Automata with Timers

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Many computer systems have timing constraints:

- Network protocols;
- Schedulers;
- Embedded systems;
- In general, real-time systems.

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In short: finite automata augmented with clocks that can be reset or used in guards along transitions and states.

BUT timed automata are hard to construct and understand.

We focus on properties that can be represented with timers: automata with timers.

Timed automata

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Timed automata

- Clocks go from 0 to infinity;
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Automata with timers

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- Well-known model.

Automata with timers

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- We do not know the current value of the timers;
- Automata with timers are more restrictive;
- Future work: learning algorithm;
- This work studies some properties of automata with timers.

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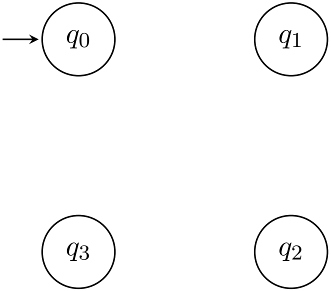

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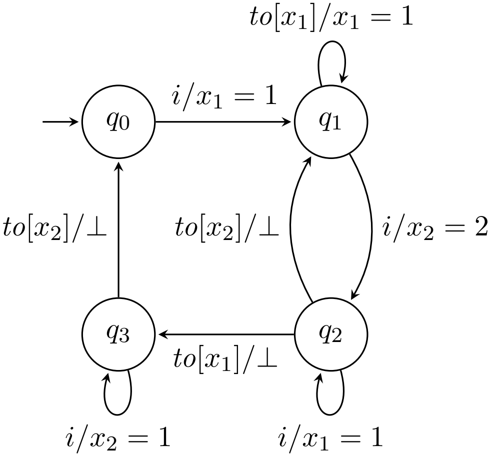

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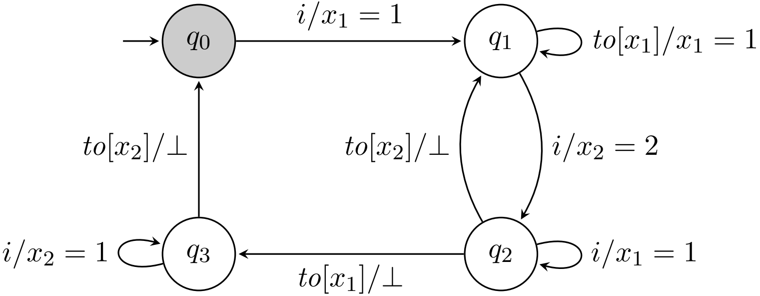
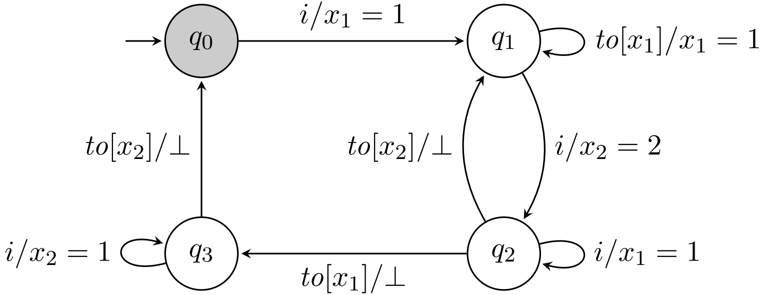


Figure 2: The same AT.

$(q_0, \emptyset)$



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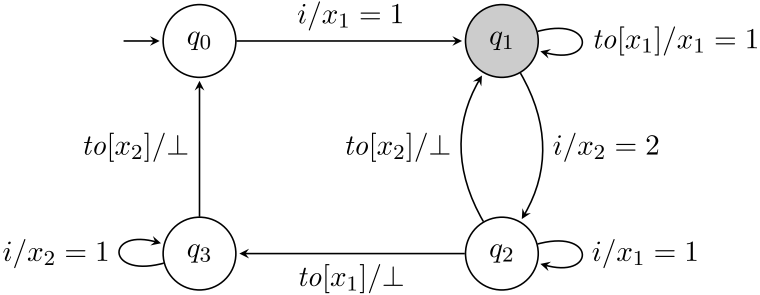
$$(q_0, \emptyset) \xrightarrow{1} (q_0, \emptyset)$$



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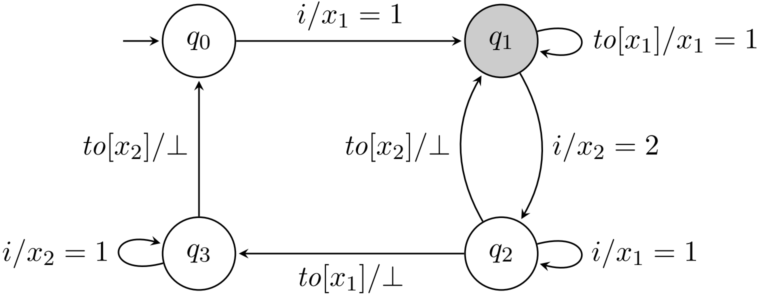
$$(q_0, \emptyset) \xrightarrow{1} (q_0, \emptyset) \xrightarrow[x_1,1]{i} (q_1, x_1 = 1)$$



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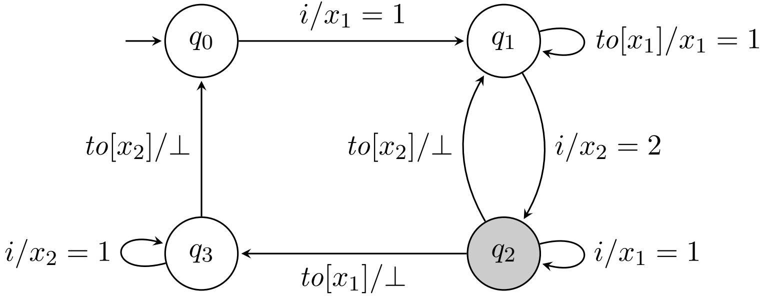

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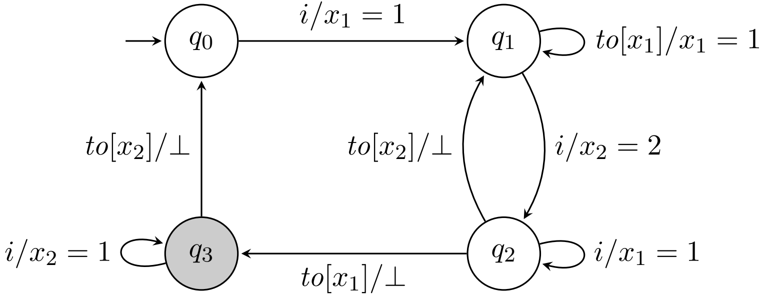

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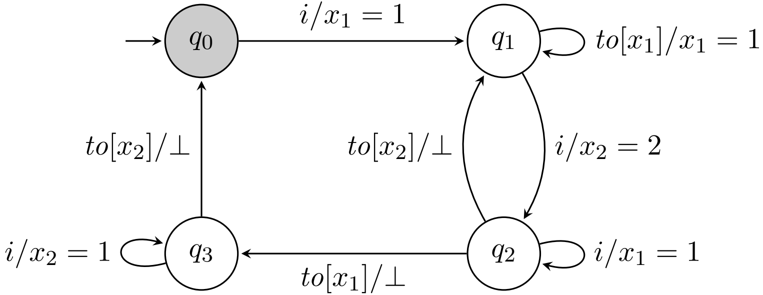

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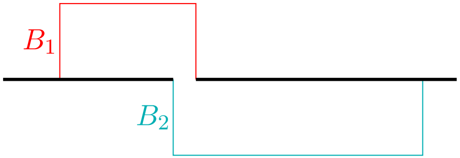

Figure 3: Block representation of the execution.

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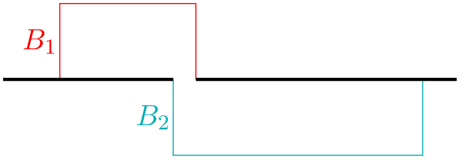

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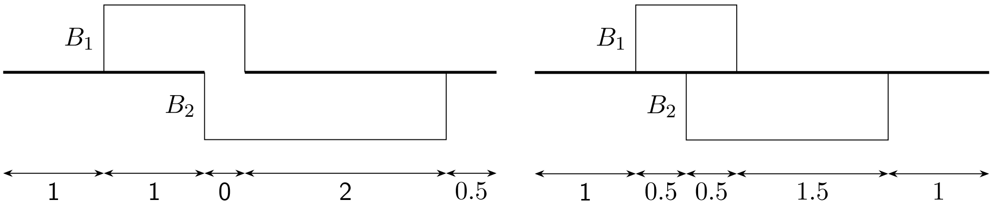
We have concurrent actions.

We can avoid this concurrency and still see the same sequence of actions.



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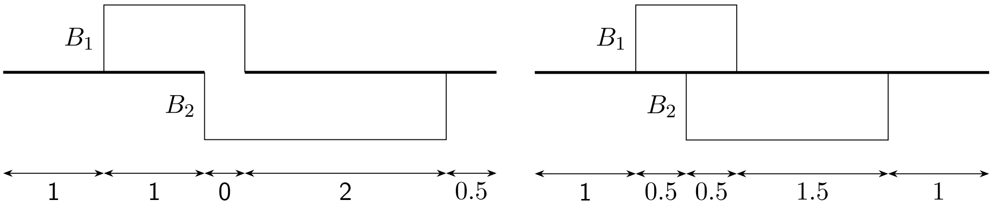

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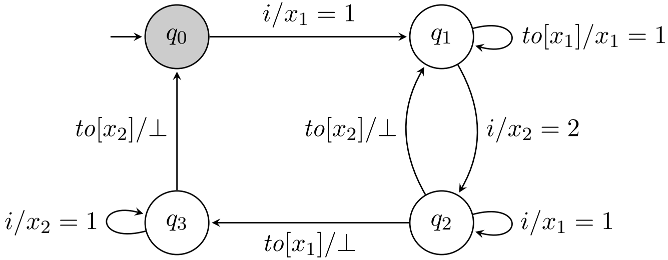
Is it always possible?



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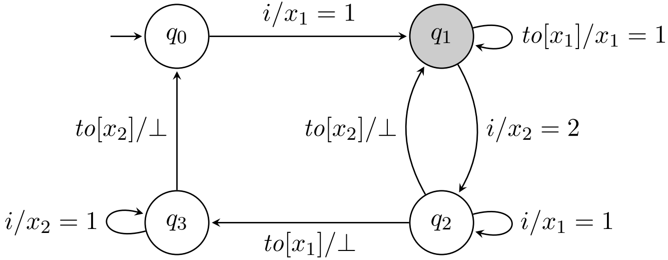

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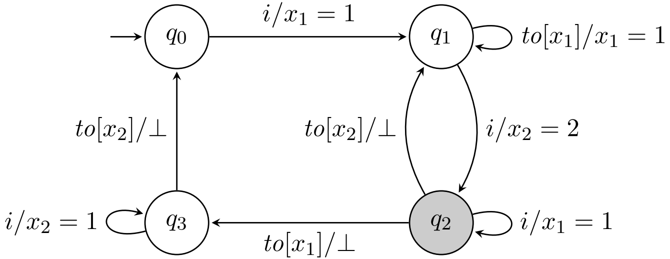

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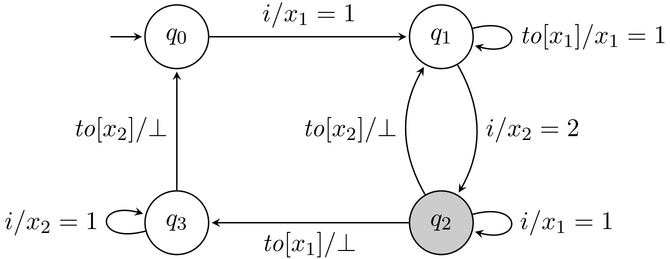

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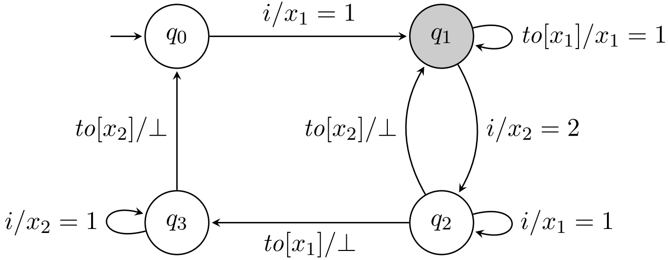

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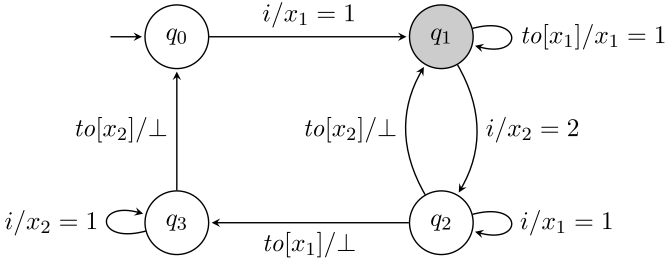

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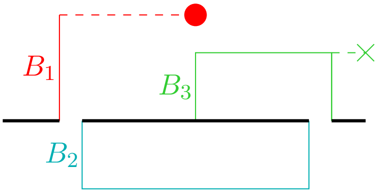

Figure 6: Block representation of the timed run.

$$
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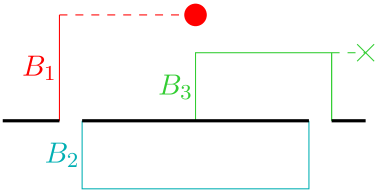

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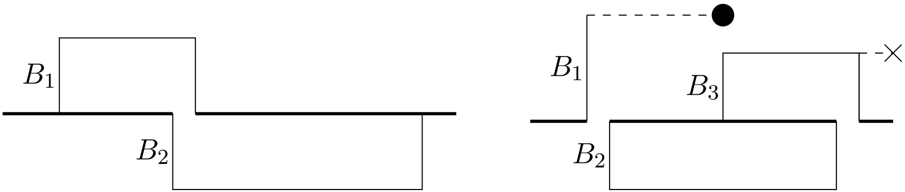
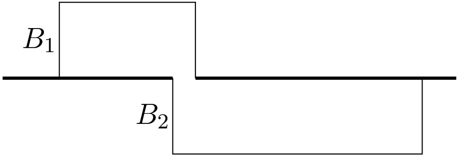


(a) Can be avoided. (b) Cannot be avoided.

Figure 7: Some concurrency can be avoided, some not.



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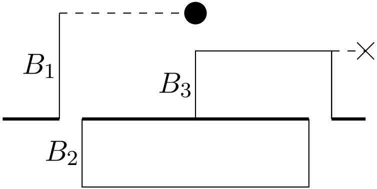

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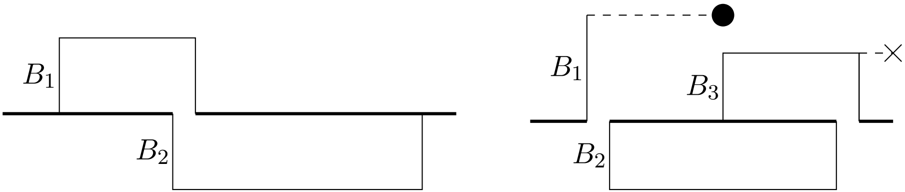
Can we characterize when it is possible to remove the concurrency?



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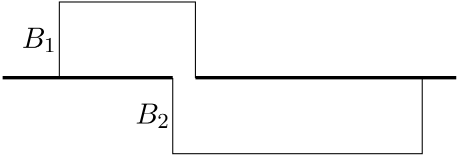
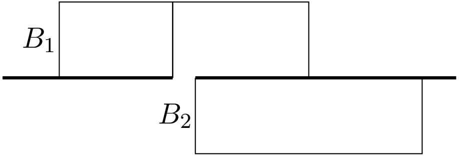
Theorem 2 (Contribution)

Deciding whether an AT contains an execution in which some concurrency cannot be avoided is PSPACE*-hard and in* 3EXP.



(a) Can be wiggled.



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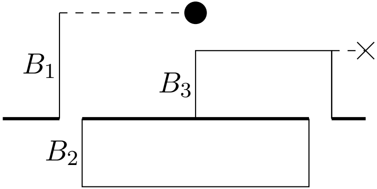
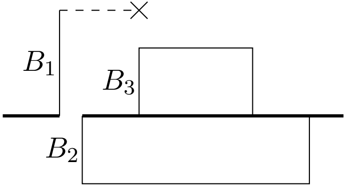
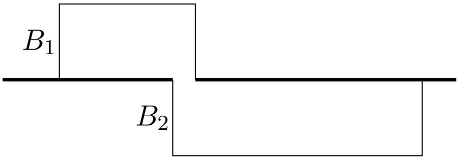


(c) Cannot be wiggled.

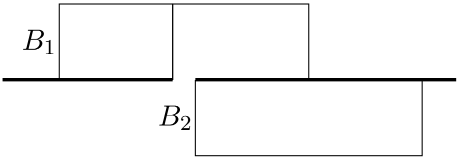


(d) Can be wiggled.

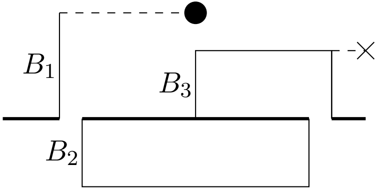
Figure 8: Not all runs can be wiggled.



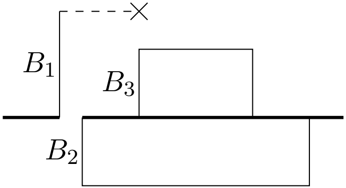
(a) $B_2 \prec B_1$.



(b) $B_1 \prec B_2$.

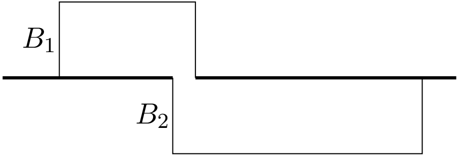


(c) $B_1 \prec B_2, B_3 \prec B_1, B_2 \prec B_3$.



(d) $B_1 \prec B_2$.

Figure 9: Define an order $\prec$ over the blocks, based on races.



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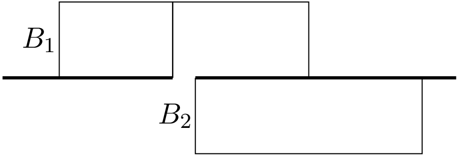

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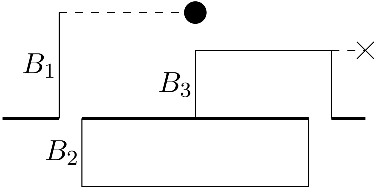
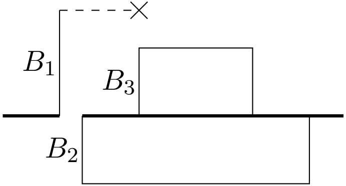


(c) $B_1 \prec B_2, B_3 \prec B_1, B_2 \prec B_3$.



(d) $B_1 \prec B_2$.

Figure 9: Define an order $\prec$ over the blocks, based on races.

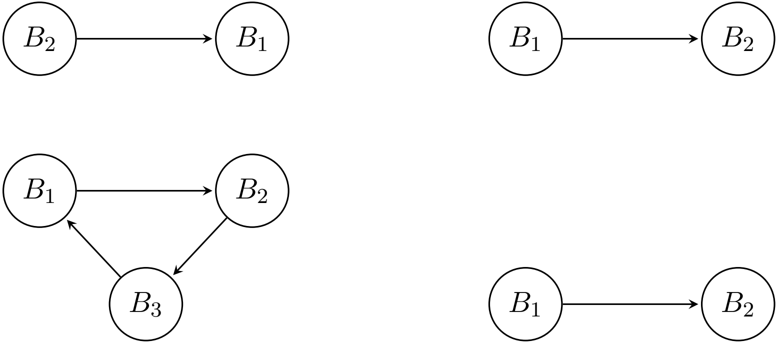


Figure 10: Block graphs defined from the blocks and $\prec$.

Theorem 3 (Contribution)

An AT contains a run that cannot be wiggled if and only if the block graph of that run is cyclic.

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- there exists a run of the region automaton that cannot be wiggled,
- i.e., there are concurrent actions inducing a cyclic block graph.

The formula can be written with three quantifiers alternations $\rightsquigarrow$ 3EXP.

Theorem 4 (Contribution)

Fix an automaton and a state q. Deciding whether there exists an execution of the automaton that reaches q is PSPACE*-complete.*

Theorem 5 (Contribution)

Deciding whether an AT contains an execution in which some concurrency cannot be avoided is PSPACE*-hard and in* 3EXP*.*

Thank you!

For all details, see Bruyère et al., “Automata with Timers”, 2023.

References I

- Angluin, Dana. "Learning Regular Sets from Queries and Counterexamples". In: *Inf. Comput.* 75.2 (1987), pp. 87–106. DOI: 10.1016/0890-5401(87)90052-6.
- Bruyère, Véronique et al. "Automata with Timers". In: *CoRR* abs/2305.07451 (2023). DOI: 10.48550/arXiv.2305.07451. arXiv: 2305.07451. URL: https://doi.org/10.48550/arXiv.2305.07451.

Proposition 6 (Contribution)

A timed run ρ can be wiggled if and only if its block graph is acyclic.

$\Rightarrow$ By contraposition, we have a cycle.
If a block has...

- A predecessor? It cannot move left.
- A successor? It cannot move right.
- Both? It cannot move at all.

Thus, ρ cannot be wiggled since we have a cycle.

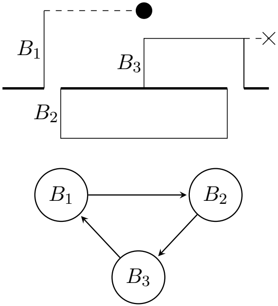


Figure 11: We have a cycle.

Proposition 6 (Contribution)

A timed run ρ can be wiggled if and only if its block graph is acyclic.

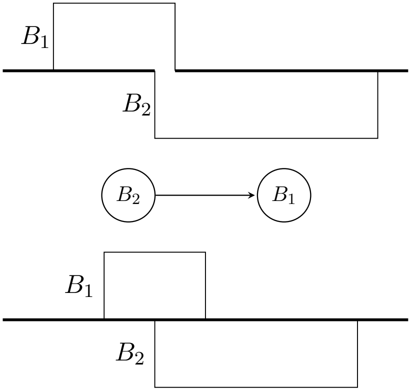
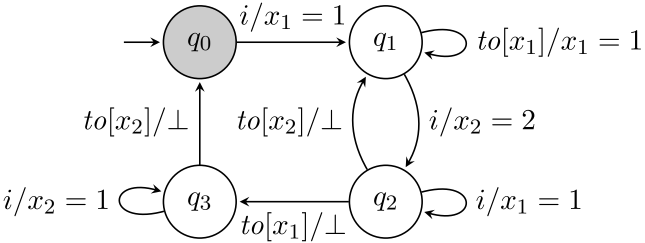
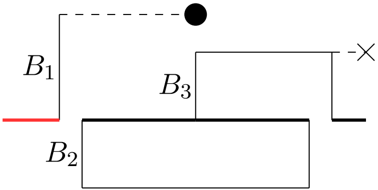


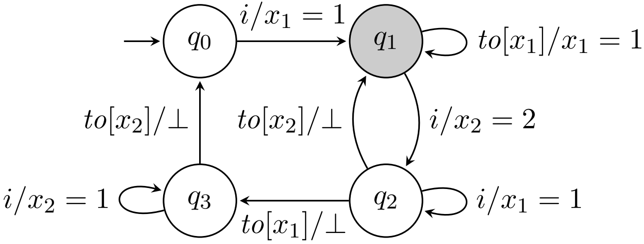
Figure 12: We change delays.

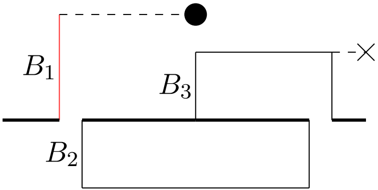
$\Leftarrow$ The graph is acyclic. Compute its topological sort and move the "last" block to the right.
$\hookrightarrow$ obtain ρ' with the same sequence of actions as ρ but ρ' contains strictly less races.
Repeat until all races are removed.



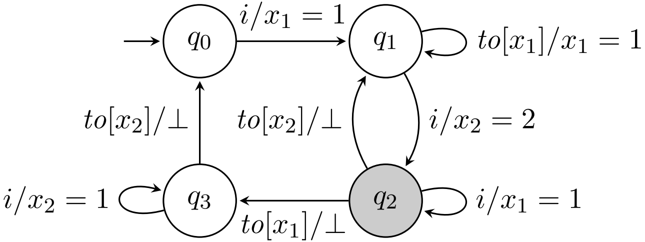


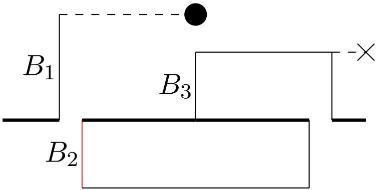
$$(q_0, \emptyset, \emptyset) \xrightarrow{\tau} (q_0, \emptyset, \emptyset)$$



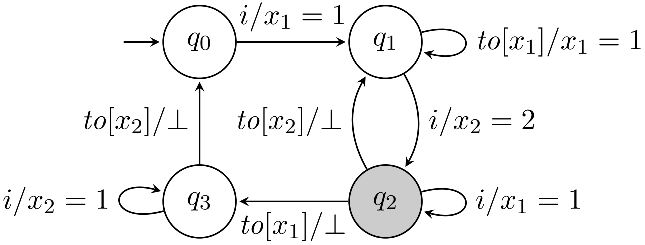


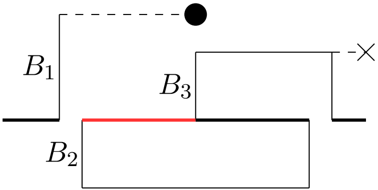
$$(q_0, \emptyset, \emptyset) \xrightarrow{\tau} (q_0, \emptyset, \emptyset) \xrightarrow{(i, x_1)} (q_1, x_1 = 1, \emptyset)$$



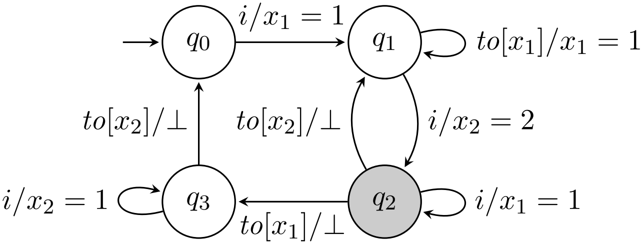


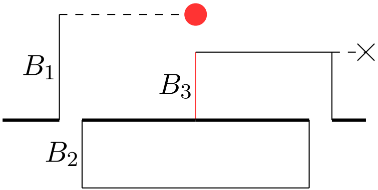
$$(q_0, \emptyset, \emptyset) \xrightarrow{\tau} (q_0, \emptyset, \emptyset) \xrightarrow{(i,x_1)} (q_1, x_1 = 1, \emptyset) \xrightarrow{(i,x_2)} (q_2, x_1 = 1 \wedge x_2 = 2, \emptyset)$$





$$(q_0, \emptyset, \emptyset) \xrightarrow{\tau} (q_0, \emptyset, \emptyset) \xrightarrow{(i, x_1)} (q_1, x_1 = 1, \emptyset) \xrightarrow{(i, x_2)} (q_2, x_1 = 1 \land x_2 = 2, \emptyset)$$
$$\xrightarrow{\tau} (q_2, 0 < x_1 < 1 \land x_2 - x_1 = 1, \emptyset) \xrightarrow{\tau} (q_2, x_1 = 0 \land x_2 = 1, \emptyset)$$





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\begin{aligned}
&(q_0, \emptyset, \emptyset) \xrightarrow{\tau} (q_0, \emptyset, \emptyset) \xrightarrow{(i, x_1)} (q_1, x_1 = 1, \emptyset) \xrightarrow{(i, x_2)} (q_2, x_1 = 1 \wedge x_2 = 2, \emptyset) \\
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\end{aligned}
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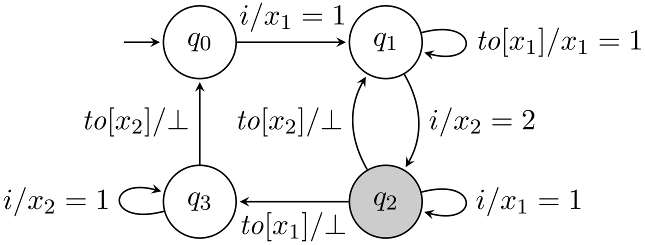

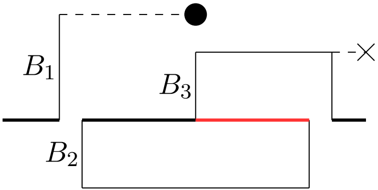

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\begin{aligned}
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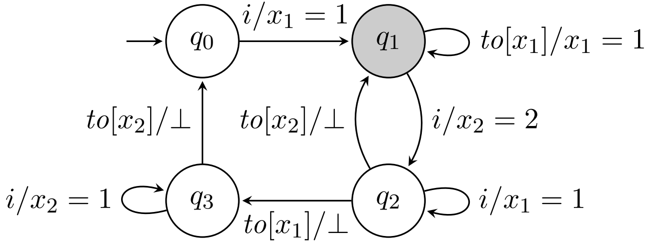

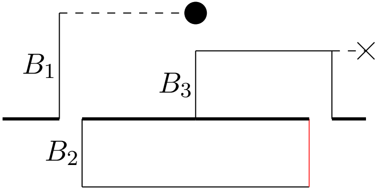

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&\quad \xrightarrow{(to[x_2],\bot)} (q_1, x_1 = 0, \emptyset)
\end{aligned}
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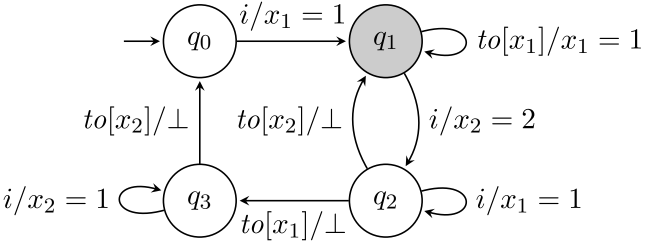

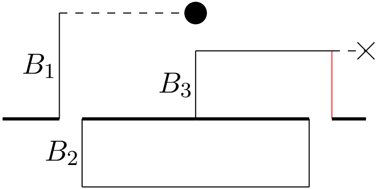

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&\xrightarrow{\tau} (q_2, 0 < x_1 = x_2 < 1, \emptyset) \xrightarrow{\tau} (q_2, x_1 = 0 = x_2, \emptyset) \\
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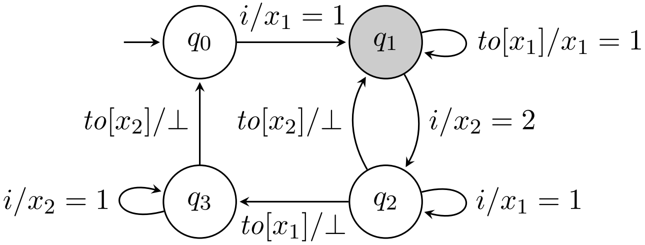
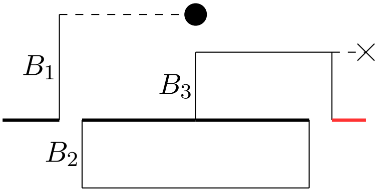
\end{aligned}
$$





$$
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\end{aligned}
$$

We need to express the following:

- Two symbols are in concurrency iff there is no τ in between.
- Two symbols are in the same block iff there is no transition using the timer of the block.
- There exists a cycle in the block graph.

The formula can be written with three quantifiers alternations $\rightsquigarrow$ 3EXP.

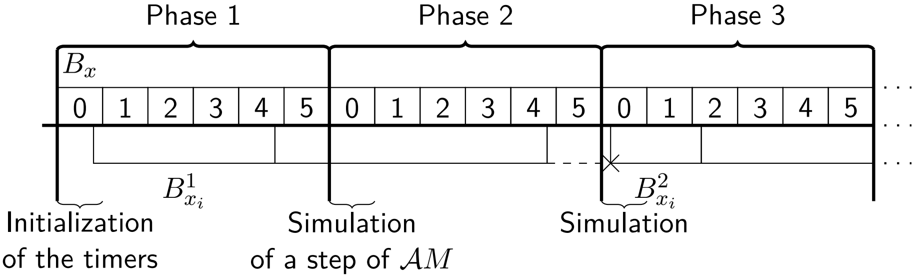

Figure 15: The beginning of a run for the reachability PSPACE-hardness proof.

Let $\mathcal{A} = (X, I, Q, q_0, \chi, \delta)$ be an automaton with timers. For a timer $x \in X$, c_x denotes the largest constant to which x is updated in $\mathcal{A}$. Let $C = \max_{x \in X} c_x$.
Two valuations κ and κ' are said *timer-equivalent*, noted $\kappa \cong \kappa'$, iff $\mathrm{dom}(\kappa) = \mathrm{dom}(\kappa')$ and the following hold for all $x_1, x_2 \in \mathrm{dom}(\kappa)$:

- $\lfloor \kappa(x_1) \rfloor = \lfloor \kappa'(x_1) \rfloor$,
- $\mathrm{frac}(\kappa(x_1)) = 0$ iff $\mathrm{frac}(\kappa'(x_1)) = 0$,
- $\mathrm{frac}(\kappa(x_1)) \leq \mathrm{frac}(\kappa(x_2))$ iff $\mathrm{frac}(\kappa'(x_1)) \leq \mathrm{frac}(\kappa'(x_2))$.

A *timer region* for $\mathcal{A}$ is an equivalence class of timer valuations induced by $\cong$. We lift the relation to configurations: $(q, \kappa) \cong (q', \kappa')$ iff $\kappa \cong \kappa'$ and $q = q'$. Finally, $[\![(q, \kappa)]\!]_{\cong}$ denotes the equivalence class of (q, κ).

We are now able to define a finite automaton called the *region automaton* of $\mathcal{A}$ and denoted $\mathcal{R}$. The alphabet of $\mathcal{R}$ is $\Sigma = \{\tau\} \cup \hat{I}$ where τ is a special symbol used in non-zero delay transitions. Formally, $\mathcal{R}$ is the finite automaton (Σ, S, s_0, Δ) where:

- $S = \{(q, \kappa) \mid q \in Q, \kappa \in \mathsf{Val}(\chi(q))\}_{/\cong}$, i.e., the quotient of the configurations by $\cong$, is the set of states,
- $s_0 = (q_0, [\![\kappa_0]\!]_{\cong})$ with κ_0 the empty valuation, is the initial state,
- the set of transitions $\Delta \subseteq S \times \Sigma \times S$ includes $([\![(q, \kappa)]\!]_{\cong}, \tau, [\![(q, \kappa')]\!]_{\cong})$ if $(q, \kappa) \xrightarrow{d} (q, \kappa')$ in $\mathcal{A}$ whenever $d > 0$, and $([\![(q, \kappa)]\!]_{\cong}, i, [\![(q', \kappa')]\!]_{\cong})$ if $(q, \kappa) \xrightarrow[u]{i} (q', \kappa')$ in $\mathcal{A}$.

Lemma 7

Let $\mathcal{A} = (X, I, Q, q_0, \chi, \delta)$ *be an automaton with timers and* $\mathcal{R}$ *be its region automaton.*

1. *The size of* $\mathcal{R}$ *is linear in* $|Q|$ *and exponential in* $|X|$. *That is,* $|S|$ *is smaller than or equal to* $|Q| \cdot |X|! \cdot 2^{|X|} \cdot (C+1)^{|X|}$.
2. *There is a timed run* ρ *of* $\mathcal{A}$ *that begins in* (q, κ) *and ends in* (q', κ') *iff there is a run* ρ' *of* $\mathcal{R}$ *that begins in* $[\![(q, \kappa)]\!]_{\cong}$ *and ends in* $[\![(q', \kappa')]\!]_{\cong}$.

Corollary 8

Let $\mathcal{A}$ be an automaton with timers and $\rho \in ptruns(\mathcal{A})$ be a padded timed run with races. Suppose that G_ρ is cyclic. Then there exists a cycle $\mathcal{C}$ in G_ρ such that

- *any block of $\mathcal{C}$ participates in exactly two races described by this cycle,*
- *for any race described by $\mathcal{C}$, exactly two blocks of $\mathcal{C}$ participate in the race,*
- *the blocks $B = (k_1 \dots k_m, \gamma)$ of $\mathcal{C}$ satisfy either $m \geq 2$, or $m = 1$ and $\gamma = \bullet$.*