k+1}) + r_{k+1}^2}{J(r_k)} = \frac{J(r_{k+1})}{J(r_k)} + \frac{r_{k+1}^2}{J(r_k)},$$

and $(I(r_{k+1})/I(r_k))_{k\geq 1}$ would be bounded, note that $J(r_k) \geq J(r_1)$ for all $k \geq 1$.

Therefore, let us show that $(J(r_{k+1})/J(r_k))_{k\geq 1}$ is indeed bounded. Let $0 < r < 1$. The change of variables $y = 2|\log(r)|x$ gives

$$J(r) = \int_1^\infty x^{-2\alpha} e^{2\log(r)x}\mathrm{d}x = \frac{1}{|\log(r)|^{-2\alpha+1} 2^{-2\alpha+1}} \int_{2|\log(r)|}^\infty y^{-2\alpha} e^{-y}\mathrm{d}y.$$

We distinguish three cases. In the first one, we show directly that the sequence $(I(r_{k+1})/I(r_k))_{k\geq 1}$ is bounded.

Case $\alpha > 1/2$

Since $\alpha > 1/2$, for every $0 < r < 1$ we have

$$\sum_{n=1}^\infty n^{-2\alpha} r^{2n} \leq \sum_{n=1}^\infty \frac{1}{n^{2\alpha}} < \infty,$$

hence the function I is bounded and increasing on $[0, 1[$, and $(I(r_{k+1})/I(r_k))_{k\geq 1}$ is bounded.

Case $0 < \alpha < 1/2$

Since $\alpha < 1/2$, the function $y \longmapsto y^{-2\alpha}e^{-y}$ is integrable on $]0,\infty[$, hence

$$J(r) \lesssim \frac{1}{|\log(r)|^{-2\alpha+1}} \int_0^\infty y^{-2\alpha}e^{-y}\mathrm{d}y \asymp \frac{1}{(1-r)^{-2\alpha+1}},$$

where we have used for the last equality the Taylor expansion of order 1 at 1 of the logarithm. Similarly, one has

$$J(r) \gtrsim \frac{1}{|\log(r)|^{-2\alpha+1}} \int_0^\infty y^{-2\alpha}e^{-y}\mathrm{d}y \asymp \frac{1}{(1-r)^{-2\alpha+1}},$$

hence $(J(r_{k+1})/J(r_k))_{k\geq 1}$ is bounded.

Case $\alpha = 1/2$

For this last case, we use the inequalities

$$2^{-1}e^{-x}\log\left(1+\frac{2}{x}\right) \leq \int_x^\infty y^{-1}e^{-y}\mathrm{d}y \leq e^{-x}\log\left(1+\frac{1}{x}\right)$$

valid for $x > 0$, see [82, Inequalities 6.8.1]. We then have

$$J(r) = \int_{2|\log(r)|}^\infty y^{-1}e^{-y}\mathrm{d}y \leq e^{2\log(r)}\log\left(1+\frac{1}{2|\log(r)|}\right).$$

Using a Taylor expansion of order 1 at 1 of the logarithm gives

$$\begin{aligned}
\log\left(1+\frac{1}{2|\log(r)|}\right) &= \log\left(1-2\log(r)\right) - \log\left(-2\log(r)\right) \\
&= \log\left(1-2\log(r)\right) - \log\big(2(1-r)+o(1-r)\big) \\
&= \log\left(1-2\log(r)\right) - \log(1-r) - \log\big(2+o(1-r)(1-r)^{-1}\big) \\
&\asymp -\log(1-r),
\end{aligned}$$

showing that $J(r) \lesssim -\log(1-r)$. Similar calculations finally yield $J(r) \asymp -\log(1-r)$, and $(J(r_{k+1})/J(r_k))_{k\geq 1}$ is bounded. Thus Theorem 3.3.4 can again be applied. The result then follows with similar estimates as at the end of the proof of Theorem 4.2.7 and by noting that $I(r) \leq J(r)+1$. □

If $\alpha = 0$, the operator T_0 is the so-called Taylor shift. This is indeed the result obtained by Mouze and Munnier in [75, Theorem 1.3 and p. 627]. They even proved that the optimal rate of growth is $r \longmapsto \sqrt{1/(1-r)}$, see [75, Theorem 1.4].

The rate of growth for the operators T_α has also been studied by Mouze and Munnier in [76] with non-probabilistic methods, see [76, Theorem 1.3 and Proposition 2.1]. When $\alpha \neq 1/2$, they showed that the optimal growth is indeed given by $(1-r)^{\alpha-1/2}$ if $\alpha < 1/2$ and by 1 if $\alpha > 1/2$. If $\alpha = 1/2$ then $\sqrt{|\log(1-r)|}$ is a minimal rate of growth, while $|\log(1-r)|$ is an admissible one. Therefore, our result suggests that $\sqrt{|\log(1-r)|}$ could be the optimal rate of growth for $T_{1/2}$, see Section 4.6.

4.2.3 Logarithmic weights

As a last example of operators on $H(\mathbb{D})$, we consider the weighted shifts T_α with respect to the basis of monomials of $H(\mathbb{D})$, with weights $w_n = (\log(n))^\alpha/(\log(n-1))^\alpha$, $n \geq 3$, $w_2 = \log(2)^\alpha$ and $w_1 = 1$, where $\alpha \in \mathbb{R}$. These operators are well-defined on $H(\mathbb{D})$. We have $\beta_n = (\log(n))^\alpha$, $n \geq 2$, and $\beta_1 = 1$. Since $\lim_{n\to\infty} |\beta_n|^{1/n} = 1$, they are also chaotic on $H(\mathbb{D})$, see Example 0.1.22.

We first need two technical lemmas.

Lemma 4.2.9. *For any $\beta > 0$, we have*

$$\limsup_{x\to 0} \int_0^\infty e^{-u}\Big(\frac{\log(2^{-1}u+2x)}{-\log(x)}+1\Big)^{-\beta} \mathrm{d}u \leq 1.$$

Proof. Let $x > 0$. By using the change of variables $t = u/(2x) + 2$, we get

$$\begin{aligned}\int_0^\infty e^{-u}\Big(\frac{\log(2^{-1}u+2x)}{-\log(x)}+1\Big)^{-\beta} \mathrm{d}u &= \int_0^\infty e^{-u}\Big(\log\big(\frac{u}{2x}+2\big)\Big)^{-\beta}(-\log(x))^\beta \mathrm{d}u \\ &= (-\log(x))^\beta 2x \int_2^\infty \frac{e^{-(t-2)2x}}{\log(t)^\beta}\mathrm{d}t. \qquad (4.2.4)\end{aligned}$$

Let $0 < \gamma < 1$ and set $A := 1/x^\gamma$. We divide the last integral on the intervals $[2, A]$ and $[A, \infty[$. First notice that

$$\begin{aligned}(-\log(x))^\beta 2x \int_A^\infty \frac{e^{-(t-2)2x}}{\log(t)^\beta}\mathrm{d}t &\leq (-\log(x))^\beta \frac{2x}{\log(A)^\beta}\int_A^\infty e^{-(t-2)2x}\mathrm{d}t \\ &\leq (-\log(x))^\beta \frac{2x}{\log(A)^\beta}\frac{e^{-A2x}e^{4x}}{2x} \\ &= \gamma^{-\beta}e^{-2x^{1-\gamma}}e^{4x},\end{aligned}$$

which converges to $\gamma^{-\beta}$ when x goes to 0.

We now show that the integral in (4.2.4) on $[2, A]$ converges to 0 when x goes to 0. For each $n \geq 1$, define

$$I_n := \int_e^A \frac{1}{\log(t)^n}\mathrm{d}t.$$

By induction, one can prove that

$$I_n = \frac{I_1}{(n-1)!} - \sum_{j=1}^{n-1}\Big(\frac{A}{\log(A)^j} - e\Big)\frac{(j-1)!}{(n-1)!} \qquad (4.2.5)$$

for every $n \geq 1$. Now, setting $n := \lfloor \beta \rfloor$, we have $\log(t)^\beta \geq \log(t)^n$ for any $t \geq e$. Therefore,

$$\int_2^A \frac{e^{-(t-2)2x}}{\log(t)^\beta}\mathrm{d}t \leq \int_2^e \frac{1}{\log(t)^\beta}\mathrm{d}t + \int_e^A \frac{1}{\log(t)^n}\mathrm{d}t.$$

Assume first that $\beta \geq 1$. By the formula (4.2.5), we have

$$(-\log(x))^\beta 2xI_n = (-\log(x))^\beta 2x\Big(\frac{I_1}{(n-1)!} - \sum_{j=1}^{n-1}\Big(\frac{A}{\log(A)^j} - e\Big)\frac{(j-1)!}{(n-1)!}\Big). \qquad (4.2.6)$$

For any $0 \le j \le n-1$, we have

$$(-\log(x))^{\beta} 2x \frac{A}{\log(A)^j} = 2|\log(x)|^{\beta-j} \frac{x^{1-\gamma}}{\gamma^j},$$

which converges to 0 as x goes to 0 since $0 < \gamma < 1$, and so does $(-\log(x))^{\beta} 2xe$. Now, we have $I_1 = \mathrm{li}(A) - \mathrm{li}(e)$, where li is the logarithmic integral, see [82, Definition 6.2.8]. Using the fact that $\mathrm{li}(A) = O(A/\log(A))$, see [82, Formula 6.12.2], we get

$$\begin{aligned}(-\log(x))^{\beta} 2xI_1 &= (-\log(x))^{\beta} 2x\big(\mathrm{li}(A) - \mathrm{li}(e)\big)\\ &\lesssim (-\log(x))^{\beta} x \frac{A}{\log(A)} = \frac{(-\log(x))^{\beta} x^{1-\gamma}}{-\log(x)\gamma},\end{aligned}$$

which converges to 0 when x goes to 0. All of this shows that the right-hand term in (4.2.6) converges to 0, and in turn that (4.2.4) converges to $\gamma^{-\beta}$ in the case $\beta \ge 1$.

If $0 < \beta < 1$ then

$$x(-\log(x))^{\beta} \int_2^A \frac{e^{-(t-2)2x}}{\log(t)^{\beta}} \mathrm{d}t \le x(-\log(x))^{\beta} \int_2^e \frac{1}{\log(t)^{\beta}} \mathrm{d}t + x(-\log(x))^{\beta} \int_e^A 1 \mathrm{d}t,$$

which again converges to 0.

We conclude that

$$\limsup_{x\to 0} \int_0^{\infty} e^{-u} \left(\frac{\log(2^{-1}u + 2x)}{-\log(x)} + 1\right)^{-\beta} \mathrm{d}u \le \frac{1}{\gamma^{\beta}}$$

for any $0 < \gamma < 1$. The results follows by taking the limit when γ goes to 1. □

Lemma 4.2.10. *For any $\beta > 0$, we have*

$$\liminf_{x\to 0} \int_0^{\infty} e^{-u} \Big(\frac{\log(2^{-1}u + 2x)}{-\log(x)} + 1\Big)^{-\beta} \mathrm{d}u \ge 1.$$

Proof. Let $\rho > 0$ and $0 < \varepsilon < 1/2$, and assume that $x > 0$ is such that $-\log(x) \ge \rho$ and $x \le \varepsilon$. Then $\log(2^{-1}u + 2x)(-\log(x))^{-1} \le 0$ if $2^{-1}u + 2\varepsilon \le 1$ and $\log(2^{-1}u + 2x)(-\log(x))^{-1} \le \log(2^{-1}u + 2\varepsilon)\rho^{-1}$ otherwise. Therefore,

$$\begin{aligned}&\liminf_{x\to 0} \int_0^{\infty} e^{-u} \Big(\frac{\log(2^{-1}u + 2x)}{-\log(x)} + 1\Big)^{-\beta} \mathrm{d}u\\ &\quad\ge \int_0^{2-4\varepsilon} e^{-u} \mathrm{d}u + \int_{2-4\varepsilon}^{\infty} e^{-u} \Big(\frac{\log(2^{-1}u + 2\varepsilon)}{\rho} + 1\Big)^{-\beta} \mathrm{d}u.\end{aligned}$$

An application of the Dominated Convergence Theorem yields the claim. □

We are now ready for the main result of this subsection.

Theorem 4.2.11. *Let $(X_n)_{n\ge 0}$ be a sequence of i.i.d. centred subgaussian random variables with full support, let $(e_n)_{n\ge 0}$ be the sequence of monomials and let $\alpha \in \mathbb{R}$. Then the random vector $\sum_{n=0}^{\infty} X_n/\beta_n e_n$ is almost surely holomorphic on $\mathbb{D}$, is* *frequently hypercyclic for the weighted shift T_α and there exists $c > 0$ such that almost surely, there exists $0 < r_0 < 1$ such that*

$$\left\| X_0 e_0 + X_1 e_1 + \sum_{n=2}^{\infty} \frac{X_n}{(\log(n))^\alpha} e_n \right\|_r \leq c\sqrt{|\log(1-r)|}\frac{|\log(1-r)|^{-\alpha}}{\sqrt{1-r}}$$

for every $r_0 \leq r < 1$.

Proof. The random vector $X_0e_0+X_1e_1+\sum_{n\geq 2} X_n/(\log(n))^\alpha e_n$ is almost surely holomorphic on $\mathbb{D}$ and frequently hypercyclic for T_α by Theorem 1.3.4.

If $\alpha \geq 0$, then $\beta_n = \log(n)^\alpha \geq n^0$ for any $n \geq 3$, and the case $\alpha = 0$ of Theorem 4.2.7 already yields that condition (i) of Theorem 3.3.4 is satisfied for the operator T_α, with the same functions A_j, $j \geq 1$, thanks to Lemma 4.1.3 and Remark 4.2.4. Since $\beta_n \geq n^\alpha$ for every $n \geq 1$ if $\alpha < 0$, the same conclusion applies to the case $\alpha < 0$. It only remains to check conditions (ii) and (iii). Choosing $(r_k)_{k\geq 1} = (1-e^{-k})_{k\geq 1}$, the proof of Theorem 4.2.7 shows that (ii) and the first part of (iii) hold. As in the proof of Theorem 4.2.7, it remains to show that $(I(r_{k+1})/I(r_k))_{k\geq 1}$ is bounded, where $I(r) := \sum_{n\geq 2} r^{2n}/(\log(n))^{2\alpha}$ for $0 < r < 1$.

Case $\alpha \geq 0$

Set $\beta := 2\alpha$ and let $0 < r < 1$. By comparing series and integrals, we get that

$$\sum_{n=2}^{\infty} \frac{r^{2n}}{(\log(n))^\beta} \leq \frac{r^4}{(\log(2))^\beta} + \sum_{n=3}^{\infty} \int_{n-1}^{n} \frac{r^{2x}}{(\log(x))^\beta}\mathrm{d}x = \frac{r^4}{(\log(2))^\beta} + \int_2^\infty \frac{r^{2x}\mathrm{d}x}{(\log(x))^\beta}$$

and

$$\sum_{n=2}^{\infty} \frac{r^{2n}}{(\log(n))^\beta} \geq \sum_{n=2}^{\infty} \int_{n}^{n+1} \frac{r^{2x}}{(\log(x))^\beta}\mathrm{d}x = \int_2^\infty \frac{r^{2x}\mathrm{d}x}{(\log(x))^\beta}.$$

The changes of variables $y = 2|\log(r)|x$ and $u = y - 4|\log(r)|$ yield for $r > e^{-1}$

$$\begin{aligned}
\int_2^\infty \frac{r^{2x}}{(\log(x))^\beta}\mathrm{d}x &= \frac{1}{2|\log(r)|}\int_{4|\log(r)|}^\infty e^{-y}\left(\log\left(\frac{y}{2|\log(r)|}\right)\right)^{-\beta}\mathrm{d}y \\
&= \frac{e^{-4|\log(r)|}}{2|\log(r)|}\int_0^\infty e^{-u}\left(\log\left(\frac{u+4|\log(r)|}{2|\log(r)|}\right)\right)^{-\beta}\mathrm{d}u \\
&= \frac{e^{-4|\log(r)|}}{2|\log(r)|}\big(-\log(|\log(r)|)\big)^{-\beta}\int_0^\infty e^{-u}\left(\frac{\log(2^{-1}u+2|\log(r)|)}{-\log(|\log(r)|)}+1\right)^{-\beta}\mathrm{d}u.
\end{aligned}$$

Lemmas 4.2.9 and 4.2.10 show that the last integral converges to 1 as r goes to 1, the case $\beta = 0$ being trivial. Therefore, we have

$$\frac{1}{|\log(r)|}\big(-\log(-\log(r))\big)^{-\beta} \lesssim I(r) \lesssim \frac{r^4}{(\log(2))^\beta} + \frac{1}{|\log(r)|}\big(-\log(-\log(r))\big)^{-\beta}, \tag{4.2.7}$$

and $\sup_{k\geq 1} I(r_{k+1})/I(r_k)$ is finite.

Case $\alpha < 0$

Set $\beta := -2\alpha$ and let $0 < r < 1$. Define the function

$$g_r :]1, \infty[\longrightarrow [0, \infty[, x \longmapsto r^{2x}(\log(x))^\beta. \tag{4.2.8}$$

It is elementary to show that g_r has a single maximum at $x_{\max}(r)$, where the equality

$$x_{\max}(r) \log(x_{\max}(r)) = -\beta/(2\log(r)) \tag{4.2.9}$$

holds. By comparing series and integrals, where we set $N := \lfloor x_{\max}(r) \rfloor$ and assume that r is so large that $N \geq 2$, we get

$$\begin{aligned}
\sum_{n\geq 2} g_r(n) &\leq \sum_{n=2}^{N-1} \int_n^{n+1} g_r(x)\mathrm{d}x + g_r(N) + g_r(N+1) + \sum_{n\geq N+2} \int_{n-1}^{n} g_r(x)\mathrm{d}x \\
&\leq \int_2^\infty g_r(x)\mathrm{d}x + 2g_r(x_{\max}(r))
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n\geq 2} g_r(n) &\geq g_r(2) + \sum_{n=3}^{N} \int_{n-1}^{n} g_r(x)\mathrm{d}x + \sum_{n\geq N+1} \int_n^{n+1} g_r(x)\mathrm{d}x \\
&= g_r(2) + \int_2^\infty g_r(x)\mathrm{d}x - \int_N^{N+1} g_r(x)\mathrm{d}x \\
&\geq g_r(2) + \int_2^\infty g_r(x)\mathrm{d}x - g_r(x_{\max}(r)).
\end{aligned}$$

The same calculations as in the previous case give, for $r > e^{-1}$,

$$\begin{aligned}
\int_2^\infty g_r(x)\mathrm{d}x &= \frac{1}{2|\log(r)|} \int_{4|\log(r)|}^\infty e^{-y} \left(\log\left(\frac{y}{2|\log(r)|}\right)\right)^\beta \mathrm{d}y \\
&= \frac{e^{-4|\log(r)|}}{2|\log(r)|} \int_0^\infty e^{-u} \left(\log\left(\frac{u + 4|\log(r)|}{2|\log(r)|}\right)\right)^\beta \mathrm{d}u \\
&= \frac{e^{-4|\log(r)|}}{2|\log(r)|} \big(-\log(|\log(r)|)\big)^\beta \int_0^\infty e^{-u} \left(\frac{\log(2^{-1}u + 2|\log(r)|)}{-\log(|\log(r)|)} + 1\right)^\beta \mathrm{d}u.
\end{aligned}$$

The integral converges to $\int_0^\infty e^{-u}\mathrm{d}u = 1$ when r goes to 1. Indeed, fix $0 < \varepsilon < 1/2$ and $\rho > 0$; we can assume that $|\log(r)| < \varepsilon$ and $-\log(|\log(r)|) \geq \rho$. Now, for any $u \geq (1-2\varepsilon)2$, we have $\log(2^{-1}u + 2\varepsilon) \geq 0$, and

$$\frac{\log(2^{-1}u + 2|\log(r)|)}{-\log(|\log(r)|)} \leq \frac{\log(2^{-1}u + 2\varepsilon)}{-\log(|\log(r)|)} \leq \frac{\log(2^{-1}u + 2\varepsilon)}{\rho}.$$

Remark also that for any $0 < u < (1-4\varepsilon)2$, we have $\log(2^{-1}u + 2|\log(r)|) \leq 0$. This allows us to use the Dominated Convergence Theorem.

Therefore, if $J(r) := \frac{1}{|\log(r)|}\big(-\log(-\log(r))\big)^{\beta}$ then we have

$$J(r) + g_r(2) - g_r(x_{\max}(r)) \lesssim I(r) \lesssim J(r) + g_r(x_{\max}(r)), \tag{4.2.10}$$

hence

$$\frac{I(r_{k+1})}{I(r_k)} \lesssim \frac{J(r_{k+1})}{J(r_k)} \frac{1 + g_{r_{k+1}}(x_{\max}(r_{k+1}))/J(r_{k+1})}{1 + g_{r_k}(2)/J(r_k) - g_{r_k}(x_{\max}(r_k))/J(r_k)}$$

for every $k \geq 1$. It is clear that $\sup_{k\geq 1} J(r_{k+1})/J(r_k) < \infty$ and $\lim_{r\to 1} g_r(2)/J(r) = 0$. We will prove that $\lim_{r\to 1} g_r(x_{\max}(r))/J(r) = 0$ and this will show that the hypotheses of Theorem 3.3.4 hold. For every $0 < r < 1$, we have

$$\frac{g_r(x_{\max}(r))}{J(r)} \asymp \frac{r^{2x_{\max}(r)}\big(\log(x_{\max}(r))\big)^{\beta}(1-r)}{|\log(1-r)|^{\beta}}.$$

By (4.2.9), we can write

$$\frac{g_r(x_{\max}(r))}{J(r)} \asymp \frac{r^{2x_{\max}(r)}}{|\log(1-r)|^{\beta}} \left(\frac{-\beta}{x_{\max}(r)2\log(r)}\right)^{\beta} (1-r).$$

Let $\varepsilon > 0$; for r close enough to 1, one has $x_{\max}(r)\log(x_{\max}(r)) \leq (x_{\max}(r))^{1+\varepsilon}$. Then, again by (4.2.9) and with a Taylor expansion of order 1 of the logarithm at 1, we get that

$$\begin{aligned}
\frac{g_r(x_{\max}(r))}{J(r)} &\lesssim \frac{r^{2(-\beta/(2\log(r)))^{1/(1+\varepsilon)}}}{|\log(1-r)|^{\beta}} \left(\frac{-1}{\log(r)}\right)^{\beta} (-\log(r))^{\beta/(1+\varepsilon)}\,(1-r) \\
&\asymp \frac{r^{2(-\beta/(2\log(r)))^{1/(1+\varepsilon)}}}{|\log(1-r)|^{\beta}} \frac{1}{(1-r)^{\beta}}\,(1-r)^{\beta/(1+\varepsilon)}\,(1-r) \\
&\leq \frac{1}{|\log(1-r)|^{\beta}}(1-r)^{-\beta+\beta/(1+\varepsilon)+1}.
\end{aligned}$$

If $\varepsilon > 0$ is chosen small enough such that $-\beta + \beta/(1+\varepsilon) + 1 > 0$, then the right-hand side converges to 0 when r goes to 1. We deduce that

$$\lim_{r\to 1} \frac{g_r(x_{\max}(r))}{J(r)} = 0. \tag{4.2.11}$$

Thus we can apply Theorem 3.3.4. As in the proof of Theorem 4.2.7, we have that $\log(A_1(r)) \lesssim |\log(1-r)|$ for $0 < r < 1$ close enough to 1. For S_f, we have for $\alpha \geq 0$, by (4.2.7),

$$\begin{aligned}
S_f(r) = \sqrt{1 + r^2 + I(r)} &\asymp r^2 + \frac{1}{\sqrt{|\log(r)|}}\big(-\log(-\log(r))\big)^{-\alpha} \\
&\asymp \frac{|\log(1-r)|^{-\alpha}}{\sqrt{1-r}},
\end{aligned}$$

and for $\alpha < 0$, by (4.2.10) and (4.2.11),

$$S_f(r) = \sqrt{1 + r^2 + I(r)} \asymp \frac{1}{\sqrt{|\log(r)|}}\big(-\log(-\log(r))\big)^{-\alpha}$$
$$\asymp \frac{|\log(1-r)|^{-\alpha}}{\sqrt{1-r}}.$$

□

4.3 Harmonic functions on the plane

This section is devoted to differential operators defined on the space of harmonic functions on $\mathbb{R}^2$ denoted by $\mathcal{H}(\mathbb{R}^2)$, endowed with the topology of local uniform convergence. The notation ∂_x (resp. ∂_y) denotes the partial derivative with respect to the first variable (resp. the second variable).

Definition 4.3.1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an infinitely differentiable function. Then f is *harmonic* if $\partial_x^2 f + \partial_y^2 f = 0$.

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $\alpha \neq 0$. Then the operator D^α defined on $\mathcal{H}(\mathbb{R}^2)$ by

$$D^\alpha(f) = \partial_x^{\alpha_1}\partial_y^{\alpha_2} f,\ \ f \in \mathcal{H}(\mathbb{R}^2),$$

is frequently hypercyclic, see [17, Theorem 4.3].

Our aim is to find an admissible rate of growth for frequently hypercyclic functions of each differential operator on $\mathcal{H}(\mathbb{R}^2)$ with a probabilistic approach. It will be in fact the same for all of them. The optimal rate of growth for the frequently hypercyclic functions of the operators ∂_x and ∂_y on $\mathcal{H}(\mathbb{R}^N)$, $N \geq 2$, has been studied in [17, Theorem 4.2] and [41, Theorem 2.1] in terms of the L^2-norm on spheres. The growth of hypercyclic functions of differential operators defined on $\mathcal{H}(\mathbb{R}^N)$ has been studied in [3] and [2].

Let us set some notations.

Definition 4.3.2. A polynomial $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is *homogeneous* of degree $m \in \mathbb{N}$ if there is $(a_{i,j})_{i+j=m} \subseteq \mathbb{R}$ such that $p(x,y) = \sum_{i+j=m} a_{i,j} x^i y^j$ for all $x, y \in \mathbb{R}$.

The space of homogeneous harmonic polynomials of degree $m \in \mathbb{N}$ is noted $\mathcal{H}_m(\mathbb{R}^2)$ and the space span$\bigcup_{m\geq 0} \mathcal{H}_m(\mathbb{R}^2)$ is dense in $\mathcal{H}(\mathbb{R}^2)$ by [5, Corollary 5.34]. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ define $|\alpha| := \alpha_1 + \alpha_2$, and set $\mathbb{N}_0^2 := \mathbb{N}^2 \setminus \{(0,0)\}$. For any $f \in \mathcal{H}(\mathbb{R}^2)$ and $r > 0$, we define

$$\|f\|_{2,r} := \left(\int_{S(r)} |f|^2 \mathrm{d}\sigma\right)^{1/2},$$

where $S(r)$ is the circle of radius r centred at the origin of $\mathbb{R}^2$ and σ is the normalized Lebesgue measure on $S(r)$.

Let $p \in \mathcal{H}(\mathbb{R}^2)$ be a polynomial. When restricted to the circle of radius $r > 0$ centred at the origin, p can be viewed as a trigonometric polynomial. Indeed, let $(x, y) \in S(r)$, then

$$p(x,y) = p\Big(r\big(\frac{e^{is}+e^{-is}}{2}\big), r\big(\frac{e^{is}-e^{-is}}{2i}\big)\Big)$$

for some $s \in [0, 2\pi]$. Thus, Lemma 3.1.13 holds for polynomials in $\mathcal{H}(\mathbb{R}^2)$. Therefore, by carefully reading the proof of Theorem 3.3.4, one can see that this result still holds for functions in $\mathcal{H}(\mathbb{R}^2)$.

Theorem 4.3.3. *Let $f = \sum_{n\geq 0} a_n e_n \in \mathcal{H}(\mathbb{R}^2)$ where $(e_n)_{n\geq 0}$ is a sequence of polynomials such that for every $n \geq 0$, the degree of e_n is at most Cn, where $C > 0$ is some constant. Set $S_f(r) := \sqrt{\sum_{n\geq 0} a_n^2 \|e_n\|_r^2}$ for any $r > 0$. Let $(X_n)_{n\geq 0}$ be a centred subgaussian sequence of independent random variables such that $\sum_{n=0}^{\infty} a_n X_n e_n$ is almost surely convergent and let $(A_j)_{j\geq 1}$ be a non-decreasing sequence of positive functions defined on $]0, \infty[$ such that A_1 is non-decreasing.*

If the conditions (i) *and* (ii) *of Theorem 3.3.4 are satisfied with $w = \infty$, then there exists $c > 0$ such that almost surely, there exists $k_0 \geq 1$ such that for every $k \geq k_0$,*

$$\Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_{r_k} \leq c\sqrt{\log(A_1(r_k))}S_f(r_k).$$

Furthermore, if the condition (iii) *holds then there exists $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\Big\|\sum_{n=0}^{\infty} a_n X_n e_n\Big\|_{r} \leq c\sqrt{\log(A_1(r))}S_f(r)$$

for every $r \geq r_0$.

4.3.1 A random vector for differential operators

First of all, we must find a frequently hypercyclic random vector for each operator D^α, $\alpha \in \mathbb{N}_0^2$. We will need the following lemma.

Lemma 4.3.4. [3, Lemma 4] *Let $m, k \in \mathbb{N}$ and $u \in \mathcal{H}_m(\mathbb{R}^2)$. Then there exists $P_{(k,0)}(u) \in \mathcal{H}_{m+k}(\mathbb{R}^2)$ such that $\partial_x^k P_{(k,0)}(u) = u$ and*

$$\|P_{(k,0)}(u)\|_{2,1} \asymp \frac{m!}{(m+k)!}\|u\|_{2,1}.$$

Furthermore, the map $P_{(k,0)} : \mathcal{H}_m(\mathbb{R}^2) \longrightarrow \mathcal{H}_{m+k}(\mathbb{R}^2)$ is linear.

Of course, this result also holds for the partial derivatives with respect to the second variable. The proof in [3, Lemma 4] gives only one inequality, but with the help of [3, Remark at p. 152], one can obtain the claim of Lemma 4.3.4. This immediately yields the next result.

Lemma 4.3.5 ([2, Lemma 2]). *Let $m \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $u \in \mathcal{H}_m(\mathbb{R}^2)$. Then the polynomial $P_{(\alpha_1,\alpha_2)}(u) := P_{(0,\alpha_2)}(P_{(\alpha_1,0)}(u)) \in \mathcal{H}_{m+|\alpha|}(\mathbb{R}^2)$ is such that $D^\alpha P_{(\alpha_1,\alpha_2)}(u) = u$ and*

$$\|P_{(\alpha_1,\alpha_2)}(u)\|_{2,1} \asymp \frac{m!}{(m+|\alpha|)!}\|u\|_{2,1}.$$

Furthermore, the map $P_{(\alpha_1,\alpha_2)} : \mathcal{H}_m(\mathbb{R}^2) \longrightarrow \mathcal{H}_{m+|\alpha|}(\mathbb{R}^2)$ is linear.

Proof. Define $P_{(\alpha_1,\alpha_2)}(u) := P_{(0,\alpha_2)}(P_{(\alpha_1,0)}(u))$. Then by Lemma 4.3.4 applied twice, we have $D^\alpha P_{(\alpha_1,\alpha_2)}(u) = u$ and

$$\|P_{(\alpha_1,\alpha_2)}(u)\|_{2,1} \asymp \frac{(m+\alpha_1)!}{(m+|\alpha|)!}\|P_{(\alpha_1,0)}(u)\|_{2,1} \asymp \frac{m!}{(m+|\alpha|)!}\|u\|_{2,1}.$$

The linearity of $P_{(\alpha_1,\alpha_2)}$ is obvious by linearity of $P_{(0,\alpha_2)}$ and $P_{(\alpha_1,0)}$. □

The bound given in the previous lemma is not exactly the one stated in [2, Lemma 2], but it can be deduced from its proof given there and Lemma 4.3.4.

Lemma 4.3.6. *The maps $P_{(1,0)}$ and $P_{(0,1)}$ commute, and for every $j \geq 0$, $P_{(j+1,0)} = P_{(j,0)}P_{(1,0)}$ and $P_{(0,j+1)} = P_{(0,j)}P_{(0,1)}$. Therefore, $P_{(i,j)}P_{(k,l)} = P_{(k,l)}P_{(i,j)}$ for every $i, j, k, l \in \mathbb{N}$. Furthermore, the maps $P_{(1,0)}$ and $P_{(0,1)}$ are injective on $\mathcal{H}_n(\mathbb{R}^2)$, for every $n \geq 1$.*

Proof. For any $n \geq 0$, define the harmonic polynomials

$$u_n := \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j} y^{2j}}{(n-2j)!(2j)!} \text{ and } v_n := \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j} y^{2j+1}}{(n-2j)!(2j+1)!}.$$

By simple calculations, one can get the formulas $\partial_x u_n = u_{n-1}, \partial_x v_n = v_{n-1}$, $\partial_y u_n = -v_{n-2}$ and $\partial_y v_n = u_n$, for any $n \geq 0$, setting $u_{-1} = v_{-2} = v_{-1} = 0$.

For any $n \geq 1$, the vectors u_n and v_{n-1} are linearly independent since $x \longmapsto u_n(x, 1)$ is a polynomial of degree n and $x \longmapsto v_{n-1}(x, 1)$ is a polynomial of degree $n-1$, hence $\mathcal{H}_n(\mathbb{R}^2) = \langle u_n, v_{n-1} \rangle$ since $\dim \mathcal{H}_n(\mathbb{R}^2) = 2$ by [5, Proposition 5.8].

Let $n \geq 1$ and $u \in \mathcal{H}_n(\mathbb{R}^2)$. Let us show that $P_{(1,0)}(u)$ is the unique polynomial belonging to $\mathcal{H}_{n+1}(\mathbb{R}^2)$ such that $\partial_x P_{(1,0)}(u) = u$. Write $P_{(1,0)}(u) = au_{n+1} + bv_n$ for some $a, b \in \mathbb{R}$. Using the previous formulas, we get $u = P_{(1,0)}(u) = au_n + bv_{n-1}$, which implies that $P_{(1,0)}(u)$ is the unique polynomial in $\mathcal{H}_{n+1}(\mathbb{R}^2)$ such that $\partial_x P_{(1,0)}(u) = u$ since u_n and v_{n-1} are linearly independent. The same result holds for $P_{(0,1)}(u)$. Now, it is readily shown that $P_{(1,0)}P_{(0,1)}(u) = P_{(0,1)}P_{(1,0)}(u)$. Indeed, $\partial_x\partial_y P_{(1,0)}P_{(0,1)}(u) = u$, which implies by uniqueness that $P_{(1,0)}(u) = \partial_y P_{(1,0)}P_{(0,1)}(u)$. Then again by uniqueness, we get $P_{(1,0)}P_{(0,1)}(u) = P_{(0,1)}P_{(1,0)}(u)$.

Now, if $u \in \mathcal{H}_0(\mathbb{R}^2) = \langle 1 \rangle$, we can assume without loss of generality that $u = 1$. In that case, one can easily see in the proof of [3, Lemma 4] that $P_{(1,0)}(1) = x$ and $P_{(0,1)}(1) = y$, or simply set $P_{(1,0)}(1) = x$ and $P_{(0,1)}(1) = y$, and Lemma 4.3.4 holds for $u = 1$ with these definitions of $P_{(1,0)}$ and $P_{(0,1)}$. We want to prove that $P_{(1,0)}P_{(0,1)}(1) = P_{(0,1)}P_{(1,0)}(1)$, that is $P_{(1,0)}(y) = P_{(0,1)}(x)$, which is equivalent to

$\partial_x P_{(0,1)}(x) = y$ by uniqueness proved above. But $P_{(0,1)}(x) = xy$ since $\partial_y(xy) = x$ and by uniqueness. Therefore, $\partial_x P_{(0,1)}(x) = \partial_x(xy) = y$.

Similarly, it is easy to show that $P_{(j+1,0)} = P_{(j,0)}P_{(1,0)}$ and $P_{(0,j+1)} = P_{(0,j)}P_{(0,1)}$ for every $j \geq 0$. □

For the next two results, for each $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, define the set

$$\begin{aligned} M_\alpha := &\left\{P_{(i,\alpha_2)}(u) \mid u \in \{1, y\}, 0 \leq i \leq \alpha_1 - 1\right\} \\ &\cup \left\{P_{(0,j)}(u) \mid u \in \{1, x\}, 0 \leq j \leq \alpha_2 - 1\right\}. \end{aligned}$$

Here, we denote by x (resp. y) the function $f \in \mathcal{H}(\mathbb{R}^2)$ defined by $f(x, y) = x$ (resp. $f(x, y) = y$), for every $(x, y) \in \mathbb{R}^2$.

Lemma 4.3.7. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. The vectors of the set M_α are linearly independent.*

Proof. The claim clearly holds if $|\alpha| = 1$. Assume that it also holds for any $\alpha \in \mathbb{N}_0^2$ such that $|\alpha| \leq m$, for some $m \geq 1$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $|\alpha| = m + 1$ and assume without loss of generality that $\alpha_1 \geq 1$.

Let $a_0, \ldots, a_{\alpha_1 - 1}, b_0, \ldots, b_{\alpha_1 - 1}, c_0, \ldots, b_{\alpha_2 - 1}, d_0, \ldots, d_{\alpha_2 - 1} \in \mathbb{R}$ be such that

$$\sum_{i=0}^{\alpha_1 - 1} (a_i P_{(i,\alpha_2)}(1) + b_i P_{(i,\alpha_2)}(y)) + \sum_{j=0}^{\alpha_2 - 1} (c_j P_{(0,j)}(1) + d_j P_{(0,j)}(x)) = 0.$$

By taking $\partial_x^{\alpha_1 - 1} \partial_y^{\alpha_2}$ on both sides of the equality, we get $a_{\alpha_1 - 1} + b_{\alpha_1 - 1} y = 0$, hence $a_{\alpha_1 - 1} = b_{\alpha_1 - 1} = 0$. The other coefficients are then also equal to zero by the induction hypothesis. □

Lemma 4.3.8. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. The subspace of $\mathcal{H}(\mathbb{R}^2)$ generated by the set $\bigcup_{n \geq 0} P_{(n\alpha_1, n\alpha_2)}(M_\alpha)$ is equal to $\operatorname{span} \bigcup_{m \geq 0} \mathcal{H}_m(\mathbb{R}^2)$, and is therefore dense in $\mathcal{H}(\mathbb{R}^2)$.*

Proof. We clearly have $\mathcal{H}_0(\mathbb{R}^2) = \langle 1 \rangle = \langle P_{(0,0)}(1) \rangle$.

Let $n \in \mathbb{N}$ and set $S_n := P_{(n\alpha_1, n\alpha_2)}$. Then

$$\left.\begin{aligned} &S_n(P_{(i,\alpha_2)}(1)) \in \mathcal{H}_{n|\alpha| + i + \alpha_2}(\mathbb{R}^2) \\ &S_n(P_{(i,\alpha_2)}(y)) \in \mathcal{H}_{n|\alpha| + i + \alpha_2 + 1}(\mathbb{R}^2) \end{aligned}\right\} \text{ if } 0 \leq i \leq \alpha_1 - 1,$$
$$\left.\begin{aligned} &S_n(P_{(0,j)}(1)) \in \mathcal{H}_{n|\alpha| + j}(\mathbb{R}^2) \\ &S_n(P_{(0,j)}(x)) \in \mathcal{H}_{n|\alpha| + j + 1}(\mathbb{R}^2) \end{aligned}\right\} \text{ if } 0 \leq j \leq \alpha_2 - 1.$$

Let $m \geq 1$. There exist $n \in \mathbb{N}$ and $0 \leq r \leq |\alpha| - 1$ such that $m = n|\alpha| + r$. Since $\dim \mathcal{H}_m(\mathbb{R}^2) = 2$ by [5, Proposition 5.8], we then have

$$\mathcal{H}_m(\mathbb{R}^2) = \begin{cases} \left\langle S_n(P_{(0,0)}(1)), S_{n-1}(P_{(\alpha_1 - 1, \alpha_2)}(y)) \right\rangle & \text{if } r = 0, n \geq 1, \\ \left\langle S_n(P_{(0,r)}(1)), S_n(P_{(0,r-1)}(x)) \right\rangle & \text{if } 1 \leq r \leq \alpha_2 - 1, \\ \left\langle S_n(P_{(0,\alpha_2)}(1)), S_n(P_{(0,\alpha_2 - 1)}(x)) \right\rangle & \text{if } r = \alpha_2, \\ \left\langle S_n(P_{(r - \alpha_2, \alpha_2)}(1)), S_n(P_{(r - \alpha_2 - 1, \alpha_2)}(y)) \right\rangle & \text{if } \alpha_2 + 1 \leq r \leq |\alpha| - 1. \end{cases}$$

Indeed, the last three cases follow by using Lemmas 4.3.6 and 4.3.7. For the first case, remark that $S_n(P_{(0,0)}(1)) = S_{n-1}(P_{(\alpha_1,\alpha_2)}(1))$, and that $P_{(\alpha_1,\alpha_2)}(1)$ and $P_{(\alpha_1-1,\alpha_2)}(y)$ are linearly independent by arguing as in the proof of Lemma 4.3.7. $\square$

Lemma 4.3.9. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, $m \in \mathbb{N}$ and $u \in \mathcal{H}_m(\mathbb{R}^2)$. The series $\sum_{n\geq 1} \sqrt{\log(n)} P_{(n\alpha_1,n\alpha_2)}(u)$ converges unconditionally in $\mathcal{H}(\mathbb{R}^2)$.*

Proof. Let $u \in \mathcal{H}_m(\mathbb{R}^2)$. Remark that $\|p\|_{2,r} = \|p\|_{2,1} r^n$ for every $r > 0$ and for any homogeneous polynomial p of degree $n \geq 0$. By using [41, Lemma 5.1] and Lemma 4.3.5, we get for every $r > 0$ that

$$\begin{aligned}\sum_{n\geq 1} \sqrt{\log(n)} \|P_{(n\alpha_1,n\alpha_2)}(u)\|_r &\lesssim \sum_{n\geq 1} \sqrt{\log(n)} \|P_{(n\alpha_1,n\alpha_2)}(u)\|_{2,2r} \\ &\lesssim \sum_{n\geq 1} \sqrt{\log(n)} \frac{m!(2r)^{n|\alpha|+m}}{(n|\alpha|+m)!} \|u\|_{2,1},\end{aligned}$$

and the series $\sum_{n\geq 1} \sqrt{\log(n)} P_{(n\alpha_1,n\alpha_2)}(u)$ is unconditionally convergent. $\square$

These lemmas and Theorem 1.2.10 allow us to obtain a frequently hypercyclic random vector for any differentiation operator.

Theorem 4.3.10. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and X be a subgaussian random variable with full support. Then the random vector*

$$\begin{aligned}v := &\sum_{u\in\{1,x\}} \sum_{j=0}^{\alpha_2-1} \sum_{n=0}^{\infty} X_{n,j,u} P_{(n\alpha_1,n\alpha_2)}(P_{(0,j)}(u)) \\ &+ \sum_{u\in\{1,y\}} \sum_{i=0}^{\alpha_1-1} \sum_{n=0}^{\infty} X_{n,i,u} P_{(n\alpha_1,n\alpha_2)}(P_{(i,\alpha_2)}(u)) \end{aligned} \tag{4.3.1}$$

is almost surely well-defined and frequently hypercyclic for D^α, where $X_{n,i,u}$, $n \in \mathbb{N}$, $0 \leq i \leq \alpha_1 - 1$, $u \in \{1, y\}$, and $X_{n,j,u}$, $n \in \mathbb{N}$, $0 \leq j \leq \alpha_2 - 1$, $u \in \{1, x\}$, are i.i.d. copies of X.

Proof. In order to use Theorem 1.2.10, we must find a sequence $(u_n)_{n\in\mathbb{Z}}$ of harmonic functions such that $D^\alpha(u_n) = u_{n-1}$. As we said in Remark 1.2.13, the proof of Theorem 1.2.10 still holds if there exist some $N \geq 1$ and a family of vectors $(u_{n,j})_{n\in\mathbb{Z},1\leq j\leq N}$ such that for every $1 \leq j \leq N$ and $n \in \mathbb{Z}$, one has $D^\alpha(u_{n,j}) = u_{n-1,j}$. Here, we take the family of sequences $(u_{n,j,u})_{n\geq 0} = (P_{(n\alpha_1,n\alpha_2)} P_{(0,j)}(u))_{n\geq 0}$, $0 \leq j \leq \alpha_2 - 1$, $u \in \{1, x\}$, and $(v_{n,i,u})_{n\geq 0} = (P_{(n\alpha_1,n\alpha_2)} P_{(i,\alpha_2)}(u))_{n\geq 0}$, $0 \leq i \leq \alpha_1 - 1$, $u \in \{1, y\}$.

Lemma 4.3.6 implies that $D^\alpha(u_{n,j,u}) = u_{n-1,j,u}$ and $D^\alpha(v_{n,i,u}) = v_{n-1,i,u}$ for every $n \geq 0$, where $u_{-1,j,u} = v_{-1,i,u} = 0$. By Lemma 4.3.8, the span of this family of vectors is dense in $\mathcal{H}(\mathbb{R}^2)$. Finally, Lemma 4.3.9 allows us to apply Theorem 1.2.10 to get the result. $\square$

4.3.2 Rate of growth

In order to find an admissible rate of growth for the frequently hypercyclic vectors of the differential operators, we will use Theorem 4.3.3.

Theorem 4.3.11. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and X be a centred subgaussian random variable with full support. Then there exists $c > 0$ such that almost surely, there exists $r_0 > 0$ such that*

$$\|v\|_r \leq c\sqrt{\log(r)}\frac{e^r}{r^{1/4}}$$

for every $r \geq r_0$, where v is defined in (4.3.1).

Proof. Let $u \in \{1, x, y\}$, $0 \leq i \leq \alpha_1 - 1$ and $0 \leq j \leq \alpha_2$, and set $m := |\alpha|$. Recall that $\|p\|_{2,r} = \|p\|_{2,1} r^n$ for every $r > 0$ and for any homogeneous polynomial p of degree $n \geq 0$. By [2, inequality (2.4)] and Lemma 4.3.5, we have

$$\|P_{(n\alpha_1, n\alpha_2)}(P_{(i,j)}(u))\|_r^2 \asymp \|P_{(n\alpha_1, n\alpha_2)}(P_{(i,j)}(u))\|_{2,r}^2 \asymp \frac{r^{2(nm+i+j+\deg(u))}}{(nm+i+j+\deg(u))!^2}. \tag{4.3.2}$$

We show that the assumptions (i) and (ii) of Theorem 3.3.4 with $A_k(r) = d^k r/m$, $r > 0$, $k \geq 1$, where $d > e$, are satisfied for the series $\sum_{n\geq 0} P_{(n\alpha_1, n\alpha_2)}(P_{(i,j)}(u))$.

Let $r > 0$. Some simple calculations yield

$$\begin{aligned}
\sum_{n \geq A_k(r)+1} \frac{r^{2(mn+i+j+\deg(u))}}{(mn+i+j+\deg(u))!^2} &\leq \sum_{n \geq A_k(r)+1} \frac{r^{2mn}}{(mn)!^2} \\
&= \sum_{n\geq 1} \frac{r^{2m(n+A_k(r))}}{(m(n+A_k(r)))!^2} \\
&\leq \frac{r^{A_k(r)}}{(mA_k(r))!^2} \sum_{n \geq 1} \frac{r^{2mn}}{(mA_k(r))^{2nm}} \\
&\asymp \frac{r^{2d^k r}}{(d^k r)!^2}.
\end{aligned}$$

By using Stirling's formula, our task is now to prove that

$$\sup_{r>0} \sum_{k\geq 1} \sqrt{\log\left(\frac{d^k r^n}{m}\right)} \frac{e^{d^k r}}{d^{k d^k r}\sqrt{d^k r}} < \infty. \tag{4.3.3}$$

By using the same arguments as in the proof of Proposition 4.1.2, it is enough to show that for every $M > e$, the function

$$f_M :]1, \infty[\longrightarrow]0, \infty[, x \longmapsto \sqrt{\log(x)} \frac{e^x}{M^x \sqrt{x}},$$

is non-increasing away from 0, uniformly in $M > e$, and converges to 0 when x goes to ∞. For any $x > 1$, we have

$$\begin{aligned}\log(f(x)) &= \frac{\log_2(x)}{2} - x\log(M) + x - \frac{\log(x)}{2} \\ &= x\Big(- \log(M) + 1 + \frac{\log_2(x) - \log(x)}{2x}\Big).\end{aligned}$$

If $M > e$ then $\lim_{x\to\infty} f(x) = 0$. For any $x > 1$, we have

$$\partial_x \log(f)(x) = \frac{1}{2\log(x)x} - \log(M) + 1 - \frac{1}{2x}.$$

Then $\partial_x \log(f)(x)$ is negative if and only if $1 < 2x\log(x)(\log(M)-1+1/(2x))$, which holds if $M > e$ and $x > e$. Setting now $M = d^k$, the Dominated Convergence Theorem allows us to conclude that (4.3.3) holds.

It remains to estimate the series

$$\sum_{u\in\{1,x\}} \sum_{j=0}^{\alpha_2-1} \sum_{n=0}^{\infty} \|P_{(n\alpha_1,n\alpha_2)}(P_{(0,j)}(u))\|_r^2 + \sum_{u\in\{1,y\}} \sum_{i=0}^{\alpha_1-1} \sum_{n=0}^{\infty} \|P_{(n\alpha_1,n\alpha_2)}(P_{(i,\alpha_2)}(u))\|_r^2 \tag{4.3.4}$$

By using (4.3.2), we see that

$$\begin{aligned}&\sum_{j=0}^{\alpha_2-1} \sum_{n=0}^{\infty}\|P_{(n\alpha_1,n\alpha_2)}(P_{(0,j)}(1))\|_r^2 + \sum_{i=0}^{\alpha_1-1} \sum_{n=0}^{\infty} \|P_{(n\alpha_1,n\alpha_2)}(P_{(i,\alpha_2)}(1))\|_r^2 \\ &\quad\asymp \sum_{j=0}^{\alpha_2-1} \sum_{n=0}^{\infty} \frac{r^{2(nm+j)}}{(nm+j)!^2} + \sum_{i=0}^{\alpha_1-1} \sum_{n=0}^{\infty} \frac{r^{2(nm+i+\alpha_2)}}{(nm+i+\alpha_2))!^2} \\ &\quad= \sum_{n=0}^{\infty} \frac{r^{2n}}{n!^2}\end{aligned}$$

The same conclusion holds for $u = x$ and $u = y$ in (4.3.4). By Lemma 4.1.8, we deduce that the series (4.3.4) is estimated by $e^{2r}/r^{1/2}$.

It is now easy to check that assumption (iii) of Theorem 3.3.4 is satisfied with $(r_k)_{k\geq 1} = (k)_{k\geq 1}$, and Theorem 4.3.3 can be applied. $\square$

4.4 Köthe sequence spaces

In this section, we study the growth of frequently hypercyclic functions of chaotic weighted shifts defined on Köthe sequence spaces, see [71, Chapter 27]. Linear dynamics of shifts on such spaces were studied in [27].

Definition 4.4.1. A *Köthe matrix* is a matrix $A = (a_{m,n})_{m,n\geq 0}$ of positive numbers such that for all $m, n \geq 0$, one has $a_{m,n} \leq a_{m+1,n}$.

Definition 4.4.2. Let $1 \leq p \leq \infty$ and A be a Köthe matrix. The *Köthe sequence space of order p* is defined as

$$\lambda^p(A) := \Big\{(x_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}} \mid \forall m \geq 0,\ \sum_{n=0}^{\infty} |x_n|^p a_{m,n} < \infty\Big\}$$

if $p < \infty$, and

$$\lambda^\infty(A) := \Big\{(x_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}} \mid \forall m \geq 0,\ \sup_{n\in\mathbb{N}} |x_n| a_{m,n} < \infty\Big\}.$$

These spaces are endowed with the seminorms $\|x\|_m := \Big(\sum_{n\geq 0} |x_n|^p a_{m,n}\Big)^{1/p}$ if $p < \infty$, and $\|x\|_m := \sup_{n\geq 0} |x_n| a_{m,n}$ if $p = \infty$, for all $m \geq 0$ and $x \in \lambda^p(A)$.

We will need a probabilistic result.

Proposition 4.4.3 ([1, Proposition 1.8]). *Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be a real matrix and let $(X_{i,j})_{1\leq i,j\leq n}$ be independent standard Gaussian real random variables. Define $G_A := (a_{i,j}X_{i,j})_{1\leq i,j\leq n}$. For every $1 < p \leq \infty$, we have*

$$\mathbb{E}\|G_A : \ell_p^n \longrightarrow \ell_1^n\| \asymp \|A \circ A : \ell_{p/2}^n \longrightarrow \ell_{1/2}^n\|^{1/2} + \|(A\circ A)^t : \ell_\infty^n \longrightarrow \ell_{p^*/2}^n\|^{1/2}.$$

Furthermore,

$$\mathbb{E}\|G_A : \ell_1^n \longrightarrow \ell_1^n\| \asymp \|A \circ A : \ell_{1/2}^n \longrightarrow \ell_{1/2}^n\|^{1/2} + \max_{1\leq j\leq n} \sqrt{\log(j+1)}\, b_j^*,$$

where $b_j := \|(a_{i,j})_{1\leq i\leq n}\|_2$, $1 \leq j \leq n$, and $(b_j^*)_{1\leq j\leq n}$ is the non-increasing rearrangement of $(b_j)_{1\leq j\leq n}$.

Lemma 4.4.4. *Let $e_1, \ldots, e_N$ be the canonical basis of ℓ_p^N, where $1 \leq p \leq \infty$ and $N \geq 1$. Let $a_1, \ldots, a_N \in \mathbb{R}$ and let $(X_i)_{i=1}^N$ be independent standard Gaussian real random variables. Then there exists a constant $c > 0$, independent of N, such that for every $R \geq 1$, one has*

$$\mathbb{P}\bigg(\Big\|\sum_{n=1}^N a_n X_n e_n\Big\|_p > c\sqrt{R}\Big(\sum_{n=1}^N |a_n|^p\Big)^{1/p}\bigg) \leq \frac{1}{e^R}$$

if $1 \leq p < \infty$, and

$$\mathbb{P}\bigg(\Big\|\sum_{n=1}^N a_n X_n e_n\Big\|_\infty > c\sqrt{\log(N+1)}\sqrt{R} \max_{1\leq n\leq N} |a_n|\bigg) \leq \frac{1}{e^R}.$$

Proof. Define $f : \mathbb{R}^N \longrightarrow [0,\infty[, x \longmapsto \|\sum_{n=1}^N a_n x_n e_n\|_p$, $X := (X_1, \ldots, X_N)$ and $S := \|(a_n)_{1\leq n\leq N}\|_p$, and let p^* be the conjugate exponent of p. We can assume that $S \neq 0$. For every $x, y \in \mathbb{R}^N$, one has that

$$\begin{aligned} |f(x) - f(y)| &\leq \Big\|\sum_{n=1}^N a_n(x_n - y_n)e_n\Big\|_p = \Big(\sum_{n=1}^N |a_n|^p |x_n - y_n|^p\Big)^{1/p} \\ &\leq \max_{1\leq n\leq N} |x_n - y_n| S \leq \|x - y\|_2 S, \end{aligned}$$

with the obvious modifications for $p = \infty$. By Markov's inequality and [24, Theorem 5.5], we get for any $\lambda > 0$ that

$$\mathbb{P}\Big(f(X) > \frac{R}{\lambda} + \frac{\lambda S^2}{2} + \mathbb{E}(f(X))\Big) \leq e^{-R}\mathbb{E}\big(e^{\lambda(f(X) - \mathbb{E}(f(X)))}\big)e^{-\frac{\lambda^2 S^2}{2}} \leq e^{-R}.$$

Choosing $\lambda = \sqrt{2R}/S$ yields

$$\mathbb{P}\Big(f(X) > \sqrt{2R}S + \mathbb{E}(f(X))\Big) \leq e^{-R}.$$

Now, notice that $\|\sum_{n=1}^{N} a_n X_n e_n\|_p = \|G_A : \ell_{p^*}^N \longrightarrow \ell_1^N\|$, where A is the diagonal matrix with entries $(a_n)_{1\leq n\leq N}$. Using Proposition 4.4.3, we get $\mathbb{E}(f(X)) \lesssim S$ if $1 \leq p < \infty$ and $\mathbb{E}(f(X)) \lesssim \sqrt{\log(N+1)}S$ if $p = \infty$, and this combined with the previous inequality concludes the proof. □

The next theorem is the main result of this section.

Theorem 4.4.5. *Let $1 \leq p \leq \infty$ and A be a Köthe matrix. Let T be a chaotic weighted shift on $\lambda^p(A)$ with weight sequence $(w_n)_{n\geq 1}$ such that $\sum_{n\geq 0} X_n e_n/\beta_n$ is almost surely convergent, where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. standard Gaussian random variables. Let $(A_j)_{j\geq 1}$ be a non-decreasing sequence of positive functions defined on $\mathbb{N}$. Assume that the following conditions hold, where $a := 1/2$ if $1 \leq p < \infty$ and $a := 1$ if $p = \infty$:*

(i) *the quantity*

$$\sup_{m\geq 0} \sum_{j\geq 1} \log(A_{j+1}(m))^a \left\| \left(\frac{a_{m,n}^{1/p}}{\beta_n}\right)_{n\geq A_j(m)+1} \right\|_{\ell^p}$$

is finite, with the usual modification for $p = \infty$,

(ii) *the family $(A_j(m)^{-1})_{j,m\geq 1}$ is q-summable for some $q > 0$.*

Then there exists $c > 0$ such that almost surely, there exists $m_0 \geq 0$ such that

$$\left\| \sum_{n=0}^{\infty} \frac{X_n}{\beta_n} e_n \right\|_m \leq c \log(A_1(m))^a \left\| \left(\frac{a_{m,n}^{1/p}}{\beta_n}\right)_{n\geq 0} \right\|_{\ell^p}$$

for every $m \geq m_0$, with the usual modification for $p = \infty$.

Proof. For every $a_0, \dots, a_N \in \mathbb{K}$ and any $m \geq 0$, we have

$$\left\| \sum_{n=0}^{N} a_n e_n \right\|_m = \|(|a_n| a_{m,n}^{1/p})_{0\leq n\leq N}\|_p.$$

Therefore, the theorem is proved by repeating verbatim the proof of Theorem 3.3.4, but by using Lemma 4.4.4 instead of Lemma 3.1.13. □

We now apply the previous theorem to the *space of rapidly decreasing sequences* s defined as

$$s := \left\{ x \in \mathbb{K}^{\mathbb{N}} \mid \text{for all } 0 \leq t < \infty, \sum_{n=0}^{\infty} |x_n| e^{t\log(n+1)} < \infty \right\}.$$

This space is a Köthe sequence space with matrix $A = (m_k^{\log(n)})_{k,n\geq 0}$, where $(m_k)_{k\geq 0}$ can be any increasing sequence of positive numbers tending to infinity and setting

$m_k^{\log(0)} = 1$ for all $k \geq 0$. By [27, Proposition 4.1] and [27, Proposition 3.3], we have $s = \lambda^p(A)$ for any $1 \leq p \leq \infty$.

We will find an admissible rate of growth for the frequently hypercyclic sequences of the operator

$$T : s \longrightarrow s, x \longmapsto ((n+1)x_{n+1})_{n\geq 0}.$$

To end this section, we will show that this operator is conjugate to a natural operator on the space of 2π-periodic infinitely differentiable functions.

First of all, let us prove that $\sum_{n\geq 0} X_n e_n/\beta_n$ is almost surely frequently hypercyclic for T, where $(e_n)_{n\geq 0}$ is the canonical basis of s, $(X_n)_{n\geq 0}$ is a sequence of i.i.d. standard Gaussian random variables and $\beta_n := n!$ for all $n \geq 0$. Let $k \geq 0$, then we have

$$\sum_{n=1}^{\infty} \sqrt{\log(n)} \Big\| \frac{e_n}{\beta_n} \Big\|_k = \sum_{n=1}^{\infty} \sqrt{\log(n)} \frac{m_k^{\log(n)}}{n!} < \infty.$$

By Theorem 1.3.1, the random series $\sum_{n\geq 0} X_n e_n/\beta_n$ is almost surely convergent and frequently hypercyclic for T. In the sequel, we will assume that $\sum_{k\geq 0} m_k^{-q}$ converges for some $q > 0$ and that, without loss of generality, $m_k \geq 1$ for all $k \geq 0$.

Let $1 \leq p < \infty$. Our aim is to prove that

$$\sup_{k\geq 0} \sum_{j\geq 1} \sqrt{\log(A_{j+1}(k))} \Bigg(\sum_{n\geq A_j(k)+1} \frac{m_k^{\log(n)}}{|\beta_n|^p} \Bigg)^{1/p}$$

is finite, where $A_j(k) := d^j m_k^{1/p}$ with $d > e$ and all $j \geq 1$, $k \geq 0$.

Some simple calculations yield

$$\begin{aligned} \sum_{n\geq A_j(k)+1} \frac{m_k^{\log(n)}}{|\beta_n|^p} \leq \sum_{n\geq A_j(k)+1} \frac{m_k^n}{n!^p} &= \sum_{n\geq 1} \frac{m_k^{n+A_j(k)}}{(n+A_j(k))!^p} \\ &\leq \frac{m_k^{A_j(k)}}{A_j(k)!^p} \sum_{n\geq 1} \frac{m_k^n}{A_j(k)^{np}} \\ &= \frac{m_k^{A_j(k)}}{A_j(k)!^p} \frac{1}{d^{jp}-1}. \end{aligned}$$

By using Stirling's formula, our task is now to prove that

$$\sup_{k\geq 0} \sum_{j\geq 1} \sqrt{\log(A_j(k))} \frac{e^{A_j(k)}}{d^{jA_j(k)}\sqrt{A_j(k)}} < \infty. \tag{4.4.1}$$

By using the same arguments as in the proof of Proposition 4.1.2, it is enough to show that for every $M > e$, the function

$$f_M :]1, \infty[\longrightarrow]0, \infty[, x \longmapsto \sqrt{\log(x)} \frac{e^x}{M^x \sqrt{x}},$$

is non-increasing away from 0, uniformly in $M > e$, and converges to 0 when x goes to ∞. This has already been done in the proof of Theorem 4.3.11. Setting now $M = d^j$, the Dominated Convergence Theorem allows us to conclude that (4.4.1) holds, and Theorem 4.4.5 can be applied to T.

For $p = \infty$, similar calculations show that

$$\sup_{k\geq 0}\sum_{j\geq 1}\log(A_{j+1}(k))\Big\|\Big(\frac{m_k^{\log(n)}}{\beta_n}\Big)_{n\geq A_j(k)+1}\Big\|_{\ell^\infty}<\infty,$$

and we can use Theorem 4.4.5.

We have thus obtained the following result.

Proposition 4.4.6. *Let $T : s \longrightarrow s, x \longmapsto ((n+1)x_{n+1})_{n\geq 0}$. Set $a := 1/2$ if $1 \leq p < \infty$ and $a := 1$ if $p = \infty$. Then the random vector $\sum_{n\geq 0} X_n e_n/n!$ is almost surely frequently hypercyclic for T, where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. standard Gaussian random variables. Furthermore, if $\sum_{k\geq 0} m_k^{-q}$ converges for some $q > 0$ then, for any $1 \leq p \leq \infty$, there exists $c > 0$ such that almost surely, there exists $k_0 \geq 0$ such that*

$$\Big\|\sum_{n=0}^{\infty}\frac{X_n}{n!}e_n\Big\|_k \leq c\log(m_k)^a\Big(\sum_{n=0}^{\infty}\frac{m_k^{\log(n)}}{n!^p}\Big)^{1/p}$$

for every $k \geq k_0$, with the usual modification for $p = \infty$.

The space s is isomorphic to many Fréchet spaces, see [71, Example 29.4(1)]. Here, we consider the space $C^\infty_{2\pi}(\mathbb{R})$ of 2π-periodic infinitely differentiable complex-valued functions on $\mathbb{R}$, endowed with the seminorms

$$\|f\|_m := \max_{0\leq k\leq m}\|f^{(k)}\|_{L^2([-\pi,\pi])},\ f \in C^\infty_{2\pi}(\mathbb{R}),\ m \in \mathbb{N}.$$

This space is isomorphic to s, see [71, Example 29.5(1)]. Every function $f \in C^\infty_{2\pi}(\mathbb{R})$ has a representation $f = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, where $(a_n)_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of f.

Let us define the operator $B : E \longrightarrow E$ defined on the subspace $E := \{f = \sum_{n\in\mathbb{N}} a_n e^{inx} \in C^\infty_{2\pi}(\mathbb{R})\}$ by

$$B(f) = e^{-ix}\partial_x f,\ f \in E.$$

This operator is a weighted shift with respect to the Fourier coefficients and with sequence of weights $(in)_{n\in\mathbb{N}}$. The space E is also isomorphic to s via the isomorphism $F : E \longrightarrow s, f = \sum_{n\in\mathbb{N}} a_n e^{inx} \longmapsto (a_n)_{n\in\mathbb{N}}$. For any $m \geq 0$ and $f = \sum_{n\in\mathbb{N}} a_n e^{inx} \in B$, we have

$$\begin{aligned}\|f\|_m^2 &= \max_{0\leq k\leq m}\|f^{(k)}\|^2_{L^2([-\pi,\pi])} = \max_{0\leq k\leq m}\Big\|\sum_{n=0}^{\infty} i^k n^k a_n e^{inx}\Big\|^2_{L^2([-\pi,\pi])}\\ &= 2\pi\max_{0\leq k\leq m}\sum_{n=0}^{\infty} n^{2k}|a_n|^2 = 2\pi\sum_{n=0}^{\infty} n^{2m}|a_n|^2 = \Big\|\sum_{n=0}^{\infty} n^m a_n e^{inx}\Big\|^2_{L^2([-\pi,\pi])}.\end{aligned}$$

By using the previous proposition, we can obtain an admissible rate of growth for the frequently hypercyclic functions of B. Notice that if $(X_n)_{n\geq 0}$ is a standard Gaussian sequence, then so is $(X_n/i^n)_{n\geq 0}$.

Proposition 4.4.7. *Let $B : E \longrightarrow E, f \longmapsto e^{-ix}\partial_x f$. Then the random vector $\sum_{n\geq 0} X_n e^{inx}/(i^n n!)$ is almost surely frequently hypercyclic for B, where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. standard Gaussian random variables. Furthermore, there exists $c > 0$ such that almost surely, there exists $k_0 \geq 0$ such that*

$$\left\| \sum_{n=0}^{\infty} n^k \frac{X_n}{i^n n!} e^{inx} \right\|_{L^2([-\pi,\pi])} \leq c\sqrt{2k} \left(\sum_{n=0}^{\infty} \frac{n^{2k}}{n!^2} \right)^{1/2}.$$

for every $k \geq k_0$.

Proof. It suffices to use Proposition 4.4.6 with the sequence $(m_k)_{k\geq 0} = (e^{2k})_{k\geq 0}$ and $p = 2$ and to notice that

$$F^{-1}\left(\sum_{n=0}^{\infty} \frac{X_n}{i^n n!} e_n \right) = \sum_{n=0}^{\infty} \frac{X_n}{i^n n!} e^{inx}$$

is frequently hypercyclic for B whenever $\sum_{n\geq 0} X_n/(i^n n!) e_n$ is so for T. $\square$

4.5 Optimality

So far we have found an admissible rate of growth for some chaotic weighted shifts on $H(\mathbb{C})$ or $H(\mathbb{D})$. A natural question is to find the optimal growth. The function $r \longmapsto \sqrt{\sum_{n\geq 0} \|e_n\|_r^2/|\beta_n|^2}$, up to a logarithmic factor, has been proved to be an admissible growth for the weighted shifts considered in the previous sections. In [34], Drasin and Saksman proved that $r \longmapsto \sqrt{\sum_{n\geq 0} r^2/n!^2}$ is in fact an admissible growth for the differentiation operator and is the optimal one, by constructing a frequently hypercyclic function with this rate of growth. In [75] and [76], Mouze and Munnier used the same construction to get a holomorphic function whose growth is the optimal one for some weighted shifts on $H(\mathbb{D})$. This construction relied on the so-called Rudin-Shapiro polynomials.

In this section, we do not pretend to obtain the optimal rate of growth for every chaotic weighted shift. Instead, we show that the above-mentioned construction can be generalized to any chaotic weighted shift on $H(\mathbb{C})$ or $H(\mathbb{D})$ in order to get a frequently hypercyclic vector. Then, it would remain to calculate the growth of this function and prove that it is optimal.

This section is divided into four parts. First, we define some notations and explain the construction of the frequently hypercyclic function. Then, this function is proved to be well-defined. Finally, we prove in two parts that it is frequently hypercyclic for the given chaotic weighted shift.

Notations. Let $T : E \longrightarrow E$ be a chaotic weighted shift on the Fréchet space $E = H(\mathbb{C})$ or $E = H(\mathbb{D})$, with respect to the basis of monomials $(e_n)_{n\geq 0}$ and with sequence of weights $(w_n)_{n\geq 1}$. As usual, we define $\beta_n := w_1 \dots w_n$ for every $n \geq 1$, and $\beta_0 := 1$.

If $q = \sum_{j=0}^{n} b_j e_j$ is a polynomial of degree $n \geq 0$, we define $\|q\|_{l^1} := \sum_{j=0}^{n} |b_j|$. Let $(q_k)_{k\geq 1}$ be a dense sequence of polynomials with rational coefficients, and let

$(l_k)_{k\geq 1}$ be a sequence of positive integers such that $\|q_k\|_{l^1} \leq l_k$ for every $k \geq 1$ and $\lim_{k\to\infty} l_k = \infty$. For each $k \geq 1$, set $q_k = \sum_{j=0}^{d_k} b_j^{(k)} e_j$, where $d_k := \deg(q_k)$, and define $\widetilde{q}_k := \sum_{j=0}^{d_k} \beta_j b_j^{(k)} e_j$.

For each positive integer $N \geq 1$, choose a finite sequence $(\varepsilon_n^N)_{n=0}^{N-1}$ of complex numbers of modulus equal to 1 and such that at least half of them are equal to 1; define $p_N := \sum_{n=0}^{N-1} \varepsilon_n^N e_n$. In [34], [75] and [76], the Rudin-Shapiro polynomials have in fact been taken. But the properties of those polynomials are only used to bound the sup-norm, which we do not perform here.

Choose any monotone sequence $(\delta_k)_{k\geq 1}$ and $(r_k)_{k\geq 1}$ of positive real numbers such that $\lim_{k\to\infty} \delta_k = 0$ and $\lim_{k\to\infty} r_k = w$, where $w := \infty$ if $E = H(\mathbb{C})$, or $w := 1$ if $E = H(\mathbb{D})$. We partition the set of positive even integers into

$$\mathcal{A}_k := \left\{2^k(2j-1) \mid j \geq 1\right\}, \; k \geq 1.$$

Let $(\alpha_k)_{k\geq 1}$, $(j_n)_{n\geq 1}$, $(a_n)_{n\geq 1}$ and $(N_n)_{n\geq 1}$ be strictly increasing sequences of positive integers satisfying the following properties: for every $k \geq 1$ and every $n \in \mathcal{A}_k$ with $a_n \geq \alpha_k$,

(1) $\alpha_k > d_k$,

(2) $\|\widetilde{q}_k\|_{l^1} \leq \alpha_k$,

(3) $\max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{j\geq j_{n_k}} \frac{r_k^j}{|\beta_j|} \leq 1/2^k$, where for each $k \geq 1$, n_k is the smallest $m \in \mathcal{A}_k$ such that $a_m \geq \alpha_k$,

(4) $\max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{j\geq \alpha_k} \frac{r_k^j}{|\beta_j|} \leq \delta_k$,

(5) $\sum_{j\geq C\alpha_k} \frac{j}{|\beta_j|} r_k^j \leq \delta_k$, where $C > 0$ is the constant in (9),

(6) $a_n \lesssim j_n$,

(7) $j_{n+1} \leq c_k^{(1)} j_n$ for some $c_k^{(1)} > 1$,

(8) $(N_n - 1)\alpha_k + d_k < j_{n+1} - j_n \leq c_k^{(2)} \lfloor N_n/2 \rfloor$ for some $c_k^{(2)} > 0$,

(9) $Ca_n \leq j_{n+2} - j_{n+1}$ for some $C > 0$,

(10) $j_n \leq c_k^{(3)} \sum_{m\in\mathcal{A}_k, a_m\geq\alpha_k}^{n-1} (j_{m+1} - j_m)$ for some $c_k^{(3)} > 0$.

The constants $c_k^{(1)}$, $c_k^{(2)}$ and $c_k^{(3)}$ in inequalities (7), (8) and (10) can depend on k.

Since T is chaotic, such a sequence $(\alpha_k)_{k\geq 1}$ can always be chosen to satisfy properties (1) to (5), given $(j_n)_{n\geq 1}$ and $(a_n)_{n\geq 1}$. Furthermore, note that the properties (6) to (10) can also always be satisfied, whether the shift is chaotic or not. For example, in [34], the sequences are set to $a_n = n/10$, $j_n = n^2$ and $N_n = \lfloor n/\alpha_k \rfloor$ for the differentiation operator, while in [75] and [76], $a_n = 2^{n-1}$, $j_n = 2^n$ and $N_n = \lfloor 2^{n-1}/\alpha_k \rfloor$ for the weighted Taylor shifts. However, that does not mean that these are good choices for every chaotic shift. These sequences should be chosen in order to ensure the right optimal rate of growth.

For each $n \geq 1$, define $I_n := \{j_n, \dots, j_{n+1} - 1\}$ and if $n \in \mathcal{A}_k$, $k \geq 1$,

$$Q_n := \sum_{j \in I_n} \frac{c_{j-j_n}^{(k)}}{\beta_j} e_{j-j_n},$$

where the sequence of coefficients $(c_i^{(k)})_{i=0}^{j_{n+1}-1-j_n}$ is such that

$$p_{N_n}(z^{\alpha_k})\widetilde{q}_k(z) = \sum_{j=0}^{j_{n+1}-1-j_n} c_j^{(k)} z^j, \ z \in \mathbb{C},$$

and define

$$P_n := \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \in \mathcal{A}_k \text{ and } a_n < \alpha_k, \\ z^{j_n} Q_n(z) & \text{if } n \in \mathcal{A}_k \text{ and } a_n \geq \alpha_k. \end{cases} \tag{4.5.1}$$

Observe that (8) ensures that the degree of $p_{N_n}(z^{\alpha_k})\widetilde{q}_k(z)$ is at most $j_{n+1} - j_n - 1$, and Q_n is well-defined. For the sake of clarity, we define for each $k \geq 1$ the set $\mathcal{B}_k := \{n \in \mathcal{A}_k \mid a_n \geq \alpha_k\}$.

Finally, define $f := \sum_{n \geq 1} P_n$. This function will be proved to be well-defined and frequently hypercyclic for T.

Proposition 4.5.1. *The function $f = \sum_{n \geq 1} P_n$ is well-defined and frequently hypercyclic for T, where the polynomials P_n, $n \geq 1$, are defined in* (4.5.1).

The function f is well-defined. Let $k \geq 1$ and $n \in \mathcal{B}_k$. We first study the blocks of coefficients of the polynomial P_n. For any $z \in \mathbb{C}$, we have

$$p_{N_n}(z^{\alpha_k})\widetilde{q}_k(z) = \left(\sum_{j=0}^{N_n - 1} \varepsilon_j^{N_n} z^{j\alpha_k}\right)\left(\sum_{l=0}^{d_k} \beta_l b_l^{(k)} z^l\right) = \sum_{j=0}^{N_n - 1} \sum_{l=0}^{d_k} \varepsilon_j^{N_n} \beta_l b_l^{(k)} z^{j\alpha_k + l}.$$

We have $(j+1)\alpha_k - j\alpha_k - d_k > 0$ for any $j \geq 0$ since $\alpha_k > d_k$ by (1). This implies that $c_{j\alpha_k + l}^{(k)} = \varepsilon_j^{N_n} \beta_l b_l^{(k)}$ for every $0 \leq j \leq N_n - 1$ and $0 \leq l \leq d_k$. Therefore,

$$Q_n = \sum_{j=0}^{N_n - 1} \sum_{l=0}^{d_k} \frac{\varepsilon_j^{N_n} \beta_l b_l^{(k)}}{\beta_{j\alpha_k + l + j_n}} e_{j\alpha_k + l}$$

and

$$P_n = \sum_{j=0}^{N_n - 1} \sum_{l=0}^{d_k} \frac{\varepsilon_j^{N_n} \beta_l b_l^{(k)}}{\beta_{j\alpha_k + l + j_n}} e_{j\alpha_k + l + j_n}. \tag{4.5.2}$$

To be clearer, the coefficients of P_n are divided into distinct blocks as follows: the first one starts from degree j_n and ends at $j_n + d_k$, then the second one starts from degree $j_n + \alpha_k$ and ends at $j_n + \alpha_k + d_k$, and so on; see Figure 4.1.

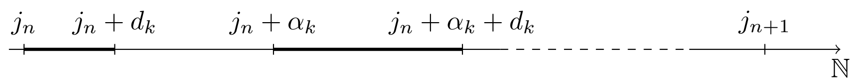

Figure 4.1: Blocks of coefficients of P_n: bold lines indicate possibly non-zero coefficients.

Let $0 < r < \omega$ and let $k_0 \geq 1$ be such that $r \leq r_{k_0}$. We have by using (4.5.2), the triangle inequality, the fact that $\|q_k\|_{l^1} \leq l_k$, $n_k = \min \mathcal{B}_k$ and (3),

$$\begin{aligned}\sum_{n\geq 1} \|P_n\|_r &\leq \sum_{k\geq 1} \sum_{n\in\mathcal{B}_k} \sum_{j=0}^{N_n-1} \sum_{l=0}^{d_k} |\varepsilon_j^{N_n}| \frac{|\beta_l||b_l^{(k)}| r^{\alpha_k j+l+j_n}}{|\beta_{\alpha_k j+l+j_n}|} \\ &\leq \sum_{k\geq 1} \max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{n\in\mathcal{B}_k} \sum_{j=0}^{N_n-1} \sum_{l=0}^{d_k} \frac{r^{\alpha_k j+l+j_n}}{|\beta_{\alpha_k j+l+j_n}|} \\ &\leq \sum_{k=1}^{k_0-1} \max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{j\geq j_{n_k}} \frac{r_{k_0}^j}{|\beta_j|} + \sum_{k\geq k_0} \max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{j\geq j_{n_k}} \frac{r_k^j}{|\beta_j|} \\ &\leq \sum_{k=1}^{k_0-1} \max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{j\geq j_{n_k}} \frac{r_{k_0}^j}{|\beta_j|} + \sum_{k\geq k_0} \frac{1}{2^k}.\end{aligned}$$

Thus the series $\sum_{n\geq 0} P_n$ converges on any disk of radius $0 < r < w$ and centred at the origin, and the function f is then well-defined.

Frequent hypercyclicity I. Let $k \geq 1$ and $n \in \mathcal{B}_k$. Define

$$B_n^{(k)} := \left\{ j_n + j\alpha_k \in I_n \mid \varepsilon_j^{N_n} = 1,\ 0 \leq j \leq N_n - 1 \right\},$$

and let $s \in B_n^{(k)}$. We are going to show that

$$\|T^s(f) - q_k\|_{r_k} \lesssim \delta_k. \tag{4.5.3}$$

Write $s = j_n + m\alpha_k$ with $m \in \{0, \ldots, N_n - 1\}$. For any $l \geq 1$ and $\widetilde{n} \in \mathcal{B}_l$ such that $\widetilde{n} < n$, we have $j_{\widetilde{n}+1} - 1 < j_n$ since $(j_{\widetilde{n}})_{\widetilde{n}\geq 1}$ is strictly increasing, and $T^s(e_j) = 0$ for any $j \in I_{\widetilde{n}}$. Therefore, by continuity and linearity of T, we have

$$T^s(f) = T^s(P_n) + \sum_{\widetilde{n}>n} T^s(P_{\widetilde{n}}) = T^s(P_n) + \sum_{\widetilde{k}\geq 1} \sum_{\widetilde{n}\in\mathcal{B}_{\widetilde{k}}, \widetilde{n}>n} \sum_{j\in I_{\widetilde{n}}} \frac{c_{j-j_{\widetilde{n}}}^{(\widetilde{k})}}{\beta_{j-s}} e_{j-s}.$$

It follows by (4.5.2) and definition of T that

$$T^s(P_n) = \sum_{j=m}^{N_n-1} \sum_{l=0}^{d_k} \frac{\varepsilon_j^{N_n} \beta_l b_l^{(k)}}{\beta_{j\alpha_k+l+j_n-s}} e_{j\alpha_k+l+j_n-s}.$$

For $j = m$ and since $\varepsilon_m^{N_n} = 1$, we have

$$\sum_{l=0}^{d_k} \frac{\varepsilon_m^{N_n} \beta_l b_l^{(k)}}{\beta_{m\alpha_k+l+j_n-j_n-m\alpha_k}} e_{m\alpha_k+l+j_n-j_n-m\alpha_k} = \sum_{l=0}^{d_k} b_l^{(k)} e_l = q_k.$$

Observe by (4.5.2) that the coefficients of P_n of degree $s + d_k + 1 \leq j \leq j_n + (m+1)\alpha_k - 1 = s - 1 + \alpha_k$ are equal to zero. Therefore,

$$T^s(f) - q_k = \sum_{j=s+\alpha_k}^{j_{n+1}-1} \frac{c_{j-j_n}^{(k)}}{\beta_{j-s}} e_{j-s} + \sum_{\widetilde{k}\geq 1} \sum_{\widetilde{n}\in\mathcal{B}_{\widetilde{k}}, \widetilde{n}\geq n+1} \sum_{j\in I_{\widetilde{n}}} \frac{c_{j-j_{\widetilde{n}}}^{(\widetilde{k})}}{\beta_{j-s}} e_{j-s}.$$

Denote by S_1 the first sum and by S_2 the second series. By the triangle inequality and the fact that $\|q_k\|_{l^1} \leq l_k$, we have

$$\begin{aligned}
\|S_1\|_{r_k} &\leq \sum_{j=s+\alpha_k}^{j_{n+1}-1} \frac{|c_{j-j_n}^{(k)}|}{|\beta_{j-s}|} r_k^{j-s} \leq \sum_{j=s+\alpha_k}^{j_{n+1}-1} \frac{\|\widetilde{q}_k\|_{l^1}}{|\beta_{j-s}|} r_k^{j-s} \\
&\leq \max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{j\geq s+\alpha_k} \frac{r_k^{j-s}}{|\beta_{j-s}|} \\
&= \max_{0\leq l\leq d_k} |\beta_l| l_k \sum_{j\geq \alpha_k} \frac{r_k^{j}}{|\beta_{j}|} \leq \delta_k,
\end{aligned} \tag{4.5.4}$$

where the last inequality holds by (4). For S_2, again by the triangle inequality we get that

$$\|S_2\|_{r_k} \leq \sum_{\widetilde{k}\geq 1} \sum_{\widetilde{n}\in\mathcal{B}_{\widetilde{k}}, \widetilde{n}\geq n+1} \sum_{j\in I_{\widetilde{n}}} \frac{|c_{j-j_{\widetilde{n}}}^{(\widetilde{k})}|}{|\beta_{j-s}|} r_k^{j-s}.$$

For every $\widetilde{k} \geq 1$, $\widetilde{n} \in \mathcal{B}_{\widetilde{k}}$ and $j \in I_{\widetilde{n}}$, by (2) and (6), one has

$$|c_{j-j_{\widetilde{n}}}^{(\widetilde{k})}| \leq \|\widetilde{q}_{\widetilde{k}}\|_{l^1} \leq \alpha_{\widetilde{k}} \leq a_{\widetilde{n}} \lesssim j_{\widetilde{n}} \leq j,$$

which implies

$$\|S_2\|_{r_k} \lesssim \sum_{\widetilde{k}\geq 1} \sum_{\widetilde{n}\in\mathcal{B}_{\widetilde{k}}, \widetilde{n}\geq n+1} \sum_{j\in I_{\widetilde{n}}} \frac{j}{|\beta_{j-s}|} r_k^{j-s} \leq \sum_{j\geq j_{n+2}} \frac{j}{|\beta_{j-s}|} r_k^{j-s}.$$

Note that $n+1$ is not in $\mathcal{A}_k$, allowing us to sum from j_{n+2}. Since $s \leq j_{n+1}$ and $C\alpha_k \leq Ca_n \leq j_{n+2} - j_{n+1}$ for some $C > 0$ by (9), we have

$$\|S_2\|_{r_k} \lesssim \sum_{j\geq C\alpha_k} \frac{j}{|\beta_j|} r_k^j \leq \delta_k, \tag{4.5.5}$$

where the last inequality holds by (5).

Combining the inequalities (4.5.4) and (4.5.5) shows (4.5.3).

Frequent hypercyclicity II. Let $k \geq 1$, we now want to show that the set $D_k := \bigcup_{n\in\mathcal{B}_k} B_n^{(k)}$ has positive lower density. This will finally conclude that the function f is frequently hypercyclic for T.

Let $N \geq 1$ be large enough. There exists $m \in \mathcal{B}_k$ such that either $j_m \leq N < j_{m+1}$ or $j_{m+1} \leq N < j_{m+2}$. Assume first that $j_m \leq N < j_{m+1}$. Then we get, since for every $n \geq 1$ at least half of the coefficients of p_{N_n} are equal to 1 and by using (8), (10) and (7),

$$\frac{\left|D_k \cap \{0,\ldots,N\}\right|}{N+1} \geq \frac{\sum_{n\in\mathcal{B}_k}^{m-1} |B_n^{(k)}|}{N+1} \gtrsim \frac{\sum_{n\in\mathcal{B}_k}^{m-1} \lfloor N_n/2 \rfloor}{N+1} \geq \frac{1}{c_k^{(2)} c_k^{(3)}} \frac{j_m}{j_{m+1}} \geq \frac{1}{c_k^{(1)} c_k^{(2)} c_k^{(3)}}.$$

Now, assume that $j_{m+1} \leq N < j_{m+2}$. By (8) and (10), since $m+1$ does not belong to $\mathcal{A}_k$ and for every $n \geq 1$ at least half of the coefficients of p_{N_n} are equal to 1, we get that

$$\frac{\left|D_k \cap \{0,\ldots,N\}\right|}{N+1} \geq \frac{\sum_{n\in\mathcal{B}_k}^{m+1} |B_n^{(k)}|}{N+1} \gtrsim \frac{1}{c_k^{(2)} c_k^{(3)}} \frac{j_{m+2}}{j_{m+2}} = \frac{1}{c_k^{(2)} c_k^{(3)}}.$$

Taking the limit as N goes to ∞ shows that the set D_k has positive lower density.

4.6 Conclusion

As discussed in Section 4.5, one natural question is to find the optimal rate of growth for chaotic weighted shifts. Theorem 3.1.17 says that the function S_f associated with the entire function f is always, up to a logarithmic factor, an admissible rate of growth that is valid outside some set of finite logarithmic measure. And by Theorem 3.3.4, the inequality can even sometimes hold everywhere. Then one may ask whether the rate of growth found in Theorem 4.1.1 holds everywhere for any chaotic weighted shift.

In [80, Proposition 6], it is proved that the logarithmic factor is optimal for the random frequently hypercyclic vector associated with the differentiation operator. But it is known by [34, Theorem 1.1] that the optimal growth for the frequently hypercyclic vectors is actually the function S_f, where f is the exponential function. With a careful reading of the proofs, one can see that this logarithmic factor comes from a probabilistic result, namely Lemma 3.1.13.

The function S_f associated with a chaotic weighted shift is also the optimal rate of growth for almost any weighted Taylor shifts considered in Subsection 4.2.2, except possibly for $T_{1/2}$, see [76, Theorem 1.3]. And [15, Theorem 6] indicates that this function could also be the optimal rate of growth for the Dunkl operator.

Therefore, we might think that for every chaotic weighted shift defined on $H(\mathbb{D})$ or $H(\mathbb{C})$, the map S_f is actually always the optimal rate of growth. This claim is supported by the construction of a frequently hypercyclic vector in Section 4.5 since this led to the proof of the optimality for the differentiation operator in [34] and for some weighted Taylor shifts in [75] and [76].

Finally, we point out that even if optimality were proved, the probabilistic methods of this chapter may still be of some interest. Recall that by Theorem 0.1.19, the chaoticity of a weighted shift is equivalent to the unconditional convergence of $\sum_{n\geq 0} e_n/\beta_n$. Therefore, our results say that a random perturbation of the coefficients of this series could nearly gives the optimal rate of growth. We might expect that a frequently hypercyclic function with the optimal growth would be more complicated to construct. Furthermore, these results also imply that, in a measure-theoretical sense, there are many functions with a quasi-optimal growth.

We then ask the following questions to conclude Chapter 4.

Question 4.6.1. *Let $T : H(\mathbb{C}) \longrightarrow H(\mathbb{C})$ be a chaotic weighted shift with weight sequence $(w_n)_{n\in\mathbb{N}_0}$. Is the map*

$$S_T : [0, \infty[\longrightarrow [0, \infty[, r \longmapsto \sqrt{\sum_{n=0}^{\infty} \frac{r^{2n}}{\prod_{j=1}^{n} w_n^2}}$$

optimal for the growth of frequently hypercyclic functions of T? If not, for which class of shifts T is the map S_T the optimal rate of growth?

Question 4.6.2. *Let $T : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ be a chaotic weighted shift with weight sequence $(w_n)_{n\in\mathbb{N}_0}$. Is the map*

$$S_T : [0, 1[\longrightarrow [0, \infty[, r \longmapsto \sqrt{\sum_{n=0}^{\infty} \frac{r^{2n}}{\prod_{j=1}^{n} w_n^2}}$$

optimal for the growth of frequently hypercyclic functions of T? If not, for which class of shifts T is the map S_T the optimal rate of growth?

Appendix

Throughout the appendix, let E be a separable Fréchet space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

A.1 Bochner spaces

In this section, we define the Bochner spaces for Fréchet space-valued functions. These spaces are defined in the same way as the classical Bochner spaces, see [50, Chapter 1] for the Banach case. We will prove that they are Fréchet spaces too. This result is stated and proved in [64, Proposition 1], but it might have been proved before.

Definition A.1.1. Let $(S, \mathcal{B}, \mu)$ be a measure space. Let $1 \leq p < \infty$. We define the space $L^p(S, \mathcal{B}, \mu; E)$ as the space of all equivalence classes of measurable functions $f : S \longrightarrow E$ for which

$$\int_S \|f\|^p \mathrm{d}\mu < \infty$$

for all continuous seminorms $\|\cdot\|$ on E. This space is also noted $L^p(S; E)$.

For a given non-decreasing sequence $(\|\cdot\|_k)_{k\geq 1}$ of seminorms of E generating its topology, we define for each real number $1 \leq p < \infty$ and integer $k \geq 1$ the seminorm $\|\cdot\|_{p,k}$ by

$$\|f\|_{p,k} := \Big(\int_S \|f\|_k^p \mathrm{d}\mu \Big)^{1/p}$$

for every $f \in L^p(S; E)$.

We now prove that the space $L^p(S; E)$ endowed with this sequence of seminorms is a Fréchet space. The proof is actually a straightforward modification of the scalar case, see [87, Theorem 3.11].

Theorem A.1.2. *Let $1 \leq p < \infty$. The space $L^p(S; E)$ endowed with the sequence of seminorms $(\|\cdot\|_{p,k})_{k\geq 1}$ is a Fréchet space.*

Proof. First we show that $(\|\cdot\|_{p,k})_{k\geq 1}$ is a separating sequence. Let $f \in L^p(S; E)$ be such that $\|f\|_{p,k} = 0$ for all $k \in \mathbb{N}_0$. Then for each $k \in \mathbb{N}_0$, there exists a set $A_k \subseteq E$ of measure zero such that $\|f\|_k = 0$ outside A_k. Set $A := \bigcup_{k\geq 1} A_k$. This set has measure zero, and $f = 0$ outside A.

We now show that every Cauchy sequence is convergent. Let $(f_n)_{n\geq 1}$ be Cauchy in $L^p(S;E)$. There exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that for all $k \geq 1$, one has

$$\|f_{n_{k+1}} - f_{n_k}\|_{p,k} \leq \frac{1}{2^k}.$$

Fix $n \geq 1$. Set $g_{k,n} := \sum_{j=1}^{k} \|f_{n_{j+1}} - f_{n_j}\|_n$ for each $k \geq 1$, and $g_n := \sum_{j\geq 1} \|f_{n_{j+1}} - f_{n_j}\|_n$. By using the Minkowski inequality in $L^p(S;\mathbb{K})$, we have $\sup_{k\geq 1} \|g_{k,n}\|_{L^p(S)} < \infty$, implying that

$$\int_S \liminf_{k\to\infty} g_{k,n}^p \mathrm{d}\mu \leq \liminf_{k\to\infty} \int_S g_{k,n}^p \mathrm{d}\mu < \infty$$

by Fatou's lemma, see [87, Lemma 1.28]. We deduce that $g_n < \infty$ almost everywhere. Therefore, almost everywhere, we have $\sum_{j\geq 1} \|f_{n_{j+1}} - f_{n_j}\|_n < \infty$ for every $n \geq 1$. Since E is complete, we deduce that

$$f_{n_k} = f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j})$$

converges to some function f almost everywhere when k goes to ∞. On a set of measure zero, we can set f and each $f_k, k \geq 1$, to 0, and $(f_{n_k})_{k\geq 1}$ converges everywhere. Thus f is measurable by Lemma 1.1.1. We must show that $f \in L^p(S;E)$ and that $(f_n)_{n\geq 1}$ converges to f in $L^p(S;E)$. Let $k \geq 1$ and $\varepsilon > 0$. There exists $N \geq 1$ such that for all $n, m \geq N$, one has $\|f_n - f_m\|_{p,k} < \varepsilon$. By Fatou's lemma, we get

$$\int_S \|f - f_m\|_k^p \mathrm{d}\mu \leq \liminf_{j\to\infty} \int_S \|f_{n_j} - f_m\|_k^p \mathrm{d}\mu \leq \varepsilon^p$$

for all $m \geq N$. We deduce that $f = f - f_m + f_m \in L^p(S;E)$, and $\lim_{m\to\infty} \|f - f_m\|_{p,k} = 0$ for every $k \geq 1$. $\square$

As in the scalar case, the previous proof shows that a Cauchy sequence in $L^p(S;E)$ has an almost everywhere convergent subsequence.

Lemma A.1.3. *Let $1 \leq p < \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions converging to f in $L^p(S;E)$. Then there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{n_k} = f$ almost everywhere.*

A.2 γ-radonifying operators

We will now state the main property of γ-radonifying operators used in this work, namely the ideal property. The proof is given in [51, Theorem 9.1.10] for maps with values in a Banach space. Since the definition given there is not Definition 0.5.8 but an equivalent one, we will give the relevant definitions for the Fréchet space case. All the proofs of the stated results remain the same as in [51], see the references given in each result.

In the remainder of this section, let $(g_n)_{n\in\mathbb{N}}$ be a standard Gaussian sequence, that is, an i.i.d. sequence of Gaussian random variables with mean 0 and variance 1.

Definition A.2.1. Let H be a separable Hilbert space and $1 \le p < \infty$ be a real number. A linear map $T : H \longrightarrow E$ is *γ-summing* if

$$\sup \mathbb{E}\left(\Big\| \sum_{j=0}^{k} g_j T(h_j) \Big\|^p\right) < \infty$$

for all continuous seminorms $\|\cdot\|$ on E, where the supremum is taken over all finite orthonormal systems $(h_j)_{j=0}^k$ in H. The space of all γ-summing operators is denoted by $\gamma_\infty^p(H, E)$.

By the Kahane-Khintchine inequalities, see [51, Theorem 6.2.6], all the spaces $\gamma_\infty^p(H, E)$, $1 \le p < \infty$, are equal. In [66, Chapitre 3, Théorème IV.1], these inequalities are stated in a Banach space framework, but the proof carries over verbatim to the Fréchet space case.

Lemma A.2.2 ([51, p. 255])**.** *Every γ-summing map $T : H \longrightarrow E$ is continuous.*

Proof. Let $\|\cdot\|$ be a continuous seminorm on E. Since T is γ-summing, there exists some $M > 0$ such that $\|T(h)\| \le M\|h\|_H$ for all $h \in H$. This yields the continuity of T. □

For the remainder of this section, let $(\|\cdot\|_n)_{n\in\mathbb{N}_0}$ be a sequence of seminorms of E generating its topology.

Definition A.2.3. Let H be a separable Hilbert space and let $1 \le p < \infty$. For each $n \in \mathbb{N}_0$ and every $T \in \gamma_\infty^p(H, E)$, we define

$$\|T\|_{\gamma_\infty^p(H,E),n} := \sup \mathbb{E}\left(\Big\| \sum_{j=0}^{k} g_j T(h_j) \Big\|_n^p\right)^{1/p},$$

where the supremum is taken over all finite orthonormal systems $(h_j)_{j=0}^k$ in H.

Endowed with the sequence of seminorms $(\|\cdot\|_{\gamma_\infty^p(H,E),n})_{n\in\mathbb{N}_0}$ defined above, the space $\gamma_\infty^p(H, E)$ is complete, as the next result says.

Proposition A.2.4 ([51, Proposition 9.1.2])**.** *Let H be a separable Hilbert space and $1 \le p < \infty$. The space $\gamma_\infty^p(H, E)$ is a Fréchet space.*

The γ-radonifying operators will be defined as the limit of finite rank operators.

Definition A.2.5. Let H be a separable Hilbert space. For $h^* \in H^*$ and $x \in E$, define the linear map

$$h^* \otimes x : H \longrightarrow E, h \longmapsto h^*(h)x.$$

A *finite rank operator* $T : H \longrightarrow E$ is a linear map of the form $T = \sum_{n=1}^N h_n^* \otimes x_n$, where $h_1^*, \dots, h_N^* \in H^*$ are orthonormal and $x_1, \dots, x_N \in E$, $N \ge 1$.

Every finite rank operator is γ-summing, see [51, Proposition 9.1.3].

Definition A.2.6. Let H be a separable Hilbert space and $1 \leq p < \infty$. The space $\gamma^p(H, E)$ is defined as the closure in $\gamma^p_\infty(H, E)$ of the finite rank operators. The elements of $\gamma^p(H, E)$ are called *γ-radonifying operators.*

As with the γ-summing operators, all the spaces $\gamma^p(H, E)$, $1 \leq p < \infty$, are equal. We now state the *ideal property.*

Theorem A.2.7 ([51, Theorem 9.1.10])**.** *Let $T : H \longrightarrow E$ be a γ-radonifying operator on a separable Hilbert space H. Let G be another separable Hilbert space, F be another separable Fréchet space, and let $S : G \longrightarrow H$ and $U : E \longrightarrow F$ be continuous and linear maps. Then the map UTS is γ-radonifying.*

The last result states that Definitions 0.5.8 and A.2.6 are equivalent.

Theorem A.2.8 ([51, Theorem 9.1.17])**.** *Let H be a separable Hilbert space and $1 \leq p < \infty$. Let $(h_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H and $T : H \longrightarrow E$ be a continuous linear map. Then $T \in \gamma^p(H, E)$ if and only if $\sum_{n\geq 0} g_n T(h_n)$ converges in $L^p(\Omega; E)$ and if and only if $\sum_{n\geq 0} g_n T(h_n)$ converges almost surely.*

A.3 Stochastic calculus in Fréchet spaces

We will give a proof of Theorems 2.1.4 and 2.1.6 in this section. We will need some results. First, we recall some topologies on a Fréchet space E and its dual E^*. Our main references are [71] and [89].

Definition A.3.1. Let $\mathcal{F}$ be the set of all finite sets of E^*. The topology generated by the seminorm system $(p_F)_{F\in\mathcal{F}}$, where

$$p_F : E \longrightarrow [0, \infty[, y \longmapsto \sup_{x^*\in F} |x^*(y)|,$$

for each $F \in \mathcal{F}$, is called the *weak topology* on E and is denoted by $\sigma(E, E^*)$.

Definition A.3.2. Let $\mathcal{F}$ be the set of all finite sets of E. The topology generated by the seminorm system $(p_F)_{F\in\mathcal{F}}$, where

$$p_F : E^* \longrightarrow [0, \infty[, x^* \longmapsto \sup_{y\in F} |x^*(y)|,$$

for each $F \in \mathcal{F}$, is called the *weak topology* on E^* and is denoted by $\sigma(E^*, E)$.

Recall that a set $A \subseteq F$ of a vector space F is *absolutely convex* if it is convex and $\lambda x \in A$ for all scalars $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$ and $x \in A$.

Definition A.3.3. Let $\mathcal{M}$ be the set of all absolutely convex and $\sigma(E^*, E)$-compact sets of E^*. The topology generated by the seminorm system $(p_M)_{M\in\mathcal{M}}$, where

$$p_M : E \longrightarrow [0, \infty[, y \longmapsto \sup_{x^*\in M} |x^*(y)|,$$

for each $M \in \mathcal{M}$, is called the *Mackey topology* on E and is denoted by $\tau(E, E^*)$.

Definition A.3.4. Let $\mathcal{M}$ be the set of all absolutely convex and $\sigma(E, E^*)$-compact sets of E. The topology generated by the seminorm system $(p_M)_{M \in \mathcal{M}}$, where

$$p_M : E^* \longrightarrow [0, \infty[, x^* \longmapsto \sup_{y \in M} |x^*(y)|,$$

for each $M \in \mathcal{M}$, is called the *Mackey topology* on E^* and is denoted by $\tau(E^*, E)$.

We will need Proposition A.3.6 in the proof of Theorem 2.1.4. It is proved in the given reference for Banach spaces, but we give the proof for the sake of completeness.

First, we need a preliminary lemma. Recall that a map $R : E^* \longrightarrow E$ is *positive* if $x^*Rx^* \geq 0$ for all $x^* \in E^*$, and is *symmetric* if $x^*Ry^* = \overline{y^*Rx^*}$ for all $x^*, y^* \in E^*$.

Lemma A.3.5 ([94, Lemma 4 and its proof]). *Let $R : E^* \longrightarrow E$ be a positive symmetric map. Then there exists a Hilbert space H_R and a continuous linear map $i_R : H_R^* \longrightarrow E$ such that $R = i_R I i_R^*$, where $I : H_R \longrightarrow H_R^*$ is the canonical conjugate-linear operator. More precisely, H_R is the completion of E^*/M under the norm induced by R, given by $\langle x^* + M, y^* + M \rangle_{H_R} = x^*Ry^*$, $x^*, y^* \in E^*$, where $M = \{x^* \in E^* \mid x^*Rx^* = 0\}$, and $i_R^* : E^* \longrightarrow H_R, x^* \longmapsto x^* + M$ is the inclusion map.*

A *positive symmetric* map $Q : E^* \longrightarrow (E^*)'$ is defined in the same way as a positive symmetric map from E^* to E: for all $x^*, y^* \in E^*$, one has $(Qx^*)(x^*) \geq 0$ and $(Qy^*)(x^*) = \overline{(Qx^*)(y^*)}$.

Proposition A.3.6 ([43, Proposition 2.2]). *Let $Q : E^* \longrightarrow (E^*)'$ be a positive symmetric map. Suppose that there exists a positive symmetric map $R : E^* \longrightarrow E$ such that $(Qx^*)(x^*) \leq x^*Rx^*$ for every $x^* \in E^*$. Then $Q(E^*) \subseteq E$.*

Proof. Let $i_R : H_R^* \longrightarrow E$ be the map given by Lemma A.3.5. Fix $x^* \in E^*$, and define the linear map

$$\phi_{x^*} : i_R^*(E^*) \longrightarrow \mathbb{K}, i_R^*(y^*) \longmapsto (Qx^*)(y^*).$$

Let $y^* \in E^*$. By the Cauchy-Schwarz inequality applied to the sesquilinear form $(x^*, y^*) \longmapsto (Qx^*)(y^*)$, the assumption and Lemma A.3.5, we get that

$$|\phi_{x^*}(i_R^*(y^*))| \leq \big((Qx^*)(x^*)\big)^{1/2}\big((Qx^*)(y^*)\big)^{1/2} \leq \|i_R^*(x^*)\|_{H_R}\|i_R^*(y^*)\|_{H_R}.$$

This shows that ϕ_{x^*} is well-defined and continuous on $(i_R^*(E^*), \|\cdot\|_{H_R})$. Therefore, we can continuously extend ϕ_{x^*} on H_R, and the extension is still noted ϕ_{x^*}. Now, for every $y^* \in E^*$, we have

$$y^*(i_R(\phi_{x^*})) = (y^* \circ i_R)(\phi_{x^*}) = \phi_{x^*}(y^* \circ i_R) = (Qx^*)(y^*)$$

We deduce that $Qx^* = i_R(\phi_{x^*}) \in E$. $\square$

The last notion we will need is the following one. Let $V \subseteq E$ be non-empty. The *polar* of V, noted V°, is the subset of E^* defined by

$$V^\circ := \Big\{x^* \in E^* \mid |x^*(y)| \leq 1 \text{ for all } y \in V\Big\}.$$

By convention, the inner product on a vector space is linear in the first argument and conjugate-linear in the second argument.

Proof of Theorem 2.1.4. The implication (i) $\Longrightarrow$ (ii) is trivial.

Assume that (ii) holds. Then for all $x^* \in E^*$, the random variable $x^*(Y)$ is Gaussian, hence Y is Gaussian. Let Q be the covariance operator of its distribution μ. For every $x^* \in E^*$, we have by the Itô isometry, see Theorem 0.4.4 for the real case and Lemma 0.4.7 for the complex case,

$$x^*Qx^* = \int_E |x^*|^2 \mathrm{d}\mu = \int_\Omega |x^*(Y)|^2 \mathrm{d}\mathbb{P} = c\int_I |x^*(\phi(t))|^2 \mathrm{d}t.$$

This shows (iii).

Now assume that (iii) holds. Set $H := (\overline{E^*})^*$, where the closure is taken in the space $L^2(\mu)$. It is a separable space by [50, Proposition 1.2.29]. Take the γ-radonifying operator $T := K : H \longrightarrow E$ given by Theorem 0.5.9. From $Q = KJK^*$, where $J : H^* \longrightarrow H$ is the canonical conjugate-linear identification operator, and by hypothesis, it follows that

$$\begin{aligned} c\int_I |x^*\big(\phi(t)\big)|^2 \mathrm{d}t = x^*Qx^* &= (x^* \circ K)JK^*(x^*) \\ &= \|K^*(x^*)\|_H^2 = \|T^*(x^*)\|_H^2 \end{aligned}$$

for all $x^* \in E^*$, and (iv) holds.

We now show that (iv) $\Longrightarrow$ (v). Define the conjugate-linear map $Q : E^* \longrightarrow (E^*)'$ by

$$Qx^* : E^* \longrightarrow \mathbb{K}, y^* \longmapsto \int_I y^*\big(\phi(t)\big)\overline{x^*\big(\phi(t)\big)}\mathrm{d}t$$

for all $x^* \in E^*$. It is well-defined since ϕ is weakly L^2, and is clearly a positive symmetric map. Let $J : H^* \longrightarrow H$ be the canonical conjugate-linear identification operator. We finish the proof of the implication in five steps.

(a) By hypothesis, we have for all $x^* \in E^*$ that

$$c\int_I |x^*\big(\phi(t)\big)|^2 \mathrm{d}t \le \|T^*(x^*)\|_H^2 = (T^*(x^*))(JT^*(x^*)) = x^*(TJT^*(x^*)).$$

Hence, setting $R := TJT^* : E^* \longrightarrow E$, we have

$$c(Qx^*)(x^*) \le x^*Rx^* \tag{A.3.1}$$

for all $x^* \in E^*$. By Proposition A.3.6, we conclude that $Q(E^*) \subseteq E$.

(b) By hypothesis and Theorem 0.5.9, R is a Gaussian covariance operator. Then by Theorem 0.5.12, inequality (A.3.1) and since $Q(E^*) \subseteq E$, Q is also a Gaussian covariance operator.

(c) Let us prove that I_ϕ takes values in E. Define $G := \{x^* \circ \phi \mid x^* \in E^*\} \subseteq L^2(I)$. Let $f \in L^2(I)$. Then

$$\begin{aligned} f \in \operatorname{Ker}(I_\phi) &\iff I_\phi(f) = 0 \\ &\iff \forall x^* \in E^*, I_\phi(f)(x^*) = 0 \\ &\iff \forall x^* \in E^*, \int_I x^*(\phi(t))\overline{f(t)}\mathrm{d}t = 0 \\ &\iff f \in G^\perp. \end{aligned}$$

Therefore, $\mathrm{Ker}(I_\phi) = G^\perp$, and $L^2(I) = \overline{G} \oplus \mathrm{Ker}(I_\phi)$. Since $\mathrm{Im}(Q) \subseteq E$ by (a), we get that $I_\phi(G \oplus \mathrm{Ker}(I_\phi)) \subseteq E$.

It remains to prove that $I_\phi(\overline{G}) \subseteq E$. Let $g \in \overline{G}$. There exists a sequence $(g_n)_{n\geq 1} \subseteq G$ converging to g in $L^2(I)$. Since $I_\phi(G) \subseteq E$, for each $n \geq 1$, there exists $x_n \in E$ such that $I_\phi(g_n) = x_n$. We show that $(x_n)_{n\geq 1}$ is Cauchy in E. Let $\|\cdot\|$ be a continuous seminorm on E. For all $x^* \in E^*$ and every integers $n, m \geq 1$, we have

$$\begin{aligned}|x^*(x_n - x_m)| &= |(I_\phi(g_n) - I_\phi(g_m))(x^*)| \\ &= \Big| \int_I x^*(\phi(t))\overline{g_n(t)}\mathrm{d}t - \int_I x^*(\phi(t))\overline{g_m(t)}\mathrm{d}t \Big| \\ &\leq \|x^*\phi\|_{L^2(I)} \|g_n - g_m\|_{L^2(I)},\end{aligned}$$

where the last inequality is justified by the Cauchy-Schwarz inequality. Define the set $V := \{x \in E, \|x\| < 1\}$. By [88, Theorems 3.15 and 3.16], the polar of V is $\sigma(E^*, E)$-compact and metrizable in the $\sigma(E^*, E)$ topology, hence it is sequentially complete and bounded for $\sigma(E^*, E)$. On the other hand, the linear map $(E^*, \sigma(E^*, E)) \longrightarrow L^2(I), x^* \longmapsto x^*\phi$ has a sequentially closed graph. By [94, Lemma 3], we then get $\sup_{x^* \in V^\circ} \|x^*\phi\|_{L^2(I)} < \infty$. Therefore,

$$\|x_n - x_m\| = \sup_{x^* \in V^\circ} |x^*(x_n - x_m)| \leq \sup_{x^* \in V^\circ} \|x^*\phi\|_{L^2(I)} \|g_n - g_m\|_{L^2(I)},$$

the equality holds by [88, Theorem 1.34] and [71, Proposition 22.14]. Since $(g_n)_{n\geq 1}$ is convergent in $L^2(I)$, we conclude that $(x_n)_{n\geq 1}$ is Cauchy for $\|\cdot\|$. Since $\|\cdot\|$ was arbitrary, the sequence $(x_n)_{n\geq 1}$ is thus Cauchy in E. By completeness, there exists $x \in E$ such that $(x_n)_{n\geq 1}$ converges to x in E. Now let $x^* \in E^*$. By continuity of x^*, we have that $\lim_{n\to\infty} I_\phi(g_n)(x^*) = \lim_{n\to\infty} x^*(x_n) = x^*(x)$. On the other hand, by the Cauchy-Schwarz inequality,

$$\big|I_\phi(g_n)(x^*) - I_\phi(g)(x^*)\big| \leq \|x^*\phi\|_{L^2(I)} \|g_n - g\|_{L^2(I)},$$

implying that $\lim_{n\to\infty} I_\phi(g_n)(x^*) = I_\phi(g)(x^*)$. We conclude that $I_\phi(g)(x^*) = x^*(x)$ for all $x^* \in E^*$, and $I_\phi(g) = x \in E$.

(d) We now prove that I_ϕ is continuous. Define the linear map

$$K : E^* \longrightarrow L^2(I), x^* \longmapsto x^* \circ \phi.$$

Let $S : L^2(I)^* \longrightarrow L^2(I)$ be the canonical conjugate-linear identification operator. We first show that $K^* = I_\phi \circ S$. Let $f^* \in L^2(I)^*$ and $x^* \in E^*$. By definition of K^*, we have

$$K^*(f^*)(x^*) = f^*(K(x^*)) = \langle x^* \circ \phi, S(f^*)\rangle_{L^2(I)} = I_\phi(S(f^*))(x^*).$$

This implies that $K^*(f^*) = I_\phi(S(f^*))$, and in turn $K^* = I_\phi \circ S$. By [71, Lemma 23.28], the map $K : (E^*, \sigma(E^*, E)) \longrightarrow (L^2(I), \sigma(L^2(I), L^2(I)^*))$ is continuous if and only if $K^*(L^2(I)^*) \subseteq E$. Since $K^* = I_\phi \circ S$ and I_ϕ takes values in E by (c), K is indeed continuous with respect to the weak topologies. By [71, Lemma 23.29], K^* is continuous with respect to the Mackey topologies. Since $L^2(I)^*$ and E are Mackey spaces by [89, Subsection IV.3.4], that is, their respective original topologies are the Mackey topology, we conclude that $K^* : L^2(I)^* \longrightarrow E$ is continuous. Finally, I_ϕ is also continuous thanks to $K^* = I_\phi \circ S$.

(e) For all $x^*, y^* \in E^*$, one has

$$I_\phi\big(K(x^*)\big)(y^*) = \int_I y^*(\phi(t))\overline{K(x^*)}\mathrm{d}t = \int_I y^*(\phi(t))\overline{x^*(\phi(t))}\mathrm{d}t = y^*Qx^*.$$

Therefore, $(I_\phi \circ S) \circ S^{-1} \circ K = I_\phi \circ K = Q$. We also have $(I_\phi \circ S)^* = K$. Since Q is a covariance operator by (b), Lemma 0.5.10 applied to $I_\phi \circ S$ and $L^2(I)^*$ then implies that I_ϕ is γ-radonifying. Note that $L^2(I)$ is separable by [50, Proposition 1.2.29].

We have thus proved (v).

It remains to show (v) $\implies$ (i). Let $A \in \mathscr{B}(I)$ and define $\phi_A := \mathbf{1}_A\phi$ and

$$M_A : L^2(I) \longrightarrow L^2(I), f \longmapsto \mathbf{1}_A f.$$

For all $f \in L^2(I)$ and $x^* \in E^*$, we have

$$I_{\phi_A}(f)(x^*) = \int_I x^*(\phi_A(t))\overline{f(t)}\mathrm{d}t = \int_A x^*(\phi(t))\overline{f(t)}\mathrm{d}t = I_\phi(M_A(f))(x^*).$$

This shows that $I_{\phi_A} = I_\phi \circ M_A$. Since M_A is obviously continuous, the map I_{ϕ_A} is γ-radonifying by the ideal property, see Theorem A.2.7.

Let $(f_n)_{n\in\mathbb{N}} \subseteq L^2(I)$ be an orthonormal basis. Denote by

$$\mathcal{I} : L^2(I) \longrightarrow L^2(\Omega), f \longmapsto \int_I f(t)\mathrm{d}B_t$$

the Itô isometry. Since $(\mathcal{I}(f_n))_{n\in\mathbb{N}}$ is a standard Gaussian sequence and I_{ϕ_A} is γ-radonifying, the random series $Y_A := \sum_{n\geq 0}\mathcal{I}(f_n)I_{\phi_A}(f_n)$ converges almost surely. Let $x^* \in E^*$, we have

$$\begin{aligned}
x^*(Y_A) &= \sum_{n=0}^{\infty}\mathcal{I}(f_n)x^*(I_{\phi_A}f_n) = \sum_{n=0}^{\infty}\mathcal{I}\Big(x^*(I_{\phi_A}f_n)f_n\Big) \\
&= \sum_{n=0}^{\infty}\mathcal{I}\Big(\langle x^* \circ \phi_A, f_n\rangle_{L^2(I)}f_n\Big) = \mathcal{I}\Big(\sum_{n=0}^{\infty}\langle x^* \circ \phi_A, f_n\rangle_{L^2(I)}f_n\Big) \\
&= \int_I x^*\phi_A(t)\mathrm{d}B_t
\end{aligned}$$

almost surely, where we have used the continuity of x^* for the first equality, the definition of I_{ϕ_A} for the third one, the linearity and continuity of $\mathcal{I}$ in the fourth one, and the fact that $(f_n)_{n\in\mathbb{N}}$ is an orthonormal basis and the definition of $\mathcal{I}$ for the last one. We then deduce that $x^*(Y_A) = \int_I x^*\phi_A(t)\mathrm{d}B_t$ almost surely, for every $x^* \in E^*$. In conclusion, ϕ is stochastically integrable.

As for the Pettis integrability of ϕ, let $A \in \mathscr{B}(I)$ be a set of finite measure. Let $x^* \in E^*$. Then $\mathbf{1}_A \in L^2(I)$, $I_\phi(\mathbf{1}_A) \in E$, and

$$x^*(I_\phi(\mathbf{1}_A)) = I_\phi(\mathbf{1}_A)(x^*) = \int_I x^*(\phi(t))\mathbf{1}_A(t)\mathrm{d}t,$$

and ϕ is Pettis integrable on A. $\square$

We state two results from [96], adapted to the case of functions defined on an arbitrary interval and taking values in a Fréchet space. They will be used in the proof of Theorem 2.1.6.

Theorem A.3.7 ([96, Corollary 2.7])**.** *Let $\phi, \psi : I \longrightarrow E$ be two weakly measurable functions on an interval $I \subseteq \mathbb{R}$. Assume that ϕ is stochastically integrable and that*

$$\int_I |x^*\psi(t)|^2 \mathrm{d}t \leq \int_I |x^*\phi(t)|^2 \mathrm{d}t$$

for every $x^* \in E^*$. Then ψ is stochastically integrable and for all $1 \leq p < \infty$ and all continuous seminorms $\|\cdot\|$ on E, we have

$$\mathbb{E}\Big(\Big\| \int_I \psi(t)\mathrm{d}B_t \Big\|^p\Big) \leq \mathbb{E}\Big(\Big\| \int_I \phi(t)\mathrm{d}B_t \Big\|^p\Big).$$

Proof. By assumption and (iv) of Theorem 2.1.4, we get for all $x^* \in E^*$ that

$$c\int_I |x^*\psi(t)|^2 \mathrm{d}t \leq c\int_I |x^*\phi(t)|^2 \mathrm{d}t \leq \|T^*x^*\|_H^2.$$

Again by (iv) of Theorem 2.1.4, we can conclude that ψ is stochastically integrable.

By (iii) of Theorem 2.1.4 applied to both ϕ and ψ, we have $x^*Rx^* \leq x^*Qx^*$ for all $x^* \in E^*$, where R and Q are the covariance operators of the distributions μ_R and μ_Q of $\int_I \psi(t)\mathrm{d}B_t$ and $\int_I \phi(t)\mathrm{d}B_t$, respectively. By Theorem 0.5.12, we finally get that

$$\int_E \|x\|^p \mathrm{d}\mu_R(x) \leq \int_E \|x\|^q \mathrm{d}\mu_Q(x).$$

$\square$

The next result is a dominated convergence theorem for the stochastic integral.

Theorem A.3.8 ([96, Theorem 6.2])**.** *Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of E-valued stochastically integrable functions defined on an interval $I \subseteq \mathbb{R}$. Assume that there exists a weakly L^2 function $\phi : I \longrightarrow E$ such that for every $x^* \in E^*$, one has*

$$\lim_{n\to\infty} \int_I |x^*\phi_n(t) - x^*\phi(t)|^2 \mathrm{d}t = 0.$$

If there exists a stochastically integrable function $\psi : I \longrightarrow E$ such that for every $x^* \in E^*$ and all $n \geq 0$, one has

$$\int_I |x^*\phi_n(t)|^2 \mathrm{d}t \leq \int_I |x^*\psi(t)|^2 \mathrm{d}t,$$

then ϕ is stochastically integrable and for every $1 \le p < \infty$, we have, in $L^p(\Omega; E)$,

$$\lim_{n\to\infty} \int_I \phi_n(t)\mathrm{d}B_t = \int_I \phi(t)\mathrm{d}B_t.$$

Proof. By using the assumptions, we get that

$$\int_I |x^*\phi(t)|^2 \mathrm{d}t \le \int_I |x^*\psi(t)|^2 \mathrm{d}t$$

for all $x^* \in E^*$. By Theorem A.3.7, ϕ is stochastically integrable.

The rest of the proof follows as in [96, Theorem 6.2]. □

Proof of Theorem 2.1.6. Assume that such a sequence of step functions exists. Let $x^* \in E^*$. By Lemma 2.1.2, we have $x^*(Y) = \lim_{n\to\infty} \int_I x^*\phi_n(t)\mathrm{d}B_t$ in probability. The random vectors on the right-hand side are Gaussian, and thus the convergence takes place in $L^2(\Omega)$ by [86, Lemma 2.1]. By the Itô isometry, the sequence $(x^*\phi_n)_{n\in\mathbb{N}}$ is Cauchy in $L^2(I)$. Thus by (i), we necessarily have $\lim_{n\to\infty} x^*\phi_n = x^*\phi$ in $L^2(I)$. Again by the Itô isometry, we get that $x^*(Y) = \int_I x^*\phi(t)\mathrm{d}B_t$ almost surely. By (ii) of Theorem 2.1.4, we conclude that ϕ is stochastically integrable on I with integral Y.

Assume that ϕ is stochastically integrable. If I is a bounded interval, the result is proved in [96, Theorem 2.5] when ϕ is a Banach space-valued function, but the proof carries over verbatim to the case of Fréchet spaces. It I is unbounded, for the sake of simplicity, we assume that $I = [0, \infty[$. By Theorem 2.1.4, ϕ is Pettis integrable on any bounded intervals. For all integers $k \ge 0$, $n \ge 1$, $1 \le j \le 2^n$ and $N \ge 1$, define

$$\phi_{n,k,j} := 2^n \int_{k+\frac{j-1}{2^n}}^{k+\frac{j}{2^n}} \phi(t)\mathrm{d}t,$$

$$\phi_n^N := \sum_{k=0}^{N-1} \sum_{j=1}^{2^n} \mathbf{1}_{]k+\frac{j-1}{2^n}, k+\frac{j}{2^n}]} \phi_{n,k,j},$$

and $\varphi_N := \phi \mathbf{1}_{[0,N]}$. For all $n, N \in \mathbb{N}_0$, define also $\mathcal{G}_{n,N}$ as the finite σ-algebra on $[0, N]$ generated by the intervals of the form $]k + \frac{j-1}{2^n}, k + \frac{j}{2^n}]$, $0 \le k \le N-1$, $1 \le j \le 2^n$.

Let $x^* \in E^*$. It is an easy task to check that $\mathbb{E}(x^*\varphi_N \mid \mathcal{G}_{n,N}) = x^*\phi_n^N$ almost everywhere on $[0, N]$, for every $n, N \ge 1$, see [18, Example 10.1.2]. Since $(\mathcal{G}_{n,N})_{n\ge1}$ is increasing and $\sigma\big(\bigcup_{n\ge1} \mathcal{G}_{n,N}\big) = \mathcal{B}([0, N])$, we get that $\lim_{n\to\infty} x^*\phi_n^N = x^*\varphi_N$ in $L^2([0, N]; \mathbb{R})$ by [50, Theorem 3.3.2].

Now, define the step functions $\phi_N := \phi_N^N \mathbf{1}_{[0,N]}$ for every integer $N \ge 1$. Let us show that $\lim_{N\to\infty} x^*\phi_N = x^*\phi$ in $L^2([0, \infty[)$. Notice that $\phi_N^M = \phi_N^N$ on $[0, M]$ for any $N \ge M \ge 1$. Let $\varepsilon > 0$, and let $N_0 \ge 1$ be such that $\int_N^\infty |x^*\phi(t)|^2 \mathrm{d}t \le \varepsilon^2$ for every $N \ge N_0$. Now let $M_0 \ge N_0$ be such that $\|x^*\phi_N^{N_0} - x^*\varphi_{N_0}\|_{L^2([0,N_0])} \le \varepsilon$ for every integer $N \ge M_0$. We then have for any $N \ge M_0$,

$$\begin{aligned}\|x^*\phi_N - x^*\phi\|_{L^2([0,\infty[)} \le{}& \|x^*\phi_N^{N_0} - x^*\varphi_{N_0}\|_{L^2([0,N_0])} \\ &+ \|x^*\phi_N - x^*\phi\|_{L^2([N_0,N])} + \|x^*\phi\|_{L^2([N,\infty[)}.\end{aligned}$$

The first and third terms are smaller than ε since $N \geq M_0 \geq N_0$. For the second term, first note that

$$\|x^*\phi_N - x^*\phi\|_{L^2([N_0,N])} \leq \|x^*\phi_N\|_{L^2([N_0,N])} + \|x^*\phi\|_{L^2([N_0,\infty[)}.$$

The second term of the right-hand side is smaller than ε by definition of N_0. For the first term, notice that $x^*\phi_N = x^*\phi_N^N = \mathbb{E}(x^*\phi \mid \mathcal{G}_N)$ almost everywhere on $[N_0, N]$, where $\mathcal{G}_N$ is the finite σ-algebra on $[N_0, N]$ generated by the intervals of the form $]k + \frac{j-1}{2^N}, k + \frac{j}{2^N}]$, $N_0 \leq k \leq N-1$, $1 \leq j \leq 2^N$; see [18, Example 10.1.2]. Therefore, by a corollary of the conditional Jensen inequality, see [50, Corollary 2.6.30], we have $\|x^*\phi_N\|_{L^2([N_0,N])} \leq \|x^*\phi\|_{L^2([N_0,N])}$. We then conclude that $\|x^*\phi_N - x^*\phi\|_{L^2([0,\infty[)} \leq 4\varepsilon$, and thus $\lim_{N\to\infty} x^*\phi_N = x^*\phi$ in $L^2([0,\infty[)$

Again by [50, Corollary 2.6.30], we have $\|x^*\phi_N\|_{L^2([0,\infty[)} \leq \|x^*\phi\|_{L^2([0,\infty[)}$ for every $N \geq 1$. Since every step function is stochastically integrable, Theorem A.3.8 yields (ii) with convergence in every $L^p(\Omega; E)$, $1 \leq p < \infty$. □

Notations

Numbers

$\lfloor x \rfloor$	largest $n \in \mathbb{N}$ such that $n \leq x < n+1$
$\lceil x \rceil$	smallest $n \in \mathbb{N}$ such that $n-1 < x \leq n$
$\lvert z \rvert$	modulus of the complex number z
$\mathrm{Im}(z)$	imaginary part of z
$\mathrm{Re}(z)$	real part of z

Sets

$\lvert A \rvert$	number of elements of the set A
$\complement A$	complement of the set A
$\underline{\mathrm{dens}}(A)$	lower density of A
$\mathbf{1}_A$	characteristic function of the set A
$\mathbb{C}$	set of complex numbers
$\mathbb{D}$	open unit disk
$\mathbb{K}$	scalar field
$\mathbb{N}$	set of natural integers $0, 1, 2, \ldots$
$\mathbb{N}_0$	set of positive integers $1, 2, \ldots$
$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of integers

Vector spaces

c_0	space of null sequences
$H(\mathbb{C})$	space of entire functions
$H(\mathbb{D})$	space of holomorphic functions on $\mathbb{D}$
$\mathcal{H}(\mathbb{R}^2)$	space of harmonic functions on $\mathbb{R}^2$
$\lambda^p(A)$	Köthe sequence space of order p
ℓ_p^N	$\mathbb{R}^N$ endowed with $\lVert \cdot \rVert_p$
ℓ^p	space of p-summable sequences

Functional analysis

$\|\cdot\|_E$	norm on the vector space E
$\|\cdot\|_p$	p-norm, $1 \leq p \leq \infty$
$\langle\cdot,\cdot\rangle_H$	inner product on H (linear in the first argument)
E'	algebraic dual of the vector space E
E^*	topological dual of the Fréchet space E
$F^\perp$	orthogonal complement of F
Im	image of a function
Ker	kernel of a linear map
$\sigma(E, E^*)$	weak topology on E
$\sigma(E^*, E)$	weak topology on E^*
span	linear span
$\tau(E, E^*)$	Mackey topology on E
$\tau(E^*, E)$	Mackey topology on E^*
V°	polar of the set V

Measure theory and Probability

$\mathscr{B}(E)$	Borel sets of a topological space E
$\mathbb{E}(X)$	expectation of the random variable X
$\mathbb{E}(\cdot \mid \cdot)$	conditional expectation
Id	identity map
i.i.d.	independent and identically distributed
$L^p(S, \mathcal{A}, \mu; E)$	equivalence classes of p-integrable functions $f : S \longrightarrow E$
$L^p(S; E)$	the space $L^p(S, \mathcal{A}, \mu; E)$
$L^p(S, \mathcal{A}, \mu) = L^p(S)$	the space $L^p(S, \mathcal{A}, \mu; E)$ if $E = \mathbb{R}$ or $E = \mathbb{C}$
$L^p(S, \mu)$	the space $L^p(S)$
$\widehat{\mu}$	characteristic functional of μ
$\mathbb{P}$	probability measure

Miscellaneous

$\|f\|_r$	$\sup_{\|z\| \leq r} \|f(z)\|$ for a function f
decreasing	strictly decreasing
increasing	strictly increasing
negative	strictly negative
$o(\cdot)$	little o notation
$O(\cdot)$	big o notation
positive	strictly positive

Bibliography

[1] Radosław Adamczak, Joscha Prochno, Marta Strzelecka, and Michał Strzelecki, *Norms of structured random matrices*, Mathematische Annalen (2023).

[2] M. P. Aldred and D. H. Armitage, *Harmonic analogues of G. R. Mac Lane's universal functions. II*, J. Math. Anal. Appl. **220** (1998), no. 1, 382–395.

[3] M. P. Aldred and D. H. Armitage, *Harmonic analogues of G. R. MacLane's universal functions*, J. London Math. Soc. (2) **57** (1998), no. 1, 148–156.

[4] Richard Aron and Dinesh Markose, *On universal functions*, 2004, pp. 65–76. Satellite Conference on Infinite Dimensional Function Theory.

[5] Sheldon Axler, Paul Bourdon, and Wade Ramey, *Harmonic function theory*, second edition, Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, 2001.

[6] Catalin Badea and Sophie Grivaux, *Unimodular eigenvalues, uniformly distributed sequences and linear dynamics*, Adv. Math. **211** (2007), no. 2, 766–793.

[7] Frédéric Bayart and Sophie Grivaux, *Hypercyclicité: le rôle du spectre ponctuel unimodulaire*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 9, 703–708.

[8] Frédéric Bayart and Sophie Grivaux, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc. **358** (2006), no. 11, 5083–5117.

[9] Frédéric Bayart and Sophie Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 1, 181–210.

[10] Frédéric Bayart and Étienne Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, 2009.

[11] Frédéric Bayart and Étienne Matheron, *Mixing operators and small subsets of the circle*, J. Reine Angew. Math. **715** (2016), 75–123.

[12] Frédéric Bayart and Imre Z. Ruzsa, *Difference sets and frequently hypercyclic weighted shifts*, Ergodic Theory Dynam. Systems **35** (2015), no. 3, 691–709.

[13] B. Beauzamy, *Un opérateur, sur un espace de Banach, avec un ensemble non-dénombrable, dense, de vecteurs hypercycliques*, Seminar on the geometry of Banach spaces, Vol. I, II (Paris, 1983), 1984, pp. 149–175.

[14] Teresa Bermúdez, Antonio Bonilla, and Antonio Martinón, *On the existence of chaotic and hypercyclic semigroups on Banach spaces*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2435–2441.

[15] Luis Bernal-González and Antonio Bonilla, *Rate of growth of hypercyclic and frequently hypercyclic functions for the Dunkl operator*, Mediterr. J. Math. **13** (2016), no. 5, 3359–3372.

[16] G.D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, Acad. Sci. Paris **189** (1929), 473–475.

[17] O. Blasco, A. Bonilla, and K.-G. Grosse-Erdmann, *Rate of growth of frequently hypercyclic functions*, Proc. Edinb. Math. Soc. (2) **53** (2010), no. 1, 39–59.

[18] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007.

[19] Vladimir I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[20] José Bonet, Félix Martínez-Giménez, and Alfredo Peris, *A Banach space which admits no chaotic operator*, Bull. London Math. Soc. **33** (2001), no. 2, 196–198.

[21] José Bonet and Alfredo Peris, *Hypercyclic operators on non-normable Fréchet spaces*, J. Funct. Anal. **159** (1998), no. 2, 587–595.

[22] A. Bonilla and K.-G. Grosse-Erdmann, *Frequently hypercyclic operators and vectors*, Ergodic Theory Dynam. Systems **27** (2007), no. 2, 383–404.

[23] Johann Boos, *Classical and modern methods in summability*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. Assisted by Peter Cass, Oxford Science Publications.

[24] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart, *Concentration inequalities*, Oxford University Press, Oxford, 2013. A nonasymptotic theory of independence, With a foreword by Michel Ledoux.

[25] Valery Buldygin and Serguei Solntsev, *Asymptotic behaviour of linearly transformed sums of random variables*, Mathematics and its Applications, vol. 416, Kluwer Academic Publishers Group, Dordrecht, 1997. Translated from the 1989 Russian original by Vladimir Zaiats and revised, updated and expanded by the authors.

[26] M. Chakir and S. EL Mourchid, *Strong mixing Gaussian measures for chaotic semigroups*, J. Math. Anal. Appl. **459** (2018), no. 2, 778–788.

[27] Stéphane Charpentier, Karl Grosse-Erdmann, and Quentin Menet, *Chaos and frequent hypercyclicity for weighted shifts*, Ergodic Theory Dynam. Systems **41** (2021), no. 12, 3634–3670.

[28] José A. Conejero, *On the existence of transitive and topologically mixing semigroups*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 3, 463–471.

[29] Jose A. Conejero, V. Müller, and A. Peris, *Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup*, J. Funct. Anal. **244** (2007), no. 1, 342–348.

[30] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ, *Ergodic theory*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 245, Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.

[31] Yves Coudène, *Ergodic theory and dynamical systems*, Universitext, Springer-Verlag London, Ltd., London; EDP Sciences, [Les Ulis], 2016. Translated from the 2013 French original by Reinie Erné.

[32] Karma Dajani and Charlene Kalle, *A first course in ergodic theory*, first edition, Chapman and Hall/CRC, 2021.

[33] Jean Dieudonné, *Calcul infinitésimal*, second edition, Hermann, Paris, 1980.

[34] David Drasin and Eero Saksman, *Optimal growth of entire functions frequently hypercyclic for the differentiation operator*, J. Funct. Anal. **263** (2012), no. 11, 3674–3688.

[35] S. EL Mourchid and K. Latrach, *On the ergodic approach for the study of chaotic linear infinite-dimensional systems*, Differential Integral Equations **26** (2013), no. 11-12, 1321–1333.

[36] Klaus-Jochen Engel and Rainer Nagel, *A short course on operator semigroups*, Universitext, Springer, New York, 2006.

[37] P. Erdős and A. Rényi, *On random entire functions*, Zastos. Mat. **10** (1969), 47–55.

[38] Romuald Ernst and Augustin Mouze, *A quantitative interpretation of the frequent hypercyclicity criterion*, Ergodic Theory Dynam. Systems **39** (2019), no. 4, 898–924.

[39] Romuald Ernst and Augustin Mouze, *Frequent universality criterion and densities*, Ergodic Theory Dynam. Systems **41** (2021), no. 3, 846–868.

[40] Irene Fonseca and Giovanni Leoni, *Modern methods in the calculus of variations: L^p spaces*, Springer Monographs in Mathematics, Springer, New York, 2007.

[41] Clifford Gilmore, Eero Saksman, and Hans-Olav Tylli, *Optimal growth of harmonic functions frequently hypercyclic for the partial differentiation operator*, Proc. Roy. Soc. Edinburgh Sect. A **149** (2019), no. 6, 1577–1594.

[42] Gilles Godefroy and Joel H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), no. 2, 229–269.

[43] B. Goldys and J. M. A. M. van Neerven, *Transition semigroups of Banach space-valued Ornstein-Uhlenbeck processes*, Acta Appl. Math. **76** (2003), no. 3, 283–330.

[44] Sophie Grivaux, *A probabilistic version of the frequent hypercyclicity criterion*, Studia Math. **176** (2006), no. 3, 279–290.

[45] K.-G. Grosse-Erdmann, *Hypercyclic and chaotic weighted shifts*, Studia Math. **139** (2000), no. 1, 47–68.

[46] Karl-G. Grosse-Erdmann, *Rate of growth of hypercyclic entire functions*, Indag. Math. (N.S.) **11** (2000), no. 4, 561–571.

[47] Karl-G. Grosse-Erdmann and Alfredo Peris Manguillot, *Linear chaos*, Universitext, Springer, London, 2011.

[48] W. K. Hayman, *The local growth of power series: a survey of the Wiman-Valiron method*, Canad. Math. Bull. **17** (1974), no. 3, 317–358.

[49] W. K. Hayman, *Subharmonic functions. Vol. 2*, London Mathematical Society Monographs, vol. 20, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1989.

[50] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis, *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 63, Springer, Cham, 2016.

[51] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis, *Analysis in Banach spaces. Vol. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 67, Springer, Cham, 2017. Probabilistic methods and operator theory.

[52] Gerhard Jank and Lutz Volkmann, *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, UTB für Wissenschaft: Grosse Reihe. [UTB for Science: Large Series], Birkhäuser Verlag, Basel, 1985.

[53] Cvetan Jardas and Nikola Sarapa, *A summability method in some strong laws of large numbers*, Math. Commun. **2** (1997), no. 2, 107–124.

[54] J.-P. Kahane, *Propriétés locales des fonctions à séries de Fourier aléatoires*, Studia Math. **19** (1960), 1–25.

[55] Jean-Pierre Kahane, *Some random series of functions*, second edition, Cambridge Studies in Advanced Mathematics, vol. 5, Cambridge University Press, Cambridge, 1985.

[56] N. J. Kalton, N. T. Peck, and James W. Roberts, *An F-space sampler*, London Mathematical Society Lecture Note Series, vol. 89, Cambridge University Press, Cambridge, 1984.

[57] P. K. Kamthan and Manjul Gupta, *Sequence spaces and series*, Lecture Notes in Pure and Applied Mathematics, vol. 65, Marcel Dekker, Inc., New York, 1981.

[58] Carol Kitai, *Invariant closed sets for linear operators*, ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)–University of Toronto (Canada).

[59] Achim Klenke, *Probability theory—a comprehensive course*, third edition, Universitext, Springer, Cham, 2020.

[60] Takako Kōmura, *Semigroups of operators in locally convex spaces*, J. Functional Analysis **2** (1968), 258–296.

[61] A. O. Kurilyak, O. B. Skaskīv, and L. O. Shapovalovs'ka, *A Wiman-type inequality for functions analytic in a polydisc*, Ukraïn. Mat. Zh. **68** (2016), no. 1, 78–86.

[62] A. Kuryliak, *Subnormal independent random variables and Levy's phenomenon for entire functions*, Mat. Stud. **47** (2017), no. 1, 10–19.

[63] Andriy Kuryliak, Oleh Skaskiv, and Severyn Skaskiv, *Lévy's phenomenon for analytic functions on a polydisc*, Eur. J. Math. **6** (2020), no. 1, 138–152.

[64] Diego S. Ledesma, *Stochastic calculus on Fréchet spaces*, Adv. Oper. Theory **6** (2021), no. 1, Paper No. 22, 31.

[65] Fernando León-Saavedra and Pilar Romero-de la Rosa, *Growth of hypercyclic entire functions for some non-convolution operators*, 2000. Preprint.

[66] Daniel Li and Hervé Queffélec, *Introduction à l'étude des espaces de Banach*, Cours Spécialisés [Specialized Courses], vol. 12, Société Mathématique de France, Paris, 2004. Analyse et probabilités. [Analysis and probability theory].

[67] Artur O. Lopes, Ali Messaoudi, Manuel Stadlbauer, and Victor Vargas, *Invariant probabilities for discrete time linear dynamics via thermodynamic formalism*, Nonlinearity **34** (2021), no. 12, 8359–8391.

[68] G. R. MacLane, *Sequences of derivatives and normal families*, J. Analyse Math. **2** (1952), 72–87.

[69] Elisabetta M. Mangino and Marina Murillo-Arcila, *Frequently hypercyclic translation semigroups*, Studia Math. **227** (2015), no. 3, 219–238.

[70] Elisabetta M. Mangino and Alfredo Peris, *Frequently hypercyclic semigroups*, Studia Math. **202** (2011), no. 3, 227–242.

[71] Reinhold Meise and Dietmar Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press, Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

[72] Quentin Menet, *Linear chaos and frequent hypercyclicity*, Trans. Amer. Math. Soc. **369** (2017), no. 7, 4977–4994.

[73] T. K. Subrahmonian Moothathu, *Two remarks on frequent hypercyclicity*, J. Math. Anal. Appl. **408** (2013), no. 2, 843–845.

[74] A. Mouze and V. Munnier, *On random frequent universality*, J. Math. Anal. Appl. **412** (2014), no. 2, 685–696.

[75] Augustin Mouze and Vincent Munnier, *Frequent hypercyclicity of random holomorphic functions for Taylor shifts and optimal growth*, J. Anal. Math. **143** (2021), no. 2, 615–637.

[76] Augustin Mouze and Vincent Munnier, *Growth of frequently hypercyclic functions for some weighted Taylor shifts on the unit disc*, Canad. Math. Bull. **64** (2021), no. 2, 264–281.

[77] M. Murillo-Arcila and A. Peris, *Strong mixing measures for linear operators and frequent hypercyclicity*, J. Math. Anal. Appl. **398** (2013), no. 2, 462–465.

[78] M. Murillo-Arcila and A. Peris, *Strong mixing measures for C_0-semigroups*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **109** (2015), no. 1, 101–115.

[79] Santiago Muro, Damián Pinasco, and Martín Savransky, *Hypercyclic behavior of some non-convolution operators on $H(\mathbb{C}^N)$*, J. Operator Theory **77** (2017), no. 1, 39–59.

[80] Miika Nikula, *Frequent hypercyclicity of random entire functions for the differentiation operator*, Complex Anal. Oper. Theory **8** (2014), no. 7, 1455–1474.

[81] Bernt Øksendal, *Stochastic differential equations*, fifth edition, Universitext, Springer-Verlag, Berlin, 1998. An introduction with applications.

[82] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark (eds.), *NIST handbook of mathematical functions*, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[83] Sunao Ōuchi, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan **25** (1973), 265–276.

[84] S. Rolewicz, *On orbits of elements*, Studia Math. **32** (1969), 17–22.

[85] P. C. Rosenbloom, *Probability and entire functions*, Studies in mathematical analysis and related topics, 1962, pp. 325–332.

[86] Jan Rosiński and Zdzisław Suchanecki, *On the space of vector-valued functions integrable with respect to the white noise*, Colloq. Math. **43** (1980), no. 1, 183–201 (1981).

[87] Walter Rudin, *Real and complex analysis*, third edition, McGraw-Hill Book Co., New York, 1987.

[88] Walter Rudin, *Functional analysis*, second edition, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[89] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, second edition, Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, 1999.

[90] Stanislav Shkarin, *On the spectrum of frequently hypercyclic operators*, Proc. Amer. Math. Soc. **137** (2009), no. 1, 123–134.

[91] Stanislav Shkarin, *Hypercyclic and mixing operator semigroups*, Proc. Edinb. Math. Soc. (2) **54** (2011), no. 3, 761–782.

[92] J. Michael Steele, *Sharper Wiman inequality for entire functions with rapidly oscillating coefficients*, J. Math. Anal. Appl. **123** (1987), no. 2, 550–558.

[93] G. Erik F. Thomas, *Integration of functions with values in locally convex Suslin spaces*, Trans. Amer. Math. Soc. **212** (1975), 61–81.

[94] N. N. Vakhania and V. I. Tarieladze, *Covariance operators of probability measures in locally convex spaces*, Theory of Probability & Its Applications **23** (1978), no. 1, 1–21.

[95] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, *Probability distributions on Banach spaces*, Mathematics and its Applications (Soviet Series), vol. 14, D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian and with a preface by Wojbor A. Woyczynski.

[96] J. M. A. M. van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. **166** (2005), no. 2, 131–170.

[97] Jan van Neerven, *γ-radonifying operators—a survey*, The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, 2010, pp. 1–61.

[98] Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.

[99] Kôsaku Yosida, *Functional analysis*, sixth edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 123, Springer-Verlag, Berlin-New York, 1980.