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Consistent Couplings between a Massive Spin-3/2 Field and a Partially Massless Spin-2 Field

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Abstract: We revisit the problem of constructing consistent interactions between a massive spin-3/2 field and a partially massless graviton in four-dimensional (anti) de Sitter (A)dS$_4$ spacetime. We use the Stueckelberg formulation of the action principle for these fields and find two non-trivial cubic vertices with less than two derivatives when moving to the unitary gauge. One of the vertices is reminiscent of the minimal coupling of the massive spin-3/2 field to gravity, except that now the graviton is partially massless.

Keywords: gauge symmetries; stueckelberg formulation; partially massless graviton

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1. Introduction

The use of the Stueckelberg formulation for the problem of constructing consistent interactions between massive fields has proved very efficient, mainly through the works of Zinoviev and collaborators, see, e.g., [1–3] and references therein. Some years ago in [4], Zinoviev constructed cubic couplings between a massive spin-3/2 field and a massive spin-2 graviton around the anti-de Sitter (AdS$_4$) and the de Sitter (dS$_4$) backgrounds, with the assumption that the couplings should not bring more than one derivative. Then, the partially massless limit for the graviton was considered, resulting in the conclusion that no cubic vertex with one derivative survived in this limit.

The partially massless graviton does not exist around a Minkowski background. In order to propagate, such a particle requires a background that is either anti-de Sitter (AdS) or de Sitter (dS), depending on whether the cosmological constant Λ of the maximally-symmetric space is negative of positive, respectively. In such maximally symmetric space-times with nonvanishing cosmological constant, a partially massless graviton possesses four propagating degrees of freedom and is characterized by a well-suited mass directly related to the cosmological constant Λ, as we recall shortly and as discussed at length in [5–7], for example. Obviously, the dS$_4$ background is particularly relevant for early-Universe cosmology. The terminology "Partially Massless" (sometimes shortened by the acronym "PM" in the rest of this paper) is justified because the four helicity degrees of freedom of such a particle are intermediate between the five degrees of freedom of a massive spin-2 field and the two helicity ± 2 degrees of freedom of the massless graviton. The partially massless graviton possesses four propagating degrees of freedom corresponding to helicities ± 2 and ± 1. The corresponding PM field $k_{\mu\nu}$ satisfies the Klein-Gordon-like equation

$$(\Box - \tfrac{\tilde{m}^2 c^2}{\hbar^2})k_{\mu\nu} = 0, \quad \tilde{m}^2 := \tfrac{2}{3}\,\Lambda, \tag{1}$$

where $\Box$ is the Laplace–Beltrami operator in (A)dS$_4$. We note that the mass term is proportional to the cosmological constant Λ of the maximally symmetric background.

A physical motivation behind the search for interactions between a PM graviton and a real, massive spin-3/2 field is that there is an equal number of physical degrees of freedom for a partially massless graviton and a real, massive spin-3/2 field, suggesting a possible supersymmetry mixing these two fields. An interacting theory for the PM graviton and a real massive spin-3/2 field would therefore be reminiscent of supergravity, for a partially massless graviton instead of the massless graviton and for a massive spin-3/2 field instead of the massless gravitino of supergravity. On a more phenomenological side, the results of black-hole mergers [8] have put a lower bound on the mass of the graviton that does not prevent the graviton from being partially massless. The point is that the mass of the partially massless graviton is fixed by the cosmological constant, as recalled in Equation (1), and that the cosmological constant of our Universe is observed to be very small indeed [9]. Numerically, we have $\Lambda \sim 10^{-66}(\mathrm{eV})^2/\mathrm{c}^4$, while it has been observed that the graviton squared mass m_G^2 is bounded from above by the numerical value of the order $10^{-52}(\mathrm{eV})^2/\mathrm{c}^4$. This leaves enough room for a partially massless graviton, with fourteen orders of magnitude.

We wish to revisit the question of possible couplings between the PM graviton and a real, massive spin-3/2 field, this time taking the spin-2 field as partially massless from the very beginning of the analysis. Indeed, the operations of introducing interactions and taking a partially massless limit do not commute, in general. In fact, in this paper, we report a coupling between a partially massless spin-2 field and a massive spin-3/2 field that seems to have gone unnoticed in previous investigations, as far as we could see.

In order to build consistent vertex involving massive fields (in the present case, the massive spin-3/2 field), we use the method proposed in [10] that combines the cohomological reformulation of the Noether method for gauge systems [11,12] with the Stueckelberg formulation for massive fields. The Stuckelberg formulation [13] of theories for massive fields proves to be very useful, in the sense that it brings gauge invariance that controls the degrees of freedom and hence the possible interactions to be added to the free theory. The Stueckelberg formalism has been widely used since its invention; see [14] for a pedagogical review of the Stueckelberg formalism. The advantage of the method [10] is that it exploits the gauge structure of massless theories to describe interactions for massive fields. It proved useful in showing that the de Rham, Gabadadze, and Tolley (dRGT) gravity (see, e.g., [15] for a review) can be recast in a frame where the Einstein–Hilbert structure disappears, leaving only a Born–Infeld-like theory, with vertices obtained by contraction of products of the manifestly gauge-invariant field strengths in the Stueckelberg formulation of the free massive theory. In particular, in [10], the full list of cubic vertices of massive dRGT gravity theory was recovered, showing the usefulness of the method.

In this paper, we use the cohomological method of [10] for the search of consistent couplings between a massive spin-3/2 field and a partially massless (PM) spin-2 field, also called PM graviton. As we wrote above, the spacetime backgrounds considered in this paper are the anti-de Sitter (AdS) and the de Sitter (dS) geometries where the cosmological constant Λ is negative and positive, respectively. In its PM phase, the graviton therefore propagates four degrees of freedom, exactly like the massive spin-3/2 field that carries four degrees of freedom. It is therefore natural to ask whether it is possible to elaborate a consistent gauge theory in which a PM graviton $k_{\mu\nu}$ and a massive spin-3/2 field ψ_μ are involved.

We study the problem in both dS ($\Lambda > 0$) and AdS ($\Lambda < 0$) backgrounds at a stroke, through the use of parameter σ that takes the value of $+1$ in AdS and -1 in dS.

2. Main Results

The main results we report in this paper consist in the construction of two vertices expressed in the Stueckelberg formulation for both the PM spin-2 and the massive spin-3/2 fields. In the unitary gauge where the Stueckelberg fields are set to zero, the first vertex is proportional to

$$\ell^{(1)} = 2\,\nabla_{[\mu}k_{\nu]\rho}\,\overline{\psi}^{\mu}\,\gamma^{\rho}\,\psi^{\nu}, \tag{2}$$

where the spinor field ψ_μ satisfies the Majorana reality condition and denotes the field for the massive spin-3/2 particle (the spinor indices are left implicit), the Lorentz-covariant derivative for the background geometry (we use conventions whereby the Lorentz-covariant derivative satisfies $[\nabla_\mu, \nabla_\nu]V^\rho = -2\sigma\,\lambda^2\,\delta^\rho{}_{[\mu}\,V_{\nu]}$, where $\sigma = \pm 1$. In other terms, the cosmological constant is $\Lambda = -3\,\sigma\,\lambda^2$ in four dimensions, where $\sigma = -1$ corresponds to dS_4 and $\sigma = 1$ to AdS_4. On a Dirac spinor ψ, we have $[\nabla_\mu, \nabla_\nu]\psi = -\frac{1}{2}\,\sigma\,\lambda^2\,\gamma_{\mu\nu}\,\psi$) is denoted by symbol ∇_μ, and the symmetric rank-two tensor field $k_{\mu\nu} = k_{\nu\mu}$ represents the PM spin-2 field. Throughout this paper, spacetime indices between square brackets are antisymmetrised with strength one. For example, one has $\nabla_{[\mu}k_{\nu]\rho} = \frac{1}{2}\,(\nabla_\mu k_{\nu\rho} - \nabla_\nu k_{\mu\rho})$ and $2\,\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The components of the (A)dS background metric are denoted as $\overline{g}_{\mu\nu}$. As usual, the four Dirac matrices are denoted by γ^a, $a = 0,1,2,3$, and $\gamma^\mu := \bar{e}^\mu{}_a\,\gamma^a$ featuring the components of the background (A)dS vierbein.

The other coupling is more interesting. In the unitary gauge, it reads

$$\ell^{(2)} = k_{\mu\nu}\,\mathcal{T}^{\mu\nu}, \quad \mathcal{T}^{\mu\nu} = \omega\,\overline{\psi}_\rho\,\gamma^{\rho(\mu}\,\psi^{\nu)} + \overline{\psi}_\sigma\,\gamma^{\rho\sigma(\mu}\,\nabla_\rho\psi^{\nu)}, \quad \omega := \sqrt{m^2 + \sigma\lambda^2}\,, \tag{3}$$

where the real parameter m is the mass of the spin-3/2 field in AdS, in the sense that the limit $m \to 0$ is the limit where the spin-3/2 field enjoys a gauge symmetry that removes the helicity $\pm 1/2$ degrees of freedom, leaving only the helicity $\pm 3/2$ degrees of freedom on shell. Tensor $\mathcal{T}^{\mu\nu}$ is traceless and divergenceless on shell:

$$\overline{g}_{\mu\nu}\,\mathcal{T}^{\mu\nu} \approx 0, \qquad \nabla_\mu \mathcal{T}^{\mu\nu} \approx 0, \tag{4}$$

where a weak equality is an equality that holds on the solutions of the field equations for the free theory. The above vertex $\ell^{(2)} = k_{\mu\nu}\,\mathcal{T}^{\mu\nu}$ induces a deformation of the gauge transformations on the physical fields $(k_{\mu\nu}, \psi_\mu)$ in the unitary gauge, given by

$$\delta_1\psi_\mu = -\psi^\nu\,\nabla_\mu\nabla_\nu\xi + \sigma\,\lambda^2\,\psi_\mu\,\xi, \qquad \delta_1 k_{\mu\nu} = 0, \tag{5}$$

where we recall that the free, quadratic action $S_0[k,\psi]$ is invariant under [5–7]

$$\delta_0 k_{\mu\nu} = \nabla_\mu\nabla_\nu\xi - \sigma\,\lambda^2\,\overline{g}_{\mu\nu}\,\xi, \qquad \delta_0\psi_\mu = 0. \tag{6}$$

From the knowledge of the quadratic and cubic actions $S_0[k,\psi]$ and $S_1[k,\psi] = \int d^4x\sqrt{\bar{g}}\,\ell^{(2)}$ in the unitary gauge, we readily find the consistency of the deformation reported above:

$$\delta_0 S_1[k,\psi] + \delta_1 S_0[k,\psi] = 0. \tag{7}$$

As far as deformation $\ell^{(2)}$ is concerned, we note from (5) and (6) that the transformation of the massive spin-3/2 field can be written as $\delta_1\psi_\mu = -\,\psi^\nu\,\delta_0 k_{\mu\nu}$, from which it is tempting to view the contravariant spinor ψ^μ as a gauge-invariant quantity, defining the covariant field as $\psi_\mu := \psi^\nu(\overline{g}_{\mu\nu} - \kappa\,k_{\mu\nu} + \mathcal{O}(\kappa^2))$, where κ is the deformation parameter that we take with units of length that defines perturbative expansion $S[\phi] = S_0[\phi] + \kappa\,S_1[\phi] + \mathcal{O}(\kappa^2)$, $\delta\phi = \delta_0\phi + \kappa\,\delta_1\phi + \mathcal{O}(\kappa^2)$ such that $\delta S[\phi] = 0 + \mathcal{O}(\kappa^2)$. In this sense, like in Riemannian geometry, it would appear that metric $g_{\mu\nu} := \overline{g}_{\mu\nu} - \kappa\,k_{\mu\nu} + \mathcal{O}(\kappa^2)$ could be defined in terms of the (A)dS background metric and the PM spin-2 field $k_{\mu\nu}$.

In the following section, we present the two deformations reported above in their Stueckelberg form, and explain how the unitary gauge at first order in perturbation can be reached, thereby reproducing the main results presented above.

3. Consistent Couplings in the Stueckelberg Formulation

In this section, we first spell out the free model, then exhibit the first-order interactions we found, and finally explain the ways in which the unitary gauge at first order in deformation can be reached.

3.1. The Free Model

We want to investigate the couplings between a massive spin-3/2 field and a PM spin-2 field. Our starting point is the Stueckelberg formulation for these models. The Stueckelberg action for a massive spin-3/2 field and a PM spin-2 field in (A)dS$_4$ reads

$$\begin{aligned} S_0[k_{\mu\nu}, B_\mu, \psi_\mu, \chi] = \int d^4x\, \sqrt{-\bar{g}}\, \Big(&-\frac{1}{2}\nabla_\rho k^{\mu\nu}\nabla^\rho k^{\mu\nu} + \nabla_\rho k^{\mu\nu}\nabla_\mu k^\rho{}_\nu - \nabla_\mu k \nabla_\nu k^{\mu\nu} \\ &+ \frac{1}{2}\nabla_\mu k \nabla^\mu k + \frac{\sigma}{4} F_{\mu\nu}F^{\mu\nu} - 2\sigma\lambda^2 (k_{\mu\nu}k^{\mu\nu} - \frac{1}{4}k^2) \\ &+ 2\lambda\, [k\nabla_\mu B^\mu - k^{\mu\nu}\nabla_\mu B_\nu] + 3\lambda^2\, B^\mu B_\mu \Big) \\ &+ \Big(-\frac{1}{2}\,\overline{\psi}_\mu\, \gamma^{\mu\nu\rho}\, \nabla_\nu \psi_\rho + \frac{\omega}{2}\, \overline{\psi}_\mu\, \gamma^{\mu\nu}\, \psi_\nu - \frac{3\omega}{2}\, \overline{\chi}\chi \\ &- \frac{3}{4}\, \overline{\chi}\, \gamma^\mu\, \nabla_\mu \chi - \frac{3m}{2}\, \overline{\psi}_\mu\, \gamma^\mu\, \chi \Big), \end{aligned} \tag{8}$$

where $F_{\mu\nu} = 2\,\nabla_{[\mu}B_{\nu]} \equiv \nabla_\mu B_\nu - \nabla_\nu B_\mu$ is the Abelian field strength for the $U(1)$ vector gauge field B_μ. For the spin-2 sector, we use the conventions of [16]. The vector field B_μ and the Majorana spinor χ are, respectively, the Stueckelberg companions of $k_{\mu\nu}$ and ψ_μ. The Stueckelberg action (8) is invariant under the following Abelian gauge transformations:

$$\begin{aligned} &\delta_0 k_{\mu\nu} = 2\,\nabla_{(\mu}\varepsilon_{\nu)} + \lambda\, \bar{g}_{\mu\nu}\, \pi, \quad \delta_0 B_\mu = \nabla_\mu \pi + 2\sigma\lambda\, \varepsilon_\mu, \\ &\delta_0 \psi_\mu = \nabla_\mu \theta + \frac{\omega}{2}\, \gamma_\mu\, \theta, \quad \delta_0 \chi = m\,\theta. \end{aligned} \tag{9}$$

The rationale behind these transformations is as follows. First, the gauge transformation law of the Stueckelberg field χ shows that, as long as the mass parameter m is nonzero, one can gauge fix χ to zero by using gauge parameter $\theta(x)$. Once this is achieved and the gauge where $\chi = 0$ is reached, the field ψ_μ becomes gauge invariant, as it should for a massive field. Second, the gauge transformation law of the Stueckelberg field B_μ shows that, as long as the cosmological constant of the (A)dS background is nonvanishing, one can gauge fix B_μ to zero by using gauge parameter $\varepsilon_\mu(x)$. There is, however, a residual ε_μ-gauge transformation whereby gauge $B_\mu = 0$ is preserved when a π-gauge transformation of field B_μ is accompanied with an $\bar{\varepsilon}_\mu$-transformation with parameter $\bar{\varepsilon}_\mu = -\frac{\sigma}{2\lambda}\,\nabla_\mu \pi$. As a result, in the gauge where $B_\mu = 0$, the residual gauge transformation of the PM gauge field $k_{\mu\nu}$ becomes $\delta_0 k_{\mu\nu} = \nabla_\mu \nabla_\nu \xi - \sigma\lambda^2\, \bar{g}_{\mu\nu}\, \xi$, where we define $\xi := -\frac{\sigma}{\lambda}\,\pi$. One thereby recovers the well-known transformation law (6) of the PM spin-2 field, offering the rationale behind gauge transformations (9).

It is useful to introduce the gauge-invariant quantities

$$\begin{aligned} \mathbf{\Psi}_\mu &:= \psi_\mu - \tfrac{1}{m}\,\nabla_\mu \chi - \tfrac{\omega}{2m}\,\gamma_\mu\,\chi, \\ K_{\mu\nu\rho} &:= 2\,\nabla_{[\mu}k_{\nu]\rho} + 2\,\lambda\, \bar{g}_{\rho[\mu}B_{\nu]} - \tfrac{\sigma}{2\lambda}\,\nabla_\rho F_{\mu\nu}, \end{aligned} \tag{10}$$

in terms of which free action (8) can be written in a manifestly gauge invariant way:

$$\begin{aligned} S_0[k_{\mu\nu}, B_\mu, \psi_\mu, \chi] &= \int d^4x\, \sqrt{-\bar{g}}\, L_0 \\ &= \frac{1}{2}\int d^4x\, \sqrt{-\bar{g}}\, \Big(-\overline{\mathbf{\Psi}}_\mu\, \gamma^{\mu\nu\rho}\, \nabla_\nu \mathbf{\Psi}_\rho + \omega\, \overline{\mathbf{\Psi}}_\mu\, \gamma^{\mu\nu}\, \mathbf{\Psi}_\nu - \tfrac{1}{2}\, K_{\mu\nu\rho}K^{\mu\nu\rho} + K^\mu K_\mu \Big), \end{aligned} \tag{11}$$

where $K^\mu = \bar{g}^{\alpha\beta}K^\mu{}_{\alpha\beta}$. The equations of motion obtained by extremising the above action with respect to the fields $k_{\mu\nu}$ and ψ_μ are, respectively,

$$\nabla_\rho K^{\rho(\mu\nu)} - \bar{g}^{\mu\nu}\nabla_\rho K^\rho + \nabla^{(\mu}K^{\nu)} \approx 0, \tag{12}$$

$$\nabla_\nu \overline{\mathbf{\Psi}}_\rho\, \gamma^{\mu\nu\rho} + \omega\, \overline{\mathbf{\Psi}}_\nu\, \gamma^{\mu\nu} \approx 0. \tag{13}$$

These equations are useful in checking the consistency of vertex $\ell^{(2)}$. We notice that, setting to zero the Stueckelberg field χ in Equation (13) and taking the flat limit whereby $\omega \to m$, we reproduce the well-known Rarita–Schwinger equation $\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - m\gamma^{\mu\nu}\psi_\nu = 0$ for the spinor field ψ_μ, using the fact that it obeys the Majorana reality condition.

3.2. Interactions to First Order

We now report our main findings obtained following the method proposed in [10] for constructing interactions of massive fields in the Stueckelberg formulation. For the sake of conciseness, in this paper, we refrain from reviewing this method and instead spell out the results that can be checked without referring to the formalism developed in [10].

- In the Stueckelberg formulation, the deformation $L_1^{(1)}$ of the Lagrangian that corresponds to the first vertex $\ell^{(1)}$ presented in Introduction reads

$$L_1^{(1)} = \overline{\mathbf{\Psi}}_\sigma \, \gamma^\rho \, \mathbf{\Psi}_\alpha \, K^{\sigma\alpha}{}_\rho, \tag{14}$$

up to trivial field redefinitions. The coefficient in front of it is, at this stage, arbitrary.
- In the Stueckelberg formulation, the deformation $L_1^{(2)}$ of the Lagrangian that corresponds to the second vertex $\ell^{(2)}$ presented in Introduction reads

$$\begin{aligned} L_1^{(2)} = {} & \omega \, \overline{\mathbf{\Psi}}_\mu \, \gamma^{\mu\nu} \, \mathbf{\Psi}^\rho \, k_{\rho\nu} - \tfrac{\omega^2}{m} \, \overline{\mathbf{\Psi}}^\mu \, \gamma^\nu \, \chi \, k_{\mu\nu} + \tfrac{\omega^2}{m} \, \overline{\mathbf{\Psi}}_\mu \, \gamma^\mu \, \chi \, k \\ & + \overline{\mathbf{\Psi}}_\sigma \, \gamma^{\sigma\nu\rho} \, \nabla_\rho \mathbf{\Psi}^\mu \, k_{\mu\nu} - \tfrac{2\,\omega}{m} \, \nabla_{[\rho} \overline{\mathbf{\Psi}}_{\sigma]} \, \gamma^{\sigma\alpha} \chi \, k^\rho{}_\alpha - \tfrac{\omega}{m} \, \nabla_\rho \overline{\mathbf{\Psi}}_\sigma \, \gamma^{\rho\sigma} \chi \, k \\ & - \tfrac{\sigma\,\omega}{2\lambda} \, \overline{\mathbf{\Psi}}_\sigma \gamma^{\rho\sigma} \mathbf{\Psi}^\alpha F_{\rho\alpha} - \tfrac{\sigma}{2\lambda} \, \overline{\mathbf{\Psi}}_\sigma \gamma^{\alpha\rho\sigma} \nabla_\alpha \mathbf{\Psi}^\mu F_{\rho\mu}. \end{aligned} \tag{15}$$

On the contrary to the free Stueckelberg theory where the flat limit is smooth, in the interacting case, we cannot take limit $\lambda \to 0$ as the vertex is non-analytical in constant λ.
- The above vertex induces a deformation of the gauge transformations given by

$$\begin{aligned} \delta_1\psi_\mu = {} & -\omega \, k_{\mu\nu} \, \gamma^\nu \, \theta + \omega \, \gamma_\mu \, \psi^\nu \, \varepsilon_\nu - \tfrac{\omega\,\lambda}{m} \, \gamma_\mu \, \chi \, \pi - \lambda \, \psi_\mu \, \pi \\ & - \tfrac{\omega}{m} \, \gamma^\nu \, \nabla_\nu \varepsilon_\mu \, \chi - \tfrac{\omega}{m} \, \gamma_\mu \, \nabla_\nu \chi \, \varepsilon^\nu + 2 \, \nabla_\mu \psi_\nu \, \varepsilon^\nu - \tfrac{\lambda}{m} \, \chi \, \nabla_\mu \pi \\ & - \tfrac{2}{m} \, \nabla_\mu \nabla_\nu \chi \, \varepsilon^\nu, \end{aligned} \tag{16}$$

$$\delta_1\chi = 2\, m \, \psi_\mu \, \varepsilon^\mu - 2\, \varepsilon^\mu \, \nabla_\mu \chi - \lambda \, \chi \, \pi, \tag{17}$$

$$\delta_1 k_{\mu\nu} = 0, \quad \delta_1 B_\mu = 0. \tag{18}$$

The corresponding gauge algebra is

$$[\delta_\theta, \delta_\varepsilon]\varphi = \delta_{\tilde{\theta}}\varphi, \qquad \tilde{\theta} = 2\,\omega\,\gamma^\mu \theta\,\varepsilon_\mu, \tag{19}$$

$$[\delta_\theta, \delta_\pi]\varphi = \delta_{\overline{\theta}}\varphi, \qquad \overline{\theta} = -2\,\lambda\,\theta\,\pi. \tag{20}$$

The redefinition of the gauge parameters that trivializes the gauge algebra is

$$\theta \quad \to \quad \theta - \kappa \, \frac{\omega}{m} \, \gamma^\mu \, \chi \, \varepsilon_\mu + \kappa \, \frac{\lambda}{m} \, \chi \, \pi. \tag{21}$$

In order to express our results in the unitary gauge, we first need to explain how to reach the unitary gauge in perturbation.

3.3. Reaching the Unitary Gauge at First Order in Deformation

The starting point is a free theory $S_0[\varphi^i, \chi^I]$ with a spectrum of fields (φ^i, χ^I) such that the latter are Stueckelberg companions of the former. In other words, we consider an action $S_0 = \int d^n x\, \mathcal{L}_0$ that is invariant under the gauge transformations

$$\delta_0 \varphi^i = {R_0}^i{}_I\, \varepsilon^I + {R_0}^i{}_\alpha\, \epsilon^\alpha, \tag{22}$$

$$\delta_0 \chi^I = m_I\, \varepsilon^I + {R_0}^I{}_\alpha\, \epsilon^\alpha, \tag{23}$$

where we use De Witt's condensed notation. The gauge invariance under the Stueckelberg gauge parameters ε^I implies the Noether identities

$$\frac{\delta \mathcal{L}_0}{\delta \chi^I} \equiv -\frac{1}{m_I}\, R_0^{+i}{}_I \frac{\delta \mathcal{L}_0}{\delta \varphi^i}, \tag{24}$$

where the operator $R_0^{+i}{}_I$ denotes the adjoint of ${R_0}^i{}_I$.

We assume we also have a consistent, first order deformation of the action and gauge transformations, i.e., functional $S_1[\varphi^i, \chi^I] = \int d^n x\, \mathcal{L}_1$ and gauge transformation laws

$$\delta_1 \varphi^i = {R_1}^i{}_I(\varphi, \chi)\, \varepsilon^I + {R_1}^i{}_\alpha(\varphi, \chi)\, \epsilon^\alpha, \tag{25}$$

$$\delta_1 \chi^I = {R_1}^I{}_J(\varphi, \chi)\, \varepsilon^J + {R_1}^I{}_\alpha(\varphi, \chi)\, \epsilon^\alpha, \tag{26}$$

such that

$$\delta_1 S_0[\varphi^i, \chi^I] + \delta_0 S_1[\varphi^i, \chi^I] = 0. \tag{27}$$

Upon expanding the latter equation using (22)–(26), we obtain the following Noether identity associated with the gauge parameters ε^I:

$$\frac{\delta \mathcal{L}_1}{\delta \chi^I} \equiv -\frac{1}{m_I}\left(R_0^{+i}{}_I \frac{\delta \mathcal{L}_1}{\delta \varphi^i} + [R_1^{+i}{}_I(\varphi, \chi) - \frac{1}{m_J}\, R_1^{+J}{}_I(\varphi, \chi) R_0^{+i}{}_J]\, \frac{\delta \mathcal{L}_0}{\delta \varphi^i}\right). \tag{28}$$

Inserting this expression for $\frac{\delta \mathcal{L}_1}{\delta \chi^I}$ in Equation (27) yields

$$0 = \int d^n x\, \epsilon^\alpha \left[\mathcal{R}_0^{+i}{}_\alpha \frac{\delta \mathcal{L}_1}{\delta \varphi^i} + \mathcal{R}_1^{+i}{}_\alpha(\varphi, \chi)\, \frac{\delta \mathcal{L}_0}{\delta \varphi^i}\right], \tag{29}$$

where

$$\mathcal{R}_0{}^i{}_\alpha = {R_0}^i{}_\alpha - \frac{1}{m_I}\, {R_0}^i{}_I\, {R_0}^I{}_\alpha, \tag{30}$$

$$\begin{aligned}\mathcal{R}_1{}^i{}_\alpha(\varphi, \chi) = {} & {R_1}^i{}_\alpha(\varphi, \chi) - \frac{1}{m_I}\, {R_1}^i{}_I(\varphi, \chi)\, {R_0}^I{}_\alpha - \frac{1}{m_I}\, {R_0}^i{}_I\, {R_1}^I{}_\alpha(\varphi, \chi) \\ & + \frac{1}{m_I m_J}\, {R_0}^i{}_J\, {R_1}^J{}_I(\varphi, \chi) {R_0}^I{}_\alpha.\end{aligned} \tag{31}$$

Equation (29) expresses the ϵ^α-gauge invariance of action $S[\varphi^i, \chi^I] = S_0[\varphi^i, \chi^I] + g\, S_1[\varphi^i, \chi^I]$ to the first order in perturbation; here, g denotes the coupling constant used in perturbation. This equation is valid for an arbitrary field configuration, in particular it is valid when we set Stueckelberg fields χ^I to zero. Using the following obvious equality

$$\frac{\delta \mathcal{L}_1}{\delta \varphi^i}(\varphi, \chi = 0) = \frac{\delta \check{\mathcal{L}}_1}{\delta \varphi^i}(\varphi), \qquad \check{\mathcal{L}}_1 := \mathcal{L}_1|_{\chi=0}, \tag{32}$$

we obtain

$$0 = \int d^n x \, \epsilon^\alpha \left[\mathcal{R}_0^{+i}{}_\alpha \frac{\delta \check{\mathcal{L}}_1}{\delta \varphi^i} + \check{\mathcal{R}}_1^{+i}{}_\alpha(\varphi) \frac{\delta \check{\mathcal{L}}_0}{\delta \varphi^i} \right],$$
$$\check{\mathcal{L}}_0 := \mathcal{L}_0|_{\chi=0}, \quad \check{\mathcal{R}}_1{}^i{}_\alpha(\varphi) := \mathcal{R}_1{}^i{}_\alpha(\varphi, \chi = 0). \tag{33}$$

In its turn, the latter equation expresses the gauge invariance, up to first order in perturbation, of the reduced action $\check{S}[\varphi^i] = S_0[\varphi^i, \chi^I = 0] + g\, S_1[\varphi^i, \chi^I = 0]$ under the gauge transformations

$$\delta_\epsilon \varphi^i = [\mathcal{R}_0{}^i{}_\alpha + g\, \check{\mathcal{R}}_1{}^i{}_\alpha(\varphi)]\, \epsilon^\alpha + \mathcal{O}(g^2). \tag{34}$$

In the particular case studied in this paper where we have physical fields $\varphi^i = \{k_{\mu\nu}, \psi_\mu\}$ and Stueckelberg fields $\chi^I = \{B_\mu, \chi\}$ with gauge transformations at zeroth and first order given in (9) and (16)–(18), respectively, we find Equation (7). As a consequence, the reduced action is invariant under (5) and (6) as announced in the introduction, where we rename the scalar parameter π into ξ, absorbing in it the constant factor $-\frac{\sigma}{\lambda}$.

From the above-derived Formula (34) for gauge transformations $\delta_\epsilon \varphi^i$ that leave invariant the reduced action, we can make an observation on the corresponding transformations of Stueckelberg field strengths

$$\Phi^i := \varphi^i - \tfrac{1}{m_J} R_0{}^i{}_J \, \chi^J. \tag{35}$$

As is well-known, these quantities are invariant under Stueckelberg transformations

$$\delta_0^\varepsilon \varphi^i = R_0{}^i{}_I \, \varepsilon^I, \qquad \delta_0^\varepsilon \chi^I = m_I \, \varepsilon^I. \tag{36}$$

Under complete transformation laws (22), (23), (25) and (26), Stueckelberg field strengths Φ^i transform as

$$\begin{aligned} \delta \Phi^i = {} & \mathcal{R}_0{}^i{}_\alpha \epsilon^\alpha + g \Big(R_1{}^i{}_\alpha(\varphi, \chi) \epsilon^\alpha + R_1{}^i{}_I(\varphi, \chi) \varepsilon^I \\ & - \frac{1}{m_I} R_0{}^i{}_I \, R_1{}^I{}_\alpha(\varphi, \chi) \epsilon^\alpha - \frac{1}{m_J} R_0{}^i{}_J \, R_1{}^J{}_I(\varphi, \chi) \varepsilon^I \Big) + \mathcal{O}(g^2). \end{aligned} \tag{37}$$

If, on the right-hand-side of the above formula, we set $\chi^I = 0$ and $\varepsilon^I = -\frac{1}{m_I} R_0{}^I{}_\alpha \, \epsilon^\alpha$ for the residual $\varepsilon^I(\epsilon)$ parameters that preserve the unitary gauge $\chi^I = 0$ at the zeroth order in perturbation, it turns out that we exactly recover the expression for $\delta_\epsilon \varphi^i$ transformations (34).

In other words, we could turn the argument around and obtain a heuristic way of producing the right-hand side of Formula (34) by demanding that the operations of setting fields χ^I to zero and performing gauge transformations commute on the Stueckelberg field strengths, i.e., imposing $(\delta \Phi^i)|_{\chi=0,\, \varepsilon=\varepsilon(\epsilon)} = \delta_\epsilon (\Phi^i|_{\chi=0})$.

4. Conclusions and Outlook

The first vertex, $L_1^{(1)}$, that we presented above in (14) does not deform Stueckelberg gauge transformations (9). It is exactly invariant under the latter transformations. The second vertex, $L_1^{(2)}$, given in (15) is more interesting in the sense that it truly deforms the gauge transformations given in (9). The first term on the right-hand side of (16) is reminiscent of the local supersymmetry transformations in the AdS background, offering the minimal deformation of the mass-like term on the right-hand side of $\delta_0 \psi_\mu = \nabla_\mu \theta + \frac{\omega}{2} \gamma_\mu \theta$; see (9). Correspondingly, the first term on the right-hand side of (15) is the minimal deformation of the mass-like term for the spinor $\boldsymbol{\Psi}_\mu$ in the free action (11). However, on the contrary to the situation in supergravity theories, there is no deformation proportional to the linearised "spin-connection" $\nabla_{[\mu} k_{\nu]\rho}$ in (15) or in (16).

It will be interesting to investigate the consistent interactions among the fields of the enlarged spectra given in [17]. There, it was shown that the doublet $(k_{\mu\nu}, \psi_\mu)$ consisting of a PM spin-2 and a massive spin-3/2 field studied in the present paper must be completed with a massless spin-$(3/2, 1)$ doublet in order to carry the action of supersymmetry. We hope to report soon on the interactions among these four fields. In paper [18], on the other hand, it was found that the partially massless doublet $(5/2, 2)$ can be completed with a massless doublet $(2, 3/2)$ in order to carry the action of supersymmetry. We intend to investigate the consistent couplings among those fields in future work.

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