Quantitative Reachability Stackelberg-Pareto Synthesis is NEXPTIME-Complete*

Thomas Brihaye ⊠�©

University of Mons, Belgium

Véronique Bruyère 🖂 🧥 📵

University of Mons, Belgium

Gaspard Reghem

□

ENS Paris-Saclay, Université Paris-Saclay, France

Abstract

In this paper, we deepen the study of two-player Stackelberg games played on graphs in which Player 0 announces a strategy and Player 1, having several objectives, responds rationally by following plays providing him Pareto-optimal payoffs given the strategy of Player 0. The Stackelberg-Pareto synthesis problem, asking whether Player 0 can announce a strategy which satisfies his objective, whatever the rational response of Player 1, has been recently investigated for ω -regular objectives. We solve this problem for weighted graph games and quantitative reachability objectives such that Player 0 wants to reach his target set with a total cost less than some given upper bound. We show that it is NEXPTIME-complete, as for Boolean reachability objectives.

2012 ACM Subject Classification Software and its engineering \rightarrow Formal methods; Theory of computation \rightarrow Logic and verification; Theory of computation \rightarrow Solution concepts in game theory

Keywords and phrases Two-player Stackelberg games played on graphs, Strategy synthesis, Quantitative reachability objectives, Pareto-optimal costs

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

Formal verification, and more specifically model-checking, is a branch of computer science which offers techniques to check automatically whether a system is correct [3, 18]. This is essential for systems responsible for critical tasks like air traffic management or control of nuclear power plants. Much progress has been made in model-checking both theoretically and in tool development, and the technique is now widely used in industry.

Nowadays, it is common to face more complex systems, called *multi-agent systems*, that are composed of heterogeneous components, ranging from traditional pieces of reactive code, to wholly autonomous robots or human users. Modelling and verifying such systems is a challenging problem that is far from being solved. One possible approach is to rely on *game theory*, a branch of mathematics that studies mathematical models of interaction between agents and the understanding of their decisions assuming that they are *rational* [32, 38]. Typically, each agent (i.e. player) composing the system has his own objectives or preferences, and the way he manages to achieve them is influenced by the behavior of the other agents.

Rationality can be formalized in several ways. A famous model of agents' rational behavior is the concept of *Nash equilibrium* (NE) [31] in a multiplayer non-zero sum game graph that represents the possible interactions between the players [36]. Another model is the one of *Stackelberg games* [39], in which one designated player – the leader, announces a strategy to achieve his goal, and the other players – the followers, respond rationally with an optimal

^{*} Thomas Brihaye – Partly supported by the F.R.S.- FNRS under grant n°T.0027.21. Véronique Bruyère – Partly supported by the F.R.S.- FNRS under grant n°T.0023.22.

response depending on their goals (e.g. with an NE). This framework is well-suited for the verification of correctness of a controller intending to enforce a given property, while interacting with an environment composed of several agents each having his own objective. In practical applications, a strategy for interacting with the environment is committed before the interaction actually happens.

Our contribution. In this paper, we investigate the recent concept of two-player Stackelberg games, where the environment is composed of one player aiming at satisfying several objectives, and its related Stackelberg-Pareto synthesis (SPS) problem [12, 14]. In this framework, for Boolean objectives, given the strategy announced by the leader, the follower responses rationally with a strategy that ensures him a vector of Boolean payoffs that is Pareto-optimal, that is, with a maximal number of satisfied objectives. This setting encompasses scenarios where, for instance, several components of the environment can collaborate and agree on trade-offs. The SPS problem is to decide whether the leader can announce a strategy that guarantees him to satisfy his own objective, whatever the rational response of the follower.

The SPS problem has been solved in [14] for ω -regular objectives. We here solve this problem for weighted game graphs and quantitative reachability objectives for both players. Given a target of vertices, the goal is to reach this target with a cost as small as possible. In this quantitative context, the follower responds to the strategy of the leader with a strategy that ensures him a Pareto-optimal cost vector given his series of targets. The aim of the leader is to announce a strategy in a way to reach his target with a total cost less than some given upper bound, whatever the rational response of the follower. We show that the SPS problem is NEXPTIME-complete (Theorem 2), as for Boolean reachability objectives.

It is well-known that moving from Boolean objectives to quantitative ones allows to model richer properties. This paper is a first step in this direction for the SPS problem for two-player Stackelberg games with multiple objectives for the follower. Our proof follows the same pattern as for Boolean reachability [14]: if there is a solution to the SPS problem, then there is one that is finite-memory whose memory size is at most exponential. The non-deterministic algorithm thus guesses such a strategy and checks whether it is a solution. However, a crucial intermediate step is to prove that if there exists a solution, then there exists one whose Pareto-optimal costs for the follower are exponential in the size of the instance (Theorem 6). The proof of this non trivial step (which is meaningless in the Boolean case) is the main contribution of the paper. Given a solution, we first present some hypotheses and techniques that allow to locally modify it into a solution with smaller Pareto-optimal cost vectors. We then conduct a proof by induction on the number of follower's targets, to globally decrease the cost vectors and to get an exponential number of Pareto-optimal cost vectors. The NEXPTIME-hardness of the SPS problem is trivially obtained by reduction from this problem for Boolean reachability. Indeed, the Boolean version is equivalent to the quantitative one with all weights put to zero and with the given upper bound equal to zero. Notice that the two versions differ: we exhibit an example of game that has a solution to the SPS problem for quantitative reachability, but none for Boolean reachability.

Related work. During the last decade, multiplayer non-zero sum games and their applications to reactive synthesis have raised a growing attention, see for instance the surveys [4, 11, 24]. When several players (like the followers) play with the aim to satisfy their objectives, several solution concepts exist such as NE, subgame perfect equilibrium (SPE) [33], secure equilibria [16, 17], or admissibility [2, 5]. Several results have been obtained, for Boolean and quantitative objectives, about the constrained existence problem which consists in deciding whether there exists a solution concept such that the payoff obtained by each player is larger than some given threshold. Let us mention [19, 36, 37] for results on the

constrained existence for NEs and [7, 8, 10, 35] for SPEs. Some of them rely on a recent elegant characterization of SPE outcomes [6, 22].

Stackelberg games with several followers have been recently studied in the context of rational synthesis: in [21] in a setting where the followers are cooperative with the leader, and later in [29] where they are adversarial. Rational responses of the followers are, for instance, to play an NE or an SPE. The rational synthesis problem and the SPS problem are incomparable, as illustrated in [34, Section 4.3.2]: in rational synthesis, each component of the environment acts selfishly, whereas in SPS, the components cooperate in a way to obtain a Pareto-optimal cost. In [30], the authors solve the rational synthesis problem that consists in deciding whether the leader can announce a strategy satisfying his objective, when the objectives of the players are specified by LTL formulas. Complexity classes for various ω -regular objectives are established in [19] for both cooperative and adversarial settings. Extension to quantitative payoffs, like mean-payoff or discounted sum, is studied in [25, 26] in the cooperative setting and in [1, 20] in the adversarial settings.

The concept of rational verification has been introduced in [27], where instead of deciding the existence of a strategy for the leader, one verifies that some given leader's strategy satisfies his objective, whatever the NE responses of the followers. An algorithm and its implementation in the EVE system are presented in [27] for objectives specified by LTL formulas. This verification problem is studied in [28] for mean-payoff objectives for the followers and an omega-regular objective for the leader, and it is solved in [9] for both NE and SPE responses of the followers and for a variety of objectives including quantitative objectives. The Stackelberg-Pareto verification problem is solved in [15] for some ω -regular or LTL objectives.

Structure of the paper. In Section 2, we introduce the concept of Stackelberg-Pareto games with quantitative reachability costs. We also recall several useful related notions. In Section 3, we show that if there exists a solution to the SPS problem, then there exists one whose Pareto-optimal costs are exponential in the size of the instance. In Section 4, we prove that the SPS problem is NEXPTIME-complete by using the result of the previous section. Finally, we give a conclusion and some future work.

2 Preliminaries and Studied Problem

We introduce the concept of Stackelberg-Pareto games with quantitative reachability costs. We present the related Stackelberg-Pareto synthesis problem and state our main result.

2.1 Graph Games

Game arenas. A game arena is a tuple $A = (V, V_0, V_1, E, v_0, w)$ where: (1) (V, E) is a finite directed graph with V as set of vertices and E as set of edges (it is supposed that every vertex has a successor), (2) V is partitioned as $V_0 \cup V_1$ such that V_0 (resp. V_1) represents the vertices controlled by Player 0 (resp. Player 1), (3) $v_0 \in V$ is the initial vertex, and (4) $w: E \to \mathbb{N}$ is a weight function that assigns a non-negative integer² to each edge, such that $W = \max_{e \in E} w(e)$ denotes the maximum weight. An arena A is binary if $w(e) \in \{0,1\}$ for all $e \in E$.

Plays and histories. A play in an arena A is an infinite sequence of vertices $\rho = \rho_0 \rho_1 \dots \in V^{\omega}$ such that $\rho_0 = v_0$ and $(\rho_k, \rho_{k+1}) \in E$ for all $k \in \mathbb{N}$. Histories are finite sequences

² Notice that null weights are allowed.

 $h = h_0 \dots h_k \in V^+$ defined similarly. We denote $\mathsf{last}(h)$ the last vertex h_k of the history h and by |h| its length (equal to k). Let Play_A denote the set of all plays in A, Hist_A the set of all histories in A and Hist_A^i the set of all histories in A ending on a vertex in V_i , i = 0, 1. The mention of the arena will be omitted when it is clear from the context. If a history h is prefix of a play ρ , we denote it by $h \sqsubseteq \rho$. Given a play $\rho = \rho_0 \rho_1 \dots$, we denote by $\rho_{\leq k}$ the prefix $\rho_0 \dots \rho_k$ of ρ , and by $\rho_{\geq k}$ its suffix $\rho_k \rho_{k+1} \dots$ We also write $\rho_{[k,\ell]}$ for $\rho_k \dots \rho_\ell$. The weight of $\rho_{[k,\ell]}$ is equal to $w(\rho_{[k,\ell]}) = \sum_{j=k}^{\ell-1} w(\rho_j, \rho_{j+1})$.

Strategies. Let $i \in \{0,1\}$, a strategy for Player i is a function σ_i : Hist $^i \to V$ assigning to each history $h \in \mathsf{Hist}^i$ a vertex $v = \sigma_i(h)$ such that $(\mathsf{last}(h), v) \in E$. We denote by Σ_i the set of all strategies for Player i. We say that a strategy σ_i is memoryless if for all $h, h' \in \mathsf{Hist}^i$, if $\mathsf{last}(h) = \mathsf{last}(h')$, then $\sigma_i(h) = \sigma_i(h')$. A strategy is considered finite-memory if it can be encoded by a Mealy machine and its memory size is the number of states of the machine [23].

A play ρ is consistent with a strategy σ_i if for all $k \in \mathbb{N}$, $\rho_k \in V_i$ implies that $\rho_{k+1} = \sigma_i(\rho_{\leq k})$. Consistency is extended to histories as expected. We denote Play_{σ_i} (resp. Hist_{σ_i}) the set of all plays (resp. histories) consistent with σ_i . Given a couple of strategies (σ_0, σ_1) for Players 0 and 1, there exists a single play that is consistent with both of them, that we denote by $\mathsf{out}(\sigma_0, \sigma_1)$ and call the outcome of (σ_0, σ_1) .

Reachability costs. Given an arena A, let us consider a subset $T \subseteq V$ of vertices called target. We say that a play $\rho = \rho_0 \rho_1 \dots visits$ the target T, if $\rho_k \in T$ for some k. We define a cost function $\mathsf{cost}_T \colon \mathsf{Play} \to \overline{\mathbb{N}}$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, that assigns to every play ρ the quantity $\mathsf{cost}_T(\rho) = \min\{w(\rho_{\leq k}) \mid \rho_k \in T\}$, that is, the weight to the first visit of T if ρ visits T, and ∞ otherwise. The cost function is extended to histories in the expected way.

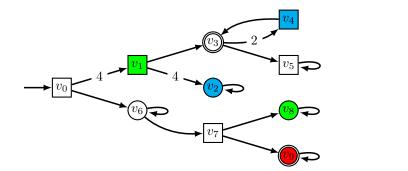
2.2 Stackelberg-Pareto Synthesis Problem

Stackelberg-Pareto games. Let $t \in \mathbb{N} \setminus \{0\}$, a Stackelberg-Pareto reachability game (SP game) is a tuple $G = (A, T_0, T_1, \dots, T_t)$ where A is a game arena and T_i are targets for all $i \in \{0, \dots, t\}$, such that T_0 is Player 0's target and T_1, \dots, T_t are the t targets of Player 1. When A is binary, we say that G is binary. The dimension t of G is the number of Player 1's targets, and we denote by Games_t (resp. $\mathsf{BinGames}_t$) the set of all (resp. binary) SP games with dimension t. The notations Play_G and Hist_G may be used instead of Play_A and Hist_A .

To distinguish the two players with respect to their targets, we introduce the following terminology. The cost of a play ρ is the tuple $cost(\rho) \in \overline{\mathbb{N}}^t$ such that $cost(\rho) = (cost_{T_1}(\rho), \ldots, cost_{T_t}(\rho))$. The value of a play ρ is a non-negative integer or ∞ defined by $val(\rho) = cost_{T_0}(\rho)$. The value can be viewed as the score of Player 0 and the cost as the score of Player 1. Both functions are extended to histories in the expected way. In the sequel, given a $cost c \in \overline{\mathbb{N}}^t$, we denote by c_i the i-th component of c and by c_{min} the component of c that is minimum, i.e. $c_{min} = min\{c_i \mid i \in \{1, \ldots, t\}\}$.

In an SP game, Player 0 wishes to minimize the value of a play with respect to the usual order < on \mathbb{N} extended to $\overline{\mathbb{N}}$ such that $n < \infty$ for all $n \in \mathbb{N}$. To compare the costs of Player 1, the following component-wise order is introduced. Let $c, c' \in \overline{\mathbb{N}}^t$ be two costs, we say that $c \leq c'$ if $c_i \leq c'_i$ for all $i \in \{1, \ldots, t\}$. Moreover, we write c < c' if $c \leq c'$ and $c \neq c'$. Notice that the order defined on costs is not total. Given two plays with respective costs c and c', if c < c', then Player 1 prefers the play with lower cost c.

Stackelberg-Pareto synthesis problem. Given an SP game and a strategy σ_0 for Player 0, we consider the set C_{σ_0} of costs of plays consistent with σ_0 that are Pareto-optimal for Player 1, i.e., minimal with respect to the order \leq on costs. Hence, $C_{\sigma_0} = \min\{ \mathsf{cost}(\rho) \mid \rho \in \mathsf{Play}_{\sigma_0} \}$.



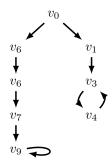


Figure 1 Arena A (on the left) – Witness tree (on the right)

Notice that C_{σ_0} is an antichain. A cost c is said to be σ_0 -fixed Pareto-optimal if $c \in C_{\sigma_0}$. Similarly, a play is said to be σ_0 -fixed Pareto-optimal if its cost is σ_0 -fixed Pareto-optimal. We will omit the mention of σ_0 when it is clear from context.

The problem we study is the following one: given an SP game G and a bound $B \in \mathbb{N}$, is there a strategy σ_0 for Player 0 such that, for all strategies σ_1 for Player 1, if the outcome $\operatorname{out}(\sigma_0, \sigma_1)$ is Pareto-optimal, then the value of the outcome is below B. It is equivalent to say that for all $\rho \in \operatorname{Play}_{\sigma_0}$, if $\operatorname{cost}(\rho)$ is σ_0 -fixed Pareto-optimal, then $\operatorname{val}(\rho)$ is below B.

▶ **Problem 1.** The Stackelberg-Pareto Synthesis problem (SPS problem) is to decide, given an SP game G and a bound B, whether

$$\exists \sigma_0 \in \Sigma_0, \forall \sigma_1 \in \Sigma_1, cost(out(\sigma_0, \sigma_1)) \in C_{\sigma_0} \Rightarrow val(out(\sigma_0, \sigma_1)) \leq B. \tag{1}$$

Any strategy σ_0 satisfying (1) is called a *solution* and we denote it by $\sigma_0 \in SPS(G, B)$. Our main result is the following theorem.

▶ **Theorem 2.** The SPS problem is NEXPTIME-complete.

The non-deterministic algorithm is exponential in the number of targets t and in the size of the binary encoding of the maximum weight W and the bound B. The general approach to obtain NEXPTIME-membership is to show that when there is a solution $\sigma_0 \in \operatorname{SPS}(G,B)$, then there exists one that is finite-memory and whose memory size is exponential. An important part of this paper is devoted to this proof. Then we show that such a strategy can be guessed and checked to be a solution in exponential time.

Example. To provide a better understanding of the SPS problem, let us solve it on a specific example. The arena A is displayed on Figure 1 where the vertices controlled by Player 0 (resp. Player 1) are represented as circles (resp. squares). The weights are indicated only if they are different from 1 (e.g., the edge (v_0, v_6) has a weight of 1). The initial vertex is v_0 . The target of Player 0 is $T_0 = \{v_3, v_9\}$ and is represented by doubled vertices. Player 1 has three targets: $T_1 = \{v_1, v_8\}$, $T_2 = \{v_9\}$ and $T_3 = \{v_2, v_4\}$, that are represented using colors (green for T_1 , red for T_2 , blue for T_3). Let us exhibit a solution σ_0 in SPS(G, 5).

We define σ_0 as the strategy that always moves from v_3 to v_4 , and that loops once on v_6 and then moves to v_7 . The plays consistent with σ_0 are $v_0v_1v_2^{\omega}$, $v_0v_1(v_3v_4)^{\omega}$, $v_0v_6v_6v_7v_8^{\omega}$, and $v_0v_6v_6v_7v_9^{\omega}$. The Pareto-optimal plays are $v_0v_1(v_3v_4)^{\omega}$ and $v_0v_6v_6v_7v_9^{\omega}$ with respective costs $(4, \infty, 7)$ and $(\infty, 4, \infty)$, and they both yield a value less than or equal to 5. Notice that σ_0 has to loop once on v_6 , i.e., it is not memoryless⁴, otherwise the consistent play $v_0v_6v_7v_8^{\omega}$

⁴ One can prove that there exists no memoryless solution.

has a Pareto-optimal cost of $(3, \infty, \infty)$ and an infinite value.

Interestingly, the Boolean version of this game does not admit any solution. In this case, given a target, the player's goal is simply to visit it (and not to minimize the cost to reach it). That is, the Boolean version is equivalent to the quantitative one with all weights and the bound B put to zero. In the example, the play $v_0v_1v_2^{\omega}$ is Pareto-optimal (with visits to T_1 and T_3), whatever the strategy of Player 0, and this play does not visit Player 0's target.

Witnesses. An important tool for solving the SPS problem is the concept of witness [14]. Given a solution σ_0 , for all $c \in C_{\sigma_0}$, we can choose arbitrarily a play ρ called witness of the cost c such that $cost(\rho) = c$. The set of all chosen witnesses is denoted by Wit_{σ_0} , whose size is the size of C_{σ_0} . Since σ_0 is a solution, the value of each witness is below B. We define the length of a witness ρ as the length $|\rho| = \min\{|h| \mid h \sqsubseteq \rho \land \mathsf{cost}(h) = \mathsf{cost}(\rho) \land \mathsf{val}(h) = \mathsf{val}(\rho)\}$.

It is useful to see the set Wit_{σ_0} as a tree composed of $|Wit_{\sigma_0}|$ branches. Moreover, given $h \in \mathsf{Hist}_{\sigma_0}$, we write $\mathsf{Wit}_{\sigma_0}(h)$ the set of witnesses for which h is a prefix, i.e., $\mathsf{Wit}_{\sigma_0}(h) =$ $\{\rho \in \mathsf{Wit}_{\sigma_0} \mid h \sqsubseteq \rho\}$. Notice that $\mathsf{Wit}_{\sigma_0}(h) = \mathsf{Wit}_{\sigma_0}$ when $h = v_0$, and that the size of $Wit_{\sigma_0}(h)$ decreases as the size of h increases, until it contains a single play or becomes empty.

The following notions about the tree Wit_{σ_0} will be useful. We say that a history h is a branching point if there are two witnesses whose greatest common prefix is h, that is, there exists $v \in V$ such that $0 < |Wit_{\sigma_0}(hv)| < |Wit_{\sigma_0}(h)|$. We define the following equivalence relations \sim on histories that are prefixes of a witness: $h \sim h'$ if and only if $(\mathsf{val}(h), \mathsf{cost}(h), \mathsf{Wit}_{\sigma_0}(h)) = (\mathsf{val}(h'), \mathsf{cost}(h'), \mathsf{Wit}_{\sigma_0}(h')).$ Notice that if $h \sim h'$, then either $h \sqsubseteq h'$ or $h' \sqsubseteq h$ and no new target is visited and no branching point is crossed from the shortest history to the longest one. We call region of h its equivalence class. This leads to a region decomposition of each witness, such that the first region is the region of the initial state v_0 and the last region is the region of $h \sqsubseteq \rho$ such that $|h| = \mathsf{length}(\rho)$. A deviation is a history hv with $h \in \mathsf{Hist}^1$ and $v \in V$, such that h is prefix of some witness, but hv is prefix of no witness.

We illustrate these notions on the previous example and its solution σ_0 . A set of witnesses is $\mathsf{Wit}_{\sigma_0} = \{v_0v_1(v_3v_4)^\omega, v_0v_6v_6v_7v_9^\omega\}$ depicted on Figure 1. We have that $\mathsf{length}(v_0v_6v_6v_7v_9^\omega) = \mathsf{length}(v_0v_6v_6v_7v_9^\omega)$ $|v_0v_6v_6v_7v_9|=4$, v_0 is a branching point, $v_0v_1v_2$ is a deviation, and the region decomposition of the witness $v_0v_6v_6v_7v_9^{\omega}$ is $\{v_0\}$, $\{v_0v_6, v_0v_6v_6, v_0v_6v_6v_7\}$, $\{v_0v_6v_6v_7v_9^k \mid k \geq 1\}$.

Reduction to binary arenas. Working with general arenas requires to deal with the parameter W in most of the proofs. To simplify the arguments, we reduce the SPS problem to binary arenas, by replacing each edge with a weight $w \geq 2$ by a path of w edges of weight 1. This (standard) reduction is exponential, but only in the size of the binary encoding of W.

- ▶ **Lemma 3.** Let $G = (A, T_0, ..., T_t)$ be an SP game and $B \in \mathbb{N}$. Then one can construct in exponential time an SP game $G' = (A', T_0, \dots, T_t)$ with a binary arena A' such that
- the set of vertices V' of A' contains V and has size $|V'| \leq |V| \cdot W$,
- there exists a solution in SPS(G, B) if and only if there exists a solution in SPS(G', B).

Proof. Let $X = \{e \in E \mid w(e) \geq 2\}$ be the set of edges of A with weight at least 2. For each $e \in X$, we replace e by a succession of w(e) - 1 new vertices linked by w(e) new edges of weight 1, and for each $e \in E \setminus X$, we keep e unmodified. In this way, we get a directed graph (V', E') with a weight function $w' : E' \to \{0, 1\}$ and with a size $|V'| \leq |V| \cdot W$. Notice that if $X = \emptyset$, the arena is already binary. The new arena $A' = (V', V'_0, V'_1, E', v_0, w')$ has the same initial vertex v_0 as A, and a partition $V_0' \cup V_1'$ of V' such that each new vertex is added⁵ to

⁵ Each new vertex could be added to V_1 as it has a unique successor.

 V_0 , hence $V_0 \subseteq V_0'$, $V_1 = V_1'$. The new SP game $G' = (A', T_0, \dots, T_t)$ keeps the same targets as in G.

Clearly, there is a trivial bijection f from Play_A to $\mathsf{Play}_{A'}$. Indeed, it suffices to replace all edges e of a play with weight $w(e) \geq 2$ by the corresponding new path composed of w(e) edges of weight 1. This bijection preserves the cost and the value of the plays. Moreover, there also exists a bijection g from the set of strategies on A to the set of strategies on A' as each new vertex has a unique successor. Notice that g is coherent with f, i.e., $\mathsf{out}(g(\sigma_0),g(\sigma_1))=f(\mathsf{out}(\sigma_0,\sigma_1))$ for all $\sigma_0\in\Sigma_0$ and all $\sigma_1\in\Sigma_1$. Therefore, for all $\sigma_0\in\Sigma_0$, we get that $\sigma_0\in\mathsf{SPS}(G,B)$ if and only if $g(\sigma_0)\in\mathsf{SPS}(G',B)$.

The transformation of the arena A into a binary arena A' has consequences on the size of the SPS problem instance. Since the weights are encoded in binary, the size |V'| could be exponential in the size |V| of the original instance. However, this will have no impact on our main result because |V| never appears in the exponent in our calculations (this will be detailed in the proof of Theorem 2).

3 Bounding Pareto-Optimal Payoffs

In this section, we show that if there exists a solution to the SPS problem, then there exists one whose Pareto-optimal costs are exponential in the size of the instance (see Theorem 6 below). It is a *crucial step* to prove that the SPS problem is in NEXPTIME. This is the main contribution of this paper.

3.1 Improving a Solution

We begin by presenting some techniques that allow to modify a solution to the SPS problem into a solution with smaller Pareto-optimal costs.

Order on strategies and subgames. Given two strategies σ_0, σ'_0 for Player 0, we say that $\sigma'_0 \leq \sigma_0$ if for all $c \in C_{\sigma_0}$, there exists $c' \in C_{\sigma'_0}$ such that $c' \leq c$. This relation \leq on strategies is a preorder (it is reflexive and transitive). We also define $\sigma'_0 < \sigma_0$ when $\sigma'_0 \leq \sigma_0$ and $C_{\sigma'_0} \neq C_{\sigma_0}$, and we say that σ'_0 is better than σ_0 whenever $\sigma'_0 \leq \sigma_0$. In the sequel, we modify solutions σ_0 to the SPS problem to get better solutions $\sigma'_0 \leq \sigma_0$, and we say that σ'_0 improves the given solution σ_0 .

A subgame of an SP game G is a couple (G,h), denoted $G_{|h}$, where $h \in \mathsf{Hist}$. In the same way that G can be seen as the set of its plays, $G_{|h}$ is seen as the restriction of G to plays with prefix h. In particular, we have $G_{|v_0} = G$ where v_0 is the initial vertex of G. The value and cost of a play ρ in $G_{|h}$ are the same as those of ρ as a play in G. The dimension of $G_{|h}$ is the dimension of G minus the number of targets visited G by G such that G is that G is the dimension of G minus the number of targets visited G by G such that G is the dimension of G minus the number of targets visited G by G such that G is the dimension of G minus the number of targets visited G by G such that G is the dimension of G minus the number of targets visited G by G such that G is the dimension of G minus the number of targets visited G by G is the dimension of G minus the number of targets visited G by G is the dimension of G minus the number of targets visited G by G is the dimension of G is the dimension of G minus the number of targets visited G by G is the dimension of G is the dimension of G minus the number of targets visited G is the dimension of G.

A strategy for Player 0 on $G_{|h}$ is a strategy τ_0 that is only defined for the histories $h' \in \mathsf{Hist}$ such that $h \sqsubseteq h'$. We denote $\Sigma_{0|h}$ the set of those strategies. Given a strategy σ_0 for Player 0 in G and $h \in \mathsf{Hist}_{\sigma_0}$, we denote the restriction of σ_0 to $G_{|h}$ by the strategy $\sigma_{0|h}$. Moreover, given $\tau_0 \in \Sigma_{0|h}$, we can define a new strategy $\sigma_0[h \to \tau_0]$ from σ_0 as the strategy on G which consists in playing the strategy σ_0 everywhere, except in the subgame $G_{|h}$ where τ_0 is played. That is, $\sigma_0[h \to \tau_0](h') = \sigma_0(h')$ if $h \not\sqsubseteq h'$, and $\sigma_0[h \to \tau_0](h') = \tau_0(h')$ otherwise.

As done with SPS(G, B), we denote by $SPS(G_{|h}, B)$ the set of all solutions $\tau_0 \in \Sigma_{0|h}$ to the SPS problem for the subgame $G_{|h}$ and the bound B.

⁷ Notice that we do not include last(h) in h', as it can be seen as the initial vertex of $G_{|h}$.

Improving a solution. A natural way to improve a strategy is to improve it on a subgame. Moreover, if it is a solution to the SPS problem, it is also the case for the improved strategy.

▶ Lemma 4. Let G be a binary SP game, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G,B)$ be a solution. Consider a history $h \in \mathsf{Hist}_{\sigma_0}$ and a strategy $\tau_0 \in \Sigma_{0|h}$ in the subgame $G_{|h}$ such that $\tau_0 < \sigma_{0|h}$ and $\tau_0 \in SPS(G_{|h}, B)$. Then the strategy $\sigma'_0 = \sigma_0[h \to \tau_0]$ is a solution in SPS(G, B) and $\sigma_0' < \sigma_0$.

Proof. Let us first prove that $\sigma'_0 \leq \sigma_0$, that is, for all $c \in C_{\sigma_0}$, there exists $c' \in C_{\sigma'_0}$ such that $c' \leq c$. Let $c \in C_{\sigma_0}$ and $\rho \in \mathsf{Play}_{G,\sigma_0}$ be such that $\mathsf{cost}(\rho) = c$. If $h \not\sqsubseteq \rho$, then ρ is also consistent with σ'_0 , thus there exists $c' \in C_{\sigma'_0}$ such that $c' \leq c = \mathsf{cost}(\rho)$ by definition of $C_{\sigma'_0}$. Otherwise, $h \sqsubseteq \rho$. Hence ρ is a play in the subgame $G_{|h}$ with a Pareto-optimal cost $c \in C_{\sigma_{0|h}}$. By hypothesis, we have $\tau_0 \leq \sigma_{0|h}$, therefore there exists $c' \in C_{\tau_0}$ such that $c' \leq c$. As $c' \in C_{\tau_0}$, there exists $\rho' \in \mathsf{Play}_{G_{|h},\tau_0}$ such that $h \sqsubseteq \rho'$ and $\mathsf{cost}(\rho') = c'$. By definition of σ_0' , we also have that $\rho' \in \mathsf{Play}_{G,\sigma_0'}$. Hence, by definition of $C_{\sigma_0'}$, there exists $c'' \in C_{\sigma_0'}$ such that $c'' \leq c'$, and thus $c'' \leq c$. We have thus proved that $\sigma'_0 \leq \sigma_0$. Notice that by the previous arguments, we have $\sigma_0' < \sigma_0$ as $\tau_0 < \sigma_{0|h}$.

Let us now show that $\sigma'_0 \in SPS(G, B)$. Let $\rho' \in Play_{\sigma'_0}$ be such that $c' = cost(\rho') \in C_{\sigma'_0}$. We have to prove that $\operatorname{val}(\rho') \leq B$. If $h \not\sqsubseteq \rho'$, then $\rho' \in \operatorname{Play}_{\sigma_0}$, thus there exists $c \in C_{\sigma_0}$ such that $c \leq c'$ by definition of C_{σ_0} . Since $\sigma'_0 \leq \sigma_0$ by the first part of the proof, it follows that $c = c' = \mathsf{cost}(\rho') \in C_{\sigma_0}$. Now, recall that σ_0 is a solution in $\mathsf{SPS}(G,B)$, implying that $\operatorname{\mathsf{val}}(\rho') \leq B$. If $h \sqsubseteq \rho'$, then $\rho' \in \operatorname{\mathsf{Play}}_{G_{1h}, \tau_0}$. As $c' \in C_{\sigma'_0}$, we have $c' \in C_{\tau_0}$ (it is not possible to have $c'' \in C_{\tau_0}$ such that c'' < c' by definition of σ'_0). As $\tau_0 \in SPS(G_{|h}, B)$, it follows that $\operatorname{val}(\rho') \leq B$. In any case, $\operatorname{val}(\rho') \leq B$ showing that σ'_0 is a solution to the SPS problem.

Another way to improve solutions to the SPS problem is to delete some particular cycles occurring in witnesses as explained in the next lemma.

▶ Lemma 5. Let G be a binary SP game, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a solution. Suppose that in a witness $\rho = \rho_0 \rho_1 \ldots \in Wit_{\sigma_0}$, there exist $m, n \in \mathbb{N}$ such that

```
m < n < length(\rho) \ and \ \rho_m = \rho_n
```

- $\rho < m$ and $\rho < n$ belong to the same region, and
- if $val(\rho_{\leq m}) = \infty$, then the weight $w(\rho_{[m,n]})$ is null.

Then the strategy $\sigma'_0 = \sigma_0[\rho_{\leq m} \to \sigma_0|_{\rho_{\leq n}}]$ is a solution in SPS(G, B) such that $\sigma'_0 \leq \sigma_0$.

The first condition means that $\rho_{[m,n]}$ is a cycle and that it appears before the last visit of a target by ρ . The second one says that $\rho_{\leq m} \sim \rho_{\leq n}$, i.e., no new target is visited and no branching point is crossed from history $\rho_{\leq m}$ to history $\rho_{\leq n}$. The third one says that if $\rho_{\leq m}$ does not visit Player 0's target, then the cycle $\rho_{[m,n]}$ must have a null weight. The new strategy σ'_0 is obtained from σ_0 by playing after $\rho_{\leq m}$ as playing after $\rho_{\leq n}$ (thus deleting the cycle $\rho_{[m,n]}$).

From now on, we say that we can eliminate cycles according to this lemma⁸ without explicitly building the new strategy. We also say that a solution σ_0 is without cycles if it does not satisfy the hypotheses of Lemma 5, i.e., if it is impossible to eliminate cycles to get a better solution.

Proof of Lemma 5. Let $g = \rho_{\leq m}$ and $h = \rho_{\leq n}$. We first prove that $\sigma'_0 \leq \sigma_0$. For this purpose, we introduce the following notation: for each play $\pi = \pi_0 \pi_1 \ldots \in \mathsf{Play}_G$ such

⁸ These are the cycles satisfying the lemma, and not just any cycle.

that $h \sqsubseteq \pi$, we denote by $\bar{\pi}$ the play $g\pi_{\geq n+1}$, that is, we delete the cycle $\rho_{[m,n]}$ in π . Let $c \in C_{\sigma_0}$ and $\pi \in \mathsf{Wit}_{\sigma_0}$ be a witness with $\mathsf{cost}(\pi) = c$. Let us prove that there exists $c' \in C_{\sigma'_0}$ such that $c' \leq c$. If $g \not\sqsubseteq \pi$, then $\pi \in \mathsf{Play}_{\sigma'_0}$ by definition of σ'_0 , and thus there exists $c' \in C_{\sigma'_0}$ such that $c' \leq c$ by definition of $C_{\sigma'_0}$. If $g \sqsubseteq \pi$, as $g \sim h$ by hypothesis (in particular, $\mathsf{Wit}_{\sigma_0}(g) = \mathsf{Wit}_{\sigma_0}(h)$), then $h \sqsubseteq \pi$. Then, $\bar{\pi} \in \mathsf{Play}_{\sigma'_0}$ with $\mathsf{cost}(\bar{\pi}) = \mathsf{cost}(\pi) - w(\rho_{[m,n]}) \leq \mathsf{cost}(\pi)$. By definition of $C_{\sigma'_0}$, there exists $c' \in C_{\sigma'_0}$ such that $c' \leq \mathsf{cost}(\bar{\pi})$, and therefore $c' \leq \mathsf{cost}(\pi) = c$. Hence $\sigma'_0 \leq \sigma_0$.

We then prove that σ'_0 is a solution to the SPS problem. We have to show that each play $\pi' \in \mathsf{Play}_{\sigma'_0}$ with a cost $c' \in C_{\sigma'_0}$ has a value $\mathsf{val}(\pi') \leq B$. If $g \not\sqsubseteq \pi'$, then $\pi' \in \mathsf{Play}_{\sigma_0}$. Notice that $c' = \mathsf{cost}(\pi') \in C_{\sigma_0}$ because $\sigma'_0 \leq \sigma_0$. Therefore, as $\sigma_0 \in \mathsf{SPS}(G,B)$, we get that $\mathsf{val}(\pi') \leq B$. If $g \sqsubseteq \pi'$, then either $\mathsf{val}(g) < \infty$, or $\mathsf{val}(g) = \infty$ in which case the weight of $\rho_{[m,n]}$ is null by hypothesis. In the first case, $\mathsf{val}(g) \leq B$ because g is prefix of the witness $\rho \in \mathsf{Wit}_{\sigma_0}$ and $\sigma_0 \in \mathsf{SPS}(G,B)$. This shows that $\mathsf{val}(\pi') \leq B$. In the second case, we consider $\pi \in \mathsf{Play}_{\sigma_0}$ such that $\bar{\pi} = \pi'$. Notice that $\mathsf{cost}(\pi) = c'$ because $g \sim h$ (in particular, $\mathsf{cost}(g) = \mathsf{cost}(h)$) and $\rho_{[m,n]}$ has a null weight. We get that $c' = \mathsf{cost}(\pi) \in C_{\sigma_0}$ since $\sigma'_0 \leq \sigma_0$. It follows that $\mathsf{val}(\pi) \leq B$ as $\sigma_0 \in \mathsf{SPS}(G,B)$. As $g \sim h$ (in particular, $\mathsf{val}(g) = \mathsf{val}(h)$), π' visits Player 0's target outside $\rho_{[m+1,n-1]}$, hence $\mathsf{val}(\pi') \leq B$. Therefore $\sigma'_0 \in \mathsf{SPS}(G,B)$.

Crucial step. We can now state the theorem announced at the beginning of Section 3.

▶ Theorem 6. Let $G \in BinGames_t$ be a binary SP game with dimension $t, B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a solution. Then there exists a solution $\sigma'_0 \in SPS(G, B)$ without cycles such that $\sigma'_0 \leq \sigma_0$, and

$$\forall c' \in C_{\sigma'_0}, \forall i \in \{1, \dots, t\}: \quad c'_i \le 2^{\Theta(t^2)} \cdot |V|^{\Theta(t)} \cdot (B+3) \quad \lor \quad c'_i = \infty$$
 (2)

In case of any general SP game $G \in \mathsf{Games}_t$, the same result holds with |V| replaced by $|V| \cdot W$ in the inequality.

In view of this result, a solution to the SPS problem is said to be *bounded* when its Pareto-optimal costs are bounded as stated in the theorem.

The theorem is proved by induction on the dimension t, with the calculation of a function f(B,t) depending on both B and t, that bounds the components $c'_i \neq \infty$. This function is defined by induction on t through the proofs, and afterwards made explicit and upper bounded by the bound given in Theorem 6. Notice that the function f can be considered as increasing in t.¹⁰ The proof of Theorem 6 is detailed in the next sections for binary SP games; it is then easily adapted to any SP games by Lemma 3.

3.2 Dimension One

We begin the proof of Theorem 6 with the case t = 1. In this case, the order on costs is total.

▶ Lemma 7. Let $G \in BinGames_1$ be a binary SP game with dimension 1, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a solution. Then there exists a solution $\sigma_0' \in SPS(G, B)$ without cycles such that $\sigma_0' \leq \sigma_0$ and

$$\forall c' \in C_{\sigma_0'}: \quad c' \le f(B, 1) = B + |V| \quad \lor \quad c' = \infty. \tag{3}$$

Notice that f(B,1) respects the bound given in Theorem 6 when t=1.

¹⁰We could artificially duplicate some targets in a way to increase the dimension.

Proof. Player 1 has only one target, thus C_{σ_0} is a singleton, say $C_{\sigma_0} = \{c\}$. If $c \leq B + |V|$ or $c = \infty$, it is trivial (we eliminate cycles if necessary). Therefore, let us suppose that $B + |V| < c < \infty$. Let $\rho \in \mathsf{Wit}_{\sigma_0}$ be such that $\mathsf{cost}(\rho) = c$. Let h be the history of maximal length such that $h \sqsubseteq \rho$ and w(h) = B. Notice that h exists as the arena is binary and $B + |V| < c < \infty$. Since $\sigma_0 \in \mathsf{SPS}(G, B)$ and $\rho \in \mathsf{Wit}_{\sigma_0}$, Player 0's target is visited by h and Player 1's target is visited by ρ at least |V| + 1 vertices after h. Hence, ρ performs a cycle between the two visits, that satisfies the hypotheses of Lemma 5. We can thus eliminate this cycle and create a better solution. We repeat this process until $c \leq B + |V|$ and the termination is guaranteed by the strict reduction of $\mathsf{length}(\rho)$.

3.3 Bounding the Pareto-Optimal Costs

We now proceed to the case of dimension t+1, with $t \ge 1$. The next lemma is proved by using the *induction hypothesis*. Recall that c_{min} is the minimum component of the cost c.

▶ Lemma 8. Let $G \in BinGames_{t+1}$ be a binary SP game with dimension t+1, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a solution. Then there exists a solution $\sigma'_0 \in SPS(G, B)$ without cycles such that $\sigma'_0 \leq \sigma_0$, and

$$\forall c' \in C_{\sigma'_0}, \forall i \in \{1, \dots, t+1\}: \quad c'_i \le \max\{c'_{min}, B\} + 1 + f(0, t) \quad \lor \quad c'_i = \infty.$$
(4)

Proof. In this proof, we assume that Theorem 6 is true for all dimensions $\leq t$, by induction hypothesis. Let us suppose that there exists $c \in C_{\sigma_0}$ such that for some $i \in \{1, \ldots, t+1\}$:

$$\max\{c_{min}, B\} + 1 + f(0, t) < c_i < \infty \tag{5}$$

(if such a cost c does not exist, the proof is trivial by eliminating cycles if necessary). Let $\rho \in \mathsf{Wit}_{\sigma_0}$ be a witness with $\mathsf{cost}(\rho) = c$. We define the history h of minimal length such that

$$h \sqsubseteq \rho \text{ and } w(h) = \max\{c_{min}, B\} + 1.$$
 (6)

Notice that h exists by definition of c_i and as the arena is binary. As $\sigma_0 \in SPS(G, B)$, by definition of h, Player 0's target and at least one target of Player 1 are visited by \bar{h} such that $h = \bar{h}last(h)$. Therefore the subgame $G_{|h}$ has dimension $k \leq t$. Notice that k > 0 by definition of c_i , see (5).

Let us consider the SP game \bar{G} with the same arena as G but with the initial vertex last(h) (instead of v_0) and with the targets visited by \bar{h} removed from Player 1's set of targets and with no target for Player 0 (since \bar{h} has visited this target). It has dimension k. We also consider the strategy $\bar{\sigma}_0$ for Player 0 in \bar{G} constructed from the strategy $\sigma_{0|h}$ in $G_{|h}$ as follows: $\bar{\sigma}_0(g) = \sigma_{0|h}(\bar{h}g)$ for all histories $g \in \operatorname{Hist}_{\bar{G}}$ (Player 0 plays in \bar{G} from last(h) as he plays in $G_{|h}$ from h). We have that $\bar{\sigma}_0 \in \operatorname{SPS}(\bar{G},0)$ (the bound is equal to 0 as Player 0 has no target). We can thus apply the induction hypothesis: by Theorem 6, there exists $\bar{\tau}_0 \in \operatorname{SPS}(\bar{G},0)$ without cycles such that $\bar{\tau}_0 \leq \bar{\sigma}_0$ and for all $\bar{c} \in C_{\bar{\tau}_0}$ and all $j \in \{1,\ldots,k\}, \bar{c}_j \leq f(0,k)$ or $\bar{c}_j = \infty$. Notice that one can choose for $\bar{\sigma}_0$ a set of witnesses derived from those of σ_0 having h as prefix: Wit $\bar{\sigma}_0 = \{\bar{\pi} \in \operatorname{Play}_{\bar{G}} \mid \bar{h}\bar{\pi} \in \operatorname{Wit}_{\sigma_0}\}$. In particular, there exists $\bar{\rho} \in \operatorname{Wit}_{\bar{\sigma}_0}$ such that $\rho = \bar{h}\bar{\rho}$. By (5) and (6), we have that $\operatorname{cost}_{T_i}(\bar{\rho}) > f(0,t)$ ($\operatorname{cost}_{T_i}(\bar{\rho})$ and $\operatorname{cost}_{T_i}(\rho)$ differ by w(h)). As $\bar{c}_i \leq f(0,k) \leq f(0,t)$, it follows that $\bar{\tau}_0 < \bar{\sigma}_0$.

We now want to transfer the previous solution $\bar{\tau}_0$ in \bar{G} to the subgame $G_{|h}$ in a way to apply Lemma 4 and thus obtain the desired strategy σ'_0 in G. Recall that $\bar{\sigma}_0$ was constructed from $\sigma_{0|h} \in \Sigma_{0|h}$. Let us conversely define the strategy $\tau_0 \in \Sigma_{0|h}$ from $\bar{\tau}_0$: $\tau_0(\bar{h}g) = \bar{\tau}_0(g)$ for all histories $\bar{h}g \in \mathsf{Hist}_G$. Moreover, we can choose the following sets of witnesses for $\sigma_{0|h}$ and τ_0 : Wit $\sigma_{0|h} = \{\bar{h}\bar{\pi} \mid \bar{\pi} \in \mathsf{Wit}_{\bar{\sigma}_0}\}$ and Wit $\tau_0 = \{\bar{h}\bar{\pi} \mid \bar{\pi} \in \mathsf{Wit}_{\bar{\tau}_0}\}$. It follows from $\bar{\tau}_0 < \bar{\sigma}_0$ that

 $\tau_0 < \sigma_{0|h}$ (again, the Pareto-optimal costs of $\bar{\tau}$ and τ differ by w(h), and so do the ones of $\bar{\sigma}_0$ and σ_0). Moreover $\tau_0 \in \mathrm{SPS}(G,B)$ since h visits Player 0's target. Hence, by Lemma 4, the strategy $\sigma_0' = \sigma_0[h \to \tau_0]$ is a solution in $\mathrm{SPS}(G,B)$ and $\sigma_0' < \sigma_0$.

We repeat the process described above as long as there remain costs $c \in C_{\sigma_0}$ that are too large. The process terminates as we are always building strictly better strategies. If the resulting strategy is not without cycles, we can eliminate them, one by one, to get a better strategy by applying Lemma 5. This second process also terminates by the strict reduction of the length of the witnesses of the intermediate solutions.

3.4 Bounding the Minimum Component of Pareto-Optimal Costs

To prove Theorem 6, in view of Lemma 8, our last step is to provide a bound on c_{min} , the minimum component of each Pareto-optimal cost $c \in C_{\sigma_0}$. Notice that if $c_{min} = \infty$, then all the components of c are equal to ∞ . In this case, $C_{\sigma_0} = \{(\infty, \ldots, \infty)\}$, i.e., there is no play in Play_{σ_0} visiting Player 1's targets. The bound on c_{min} is provided in Lemma 10, when $C_{\sigma_0} \neq \{(\infty, \ldots, \infty)\}$. It depends on $|C_{\sigma_0}|$, a bound of which is first given in the next lemma.

▶ **Lemma 9.** Let $G \in BinGames_{t+1}$ be a binary SP game with dimension t+1, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a solution satisfying (4). Suppose that $C_{\sigma_0} \neq \{(\infty, ..., \infty)\}$. Then

$$|C_{\sigma_0}| \le (f(0,t) + B + 3)^{t+1}.$$
 (7)

Proof. let σ_0 be a solution such that for all $c \in C_{\sigma_0}$, for all $i \in \{1, \ldots, t+1\}$, $c_i \le \max\{c_{min}, B\} + 1 + f(0, t)$ or $c_i = \infty$. Therefore, we can write each $c \in C_{\sigma_0}$ as $c = c_{min}(1, \ldots, 1) + d$ with $d_i \in \{0, \ldots, B+1+f(0,t)\} \cup \{\infty\}$ for all i. If two costs $c, c' \in C_{\sigma_0}$ are such that $c = c_{min}(1, \ldots, 1) + d$ and $c' = c'_{min}(1, \ldots, 1) + d$, with the same vector d, then they are comparable. Hence, as C_{σ_0} is an antichain, its size is bounded by the number of vectors d, that is, by $(f(0,t) + B + 3)^{t+1}$.

In the next lemma, we give the announced bound on the minimum component of Paretooptimal costs.

▶ Lemma 10. Let $G \in BinGames_{t+1}$ be a binary SP game with dimension t+1, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a solution without cycles and satisfying (4). Suppose that $C_{\sigma_0} \neq \{(\infty, ..., \infty)\}$. Then,

$$\forall c \in C_{\sigma_0}: \ c_{min} \le B + 2^{t+1} (|V| \cdot \log_2(|C_{\sigma_0}|) + 1 + f(0, t))$$
(8)

The proof of this lemma requires the next property about trees that is easily established. We recall the notion of depth of a node in a tree: the root has depth 0, and if a node has depth d, then its sons have depth d+1.

▶ Lemma 11. Let $n, \ell \in \mathbb{N}$ be such that $\ell \geq 1$. Consider a finite tree with at most n leaves such that there are at most $\ell - 1$ consecutive nodes with degree one along any branch of the tree. Then the leaves with the smallest depth have a depth bounded by $\ell \cdot \log_2(n)$. \blacktriangleleft

Proof of Lemma 10. Let $\sigma_0 \in SPS(G, B)$ be a solution without cycles and satisfying (4). Suppose that $C_{\sigma_0} \neq \{(\infty, \dots, \infty)\}$ and let Wit_{σ_0} be a set of witnesses for C_{σ_0} . Assume by contradiction that there exists $d \in C_{\sigma_0}$ such that

$$B + 2^{t+1} (\delta + 1 + f(0,t)) < d_{min} < \infty.$$
(9)

with $\delta = |V| \cdot \log_2(|C_{\sigma_0}|)$. Let $\rho \in \text{Wit}_{\sigma_0}$ be a witness such that $\text{cost}(\rho) = d$. We are going to build a finite sequence of Pareto-optimal costs $(c^{(k)})_{k \in \{0,...,2^{t+1}\}}$ such that, for all $k \in \{0,...,2^{t+1}-1\}$,

$$\max\{c_i^{(k)} \mid c_i^{(k)} \neq \infty\} < c_{min}^{(k+1)} \tag{10}$$

By the pigeonhole principle, there exist $k, k' \in \{0, \dots, 2^{t+1}\}$ such that k < k' and $\{i \in \{1, \dots, t+1\} \mid c_i^{(k)} = \infty\} = \{i \in \{1, \dots, t+1\} \mid c_i^{(k')} = \infty\}$ (a cost component is either finite or infinite). It follows from (10) that $c^{(k)} < c^{(k')}$. This is impossible as C_{σ_0} is an antichain.

To build the sequence $(c^{(k)})_{k \in \{0,\dots,2^{t+1}\}}$, we consider the tree \mathcal{T} of witnesses of Wit $_{\sigma_0}$ truncated at the first visit of a target of Player 1. This truncated tree is finite: its leaves correspond to the histories g such that $w(g) = c_{min}$, with $c \in C_{\sigma_0}$, the first visit of g of some target of Player 1 is in its last vertex $\mathsf{last}(g)$, and g visits Player 0's target such that $\mathsf{val}(h) \leq B$. (Notice that one leaf of \mathcal{T} corresponds to some g such that $w(g) = d_{min}$.) By recalling the region decomposition of the witnesses, \mathcal{T} has at most $|C_{\sigma_0}|$ leaves, its internal nodes with degree ≥ 2 correspond to histories that are branching points, and any two internal nodes between two consecutive branching points (with respect to the order \sqsubseteq) are in the same region.

Let us construct the first Pareto-optimal cost $c^{(0)}$. Let h_0 be the history of maximal length such that $h_0 \sqsubseteq \rho$ and $w(h_0) = B$. This history h_0 exists because the arena is binary and $cost(\rho) = d$ with d_{min} satisfying (9). We consider the subtree \mathcal{T}_0 of \mathcal{T} rooted in the last vertex of h_0 . By the region decomposition of Wit σ_0 and as σ_0 is without cycle in the sense of Lemma 5, we can apply Lemma 11 to the subtree \mathcal{T}_0 with parameters $n = |C_{\sigma_0}|$ and $\ell = |V|$. It follows that any leaf of \mathcal{T}_0 with the smallest depth has a depth $\leq \delta = |V| \cdot \log_2(|C_{\sigma_0}|)$. We set $c^{(0)}$ as the cost of the witness associated with one of these leaves. We get that $c_{min}^{(0)} \leq B + \delta$ because \mathcal{T}_0 is the subtree rooted at h_0 with $w(h_0) = B$. As σ_0 satisfies (4) by hypothesis, we get that $\max\{c_i^{(0)} \mid c_i^{(0)} \neq \infty\} \leq \max\{c_{min}^{(0)}, B\} + 1 + f(0, t)$, that is,

$$\max\{c_i^{(0)} \mid c_i^{(0)} \neq \infty\} \le B + (\delta + 1 + f(0, t)). \tag{11}$$

Let us construct the second Pareto-optimal cost $c^{(1)}$ as we did for $c^{(0)}$. Let h_1 be the history of maximal length such that $h_1 \sqsubseteq \rho$ and $w(h_1) = B + \delta + 1 + f(0,t)$ (notice that we use the bound of (11)). This history h_1 exists for the same reasons as for h_0 . Let \mathcal{T}_1 be the subtree rooted in the last vertex of h_1 (it is a subtree of \mathcal{T}_0). We can apply Lemma 11 as for \mathcal{T}_0 : any leaf of \mathcal{T}_1 with the smallest depth has a depth $\leq \delta$. Thus, we set $c^{(1)}$ as the cost of the witness associated with one of these leaves. We get that $c^{(1)}_{min} \leq B + \delta + 1 + f(0,t) + \delta$ by definition of $w(h_1)$. By (4), we get

$$\max\{c_i^{(1)} \mid c_i^{(1)} \neq \infty\} \le B + 2(\delta + 1 + f(0, t)). \tag{12}$$

We also have that $\max\{c_i^{(0)} \mid c_i^{(0)} \neq \infty\} < c_{min}^{(1)}$ as required in (10). Indeed $c^{(1)}$ and d correspond to two different leaves of \mathcal{T}_1 , and thus $c^{(1)}$ does not correspond to the root of \mathcal{T}_1 . By definition of h_1 , we get that $c_{min}^{(1)} > w(h_1) = B + \delta + 1 + f(0,t)$, and thus $c_{min}^{(1)} > \max\{c_i^{(0)} \mid c_i^{(0)} \neq \infty\}$ by (11).

As d_{min} satisfies (9), we can repeat this process to construct the next costs $c^{(2)}, c^{(3)}, \ldots$, until the last cost $c^{(2^{t+1})}$. This concludes the proof.

3.5 **Proof of Theorem 6**

Finally, we gather all our results and combine them, in a way to complete the proof of Theorem 6. Thanks to Lemmas 7-10, calculations can be done in a way to have an explicit formula for f(B,t) and a bound on its value.

Proof of Theorem 6. Let G be a binary SP game with dimension $t, B \in \mathbb{N}$, and $\sigma_0 \in$ SPS(G,B) be a solution. We assume that $C_{\sigma_0} \neq \{(\infty,\ldots,\infty)\}$, as Theorem 6 is trivially true in case $C_{\sigma_0} = \{(\infty, \dots, \infty)\}$. By Lemmas 7-10, there exists $\sigma'_0 \in SPS(G, B)$ without cycles such that $\sigma'_0 \leq \sigma_0$ and

$$\forall c' \in C_{\sigma'_o}, \forall i \in \{1, \dots, t\}: c'_i \leq f(B, t) \lor c'_i = \infty,$$

where

- \blacksquare in dimension t = 1: $f(B, 1) \leq B + |V|$,
- in dimension t+1 (under the induction hypothesis for t):
 - (a) $f(B, t+1) \le \max\{c'_{min}, B\} + 1 + f(0, t),$
 - (b) $c'_{min} \leq B + 2^{t+1} (|V| \cdot \log_2(|C'_{\sigma_0}|) + 1 + f(0, t)),$ (c) $|C'_{\sigma_0}| \leq (f(0, t) + B + 3)^{t+1}.$

We have to prove that $f(B,t) < 2^{\Theta(t^2)}|V|^{\Theta(t)}(B+3)$ for all t > 1. This is true when t = 1. Let us suppose that it is true for t, and let us prove that it remains true for t+1. Let us begin with the factor $|V| \cdot \log_2(|C'_{\sigma_0}|) + 1 + f(0,t)$ appearing in the bound on c'_{min} :

$$\begin{split} |V| \cdot \log_2(|C'_{\sigma_0}|) + 1 + f(0,t) &\leq \\ &\leq |V| \cdot (t+1) \cdot \log_2(f(0,t) + B + 3) + 1 + f(0,t) \quad \text{by (c)} \\ &\leq |V| \cdot (t+1) \cdot (f(0,t) + B + 3) + 1 + f(0,t) \quad \text{as } \log_2(x) \leq x \\ &\leq (f(0,t) + B + 3) \cdot |V| \cdot (t+2) \quad \text{(d)}. \end{split}$$

Let us now compute a bound on f(B, t + 1):

$$\begin{split} &f(B,t+1) \leq \max\{c'_{min},B\} + 1 + f(0,t) & \text{by (a)} \\ &\leq B + 2^{t+1} \Big((f(0,t) + B + 3) \cdot |V| \cdot (t+2) + 1 + f(0,t) \Big) & \text{by (b) and (d)} \\ &\leq 2^{t+1} \big(f(0,t) + B + 3 \big) \cdot |V| \cdot (t+3) & \text{(e)}. \end{split}$$

It follows that f(B, t+1) can be computed thanks to f(0,t) bounded by induction:

$$\begin{split} &f(B,t+1) \leq \\ &\leq 2^{t+1} \left(3 \cdot 2^{\Theta(t^2)} \cdot |V|^{\Theta(t)} + B + 3 \right) \cdot |V| \cdot (t+3) & \text{by (e) and (2)} \\ &\leq 2^{\Theta((t+1)^2)} \cdot |V|^{\Theta(t+1)} \cdot (B+3) & \text{as } t+3 \leq 2^{t+3}. \end{split}$$

This completes the proof of Theorem 6 by induction on t.

Thanks to Lemmas 3, 9 and Theorem 6, we easily get a bound for $|C_{\sigma_0}|$ depending on G and B, as stated in the next proposition.

▶ Proposition 12. For all games $G \in \mathsf{Games}_t$ and for all bounded¹³ solutions $\sigma_0 \in SPS(G, B)$, the size $|C_{\sigma_0}|$ is either equal to 1 or bounded exponentially by $2^{\Theta(t^3)} \cdot (|V| \cdot W)^{\Theta(t^2)} \cdot (B+3)^{\Theta(t)}$.

In the sequel, we use the same notation f(B,t) for any SP games, having in mind that |V| has to be multiplied by W when the game arena is not binary.

¹³The notion of bounded solution has been defined below Theorem 6.

4 Complexity of the SPS Problem

In this section, we prove our main result (Theorem 2). It follows the same pattern as for Boolean reachability [14], however it requires the results of Section 3 (which is meaningless in the Boolean case) and some modifications to handle quantitative reachability. For this purpose, we first show that if there exists a solution to the SPS problem, then there is one that is finite-memory and whose memory size is bounded exponentially.

▶ Proposition 13. Let G be an SP game, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a solution. Then there exists a bounded solution $\sigma'_0 \in SPS(G, B)$ such that σ'_0 is finite-memory and its memory size is bounded exponentially.

When $C_{\sigma_0} \neq \{(\infty, ..., \infty)\}$, the proof of this proposition is based on the following principles, that are detailed below (the case $C_{\sigma_0} = \{(\infty, ..., \infty)\}$ is treated separately).

- We first transform the arena of G into a binary arena and adapt the given solution $\sigma_0 \in SPS(G, B)$ to the new game. We keep the same notations G and σ_0 . We can suppose that σ_0 is bounded by Theorem 6. We consider a set of witnesses Wit σ_0 .
- We show that at any deviation¹⁴, Player 0 can switch to a *punishing strategy* that imposes that the consistent plays π either satisfy $val(\pi) \leq B$ or $cost(\pi)$ is not Pareto-optimal. Moreover, this punishing strategy is finite-memory with an exponential memory.
- We then show how to transform the witnesses into lassos, and how they can be produced by a finite-memory strategy with exponential memory. We also show that we need at most exponentially many different punishing strategies.

In this way, we get a strategy solution to the SPS problem whose memory size is exponential.

Punishing strategies. Let σ_0 be a bounded solution to the SPS Problem. By Theorem 6, we get that $c_i \leq f(B,t)$ or $c_i = \infty$ for all $c \in C_{\sigma_0}$ and all $i \in \{1,\ldots,t\}$. Moreover, if a play ρ is Pareto-optimal, then $\mathsf{val}(\rho) \leq B$. We define for each history $g \in \mathsf{Hist}_{\sigma_0}$ its record $\mathsf{rec}(h) = (w(h), \mathsf{val}(h), \mathsf{cost}(h))$ whose values are $\mathit{truncated}$ to \top if they are greater than f(B,t).

We define for each deviation hv a punishing strategy $\tau_{v,\mathsf{rec}(hv)}^{\mathsf{Pun}}$ as stated in the next lemma.

▶ **Lemma 14.** Let G be an SP game, $B \in \mathbb{N}$, and $\sigma_0 \in SPS(G, B)$ be a bounded solution. Suppose $C_{\sigma_0} \neq \{(\infty, \dots, \infty)\}$. Let hv be a deviation such that $val(h) = \infty$ (resp. $val(h) < \infty$). Then there exists a finite-memory strategy $\tau_{v, rec(hv)}^{Pun}$ with an exponential memory size (resp. with size 1) such that $\sigma'_0 = \sigma_0[hv \to \tau_{v, rec(hv)}^{Pun}]$ is also a solution in SPS(G, B).

Proof. Let hv be a deviation such that $\operatorname{val}(h) = \infty$, that is, h does not visit Player 0's target. As σ_0 is a solution, notice that $\sigma_{0|hv}$ imposes to each consistent play π in $G_{|hv}$ to satisfy $\operatorname{val}(\pi) \leq B$ or $\operatorname{cost}(\pi) > c$ for some $c \in C_{\sigma_0}$. We are going to replace $\sigma_{0|hv}$ by a winning strategy in a zero-sum game H with an exponential arena and an omegaregular objective that is equivalent to what $\sigma_{0|hv}$ imposes to plays. The arena of H is the arena of G extended with the record $\operatorname{rec}(g)$ of the current history g. More precisely, vertices are of the form $(v, (m_1, m_2, m_3))$ with $v \in V$, $m_1, m_2 \in \{0, \ldots, f(B, t)\} \cup \{\top\}$, and $m_3 \in (\{0, \ldots, f(B, t)\} \cup \{\top\})^t$, such that, whenever v belongs to some target, the weight component m_1 allows to update the (truncated) val component m_2 and the (truncated) cost component m_3 . The initial vertex of H is equal to $(v, \operatorname{rec}(hv))$. This arena is finite and of exponential size by the way the values of (m_1, m_2, m_3) are truncated and by Theorem 6.

¹⁴We recall that a deviation is a history hv with $h \in \mathsf{Hist}^1$, $v \in V$, such that h is prefix of some witness, but hv is prefix of no witness.

The omega regular objective of H is the disjunction between (1) the reachability objective $\{(v,(m_1,m_2,m_3))\mid m_2\leq B\}$, and (2) the safety objective $\{(v,(m_1,m_2,m_3))\mid m_3>c$ for some $c\in C_{\sigma_0}\}$. It is known, see e.g. [13], that if there exists a winning strategy for zero-sum games with an objective which is the disjunction of a reachability objective and a safety objective, then there exists one that is memoryless. This is the case here for the extended game H: as $\sigma_{0|hv}$ is winning in H, there exists a winning memoryless strategy in H, and thus a winning finite-memory strategy $\tau_{v,\text{rec}(hv)}^{\text{Pun}}$ with exponential size in the original game. Therefore, the strategy $\sigma_0[hv\to\tau_{v,\text{rec}(hv)}^{\text{Pun}}]$ is again a solution in SPS(G,B).

Suppose now that hv is a deviation with $val(h) < \infty$. As h already visits Player 0's target, we can use, in place of $\sigma_{0|hv}$, any memoryless strategy $\tau_{v,\text{rec}(hv)}^{\text{Pun}}$ as deviating strategy.

Lasso witnesses. To get Proposition 13, we show one can play with an exponential finite-memory strategy over the witness tree and an exponential number of deviating strategies.

Proof of Proposition 13. (1) We first suppose that $C_{\sigma_0} \neq \{(\infty, ..., \infty)\}$. According to Theorem 6, from $\sigma_0 \in SPS(G, B)$, one can construct a solution $\sigma'_0 \in SPS(G, B)$ that is bounded, thus without cycles. Consider a set of witnesses $Wit_{\sigma'_0}$ for this strategy, and the region decomposition of its witnesses. Along any witness ρ , according to Lemma 5, it is thus impossible to eliminate cycles. Recall that Lemma 5 only considers cycles implying histories in the same region, those cycles with a null weight before visiting Player 0's target; moreover, the last region of ρ is excluded. Let us study the memory used by σ'_0 . Along a witness ρ :

- As $\mathsf{val}(\rho) \leq B$ and there is no cycle with a null weight before visiting Player 0's target, it follows that the smallest h such that $\mathsf{val}(h) = \mathsf{val}(\rho)$ has a length $|h| \leq |V| \cdot B \cdot W$ (the presence of W comes from the fact that the arena has been made binary).
- Once Player 0's target is visited and before the last region, σ'_0 is "locally" memoryless inside each region, as there is no cycle.
- In the last region, as soon as a vertex is repeated, we replace σ'_0 by a memoryless strategy that repeats this cycle forever. We then get a lasso replacing ρ that has the same value and cost. We keep the same notation ρ for this lasso. We also keep the notation Wit σ'_0 for the set of lasso witnesses. Notice that for deviations hv such that h belongs to the last region of a witness, the strategy $\tau^{\mathsf{Pun}}_{v,\mathsf{rec}(hv)}$ is memoryless (as $\mathsf{val}(h) < \infty$, see Lemma 14). Hence the modification of the witnesses has no impact on the deviating strategies.

Let us study the memory necessary for σ'_0 in order to produce the new set $\operatorname{Wit}_{\sigma'_0}$: (a) The number of regions traversed by a witness ρ is bounded by $(t+2) \cdot |\operatorname{Wit}_{\sigma'_0}|$. Indeed, the number of visited targets increases from 0 to t+1 and the set $\operatorname{Wit}_{\sigma'_0}(h)$ decreases until being equal to $\{\rho\}$. As there are $|C_{\sigma'_0}|$ witnesses, the total number of traversed regions is bounded by $(t+2) \cdot |C_{\sigma'_0}|^2$ which is exponential by Proposition 12. (b) On the first regions traversed by a witness ρ , σ'_0 has to memorize the current history $h \sqsubseteq \rho$ until $|h| = |V| \cdot B \cdot W$, which needs an exponential memory. Then, on each other region traversed by ρ , σ'_0 is locally memoryless. Therefore, the memory necessary for σ'_0 to produce the lasso witnesses is exponential, as there is an exponential number of regions on which σ'_0 needs an exponential memory.

It remains to explain how to play with a finite-memory strategy in case of deviations from the witnesses. With each history g, we associate its record $\operatorname{rec}(g)$ whose second component indicates whether $\operatorname{val}(g) < \infty$ or $\operatorname{val}(g) = \infty$. Let hv be a deviation. We apply Lemma 14 to replace $\sigma'_{0|hv}$ by the punishing strategy $\tau^{\operatorname{Pun}}_{v,\operatorname{rec}(hv)}$. According to $\operatorname{val}(h)$, this punishing strategy is either finite-memory with an exponential size or it is memoryless. Notice that given a deviation hv, a punishing strategy only depends on the last vertex v of hv and its record $\operatorname{rec}(hv)$. Therefore, we have an exponential number of punishing strategies by Theorem 6.

All in all, when $C_{\sigma_0} \neq \{(\infty, ..., \infty)\}$, we get from σ_0 a solution to the SPS problem that uses a memory of exponential size.

(2) Suppose that $C_{\sigma_0} = \{(\infty, \dots, \infty)\}$. This means that each play $\rho \in \mathsf{Play}_{\sigma_0}$ visits no target of Player 1 and that σ_0 can impose to each such ρ to have a value $\mathsf{val}(\rho) \leq B$. As done in Lemma 14, we can replace σ_0 by a strategy corresponding to a winning strategy in a zero-sum game H with an exponential arena and a reachability objective. The arena of H has vertices of the form $(v, (m_1, m_2))$ with $v \in V$, $m_1, m_2 \in \{0, \dots, B\} \cup \{\top\}$, such that, whenever v belongs to Player 0's target, the weight component m_1 allows to update the val component m_2 . The initial vertex of H is equal to $(v_0, 0, \mathsf{val}(v_0))$. This arena has an exponential size and its reachability objective is $\{(v, (m_1, m_2)) \mid m_2 \leq B\}$. It is known that when there is a winning strategy in a zero-sum game with a reachability objective, then there is one that is memoryless [23]. Hence, coming back to G, σ_0 can be replaced by a finite-memory strategy with exponential memory.

NEXPTIME-completeness. We now prove that the SPS problem is NEXPTIME-complete.

Proof of Theorem 2. (1) NEXPTIME-membership. Let G be an SP game and $B \in \mathbb{N}$. Proposition 13 states the existence of a solution $\sigma_0 \in \operatorname{SPS}(G, B)$ that uses a finite memory bounded exponentially. We can guess such a strategy σ_0 as a Mealy machine \mathcal{M} with a set \mathcal{M} of memory states at most exponential in the size of the instance. This can be done in exponential time. Let us explain how to verify in exponential time that the guessed strategy σ_0 is a solution to the SPS problem.

We make the following important observation: it is enough to consider Pareto-optimal costs c whose components $c_i < \infty$ belong to $\{0, \ldots, |V| \cdot |M| \cdot t \cdot W\}$. Let us explain why:

- Consider the cartesian product $G \times \mathcal{M}$ whose infinite paths are exactly the plays consistent with σ_0 . This product has an arena of size $|V| \cdot |M|$ where Player 1 is the only player to play. The Pareto-optimal costs are among the costs of plays ρ in $G \times \mathcal{M}$ that have no cycle with positive weight between two consecutive visits of Player 1's targets¹⁶.
- Consider now a Pareto-optimal play ρ with a cycle of null weight between two consecutive visits of Player 1's targets. Then there exists another play ρ' , with the same cost as ρ , that is obtained by removing this cycle. Moreover, if $val(\rho) = \infty$, then $val(\rho') = \infty$.
- Therefore, it is enough to consider plays ρ such that for $h \sqsubseteq \rho$ with length $|h| = |V| \cdot |M| \cdot t$, we have $\mathsf{cost}(\rho) = \mathsf{cost}(h)$ (the worst case happens when ρ visit all Player 1's targets each of them separated by a longest path without cycle). It follows that $\mathsf{cost}(\rho) \le |V| \cdot |M| \cdot t \cdot W$.

From the previous observation, let us explain how to compute the set C_{σ_0} of Pareto-optimal costs, and then how to check that σ_0 is a solution.

- First, we further extend the vertices of $G \times \mathcal{M}$ to keep track of the weight, value and cost of the current history, truncated to \top when they are greater than $\alpha = \max\{B, |V| \cdot |M| \cdot t \cdot W\}$). That is, we consider an arena H whose vertices are of the form $(v, s, (m_1, m_2, m_3))$ with $v \in V$, $s \in M$, $m_1, m_2 \in \{0, \ldots, \alpha\} \cup \{\top\}$, and $m_3 \in (\{0, \ldots, \alpha\} \cup \{\top\})^t$. As in the proof of Lemma 14, whenever v belongs to some target, the weight component m_1 allows to update the value component m_2 and the cost component m_3 . The initial vertex is $(v_0, s_0, (0, \mathsf{val}(v_0), \mathsf{cost}(v_0)))$ where s_0 is the initial memory state of \mathcal{M} .
- Then, to compute C_{σ_0} , we test for the existence of a play $\rho \in \mathsf{Play}_{\sigma_0}$ with a given cost $c = \mathsf{cost}(\rho)$, beginning with the smallest possible cost $c = (0, \ldots, 0)$, and finishing with the largest possible one $c = (\top, \ldots, \top)$. Deciding the existence of such a play ρ for cost c

¹⁶ or between the initial vertex and the first visit to some Player 1's target.

corresponds to deciding the existence of a play in the extended arena H that visits some vertex $(v, s, (m_1, m_2, m_3))$ with $m_3 = c$. This corresponds to a reachability objective that can be checked in polynomial time in the size of H, thus in exponential time in the size of the given instance. Therefore, as there is at most an exponential number of costs c to consider, the set C_{σ_0} can be computed in exponential time.

- Finally, we check whether σ_0 is not a solution, i.e., there exists a play ρ in H with a cost $c \in C_{\sigma_0}$ such that $\mathsf{val}(\rho) > B$. We remove from H all vertices $(v, s, (m_1, m_2, m_3))$ such that $m_2 \leq B$, and we then check the existence of a play with a cost $c \in C_{\sigma_0}$ as done in the previous item. Checking that σ_0 is a solution can thus be done in exponential time.
- (2) NEXPTIME-hardness. In [14], the Boolean variant of the SPS problem is proved to be NEXPTIME-complete. It can be reduced to its quantitative variant by labeling each edge with a weight equal to 0 and by considering a bound B equal to 0. Hence the value and cost components are either equal to 0 or ∞ . It follows that the (quantitative) SPS problem is NEXPTIME-hard.

5 Conclusion and Future Work

In [14], the SPS problem is proved to be NEXPTIME-complete for Boolean reachability. In this paper, we proved that the same result holds for quantitative reachability (with non-negative weights). The difficult part was to show that when there exists a solution to the SPS problem, there is one whose Pareto-optimal costs are exponentially bounded.

Considering negative weights is a non-trivial work that is deferred to future work. It will require to study how cycles with a negative cost are useful to improve a solution. Considering multiple objectives for Player 0 (instead of one) is also a non-trivial problem. The order on the tuples of values becomes partial and we could consider several weight functions.

It is well-known that quantitative objectives make it possible to model richer properties than with Boolean objectives. This paper studied quantitative reachability. It would be very interesting to investigate the SPS problem for other quantitative payoffs, like mean-payoff or discounted sum.

References

- Mrudula Balachander, Shibashis Guha, and Jean-François Raskin. Fragility and robustness in mean-payoff adversarial stackelberg games. In Serge Haddad and Daniele Varacca, editors, 32nd International Conference on Concurrency Theory, CONCUR 2021, August 24-27, 2021, Virtual Conference, volume 203 of LIPIcs, pages 9:1-9:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.CONCUR.2021.9.
- 2 Dietmar Berwanger. Admissibility in infinite games. In Wolfgang Thomas and Pascal Weil, editors, STACS 2007, 24th Annual Symposium on Theoretical Aspects of Computer Science, Aachen, Germany, February 22-24, 2007, Proceedings, volume 4393 of Lecture Notes in Computer Science, pages 188-199. Springer, 2007. doi:10.1007/978-3-540-70918-3_17.
- 3 Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann. Graph games and reactive synthesis. In Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, editors, *Handbook of Model Checking*, pages 921–962. Springer, 2018. doi:10.1007/978-3-319-10575-8_27.
- 4 Romain Brenguier, Lorenzo Clemente, Paul Hunter, Guillermo A. Pérez, Mickael Randour, Jean-François Raskin, Ocan Sankur, and Mathieu Sassolas. Non-zero sum games for reactive synthesis. In Adrian-Horia Dediu, Jan Janousek, Carlos Martín-Vide, and Bianca Truthe, editors, Language and Automata Theory and Applications - 10th International Conference,

- LATA 2016, Prague, Czech Republic, March 14-18, 2016, Proceedings, volume 9618 of Lecture Notes in Computer Science, pages 3-23. Springer, 2016. doi:10.1007/978-3-319-30000-9\ 1.
- 5 Romain Brenguier, Jean-François Raskin, and Ocan Sankur. Assume-admissible synthesis. In Luca Aceto and David de Frutos-Escrig, editors, 26th International Conference on Concurrency Theory, CONCUR 2015, Madrid, Spain, September 1.4, 2015, volume 42 of LIPIcs, pages 100–113. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2015. doi:10.4230/LIPIcs. CONCUR.2015.100.
- 6 Léonard Brice, Jean-François Raskin, and Marie van den Bogaard. Subgame-perfect equilibria in mean-payoff games. In Serge Haddad and Daniele Varacca, editors, 32nd International Conference on Concurrency Theory, CONCUR 2021, August 24-27, 2021, Virtual Conference, volume 203 of LIPIcs, pages 8:1–8:17. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.CONCUR.2021.8.
- 7 Léonard Brice, Jean-François Raskin, and Marie van den Bogaard. The complexity of spes in mean-payoff games. In Mikolaj Bojanczyk, Emanuela Merelli, and David P. Woodruff, editors, 49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France, volume 229 of LIPIcs, pages 116:1–116:20. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.ICALP.2022.116.
- 8 Léonard Brice, Jean-François Raskin, and Marie van den Bogaard. On the complexity of spes in parity games. In Florin Manea and Alex Simpson, editors, 30th EACSL Annual Conference on Computer Science Logic, CSL 2022, February 14-19, 2022, Göttingen, Germany (Virtual Conference), volume 216 of LIPIcs, pages 10:1-10:17. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.CSL.2022.10.
- 9 Léonard Brice, Jean-François Raskin, and Marie van den Bogaard. Rational verification and checking for nash and subgame-perfect equilibria in graph games. CoRR, abs/2301.12913, 2023. arXiv:2301.12913, doi:10.48550/arXiv.2301.12913.
- Thomas Brihaye, Véronique Bruyère, Aline Goeminne, Jean-François Raskin, and Marie van den Bogaard. The complexity of subgame perfect equilibria in quantitative reachability games. Log. Methods Comput. Sci., 16(4), 2020. URL: https://lmcs.episciences.org/6883.
- Véronique Bruyère. Synthesis of equilibria in infinite-duration games on graphs. ACM SIGLOG News, 8(2):4–29, 2021. doi:10.1145/3467001.3467003.
- Véronique Bruyère, Baptiste Fievet, Jean-François Raskin, and Clément Tamines. Stackelberg-pareto synthesis (extended version). CoRR, abs/2203.01285, 2022. arXiv:2203.01285, doi: 10.48550/arXiv.2203.01285.
- Véronique Bruyère, Quentin Hautem, and Jean-François Raskin. Parameterized complexity of games with monotonically ordered omega-regular objectives. In Sven Schewe and Lijun Zhang, editors, 29th International Conference on Concurrency Theory, CONCUR 2018, September 4-7, 2018, Beijing, China, volume 118 of LIPIcs, pages 29:1–29:16. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.CONCUR.2018.29.
- Véronique Bruyère, Jean-François Raskin, and Clément Tamines. Stackelberg-pareto synthesis. In Serge Haddad and Daniele Varacca, editors, 32nd International Conference on Concurrency Theory, CONCUR 2021, August 24-27, 2021, Virtual Conference, volume 203 of LIPIcs, pages 27:1-27:17. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.CONCUR.2021.27.
- Véronique Bruyère, Jean-François Raskin, and Clément Tamines. Pareto-rational verification. In Bartek Klin, Slawomir Lasota, and Anca Muscholl, editors, 33rd International Conference on Concurrency Theory, CONCUR 2022, September 12-16, 2022, Warsaw, Poland, volume 243 of LIPIcs, pages 33:1–33:20. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.CONCUR.2022.33.
- Krishnendu Chatterjee and Thomas A. Henzinger. Assume-guarantee synthesis. In Orna Grumberg and Michael Huth, editors, Tools and Algorithms for the Construction and Analysis of Systems, 13th International Conference, TACAS 2007, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2007 Braga, Portugal, March 24 -

- April 1, 2007, Proceedings, volume 4424 of Lecture Notes in Computer Science, pages 261–275. Springer, 2007. doi:10.1007/978-3-540-71209-1_21.
- 17 Krishnendu Chatterjee, Thomas A. Henzinger, and Marcin Jurdzinski. Games with secure equilibria. *Theor. Comput. Sci.*, 365(1-2):67–82, 2006. doi:10.1016/j.tcs.2006.07.032.
- 18 Edmund M. Clarke, Orna Grumberg, Daniel Kroening, Doron A. Peled, and Helmut Veith. Model checking, 2nd Edition. MIT Press, 2018. URL: https://mitpress.mit.edu/books/model-checking-second-edition.
- Rodica Condurache, Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin. The complexity of rational synthesis. In Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, editors, 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy, volume 55 of LIPIcs, pages 121:1-121:15. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPIcs.ICALP.2016.121.
- 20 Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin. The adversarial Stackelberg value in quantitative games. In Artur Czumaj, Anuj Dawar, and Emanuela Merelli, editors, 47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference), volume 168 of LIPIcs, pages 127:1–127:18. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.ICALP.2020.127.
- 21 Dana Fisman, Orna Kupferman, and Yoad Lustig. Rational synthesis. In Javier Esparza and Rupak Majumdar, editors, Tools and Algorithms for the Construction and Analysis of Systems, 16th International Conference, TACAS 2010, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2010, Paphos, Cyprus, March 20-28, 2010. Proceedings, volume 6015 of Lecture Notes in Computer Science, pages 190-204. Springer, 2010. doi:10.1007/978-3-642-12002-2_16.
- János Flesch and Arkadi Predtetchinski. A characterization of subgame-perfect equilibrium plays in borel games of perfect information. *Math. Oper. Res.*, 42(4):1162–1179, 2017. doi: 10.1287/moor.2016.0843.
- 23 Erich Grädel, Wolfgang Thomas, and Thomas Wilke, editors. Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001], volume 2500 of Lecture Notes in Computer Science. Springer, 2002. doi:10.1007/3-540-36387-4.
- Erich Grädel and Michael Ummels. Solution Concepts and Algorithms for Infinite Multiplayer Games, pages 151-178. Amsterdam University Press, 2008. URL: http://www.jstor.org/stable/j.ctt46mwfz.11.
- Anshul Gupta and Sven Schewe. Quantitative verification in rational environments. In Amedeo Cesta, Carlo Combi, and François Laroussinie, editors, 21st International Symposium on Temporal Representation and Reasoning, TIME 2014, Verona, Italy, September 8-10, 2014, pages 123–131. IEEE Computer Society, 2014. doi:10.1109/TIME.2014.9.
- Anshul Gupta, Sven Schewe, and Dominik Wojtczak. Making the best of limited memory in multi-player discounted sum games. In Javier Esparza and Enrico Tronci, editors, *Proceedings Sixth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2015, Genoa, Italy, 21-22nd September 2015*, volume 193 of *EPTCS*, pages 16–30, 2015. doi:10.4204/EPTCS.193.2.
- 27 Julian Gutierrez, Muhammad Najib, Giuseppe Perelli, and Michael J. Wooldridge. Automated temporal equilibrium analysis: Verification and synthesis of multi-player games. Artif. Intell., 287:103353, 2020. doi:10.1016/j.artint.2020.103353.
- 28 Julian Gutierrez, Thomas Steeples, and Michael J. Wooldridge. Mean-payoff games with ω-regular specifications. Games, 13(1):19, 2022. doi:10.3390/g13010019.
- Orna Kupferman, Giuseppe Perelli, and Moshe Y. Vardi. Synthesis with rational environments. *Ann. Math. Artif. Intell.*, 78(1):3–20, 2016. doi:10.1007/s10472-016-9508-8.
- 30 Orna Kupferman and Noam Shenwald. The complexity of LTL rational synthesis. In Dana Fisman and Grigore Rosu, editors, *Tools and Algorithms for the Construction and Analysis of*

- Systems 28th International Conference, TACAS 2022, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2022, Munich, Germany, April 2-7, 2022, Proceedings, Part I, volume 13243 of Lecture Notes in Computer Science, pages 25-45. Springer, 2022. doi:10.1007/978-3-030-99524-9_2.
- John F. Nash. Equilibrium points in *n*-person games. In *PNAS*, volume 36, pages 48–49. National Academy of Sciences, 1950.
- 32 Martin J. Osborne and Ariel Rubinstein. A course in Game Theory. MIT Press, Cambridge, MA, 1994.
- 33 Reinhard Selten. Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit. Zeitschrift für die gesamte Staatswissenschaft, 121:301–324 and 667–689, 1965.
- 34 Clément Tamines. On Pareto-Optimality for Verification and Synthesis in Games Played on Graphs. PhD thesis, University of Mons, 2022.
- 35 Michael Ummels. Rational behaviour and strategy construction in infinite multiplayer games. In S. Arun-Kumar and Naveen Garg, editors, FSTTCS 2006: Foundations of Software Technology and Theoretical Computer Science, 26th International Conference, Kolkata, India, December 13-15, 2006, Proceedings, volume 4337 of Lecture Notes in Computer Science, pages 212–223. Springer, 2006. doi:10.1007/11944836_21.
- Michael Ummels. The complexity of Nash equilibria in infinite multiplayer games. In Roberto M. Amadio, editor, Foundations of Software Science and Computational Structures, 11th International Conference, FOSSACS 2008, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2008, Budapest, Hungary, March 29 April 6, 2008. Proceedings, volume 4962 of Lecture Notes in Computer Science, pages 20–34. Springer, 2008. doi:10.1007/978-3-540-78499-9_3.
- 37 Michael Ummels and Dominik Wojtczak. The complexity of Nash equilibria in limit-average games. In Joost-Pieter Katoen and Barbara König, editors, CONCUR 2011 Concurrency Theory 22nd International Conference, CONCUR 2011, Aachen, Germany, September 6-9, 2011. Proceedings, volume 6901 of Lecture Notes in Computer Science, pages 482–496. Springer, 2011. doi:10.1007/978-3-642-23217-6_32.
- 38 John von Neumann and Oskar Morgenstern. Theory of Games and Economic Behavior. Princeton University Press, 1944.
- 39 Heinrich Freiherr von Stackelberg. Marktform und Gleichgewicht. Wien und Berlin, J. Springer, Cambridge, MA, 1937.