

# Higher - spins from higher dualisations

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Work in collaboration with Victor Lekeu , [2012.11356]

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## ④ Conclusions and outlook

# ① Introduction

- Fierz-Pauli programme  $\rightarrow$  all possible off-shell descriptions of spin- $s$  massless field?

Interactions may not choose the most economical description

Higher-spin Gravity

Hidden symmetries of Gravity

Non-linear realisation of  $e_{11} \times l_1$

Higher, or "exotic" descriptions

Gravity in a non-standard version - duality - gauge theory - higher spin theory - infinite-dim. algebra

## ② Higher dualisations of linearised gravity and Maxwell's

An off-shell dualisation was initiated in 2001 by P. West - completed by N.B., S. Cnockaert and

M. Henneaux.

In [N.B., P. Cook, D. Ponomarev]  $\Rightarrow$  Other off-shell dualisations schemes proposed.

First, review the dualisation of [West, N.B. - Cnockaert - Henneaux]

Parent action  $S[\gamma^{ab}{}_{|d}, \omega_{abc}] = \int d^n x (\text{"}\omega\omega\text{"} + \partial_a \omega_{bc}{}^d \gamma^{ab}{}_{|d})$

$$Z = \int \mathcal{D}\omega \mathcal{D}\gamma \exp \frac{i}{\hbar} S[\omega, \gamma]$$

enforces  $\omega_{abc} = \partial_{[a} e_{b]c}$

semi-classical  $S[e_{ab}] = \text{Fierz-Pauli}$

with local Lorentz  $\delta e_{ab} = \lambda_{ab}$

Field  $\omega_{abc}$  auxiliary

$$\frac{\delta S}{\delta \omega} \approx 0 \Rightarrow \omega_{abc} \sim \partial^d \gamma_{abd|c}$$

$$S[\gamma^{ab}{}_{|d}] = \int d^n x (\partial^a \gamma_{ab|c} \partial_d \gamma^{ab|c} + \dots)$$

$$\gamma^{ab}{}_{|c} = \frac{1}{(n-3)!} \epsilon^{abcd_1 \dots d_{n-3}} C_{d[n-3]1c}$$

$$\text{Gauge inv. } \delta_\lambda \gamma^{ab}{}_{|c} = \delta_{[a}^e \lambda_{bc]} \Rightarrow \delta_\lambda C_{a[n-3]1b} = \epsilon_{a[n-3]bcd} \lambda^{cd}$$

$$C_{[n-3,1]} \rightsquigarrow C_{a_1 \dots a_{n-3}, b} = C_{[a_1 \dots a_{n-3}], b} \quad \text{s.t.} \quad C_{[a_1 \dots a_{n-3}, b]} \equiv 0$$

i.e.  $C_{[n-3,1]} \sim \begin{array}{|c|} \hline \square \\ \hline \end{array}$  of  $GL(n)$  appears in Minkowski spacetime  $\mathbb{R}^{1,n-1}$

that propagates the d.o.f. of Fierz-Pauli's graviton  $h_{ab}$  with  $\eta^{bd} K_{ab,cd}(h) = 0$ .

Note: Hull's [2001] twisted *on-shell* duality

relating

$$K_{a_1 \dots a_{n-2}, b_1 b_2}(C) := \partial_{[a_1} \partial^{[b_1} C_{a_2 \dots a_{n-2}], b_2]}$$

to

$$K_{ab,cd}(h) := -\frac{1}{2} \partial_{[a} \partial^c h^{d]}_{b]}$$

via

$$K_{[n-2,2]}(C) = *_1 K_{[2,2]}(h)$$

In same work [Hull: 2001] a field

$$D_{[n-3, n-3]} \sim \begin{array}{|c|c|} \hline & \\ \hline n-3 & n-3 \\ \hline \end{array} \text{ of } Gl(n), \quad \xrightarrow{n=5} \quad D_{[2,2]} \sim \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

was introduced with field equations  $\text{Tr}^{n-3} K_{[n-2, n-2]}(\mathbb{D}) = 0 \quad \xrightarrow{n=5} \quad \text{Tr}^2 K_{[3,3]} = 0$ .

Twisted duality with  $h_{ab}$ :  $K_{[n-2, n-2]}(\mathbb{D}) = *_1 *_2 K_{[2,2]}(h) \quad \xrightarrow{n=5} \quad K_{[3,3]}(\mathbb{D}) = *_1 *_2 K_{[2,2]}(h)$ .

$$d_1 K_{[3,3]}(\mathbb{D}) = 0 \quad \underline{\text{BII}} \quad \iff \quad d_1^+ K_{[2,2]} = 0 \quad \underline{\text{EII}}$$

also  $\text{Tr} K_{[2,2]} = 0 \quad (\text{EI}) \iff \text{Tr} *_1 *_2 K_{[3,3]} = 0 \iff \text{Tr}^2 K_{[3,3]} = 0 \quad (\text{EI}) \quad \text{EoM for } \mathbb{D}$ .

Field equations mapped to Field equations.

In presence of sources, Hull argued that  $D_{[2,2]}$  couples to the usual  $T_{ab}$  stress tensor confirming that  $D_{[2,2]}$  is, *on-shell*, an avatar of  $h_{ab}$ . *Off-shell*, it is not.

An action principle had been proposed for a  $D_{[2,2]}$  gauge field (plus other fields)

that propagates the degrees of freedom of a single graviton:

N.B, Paul Cook, Mitya Ponomarev [2012]

"Off-shell Hodge dualities in linearized gravity and  $E_{11}$ "

In the paper [2012.11356](#) (JHEP) with Victor Lekeu, we further analysed the action proposed earlier in 2012.

## 2.1) Higher dual of vector field in dimensions 3 & 4

**Idea** :  $A_b$  viewed as a  $A_{[0,1]}$  bi-form

$$A_{[0,1]} \xrightarrow{\text{higher dualise}} C_{[n-0-2,1]} \stackrel{n=4}{=} C_{[2,1]} \sim \begin{array}{|c|c|} \hline & \\ \hline \end{array}$$

$$\stackrel{n=3}{=} h_{[1,1]} \sim \begin{array}{|c|c|} \hline & \\ \hline \end{array}$$

• Starts from Maxwell and IBP :  $S[A_a] = -\frac{1}{2} \int d^n x (\underbrace{\partial_a A_b \partial^a A^b}_{\text{blue}} - \underbrace{\partial_a A^a \partial_b A^b}_{\text{purple}})$

• Parent action  $S[Y^{ab}_c, P_{a,b}] = \int d^n x (P_{a,b} \partial_c Y^{cab} - \frac{1}{2} \underbrace{P_{a,b} P^{a,b}}_{\text{blue}} + \frac{1}{2} \underbrace{P^a_a P_b^b}_{\text{purple}})$

$$\frac{\delta S[Y,P]}{\delta P_{a,b}} \approx 0 \Leftrightarrow P^{a,b} \approx \partial_c Y^{cab} - \eta^{ab} \frac{1}{n-1} \partial_c Y^{cd}_d$$

substitute to get

$$S[Y^{ab}_c] = \int d^n x \left[ \frac{1}{2} \partial_c Y^{cab} \partial_d Y^d_{a,b} - \frac{1}{2(n-1)} \partial_a Y^{ab}_b \partial^a Y_{c,d}^d \right]$$



. From

$$S[Y^{abc}] = \int d^n x \left[ \frac{1}{2} \partial_c Y^{cab} \partial_d Y^d{}_{ab} - \frac{1}{2(n-1)} \partial_a Y^{ab}{}_b \partial^c Y_{cd}{}^d \right]$$

invariant under

$$\delta Y^{abc} = \delta_c^{[a} \partial^{b]} \lambda + \partial_d Y^{abcd}$$

one decomposes

$$Y^{abc} = X^{abc} + \delta_c^{[a} Z^{b]}, \quad X^{ab}{}_b \equiv 0$$

irreducibly under  $GL(n)$ .

(A) Hodge-dualise in 4D

$$X^{abc} \xleftrightarrow{*} T_{abc} \sim \begin{array}{|c|c|} \hline a & \varepsilon \\ \hline b & \\ \hline \end{array}$$

that gauge transforms as

$$\left\{ \begin{array}{l} \delta \begin{array}{|c|c|} \hline & \\ \hline \end{array} = \delta \begin{array}{|c|c|} \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline \end{array} \\ \delta Z_a = \partial_a \lambda + \partial^b A_{ab} \end{array} \right.$$

with

for the vector

Perform some change of field variables and dualize  $Z_a \leftrightarrow \tilde{A}_a$  to get

$$S [T_{ab|c}, \tilde{A}_a] = \int d^4x \left[ \mathcal{L}^{\text{curv.}}(T_{ab,c}) + \frac{1}{4} F^{ab}(\tilde{A}) F_{ab}(\tilde{A}) + \frac{1}{2} \tilde{A}^a K_a{}''(T) \right]$$

where  $K^{a[33]}{}_{b[22]} := 6 \partial^{[a} \partial_b T^{aa]}{}_{,b]}$  curvature,  $K'' := \text{Tr}^2 K$ .

The gauge invariances are the ones expected for a Curtright field and a vector field.

The field equations give  $\bullet -\partial_a F^{ab}(\tilde{A}) + \frac{1}{2} K''{}^b = 0$  (1)

$$\bullet K''{}^{ab}{}_c + \delta_c^{[a} K''{}^{b]} - \frac{1}{2} \partial_c F^{ab} - \delta_c^{[a} \partial_d F^{b]d} = 0$$
 (2)

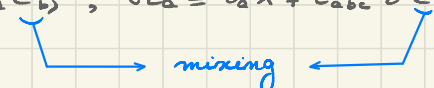
Take the trace of (2), combine with (1) to get the two equations of motion and

duality relation  $d^+ F_{[2]} = 0 = \text{Tr}^2 K_{[2,2]} \quad \& \quad \text{Tr} K_{[3,2]} = d_2 F_{[2,0]} \Rightarrow$  no doubling of d.o.f. !

③ Hodge-dualise in 3D

$S[\gamma_{ab}]$  with  $\gamma_{ab|c} = \varepsilon^{abd} h_{cd} + 2 \delta_c^{[a} z^{b]}$ ,  $h_{ab} = h_{ba}$ , giving an action  $S[h_{ab}, z_a]$

$$S[h_{ab}, z_a] = \int d^3x \left[ -\frac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \frac{1}{2} \partial_a h_{bc} \partial^b h^{ac} + \frac{1}{2} \varepsilon^{bcd} \partial^a h_{ab} F_c(z) + \frac{1}{4} F^{ab}(z) F_{ab}(z) \right]$$

invariant under  $\delta h_{ab} = 2 \partial_a \epsilon_b$ ,  $\delta z_a = \partial_a \lambda + \varepsilon_{abc} \partial^b \epsilon^c$   


- Dualise the vector  $z_a$  in 3D to a scalar. After a field redefinition, one finds

$$S[h_{ab}, \phi] = \int d^3x \left[ -\frac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \frac{1}{2} \partial_a h_{bc} \partial^b h^{ac} - \partial_a h_{bc} \partial^b h^{ac} + \partial_a h^{ab} \partial^a \phi + \frac{1}{2} \partial_a \phi \partial^a \phi + \partial_a \phi (\partial_b h^{ab} - \partial^a h) \right]$$

As consequence of field equations:

$\square \phi \approx 0 \approx R(h) \quad \& \quad R_{ab}(h) \approx \partial_a \partial_b \phi$

Field equations for propagation & duality relation = no doubling

## 2.2) Double dualisation of graviton in $n=5$

- The dualisation procedure for the double-dual graviton gives an action

$$S[D_{ab,cd}, Z^{ab|c}] = \int d^5x \mathcal{L}(\partial D, \partial Z), \quad D \sim \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

where

$$Z^{ab|c} = -Z^{ba|c}$$

$$\mathcal{L}(\partial D, \partial Z) = \mathcal{L}(\partial Z) - \mathcal{L}(\partial D) + \mathcal{L}^{\text{cross}}$$

features the complete  $D_{ab,cd}$  field, including its traceless part

- The action is gauge invariant under

$$\delta \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \partial \\ \hline \end{array}$$

$$\delta Z^{mn|a} = \lambda^{ma} + \partial^{[m} \xi^{n]} a - \frac{1}{2} \delta_a^{[m} \partial_b \xi^{n]b} + \delta_a^{[m} \partial^{n]} \chi$$

$$\text{where } \lambda_{abc} = \lambda_{[abc]}$$

- Perform the change of variable

$$Y^{ab}{}_c := Z^{ab}{}_c + \delta_c^{[a} Z^{b]} \quad \text{where } Z^b := Z^{bc}{}_c$$

to get

$$S[Y^{ab}{}_c, D_{ab,cd}] = \int d^5x \left[ \mathcal{L}^{\text{cut.}}(Y^{ab}{}_c) - \mathcal{L}(\partial\mathcal{D}) + \frac{1}{2} \epsilon_{abcde} \partial^e \mathcal{D}^{cd, mn} \left( F^{abmn}(Y) - \frac{1}{2} F^{mna1b} \right) \right]$$

that propagates a single graviton.

- Dualize  $Y^{ab}{}_c$  in  $5\mathcal{D}$  to  $f_{abc} \sim h_{ab} + B_{[ab]}$  to get (B drops out)

$$S[D_{ab,cd}, f_{ab}] = \int \left[ \mathcal{L}^{\text{FP}}(h) - \mathcal{L}(\partial\mathcal{D}) - \frac{3}{2} h_{ab} \tilde{K}^{ab}(\mathcal{D}) \right] d^5x$$

where  $\tilde{K}_{[2,2]}(\mathcal{D}) := *_1 *_2 \underbrace{d_1 d_2 D_{[2,2]}}_{K_{[3,3]}(\mathcal{D})}$ .

Field equations:

$$\text{Tr } K_{[2,2]}(h) = 0 = \text{Tr}^2 K_{[3,3]}(\mathcal{D})$$

$$\& K_{[2,2]}(h) \propto \text{Tr } K_{[3,3]}(\mathcal{D}).$$

↳ The on-shell duality relation of Hull is now derived from an action principle.

### 2.3) Higher dualisation of the graviton in 3D

Gravity in 3D is topological. Dual formulation thereof by higher dualisation  $\rightarrow$  higher spin

$$\bullet S[G_{a|bc}, D_{ab|}{}^{cd}] = \int d^3x \left[ -\frac{1}{2} G_{a|bc} G^{a|bc} + \frac{1}{2} G_{a|c}{}^c G^{a|b}{}_b - G_{a|}{}^{ab} G_{b|c}{}^c + G_{a|}{}^{ab} G^{c|}{}_c{}_b + G^d{}_{|bc} \partial^a D_{ad|}{}^{bc} \right]$$

$$\hookrightarrow \text{Algebraic symmetries: } G_{a|bc} = G_{a|cb}, \quad D_{ab|}{}^{cd} = -D_{ba|}{}^{cd} = -D_{ba|}{}^{dc}$$

$$\hookrightarrow \text{Gauge symmetries: } \delta G^a{}_{|bc} = 2 \partial^a \partial_{[b} \epsilon_{c]}, \quad \delta D_{ab|}{}^{cd} = \epsilon_{abp} \partial^p \eta^{cd} + 2 \eta^{cd} \partial_{[a} \epsilon_{b]} + 4 \delta_{[a}^{(c} \partial_{b]} \epsilon^{d)}$$

$$\bullet \frac{\delta S}{\delta D} \approx 0 \Rightarrow G_{a|bc} \approx \partial_a h_{bc} \quad \text{with} \quad h_{ab} = h_{ba} \quad \text{symmetric}$$

$$\hookrightarrow S[\partial_a h_{bc}, D_{ab|}{}^{cd}] = \int d^3x \left[ -\frac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \frac{1}{2} \partial_a h \partial^a h - \partial_a h^{ab} \partial_b h + \partial_a h^{ab} \partial^c h_{bc} \right]$$

Fierz-Pauli

$$\bullet \frac{\delta S}{\delta G_{a|bc}} \approx 0 \Rightarrow G_{a|bc} \approx \partial^d D_{da|}{}^{bc} + \text{"}\partial D\text{"} =: G_{a|bc}(\partial D)$$

$$\hookrightarrow S[G_{a|bc}(\partial D), D_{ab|}{}^{cd}] =: S[D_{ab|}{}^{cd}] \quad \text{Dual action}$$

$$D_{ab}{}^{pq} =: \epsilon_{abm} \tilde{D}^{mpq}, \quad \tilde{D}{}^{abcd} = \tilde{\varphi}{}^{acd} + 2 \epsilon^{abc} z_{b1}{}^d, \quad z_{a1}{}^a \equiv 0$$

Gl(3) decomposition

- Further decompose under SO(3):

$$\tilde{D}{}^{abc} \sim \square \otimes (\square \oplus \bullet) \simeq \square \oplus \square \oplus \square \oplus \square \in \text{SO}(3)$$

- A linear combination of the two vectors can be dualised to a scalar, in 3D.
- Combining the scalar with  $\square \rightarrow$  traceful  $h_{ab} \sim \square \in \text{Gl}(3)$
- Traceless rank-3 with remaining vector  $\rightarrow$  traceful  $\varphi_{abc} \sim \square \oplus \square \in \text{Gl}(3)$

Final spectrum:  $\{ \varphi_{abc}, h_{ab} \}$  of  $\text{Gl}(3, \mathbb{R})$ .

Gauge transformations :

$$\begin{cases} \delta \varphi_{abc} = 3 \partial_{[a} \hat{\xi}_{bc]} - \frac{2}{3} \varepsilon_{(a}{}^{pq}{}_{|}{}_{bc)} \partial_p \varepsilon_q \\ \delta h_{ab} = 2 \partial_{[a} \varepsilon_{b]} + 2 \varepsilon_{pq(a} \partial^p \hat{\xi}^q{}_{b)} \end{cases}$$

Preserve action

$$\begin{aligned} S[\varphi_{abc}, h_{ab}] = \frac{1}{2} \int d^3x & \left[ -\partial_a \varphi_{bcd} \partial^a \varphi^{bcd} + \partial^a \varphi_b \partial^c \varphi_{abc} + \partial_a \varphi^{abc} \partial^d \varphi_{bcd} - \frac{1}{7} \partial_a \varphi_b \partial^a \varphi^b - \frac{3}{28} \partial_a \varphi^a \partial^b \varphi_b \right. \\ & + \frac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \frac{1}{14} \partial_a h \partial^a h - \frac{3}{7} \partial^a h_{ab} \partial^b h^{bc} - \frac{1}{7} \partial_a h \partial_c h^{ac} \\ & \left. + \frac{10}{7} \varepsilon_{apq} \partial^b h_b{}^a \partial^p \varphi^q - 2 \varepsilon_{apq} \partial^b h^{ac} \partial^p \varphi^q{}_{bc} \right] \end{aligned}$$

- Gauge transformations entangled
- "Wrong" relative kinetic term
- $\exists$  (Abelian) Chern-Simons  $\text{Tr}(A dA)$  reformulation  $\rightarrow$  To appear hopefully soon.



## 2.4) Higher dualisations of Maxwell's in 3D

- Maxwell in 3D  $\sim$  massless scalar in 3D: 1 propagating d.o.f.
- Start from the dual action obtained by the second higher dualisation of scalar theory, i.e. the first higher dualisation of Maxwell's vector reviewed above

$$S[h_{ab}, Z_a] = \int d^3x \left[ -\frac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \frac{1}{2} \partial_a h_{bc} \partial^b h^{ac} + \frac{1}{2} \epsilon^{bcd} \partial_a h_{ab} F_{cd}(Z) + \frac{1}{4} F_{ab}(Z) F_{ab}(Z) \right]$$

We perform the higher dualisation  $\mathcal{D}$  of  $h_{ab}$  via parent action  $S[G_{a|bc}, D_{ab|}{}^{cd}, A_a]$ ,

eliminate the auxiliary field  $G_{a|bc}$  to obtain  $S[D_{ab|}{}^{cd}, A_a]$ . As before,  $\tilde{D}^{a|ij} := -\frac{1}{2} \epsilon^{abc} D_{bc|}{}^{ij}$ .

Perform a field redefinition  $\{ \underbrace{\tilde{D}_{a|cd}}_{15}, \underbrace{A_a}_{3} \} \leftrightarrow \{ \underbrace{\phi_{abc}}_{\substack{10 \\ \mathfrak{so}(3)}}, \underbrace{f_{ab}}_{5}, \underbrace{U_a}_{3}, \underbrace{A_a}_{3} \}$  where  $f_{ab} \sim \square$  of  $SO(3)$ .

s.t.  $U_a$  enters the action only via  $F_{ab}(U) = 2 \partial_{[a} U_{b]}$ . Dualise  $U_a$  in 3D to a scalar  $\sigma$  that one adds

to  $f_{ab} \rightarrow h_{ab}$  traceful.

The final action is invariant under

$$\left\{ \begin{array}{l} \delta \phi_{abc} = 3 \partial_{[a} \xi_{bc]} \\ \delta h_{ab} = 2 \partial_{[a} \epsilon_{b]} + 2 \epsilon_{pq[a} \partial^p \xi^q_{b]} \\ \delta A_a = \frac{3}{2} \partial_a \xi + \epsilon_{abc} \partial^b \epsilon^c \end{array} \right. \quad \text{Rem: } \delta(\underbrace{\phi_{abc} - \frac{2}{3} \eta_{[ab} A_{c]}}_{\varphi_{abc}}) = 3 \partial_{[a} \xi_{bc]} - \frac{2}{3} \eta_{[ab} \epsilon_{c]pq} \partial^p \epsilon^q .$$

Field equations: seemingly too many propagating d.o.f. since one finds that

- $\bar{K}_{ab}(\phi) \approx 0$  where  $\bar{K}_{ab} := \eta^{cd} \eta^{ij} K_{abicciddj}$  and  $K_{abicediej} := \delta \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$  curvature of  $\phi_{abc}$ .
- $\bar{R}^{ab}{}_{ab} \approx 0$  where  $\bar{R}{}^{ab}{}_{cd} = K_{ab}{}^{cd}(\xi) - 2(\epsilon_{abm} \partial^{[c} \psi^{d]m} + \epsilon^{cdm} \partial_{[a} \psi_{b]m})$ ,  $\psi^a_b := \partial_b \phi^a - \partial^c \phi^a{}_{bc}$   
 $\hookrightarrow$  Riemann-like curvature for  $h_{ab}$ .
- $\partial_a \tilde{F}^{ab} \approx 0$  where  $\tilde{F}_{ab} := F_{ab}(A) + \partial_{[a} \phi_{b]c} + \epsilon_{abc}(\partial_d h^{cd} - \partial^c h)$  field strength for  $A_a$

However, combining the field equations, one finds the duality relations

$$K_{abcd}(\phi) \approx -\frac{8}{21} \epsilon_{efg} \partial^g \tilde{R}_{abcd}$$

$$\tilde{R}{}^{ab}{}_{cd} \approx \frac{7}{4} \epsilon_{cdm} \partial^m \tilde{F}{}^{ab}$$

$$\Rightarrow K_{abcd}(\phi) \approx -\frac{2}{3} \epsilon_{cdp} \epsilon_{efg} \partial^p \partial^g \tilde{F}{}^{ab} \quad ,$$

All the duality relations and E.o.M come out of the action.

### ③ Higher dualisations of linearised gravity in 4D and $A_1^{+++}$

#### 3.1) The first higher dual of graviton in 4D

$$S[G_{a,bc}, D_{ab;cd}] = \int d^4x \left( -\frac{1}{2} G_{a,bc} G^{a,bc} + \dots + G_{a,bc} \partial_d D^{da;bc} \right)$$

Reproduces FP on one hand, and  $S[D_{ab;cd}]$  dual action on the other.

$$D_{ab;cd} \sim \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c & d \end{bmatrix}$$

$$D_{ab;cd} = X_{ab;cd} + 4 \delta_{[a}^c z_{b];d} \quad \text{GL}(n)\text{-irreducible for } X_{a_0;^0b} \equiv 0 \equiv z_{a;^a}$$

$$\text{Hodge-dualise in 4D: } A^{ab,cd} := -\frac{1}{2} \varepsilon^{abcd} X_{ab;cd}, \quad \hat{z}^{abc,d} := \varepsilon^{abce} z_{e;d}$$

$$A^{ab,cd} \sim \begin{bmatrix} a & c & d \\ b \end{bmatrix}, \quad \hat{z}^{abc,d} \sim \begin{bmatrix} a & d \\ b \\ c \end{bmatrix}$$

$$h_{ab} \xrightarrow{\mathcal{D}} \{A^{abc,d}, \hat{Z}^{abc,d}\}$$

with action  $S[A^{abc,d}, \hat{Z}^{abc,d}]$  invariant under

$$D_{abc,d} \sim \begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \sim \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$$

$$\bullet \delta \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} = \overset{\lambda}{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square \\ \hline \end{array}} + \overset{m}{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square \\ \hline \end{array}} \oplus \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & & \\ \hline \end{array}$$

$$\eta \sim \eta_{aa} \quad \text{and} \quad \begin{array}{|c|} \hline \bullet \\ \hline \bullet \\ \hline \end{array} \rightsquigarrow \varepsilon^{ab\bullet\bullet} \partial_{[a} \varepsilon_{b]}$$

$$\lambda_{abc} \sim \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \quad m_{ab,c} \sim \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$$

- $A^{abc,d} \rightsquigarrow$  generator  $R_{abc,d}$  at level 2 in the adjoint of  $A_1^{+++}$
- $\hat{Z}^{abc,d} \rightsquigarrow$  in  $\ell_2$  representation of  $A_1^{+++}$  at level 1
- $\lambda_{abc}$  &  $m_{ab,c} \rightsquigarrow$  in  $\ell_1$  representation at level 2.

- The field equations derived from  $S[A_{ab,cd}, \hat{\mathbb{Z}}_{abc,d}]$  imply that

$$\text{Tr}_{12}^2 \hat{G}_{[3,2,2]} \approx 0 \approx \text{Tr}_{23} \hat{G}_{[3,2,2]}$$

where  $\hat{G}_{[3,2,2]} = d_1 d_2 d_3 A_{[2,1,1]} + \dots$  the curvature

- These are exactly of the form obtained in [A. Tumanov & P. West, M. Peltit & West, K. Glannon & P. West]

when studying the Nonlinear Realisation (NLR) of  $A_1^{+++} \ltimes \mathfrak{h}_1$ .

- By taking more and more dualisations  $\mathfrak{D}^N$ ,

one recognises many fields in adjoint &  $\mathfrak{h}_2$  rep. of  $A_1^{+++}$

## higher dualisation

explicitly here. The  $GL(4)$ -irreducible field content of the new field  $D_{ab;{}^{cd,ef}}$  can be read off from its Hodge dual  $\tilde{D}^{ab;cd,ef} = \frac{1}{2} \varepsilon^{abij} D_{ij;{}^{cd,ef}}$  and the decomposition of its Young diagram:

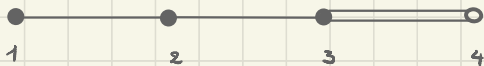
$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \sim \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}, \quad (3.51)$$

$$\Leftrightarrow \mathbb{Y}[2] \otimes \mathbb{Y}[2, 1, 1] \sim \mathbb{Y}[2, 2, 1, 1] \oplus \mathbb{Y}[3, 1, 1, 1] \oplus \mathbb{Y}[3, 2, 1] \oplus \mathbb{Y}[4, 1, 1], \quad (3.52)$$

$$\Leftrightarrow \mathbb{Y}(1, 1) \otimes \mathbb{Y}(3, 1) \sim \mathbb{Y}(4, 2) \oplus \mathbb{Y}(4, 1, 1) \oplus \mathbb{Y}(3, 2, 1) \oplus \mathbb{Y}(3, 1, 1, 1), \quad (3.53)$$

$$\Leftrightarrow D \sim A \quad \hat{Y} \quad \hat{Z} \quad \hat{W}. \quad (3.54)$$

### 3.2) A few words by a non-expert on NLR of $A_1^{+++}$

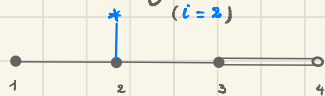


Dynkin diagram  $A_1^{+++}$

#### • Computation of fundamental representations

↳ add a new node labelled  $*$  to the Dynkin diagram and attach it to node  $i$  by a single link.

A generic root in the corresponding enlarged algebra  $A_1^{+++ (i)}$  given by



$$\alpha = m_* \alpha_* + l \alpha_i + \sum_{j=1}^3 m_j \alpha_j$$

Generators sorted by their Kac level  $l$ .

It turns out that  $l = \frac{1}{2}(\text{number of upper indices minus number of lower indices})$

One presents the  $A_3$  content of an  $A_1^{+++}$  rep.  $l_i$  level by level w.r.t.  $l$



- The notion of level is preserved by commutators, so the set of roots with  $m_* = 1$  forms a representation of  $A_1^{+++}$  —  $[m_* = 0, m_* = 1] \subset m_* = 1$  — which one can show is equivalent to the  $i^{\text{th}}$  fundamental representation denoted  $l_i$ .

$$\alpha = m_* \alpha_* + l \alpha_* + \sum_{j=1}^3 m_j \alpha_j$$

- A generic  $A_3 = sl_4$  weight can be expressed as  $\lambda = \sum_{i=1}^3 p_i \lambda_i$  where  $\lambda_i$  is the  $i^{\text{th}}$  fundamental weight.

$$\lambda = [p_1, p_2, p_3] \sim \Upsilon[3, \dots, 3, 2, \dots, 2, 1, \dots, 1] = \Upsilon[3^{\beta_1}, 2^{\beta_2}, 1^{\beta_3}] .$$

- The relationship between the permitted  $A_3$  Dynkin labels  $(p_i)_{i=1,2,3}$  of  $\lambda$  and the Kac labels  $(m_i)_{i=1,2,3}$  of the  $A_1^{+++}$  root  $\alpha$  associated with  $\lambda$  is known.

- Generators at non-negative levels:

$$R^\alpha = \left\{ \begin{array}{c} K^a_b \text{ (0)} \\ \square \otimes \square \end{array} , \begin{array}{c} R^{(ab)} \text{ (1)} \\ \square \square \end{array} , \begin{array}{c} R^{a_1 a_2, (b_1 b_2)} \text{ (2)} \\ \square \square \square \end{array} , \begin{array}{c} R^{a_1 a_2, b_1 b_2, (c_1 c_2)} \text{ (3)} \\ \square \square \square \square \end{array} , \begin{array}{c} R^{a_1 a_2 a_3, b_1 b_2, (c_1 c_2)} \text{ (3)} \\ \square \square \square \square \square \end{array} , \dots \right\}$$

$$[R^\alpha, R^\beta] = f^{\alpha\beta\gamma} R^\gamma$$

- Generators in the vector  $\ell_1$  representation:

$$L_A = \left\{ \begin{array}{c} P_a \text{ (0)} \\ \square \end{array} , \begin{array}{c} Z^a \text{ (1)} \\ \square \end{array} , \begin{array}{c} Z^{a_1 a_2 a_3} \text{ (2)} \\ \square \square \square \end{array} , \begin{array}{c} Z^{a_1 a_2, a_3} \text{ (2)} \\ \square \square \end{array} , \dots \right\}$$

$$[L_A, L_B] = 0$$

$$\sigma(\xi) = g_L(\xi) g_A(\psi(\xi)) \in A_1^{+++} \ltimes \mathfrak{l}_1$$

- non-negative level
- $g_A = \exp(\tilde{A}_a(z) R^a) = \dots \exp(A_{abcd} R^{abcd}) \exp(A_{ab} R^{ab}) \exp(h_{ab} K^a{}_b)$ ,
  - $g_L = \exp(z^a L_a) = \exp(x^a P_a) \exp(z_a Z^a) \exp(z_{abc} Z^{abc} + z_{abc} Z^{abc}) \dots$

• The fields  $A_{\pm}(z^a)$  depend on  $z^a$  coordinates.

G action  $\sigma \mapsto g_0 \sigma$   $g_0 \in G = A_1^{+++} \ltimes \mathfrak{l}_1$  rigid

$G/H$ :  $\sigma \mapsto \sigma h$   $h(z^a)$  transformation used to set the coefficient of negative-level generators to zero in  $g_A$

$H = I_2(A_1^{+++})$

$I_2(A_1^{+++})$  Cartan involution subalgebra  $\subset A_1^{+++}$   
 $E_i \mapsto -F_i$ ,  $F_i \mapsto -E_i$ ,  $H_i \mapsto -H_i$

$$\sigma(\xi) \mapsto g_0 \sigma(\xi) = \sigma(\xi') h(g_0, \xi)$$

- Field equations invariant under  $\sigma \mapsto g_0 \sigma$  and  $\sigma \mapsto \sigma h$

Maurer-Cartan 1-form  $\mathcal{V} := \sigma^{-1} d\sigma = \mathcal{V}_A + \mathcal{V}_L$ ,

$$\mathcal{V}_A = g_A^{-1} dg_A = dz^\pi G_{\pi, \alpha} R^\alpha, \quad \mathcal{V}_L = g_A^{-1} (g_L^{-1} dg_L) g_A = g_A^{-1} (dz^B L_B) g_A = dz^\pi \underbrace{E_\pi^A}_{\text{"vielbein"}} L_A.$$

$e_\mu^a = (\exp(h))_\mu^a$ .

- One derives a set of duality relations from which the E.o.M. are deduced.

- Gauge transformations  $\delta A_\alpha = C_{\alpha, \beta}^{-1} (D^\beta)_E^F \partial_F \Lambda^E$  for the linearised theory

$$C_{\alpha, \beta} \text{ Cartan-Killing metric of } A_1^{+++} \quad [R^\alpha, L_A] = - (D^\alpha)_A{}^B L_B$$

Preserve the linearised E.o.M.

- In [N.B., P.P. Cook, J. O'Connor and P. West], we started investigating

the connection between the NLR and higher dualisation programmes.

### 3.3) Higher duals of graviton in 4D

- One shows, level by level  $n$  in  $\mathcal{D}^n$ , that off-shell dualisation produces a set of extra fields closely correlated with the  $\mathfrak{l}_2$  representation.
- At the  $n^{\text{th}}$  level of higher dualisation  $\mathcal{D}^n$ , counts fields that appear in the adjoint representation at level  $n+1$  and in the  $\mathfrak{l}_2$  rep. at level  $n$ .

• In  $4D$ ,  $D^{n=0}$  :

$$C_{a1b} \sim \square \otimes \square \simeq \square \oplus \square$$

dual graviton

adj. of  $A_1^{+++}$  level 1

2-form

in  $\mathfrak{h}_2$  rep. at level 0



In the standard dualisation of gravity, the dual graviton  $\sim \mathbb{V}[\mathcal{D}-3, 1]$  and an extra  $(\mathcal{D}-2)$ -form that can be gauged away with the Hodge dual of Lorentz parameter  $\lambda_{ab}$ .

• At first  $D$  level in higher dualisation

$$D_{ab;cd} \sim \square \otimes \square \simeq \square \oplus \square$$

$A_{ab;cd}$  in adjoint  $A_1^{+++}$   
level 2

in  $\mathfrak{h}_2$ -rep. at level 1

Spectrum of fields in higher dualisation : Maximal dualisation

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \xrightarrow{\mathcal{D}} 1 \times \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \oplus 1 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \quad (3.110)$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \xrightarrow{\mathcal{D}^2} 1 \times \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array} \oplus 1 \times \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \oplus 2 \times \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \oplus 2 \times \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \oplus 1 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \\ \hline \end{array} \oplus 1 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad (3.111)$$

whereas the third level is visualised as

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \xrightarrow{\mathcal{D}^3} 1 \times \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array} \oplus 2 \times \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \oplus 3 \times \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \oplus 6 \times \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \oplus 3 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ \oplus 3 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus 1 \times \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \oplus 3 \times \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \oplus 6 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus 1 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad (3.112)$$

The irreducible fields at level four are given in column notation as

$$\begin{aligned}
 & \mathbb{Y}[2, 2, 2, 2, 1, 1] , \quad \mathbb{Y}[3, 2, 2, 2, 1] , \quad \mathbb{Y}[3, 2, 2, 1, 1, 1] , \quad \mathbb{Y}[4, 2, 2, 1, 1] , \quad \mathbb{Y}[4, 3, 1, 1, 1], \\
 & \mathbb{Y}[4, 2, 1, 1, 1, 1] , \quad \mathbb{Y}[3, 3, 2, 1, 1] , \quad \mathbb{Y}[3, 3, 1, 1, 1, 1] , \quad \mathbb{Y}[4, 3, 2, 1] , \quad \mathbb{Y}[4, 2, 2, 2] , \\
 & \mathbb{Y}[3, 3, 2, 2] , \quad \mathbb{Y}[4, 4, 1, 1] , \quad \mathbb{Y}[4, 4, 2] , \quad \mathbb{Y}[4, 3, 3] , \quad \mathbb{Y}[3, 3, 3, 1] \quad (3.113)
 \end{aligned}$$

with corresponding Young diagrams

$$(3.114)$$

and respective multiplicities in the order presented above

$$(1, 4, 3, 12, 12, 3, 8, 2, 24, 6, 6, 12, 10, 9, 6) . \quad (3.115)$$



## Minimal off-shell dualisation

After dualising the  $(n-1)^{\text{th}}$  higher dual graviton  $A^{a^1[2],a^2[2],\dots,a^{n-1}[2],c_1,c_2}$ , the set of independent fields will contain the  $n^{\text{th}}$  higher dual graviton

$$A_{[2,\dots,2,1,1]}^{(n)} \equiv A^{(n)} := A^{a^1[2],a^2[2],\dots,a^{n-1}[2],a^n[2],c_1,c_2} \sim \begin{array}{|c|c|c|c|c|c|} \hline a_1^1 & a_1^2 & \cdots & a_1^n & c_1 & c_2 \\ \hline a_2^1 & a_2^2 & \cdots & a_2^n & & \\ \hline \end{array} \quad (3.93)$$

which is a  $GL(4)$ -irreducible field of type  $\mathbb{Y}[2, \dots, 2, 1, 1] = \mathbb{Y}(n+2, 2)$ . The extra fields that are produced belong to one of the following families at the  $n^{\text{th}}$  level of higher dualisation:

$$\widehat{Y}_{[3,2,\dots,2,1,1,1]}^{(n)} \equiv \widehat{Y}^{(n)} := \widehat{Y}^{a[3],b^1[2],\dots,b^{n-2}[2],c_1,c_2,c_3} \sim \begin{array}{|c|c|c|c|c|c|c|} \hline a & b & \cdots & b & c & c & c \\ \hline a & b & \cdots & b & & & \\ \hline a & & & & & & \\ \hline \end{array} \quad (3.94)$$

$$\widehat{Z}_{[3,2,\dots,2,1]}^{(n)} \equiv \widehat{Z}^{(n)} := \widehat{Z}^{a[3],b^1[2],\dots,b^{n-1}[2],c} \sim \begin{array}{|c|c|c|c|c|c|} \hline a & b & \cdots & b & b & c & c \\ \hline a & b & \cdots & b & b & & \\ \hline a & & & & & & \\ \hline \end{array} \quad (3.95)$$

$$\widehat{W}_{[4,2,\dots,2,1,1]}^{(n)} \equiv \widehat{W}^{(n)} := \widehat{W}^{a[4],b^1[2],\dots,b^{n-2}[2],c_1,c_2} \sim \begin{array}{|c|c|c|c|c|c|} \hline a & b & \cdots & b & c & c \\ \hline a & b & \cdots & b & & \\ \hline a & & & & & \\ \hline a & & & & & \\ \hline \end{array} \quad (3.96)$$

Table 4: Extra fields from  $A_1^{+++}$  and off-shell dualisation.

label	$A_3$ weight	adj	$\ell_2$	total	maximal off-shell	net
$b$	[1, 0, 1]	0	1	1	1	0
$c_1$	[1, 0, 3]	0	1	1	1	0
$c_2$	[1, 1, 1]	1	1	2	2	0
$c_3$	[0, 0, 2]	0	1	1	2	+1
$c_4$	[0, 1, 0]	0	1	1	1	0
$c_5$	[2, 0, 0]	0	0	0	1	+1
$d_1$	[1, 1, 3]	1	1	2	2	0
$d_2$	[1, 2, 1]	2	2	4	3	-1
$d_3$	[0, 1, 2]	1	4	5	6	+1
$d_4$	[0, 2, 0]	0	2	2	3	+1
$d_5$	[2, 1, 0]	1	1	2	3	+1
$d_6$	[0, 0, 4]	0	1	1	1	0
$d_7$	[2, 0, 2]	1	1	2	3	+1
$d_8$	[1, 0, 1]	1	2	3	6	+3
$d_9$	[0, 0, 0]	0	0	0	1	+1

$\hat{\Sigma}_{abc,d}$

One can show that there is a dualisation scheme where the set of fields does not exceed the content of the adjoint and  $\ell_2$  rep. of  $A_1^{+++}$ .

At first level. Contact with Labastida formulation.

- $S[A_{ab,cd}, \hat{z}^{abc,d}]$  obtained with its gauge transformations

$$A_{ab,cd} \leftrightarrow \phi_{abc,d} \sim \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \quad \hat{z}^{abc,d} \leftrightarrow z_{a,d}$$

$$\delta \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & & \\ \hline \textcircled{a} & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \textcircled{a} \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \textcircled{a} & \textcircled{b} & \textcircled{c} \\ \hline & & \\ \hline \end{array}$$

↳ "ε... η... ∂.ε."

- $\delta \phi_{abc,d} = 3 \partial_d \lambda_{abc} - 3 \partial_{(a} \lambda_{bc)d} + 3 \partial_{(a} \mu_{bc),d} - \frac{3}{2} \eta_{cab} \varepsilon_{c,d} \varepsilon_{ij} \partial^i \varepsilon^j$

$$\delta \begin{array}{|c|c|} \hline a & d \\ \hline b & \\ \hline c & \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline \textcircled{a} & \\ \hline \end{array} + \varepsilon^{abc} \partial^d \varepsilon_c + \dots$$

- Define  $U^{ab} := -\frac{1}{2} A^{ab,c} + \hat{z}^{abc,c}$

s.t.  $\delta U^{ab} = 2 \partial^{[a} \varepsilon^{b]} + \varepsilon^{abcd} \partial_c \varepsilon_d$ ,  $z^a := \lambda^{ab}_b - \mu^{b,a}$ .

- Set  $\tilde{\gamma}^{abc,d} := \phi^{abc,d} + \frac{3}{4} \eta^{cab} \psi^{cd}$

$$\delta \tilde{\Phi}_{abc,d} = 3 \partial_d \tilde{\lambda}_{abc} - 3 \partial_{[a} \tilde{\lambda}_{bc]d} + 3 \partial_{[a} \tilde{\mu}_{bc],d} - \frac{3}{4} \eta_{cab} \epsilon_{cdpq} \partial^p \epsilon^q$$

- $\tilde{\lambda}_{abc} := \lambda_{abc} - \frac{1}{4} \eta_{cab} \zeta_c$  ,  $\tilde{\mu}_{abc} := \mu_{abc} + \frac{1}{6} (\eta_{ab} \zeta_c - \eta_{ca} \zeta_b)$

The new gauge parameters are s.t.  $\tilde{\mu}^b{}_{,a} \equiv \tilde{\lambda}_{ab}$  , as in Labastida

- Independent fields  $\{ \tilde{\Phi}_{abc,d}, U_{ab}, Z_{(a;b)} \} \rightarrow$  only  $U_{ab}$  transforms with  $\tau_a$ .

$\Leftrightarrow U_{ab}$  enters  $S[\tilde{\Phi}_{abc,d}, U_{ab}, Z_{(a;b)}]$  through  $H_{abc} = 3 \partial_{[a} U_{bc]}$ .

$\Rightarrow$  Dualise  $U_{ab} \xleftrightarrow{4D}$  scalar  $\sigma(x)$ . Combine  $Z_{(a;b)}$  with  $\sigma(x) \rightarrow \tilde{A}_{ab}(x)$  traceful

$$\delta \tilde{A}_{ab} = 2 \partial_{[a} \epsilon_{b]} - \epsilon_{cdca} \partial^c \tilde{\mu}_{,d]e}$$

entangled gauge transformations

# Field equations of $S[\tilde{\Phi}_{abc,d}, \tilde{A}_{ab}]$ and duality


Gauge-invariant tensors

$$1) K_{na, nb} := 4 \partial_{cm} \partial_{cn} \tilde{A}_{b]a]} + \text{"}\eta_{..} \partial \partial \tilde{A} \dots + \partial \partial \tilde{\Phi} \dots, \epsilon \dots \text{"}$$

where  $-\frac{7}{16} K^{ab}{}_{,ab} = \square \tilde{A}_a{}^a - \partial^\alpha \partial^\beta \tilde{A}_{\alpha\beta}$  Ricci scalar  $K$ .

$$\hookrightarrow \text{E.o.m. } -2 \frac{\delta S}{\delta \tilde{A}^{ab}} = K_{ab} - \frac{1}{2} \eta_{ab} K \approx 0 \Leftrightarrow \text{Ricci flat}$$

$$2) G_{mn, pq, d} := 4 \epsilon^{abcd} \partial_a \partial_{cm} \partial_{cp} \tilde{\Phi}_{q]n]b,c} + \text{more}$$

s.t.  $\tilde{G}_{abc, mn, pq} := \epsilon_{abcd} G_{mn, pq, d}$  is 

Find Bianchi on-shell

$$\partial_{[a} K_{bc], de} \approx 0$$

Field equations

$$K_{ab} \approx 0$$

&

$$\text{Tr}_2^2 \tilde{G}_{[3,2,2]} \approx 0$$

consequence of  $\frac{\delta S}{\delta \tilde{\varphi}^{abc,d}} \approx 0$

There is no doubling of d.o.f. because one has *on-shell duality relation*

$$\tilde{G}_{a[3],b[2],c[2]} \approx -\varepsilon_{a[3]d} \partial^d K_{b[2],c[2]}$$

$\Leftrightarrow$

$$G_{mn,pp}{}^r \approx -\partial^r K_{mn,pp}$$

between the first higher dual graviton  $\square\square$

and the (dual) graviton  $\square$  in 4D

#### ④ Conclusions and outlook

- Infinitely many off-shell covariant descriptions of linearised gravity
- Higher dualisation  $\longleftrightarrow$  higher spins
- These are relevant to  $A_4^{+++} \times \mathfrak{t}_1$  Non-linear realisation of West et collaborators  
Off-shell dualisation requires extra fields in  $\mathfrak{t}_2$  rep. not present in West et al.
- No multiplication of on-shell d.o.f. since (twisted) duality relation consequence of E.o.M

Questions: • Interactions (given by NLR equations) ?

• Repackaging to Labartida frame at higher levels.

• Relation to Bossard-Klein-Schmidt-Sezgin pseudo-Lagrangian