New higher-spin topological systems in 3D

Strange higher-spins are wild quivers

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[to appear soon]

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1. Introduction
2. Higher dualisations of linearised gravity and Maxwell's
3. First-order reformulation of the spin 2 - spin 3 systems
4. Relation with quivers

① Introduction

- Fierz-Pauli programme → all possible off-shell descriptions

  of spin-s massless field ?

  Interactions may not choose the most economical description

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Higher-spin Gravity

Hidden symmetries of Gravity

Non-linear realisation of $e_{11} \ltimes \ell_1$

Higher, or "exotic" descriptions

② Higher dualisations of linearised gravity and Maxwell's

An off-shell dualisation was initiated in 2001 by P. West completed by N.B., S. Cnockaert and M. Henneaux.

In [N.B., P. Cook, D. Ponomarev] ⟹ Other off-shell dualisations schemes proposed.

First, review the dualisation of [West, N.B.-Cnockaert-Henneaux]

Parent action

$$S[Y^{abc|}{}_d, \omega_{ab|c}] = \int d^n x \left(\text{"}\omega\,\omega\text{"} + \partial_a \omega_{bc|}{}^d \, Y^{abc|}{}_d \right)$$

$$Z = \int \mathcal{D}\omega \, \mathcal{D}Y \, \exp \tfrac{i}{\hbar} S[\omega, Y]$$

enforces $\omega_{ab|c} = \partial_{[a} e_{b]c}$

semi-classical $S[e_{ab}] =$ Fierz-Pauli

with local Lorentz $\delta e_{ab} = \lambda_{ab}$

Field $\omega_{ab|c}$ auxiliary

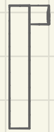
$$\frac{\delta S}{\delta \omega} \approx 0 \implies \omega_{ab|c} \sim \partial^d Y_{abd|c}$$

$$S[Y^{abc|}{}_d] = \int d^n x \left(\partial^\circ Y_{ab\circ|c} \, \partial_\bullet Y^{\bullet ab|c} + \dots \right)$$

$$Y^{abc|}{}_e = \frac{1}{(n-3)!} \varepsilon^{abc\, d_1 \dots d_{n-3}} C_{d[n-3]|e}$$

Gauge inv. $\delta_\lambda Y_{abc|}{}^e = \delta^e_{[a} \lambda_{bc]} \implies \delta_\lambda C_{a[n-3]|b} = \varepsilon_{a[n-3]bcd} \lambda^{cd}$

$$C_{[n-3,1]} \rightsquigarrow C_{a_1 \dots a_{n-3},b} = C_{[a_1 \dots a_{n-3}],b} \quad \text{s.t.} \quad C_{[a_1 \dots a_{n-3},b]} \equiv 0$$

i.e. $C_{[n-3,1]} \sim$  of $GL(n)$ appears in Minkowski spacetime $\mathbb{R}^{1,n-1}$

$n-3$

that propagates the d.o.f. of Fierz-Pauli's graviton h_{ab} with $\eta^{bd} K_{ab,cd}(h) = 0$.

Note : Hull's [2001] twisted on-shell duality

relating

$$K_{a_1 \dots a_{n-2},}{}^{b_1 b_2}(C) := \partial_{[a_1} \partial^{[b_1} C_{a_2 \dots a_{n-2}],}{}^{b_2]}$$

to

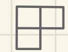
$$K_{ab,}{}^{cd}(h) := -\tfrac{1}{2}\, \partial_{[a} \partial^{c} h^{d]}{}_{b]}$$

via

$$\boxed{K_{[n-2,2]}(C) = *_1 K_{[2,2]}(h)}$$

2.1) Higher dual of vector field in dimensions 4 & 3

Idea : A_b viewed as a $A_{[0,1]}$ bi-form

$$A_{[0,1]} \xrightarrow[\text{higher dualise}]{} C_{[n-0-2\,,\,1]} \underset{n=4}{=} C_{[2,1]} \sim$$ 

$$\underset{n=3}{=} h_{[1,1]} \sim$$ 

- Starts from Maxwell and IBP : $S[A_a] = -\frac{1}{2}\int d^n x\,(\partial_a A_b\,\partial^a A^b - \partial_a A^a\,\partial_b A^b)$
- Parent action $S[Y^{ab|}{}_c\,,\,P_{a|}{}^b] = \int d^n x\,\left(P_{a|b}\,\partial_c Y^{ca|b} - \frac{1}{2}P_{a|b}P^{a|b} + \frac{1}{2}P^{a|}{}_a\,P_{b|}{}^b\right)$

$$\cdot\ \frac{\delta S[Y,P]}{\delta P_{a|b}} \approx 0 \iff P^{a|b} \approx \partial_c Y^{ca|b} - \eta^{ab}\,\frac{1}{n-1}\,\partial_c Y^{cd|}{}_d$$

substitute to get

$$\boxed{S[Y^{ab|}{}_c] = \int d^n x\left[\frac{1}{2}\,\partial_c Y^{ca|b}\,\partial_d Y^d{}_{a|b} - \frac{1}{2(n-1)}\,\partial_a Y^{ab|}{}_b\right]}$$

. From $$\boxed{S[Y^{ab|}{}_c] = \int d^n x \left[\tfrac{1}{2}\, \partial_c Y^{ca|b}\, \partial_d Y^{d}{}_{a|b} - \frac{1}{2(n-1)}\, \partial_a Y^{ab|}{}_b \right]}$$

invariant under $\delta Y^{ab|}{}_c = \delta_c^{[a}\, \partial^{b]} \lambda + \partial_d \psi^{abd|}{}_c$,

one decomposes $Y^{ab|}{}_c = X^{ab|}{}_c + \delta_c^{[a}\, Z^{b]}$, $X^{ab|}{}_b \equiv 0$

irreducibly under $Gl(n)$.

Ⓐ Hodge-dualise in 4D : $X^{ab|}{}_c \overset{*_1}{\longleftrightarrow} T_{ab|c} \sim \ytableaushort{ac,b}$ that gauge transforms as

with $$\begin{cases} \delta\ \ydiagram{2,1} = \overset{S}{\ytableaushort{\none\none,\partial}} + \overset{A}{\ytableaushort{\none\partial,\none}} \\[2ex] \delta Z_a = \partial_a \lambda + \partial^b A_{ab} \end{cases}$$ for the vector.

Perform some change of field variables and dualize $Z_a \longleftrightarrow \tilde{A}_a$ to get

$$\boxed{S[T_{ab|c}, \tilde{A}_a] = \int d^4x \left[\mathcal{L}^{\text{Curt.}}(T_{ab,c}) + \frac{1}{4} F^{ab}(\tilde{A}) F_{ab}(\tilde{A}) + \frac{1}{2} \tilde{A}^a K''_a(T) \right]}$$

where $K^{a[3]}{}_{b[2]} := 6\, \partial^{[a} \partial_{[b} T^{aa],}{}_{b]}$ curvature , $K'' := \mathrm{Tr}^2 K$.

The gauge invariances are the ones expected for a Curtright field and a vector.

The field equations give

$$-\partial_a F^{ab}(\tilde{A}) + \frac{1}{2} K''^{b} = 0 \qquad (1)$$

$$K'^{ab}{}_c + \delta^{[a}_c K''^{b]} - \frac{1}{2} \partial_c F^{ab} - \delta^{[a}_c \partial_d F^{b]d} = 0 \qquad (2)$$

Take the trace of (2), combine with (1) to get the equations of motion and duality relation

$$\boxed{d^\dagger F_{[2]} = 0 = \mathrm{Tr}^2 K_{[2,2]} \quad \& \quad \mathrm{Tr}\, K_{[3,2]} = d_2 F_{[2,0]}} \Rightarrow \text{no doubling of d.o.f. !}$$

(B) Hodge-dualise in 3D

$S[Y^{ab|}{}_c]$ with $Y^{ab|}{}_c = \varepsilon^{abd} h_{cd} + 2\,\delta_c^{[a} Z^{b]}$, $\quad h_{ab} = h_{ba}$, giving an action $S[h_{ab}, Z_a]$

$$S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \tfrac{1}{2} \partial_a h_{bc} \partial^b h^{ac} + \tfrac{1}{2} \varepsilon^{bcd} \partial^a h_{ab} F_{cd}(Z) + \tfrac{1}{4} F^{ab}(Z) F_{ab}(Z) \right]$$

invariant under $\delta h_{ab} = 2\,\partial_{(a} \epsilon_{b)}$, $\quad \delta Z_a = \partial_a \lambda + \varepsilon_{abc} \partial^b \epsilon^c$

mixing

- Dualise the vector Z_a in 3D to a scalar. After a field redefinition, one finds

$$S[h_{ab}, \phi] = \int d^3x \Big[-\tfrac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \tfrac{1}{2} \partial_a h \partial^a h - \partial_a h \partial_b h^{ab} + \partial_a h^{ab} \partial^c h_{bc}$$
$$+ \tfrac{1}{2} \partial_a \phi \partial^a \phi + \partial_a \phi \left(\partial_b h^{ab} - \partial^a h \right) \Big] .$$

As consequence of field equations:

$$\boxed{\Box \phi \approx 0 \approx R(h) \qquad \& \qquad R_{ab}(h) \approx \partial_a \partial_b \phi}$$

Field equations for propagation & duality relation = no doubling

2.2) Higher dualisations of Maxwell's in 3D

- Maxwell in 3D $\sim$ massless scalar in 3D : 1 propagating d.o.f.
- Start from the dual action obtained by the second higher dualisation of scalar theory, i.e. the first higher dualisation of Maxwell's vector reviewed above

$$S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2}\partial_a h_{bc}\partial^a h^{bc} + \tfrac{1}{2}\partial_a h_{bc}\partial^b h^{ac} + \tfrac{1}{2}\varepsilon^{bcd}\partial^a h_{ab} F_{cd}(Z) + \tfrac{1}{4} F^{ab}(Z) F_{ab}(Z) \right]$$

We perform the higher dualisation $\mathcal{D}$ of h_{ab} via parent action $S[G_{a|bc}, D_{ab|}{}^{cd}, A_a]$, eliminate the auxiliary field $G_{a|bc}$ to obtain $S[D_{ab|}{}^{cd}, A_a]$. As before, $\tilde{D}^{a|ij} := -\frac{1}{2}\varepsilon^{abc} D_{bc|}{}^{ij}$.

Perform a field redefinition $\{\underset{\underline{18}}{\tilde{D}_{a|cd}}, \underset{\underline{3}}{A_a}\} \longleftrightarrow \{\underset{\underline{10}}{\phi_{abc}}, \underset{\underline{5}}{f_{ab}}, \underset{\underline{3}}{U_a}, \underset{\underline{3}}{A_a}\}$ where $f_{ab} \sim \Box\!\Box$ of $SO(3)$.

s.t. U_a enters the action only via $F_{ab}(U) = 2\,\partial_{[a}U_{b]}$. Dualise U_a in 3D to a scalar σ that one adds to $f_{ab} \longrightarrow h_{ab}$ traceful.

The final action is invariant under

$$\begin{cases} \delta \phi_{abc} = 3\, \partial_{(a} \xi_{bc)} \\ \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} + 2\, \varepsilon_{pq(a}\, \partial^p \xi^q{}_{b)} \\ \delta A_a = \frac{3}{2}\, \partial_a \xi + \varepsilon_{abc}\, \partial^b \epsilon^c \end{cases}$$

<u>Rem</u> : $\delta(\underbrace{\phi_{abc} - \frac{2}{3}\eta_{(ab} A_{c)}}_{\varphi_{abc}}) = 3\, \partial_{(a} \hat{\xi}_{bc)} - \frac{2}{3}\eta_{(ab}\, \varepsilon_{c)pq}\, \partial^p \epsilon^q$.

<u>Field equations</u> : seemingly too many propagating d.o.f. since one finds that

- $\bar{\bar{K}}_{ab}(\phi) \approx 0$ where $\bar{\bar{K}}_{ab} := \eta^{cd}\eta^{ij} K_{ab|ci|dj}$ and $K_{ab|cd|ef} := 8 \begin{array}{|c|c|c|} \hline a & c & e \\ \hline \partial_b & \partial_d & \partial_f \\ \hline \end{array}$ curvature of ϕ_{abc} .

- $\widetilde{R}^{ab|}{}_{ab} \approx 0$ where $\widetilde{R}_{ab|}{}^{cd} = K_{ab|}{}^{cd}(h) - 2\left(\varepsilon_{abm}\, \partial^{[c} \Psi^{d]m} + \varepsilon^{cdm}\, \partial_{[a} \Psi_{b]m}\right)$, $\Psi^a{}_b := \partial_b \phi^a - \partial^c \phi^a{}_{bc}$

  $\hookrightarrow$ Riemann-like curvature for h_{ab} .

- $\partial_a \widetilde{F}^{ab} \approx 0$ where $\widetilde{F}_{ab} := F_{ab}(A) + \partial_{[a} \phi_{b]c}{}^c + \varepsilon_{abc}\left(\partial_d h^{cd} - \partial^c h\right)$ fieldstrength for A_a

However, combining the field equation, one finds the duality relations

$$\boxed{\begin{aligned} K_{ab|cd|ef}(\phi) &\approx -\frac{8}{21}\,\varepsilon_{efq}\,\partial^q\,\tilde{R}_{ab|cd} \\[2em] \tilde{R}^{ab|}{}_{cd} &\approx \frac{7}{4}\,\varepsilon_{cdm}\,\partial^m\,\tilde{F}^{ab} \end{aligned}}$$

$$\Longrightarrow\quad K_{ab|cd|ef}(\phi) \approx -\frac{2}{3}\,\varepsilon_{cdp}\,\varepsilon_{efq}\,\partial^p\partial^q\,\tilde{F}_{ab} \quad ,$$

All the duality relations and E.o.M come out of the action.

2.3) Higher dualisation of the graviton in 3D

Gravity in 3D is topological. Dual formulation thereof by higher dualisation $\to$ higher spin

- $$S[G_{a|bc}, D_{ab|}{}^{cd}] = \int d^3x \left[-\tfrac{1}{2} G_{a|bc} G^{a|bc} + \tfrac{1}{2} G_{a|c}{}^{c} G^{a|b}{}_{b} - G_{a|}{}^{ab} G_{b|c}{}^{c} + G_{a|}{}^{ab} G^{c|}{}_{cb} + G^{d}{}_{|bc}\, \partial^a D_{ad|}{}^{bc} \right]$$

$$G_{a|bc} = G_{a|cb}\,, \quad D_{ab|}{}^{cd} = -D_{ba|}{}^{cd} = -D_{ba|}{}^{dc}$$

Gauge-invariant under

$$\delta G^{a}{}_{|bc} = 2\,\partial^a \partial_{(b} \epsilon_{c)}\,, \quad \delta D_{ab|}{}^{cd} = \varepsilon_{abp}\, \partial^p v^{cd} + 2\,\eta^{cd}\, \partial_{[a}\epsilon_{b]} + 4\, \delta^{(c}_{[a}\, \partial_{b]} \epsilon^{d)}\,.$$

- $\dfrac{\delta S}{\delta D} \approx 0 \implies G_{a|bc} \approx \partial_a h_{bc}$ with $h_{ab} = h_{ba}$ symmetric

$\hookrightarrow$ $$S[\partial_a h_{bc}, D_{ab|}{}^{cd}] = \int d^3x \left[-\tfrac{1}{2} \partial_a h_{bc}\, \partial^a h^{bc} + \tfrac{1}{2} \partial_a h\, \partial^a h - \partial_a h^{ab}\, \partial_b h + \partial_a h^{ab}\, \partial^c h_{bc} \right]$$

Fierz-Pauli

- $\dfrac{\delta S}{\delta G_{a|bc}} \approx 0 \implies G_{a|bc} \approx \partial^d D_{da|bc} + \partial D =: G_{a|bc}(\partial D)$

$\hookrightarrow$ $S[G_{a|bc}(\partial D), D_{ab|}{}^{cd}] =: S[D_{ab|}{}^{cd}]$ Dual action

$$D_{ab|}{}^{pq} =: \varepsilon_{abm}\, \tilde{D}^{m|pq} \quad , \quad \tilde{D}^{a|cd} = \tilde{\varphi}^{acd} + 2\, \varepsilon^{ab(c}\, Z_{b|}{}^{d)} \quad , \quad Z_{a|}{}^{a} \equiv 0$$

Gl(3) decomposition

- Further decompose under SO(3) :

$$\tilde{D}^{a|bc} \sim \square \otimes \left(\square\square \oplus \bullet \right) \simeq \square\square\square \oplus \square\square \oplus \square \oplus \square \qquad \in SO(3)$$

- A linear combination of the two vectors can be dualised to a scalar, in 3D.
- Combining the scalar with $\square\square \longrightarrow$ traceful $h_{ab} \sim \square\square \in Gl(3)$
- Traceless rank-3 with remaing vector $\longrightarrow$ traceful $\varphi_{abc} \sim \square\square\square \in Gl(3)$

- Final spectrum : $\{ \varphi_{abc} \,,\, h_{ab} \}$.

- **Gauge transformations** :

$$\begin{cases} \delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - \frac{2}{3}\, \varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q \,, \\ \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} + 2\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)} \,, \end{cases}$$

- **Action** :

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2} \int d^3x \Big[-\partial_a \varphi_{bcd}\, \partial^a \varphi^{bcd} + \partial^a \varphi^b\, \partial^c \varphi_{abc} + \partial_a \varphi^{abc}\, \partial^d \varphi_{bcd} - \frac{1}{7}\, \partial_a \varphi_b\, \partial^a \varphi^b - \frac{31}{28}\, \partial_a \varphi^a\, \partial^b \varphi_b$$
$$+ \frac{1}{2}\, \partial_a h_{bc}\, \partial^a h^{bc} + \frac{1}{14}\, \partial_a h\, \partial^a h - \frac{3}{7}\, \partial^a h_{ab}\, \partial_c h^{bc} - \frac{1}{7}\, \partial^a h\, \partial_c h_a{}^c$$
$$+ \frac{10}{7}\, \varepsilon_{apq}\, \partial^b h_b{}^a\, \partial^p \varphi^q - 2\, \varepsilon_{apq}\, \partial^b h^{ac}\, \partial^p \varphi^q{}_{bc} \Big]$$

- Gauge transformations entangled
- "Wrong" relative kinetic term

2.4) Relation with the triplet spin-2

The action $S[h_{ab}, Z_a]$ derived above is a member of the 1-parameter family

$$S[h_{ab}, A_a] = \frac{1}{2}\int d^3x \, \Big[-\partial_a h_{bc}\,\partial^a h^{bc} + (\alpha+2)\,\partial\cdot h_a\,\partial\cdot h^a - (\alpha+1)\,\partial^a h\,(2\,\partial\cdot h_a - \partial_a h)$$

$$- \frac{\alpha}{2} F_{ab}(A)\,F^{ab}(A) - \alpha\,\varepsilon_{abc}\,\partial\cdot h^a\,F^{bc}(A)\Big]$$

invariant under

$$\delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} \qquad , \qquad \delta A_a = \partial_a\lambda + \varepsilon_{abc}\,\partial^b\epsilon^c \quad .$$

The case obtained by off-shell dualisation corresponds to $\alpha = -1$.

. As above, one dualises the vector $A_a \to \varphi$ scalar in 3D :

$$S[h_{ab}, \varphi] = \frac{1}{2}\int d^3x \, \Big[-\partial_a h_{bc}\,\partial^a h^{bc} + 2\,\partial\cdot h_a\,\partial\cdot h^a - (\alpha+1)\,\partial^a h\,(2\,\partial\cdot h_a - \partial_a h)$$

$$- 4\alpha\,(\partial_a\varphi\,\partial^a\varphi - \varphi\,\partial^a\partial^b h_{ab})\Big]$$

$$S[h_{ab},\varphi] = \frac{1}{2}\int d^3x\, [-\partial_a h_{bc}\,\partial^a h^{bc} + 2\,\partial\cdot h_a\,\partial\cdot h^a - (\alpha+1)\,\partial^a h\,(2\,\partial\cdot h_a - \partial_a h)$$

$$-4\alpha\,(\partial_a\varphi\,\partial^a\varphi - \varphi\,\partial^a\partial^b h_{ab})] \qquad (*)$$

Invariant under $\delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)}$, $\delta\varphi = -\partial^a\epsilon_a$

This is equivalent, when $\alpha = 1$, to the triplet system

$$S[h_{ab}, C_a, D] = \int d^n x\, [-\tfrac{1}{2}\,\partial_a h_{bc}\,\partial^a h^{bc} + 2\,C^a\,\partial\cdot h_a + 2\,\partial\cdot C\,D + \partial_a D\,\partial^a D - C^a C_a]$$

Invariant under $\delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)}$, $\delta C_a = \Box\epsilon_a$, $\delta D = \partial\cdot\epsilon$

The field C_a is auxiliary, its e.o.m. give $C_a = \partial\cdot h_a - \partial_a D$.

Substituting inside the triplet action $S[h_{ab}, C_a, D]$ reproduces the action $(*)$

for $n = 3$, $\alpha = -1$ and $D = -\varphi$

2.5) Spin 3 / Spin 2 system

The action shown above

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2}\int d^3x \Big[-\partial_a \varphi_{bcd}\, \partial^a \varphi^{bcd} + \partial^a \varphi^b\, \partial^c \varphi_{abc} + \partial_a \varphi^{abc}\, \partial^d \varphi_{bcd} - \frac{1}{7}\partial_a \varphi_b\, \partial^a \varphi^b - \frac{31}{28}\partial_a \varphi^a\, \partial^b \varphi_b$$

$$+ \frac{1}{2}\partial_a h_{bc}\, \partial^a h^{bc} + \frac{1}{14}\partial_a h\, \partial^a h - \frac{3}{7}\partial^a h_{ab}\, \partial_c h^{bc} - \frac{1}{7}\partial^a h\, \partial_c h_a{}^c$$

$$+ \frac{10}{7}\varepsilon_{apq}\, \partial^b h_b{}^a\, \partial^p \varphi^q - 2\,\varepsilon_{apq}\, \partial^b h^{ac}\, \partial^p \varphi^q{}_{bc} \Big]$$

invariant under $\delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - \frac{2}{3}\varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q$, $\delta h_{ab} = 2\, \partial_{(a}\epsilon_{b)} + 2\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)}$

is a member of the family

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2}\int d^3x \Big[a_0\, \partial_a \varphi_{bcd}\, \partial^a \varphi^{bcd} + a_1\, \partial^a \varphi^b\, \partial^c \varphi_{abc} + a_2\, \partial_a \varphi^{abc}\, \partial^d \varphi_{bcd} + a_3\, \partial_a \varphi_b\, \partial^a \varphi^b + a_4\, \partial_a \varphi^a\, \partial^b \varphi_b$$

$$+ b_0\, \partial_a h_{bc}\, \partial^a h^{bc} + b_1\, \partial_a h\, \partial^a h + b_2\, \partial^a h_{ab}\, \partial_c h^{bc} + b_3\, \partial^a h\, \partial_c h_a{}^c$$

$$+ c_1\, \varepsilon_{apq}\, \partial^b h_b{}^a\, \partial^p \varphi^q + c_2\, \varepsilon_{apq}\, \partial^b h^{ac}\, \partial^p \varphi^q{}_{bc} \Big]$$

invariant under

$$\delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - 3x\, \varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q \quad , \quad \delta h_{ab} = 2\, \partial_{(a}\epsilon_{b)} - 2z\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)}$$

- We find that $z = 0$ iff $x = 0$.

  In that case, $c_1 = 0 = c_2$.

  The action is the sum or difference of Fronsdal and Fierz-Pauli actions

  $\hookrightarrow$ We reject that case, hence $x \neq 0$ and $z \neq 0$

- For the traceless part $\hat{\varphi}_{abc}$ to appear in the action,

  a_0, a_1, a_2 and c_2 cannot all vanish

  and as a result we find the $b_0 \neq 0$.

- Finaly, the parameters are fixed uniquely in terms of z and $\gamma = \text{sign}(b_0)$

in the case $a_0 = -1$ that one reaches by rescaling φ_{abc}. One parameter and a sign.

$$x = -\frac{2\gamma z}{9(3\gamma z^2 - 2)} \qquad \gamma = \pm 1$$

For $\gamma = +1$, $z = \pm\sqrt{\frac{2}{3}}$ rejected since it amounts to removing the piece $\partial_{(a}\epsilon_{b)}$ in δh_{ab}.

- Rem. : There is an isolated point where $a_0 = 0$.

By rescaling φ_{abc} so that $a_2 = -1$, all the other coefficients are fixed uniquely.

- One shows that these systems are not equivalent to any known indecomposable

(triplet-like) systems.

3 First-order reformulation of the spin 2 - spin 3 systems

3.1) In flat space

A theorem [M. Grigoriev, K. Mkrtchyan, E. Skvortsov 2005] states that all topological systems in 3D are equivalent to Chern-Simons-like models.

We find that, with the Lorentz-valued one-forms $(e^a, \omega^a, E^{aa}, \Omega^{aa})$, the action

$$S = \int_{M_3} \left[\omega_a \left(de^a - \tfrac{1}{2} \varepsilon^{apq} h_p \omega_q \right) + 2z\, \omega_a h_b \Omega^{ab} + \frac{2z}{3x} \Omega_{aa} \left(dE^{aa} + \tfrac{2}{3} \varepsilon^{abc} h_b \Omega_c{}^a \right) \right]$$

invariant under

$$\delta e^a = d\epsilon^a - \varepsilon^{abc} h_b \tilde{\Lambda}_c + 2z\, h_b \tilde{\alpha}^{ab} , \qquad \delta \omega^a = d\tilde{\Lambda}^a ,$$

$$\delta E^{aa} = d\xi^{aa} + \tfrac{4}{3} h_b \varepsilon^{abc} \tilde{\alpha}_c{}^a - 3x \left(h^a \tilde{\Lambda}^a - \tfrac{1}{3} \eta^{aa} h^b \tilde{\Lambda}_b \right) , \qquad \delta \Omega^{aa} = d\tilde{\alpha}^{aa}$$

This action exactly reproduces the 2^{nd} order action $S[h_{ab}, \varphi_{abc}]$

presented above upon (i) eliminating the auxiliary fields (ω^a, Ω^{aa}).

(ii) fixing the Lorentz gauges where $e_{[a,b]} \overset{(*)}{=} 0$ & $E_{a,bc}\big|_{\begin{array}{|c|c|}\hline b & c \\ \hline a \\ \cline{1-1}\end{array}} \overset{(*)}{=} 0$

Recall

$$\delta e_{a,b} = \partial_a \epsilon_b - \Lambda_{ab} + 2z\, \tilde{\alpha}_{ab}$$

$$\delta E_{a,bc} = \partial_a \xi_{bc} - \alpha_{bc,a} + x\,(\eta_{bc} \tilde{\Lambda}_a - 3\, \eta_{a(b} \tilde{\Lambda}_{c)})$$

where

$$\tilde{\alpha}_{ab} := \tfrac{1}{2}\, \varepsilon_{apq}\, \alpha_b{}^{p,q} \quad , \quad \tilde{\Lambda}_a := \tfrac{1}{2}\, \varepsilon_{abc}\, \Lambda^{bc} \quad , \quad \alpha_{bc,a} \sim \begin{array}{|c|c|}\hline b & c \\ \hline a \\ \cline{1-1}\end{array}$$

The residual gauge transformations are $\alpha^{res.}_{a,bc} = \partial_a \xi_{bc}\big|_{\begin{array}{|c|c|}\hline b & c \\ \hline a \\ \cline{1-1}\end{array}}$, $\Lambda^{res.}_{ab} = \partial_{[a} \epsilon_{b]}$

In this gauge, the fields $\varphi_{abc} := 3\, E_{(a,bc)}$, $h_{ab} = 2\, e_{(a,b)}$ transform

$$\delta \varphi_{abc} = 3\, \partial_{(a} \hat{\xi}_{bc)} - 3\, x\, \varepsilon_{(a}{}^{pq}\, \eta_{bc)}\, \partial_p \epsilon_q \quad , \quad \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} - 2\, z\, \varepsilon_{pq(a}\, \partial^p \hat{\xi}^q{}_{b)}$$

One can express the action as

$$S = \int_{M_3} \left[\omega_a R^a(e) + e_a R^a(\omega) + \frac{2z}{3x} \left(\Omega_{ab} R^{ab}(E) + E_{ab} R^{ab}(\Omega) \right] \qquad (*)$$

where $R^a(e) := de^a - \varepsilon^{apq} h_p \omega_q + 2z\, h_b \Omega^{ab}$, $R^a(\omega) := d\omega^a$,

$$R^{aa}(E) := dE^{aa} + \frac{4}{3} h_p \varepsilon^{pqa} \Omega_q{}^a - 3x \left(h^a \omega^a - \frac{1}{3} \eta^{aa} h^b \omega_b \right) , \quad R^{aa}(\Omega) = d\Omega^{aa} .$$

3.2) Extension to $(A)dS_3$ $\qquad \nabla^2 V^a = -\sigma h^a h_b V^b$, $\sigma = +1$ for AdS_3 , $\sigma = -1$ for dS_3 .

We can keep the action in the same form $(*)$ as above for the deformed curvatures

$$\begin{aligned}
R^a(e) &= \nabla e^a + \lambda x_1 \, \varepsilon^{abc} \, h_b \, e_c + x_2 \varepsilon^{abc} \, h_b \, \omega_c + \lambda x_3 \, h_b E^{ab} + x_4 \, h_b \, \Omega^{ab} \, , \\
R^a(\omega) &= \nabla \omega^a + \lambda^2 x_5 \, \varepsilon^{abc} h_b e_c + \lambda x_6 \, \varepsilon^{abc} \, h_b \, \omega_c + \lambda^2 x_7 \, h_b \, E^{ab} + \lambda x_8 \, h_b \, \Omega^{ab} \, , \\
R^{aa}(E) &= \nabla E^{aa} + \lambda x_9 \left(h^a e^a - \tfrac{1}{3} \eta^{aa} \, h^b \, e_b \right) + x_{10} \left(h^a \omega^a - \tfrac{1}{3} \eta^{aa} \, h^b \, \omega_b \right) \\
&\quad + \lambda x_{11} \, h_b \, \varepsilon^{abc} \, E_c{}^a + x_{12} \, h_b \, \varepsilon^{abc} \, \Omega_c{}^a \, , \\
R^{aa}(\Omega) &= \nabla \Omega^{aa} + \lambda^2 x_{13} \left(h^a e^a - \tfrac{1}{3} \eta^{aa} \, h^b \, e_b \right) + \lambda x_{14} \left(h^a \omega^a - \tfrac{1}{3} \eta^{aa} \, h^b \, \omega_b \right) \\
&\quad + \lambda^2 x_{15} \, h_b \, \varepsilon^{abc} \, E_c{}^a + \lambda x_{16} \, h_b \, \varepsilon^{abc} \, \Omega_c{}^a \, .
\end{aligned}$$

The corresponding gauge transformations being

$$\boxed{\begin{aligned}\delta\begin{pmatrix} e^a \\ \lambda\omega^a \end{pmatrix} &= \nabla\begin{pmatrix} \xi^a \\ \lambda\tilde{\Lambda}^a \end{pmatrix} + A\,\varepsilon^{abc}h_b\begin{pmatrix} \xi_c \\ \lambda\tilde{\Lambda}_c \end{pmatrix} + B\,h_b\begin{pmatrix} \xi^{ab} \\ \lambda\tilde{\alpha}^{ab} \end{pmatrix}\,, \\ \delta\begin{pmatrix} E^{aa} \\ \lambda\Omega^{aa} \end{pmatrix} &= \nabla\begin{pmatrix} \xi^{aa} \\ \lambda\tilde{\alpha}^{aa} \end{pmatrix} + C\,\left(h^a\delta^a_b - \tfrac{1}{3}\eta^{aa}h_b\right)\begin{pmatrix} \xi^b \\ \lambda\tilde{\Lambda}^b \end{pmatrix} + D\,\varepsilon^{abc}h_b\begin{pmatrix} \xi_c{}^a \\ \lambda\tilde{\alpha}_c{}^a \end{pmatrix}\,,\end{aligned}}$$

with the numerical matrices explicitly given by

$$A = \begin{pmatrix} x_1 & x_2 \\ x_5 & x_6 \end{pmatrix}\,, \quad B = \begin{pmatrix} x_3 & x_4 \\ x_7 & x_8 \end{pmatrix}\,, \quad C = \begin{pmatrix} x_9 & x_{10} \\ x_{13} & x_{14} \end{pmatrix}\,, \quad D = \begin{pmatrix} x_{11} & x_{12} \\ x_{15} & x_{16} \end{pmatrix}\,.$$

The requirement of gauge invariance of the curvatures gives sixteen quadratic equations for the parameters x_i. In matrix form, they read

$$\boxed{\begin{aligned} A^2 - \tfrac{5}{6}\,BC &= \sigma I\,, \\ \tfrac{1}{2}\,D^2 + CB &= 2\sigma I\,, \\ -\tfrac{3}{2}\,DC + CA &= 0\,, \\ -\tfrac{3}{2}\,BD + AB &= 0\,. \end{aligned}}$$

A particularly simple solution is given by

$$A = \begin{pmatrix} 0 & 1 \\ \frac{9\sigma}{4} & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 1 \\ \frac{3\sigma}{2} & 0 \end{pmatrix} = C , \quad D = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} .$$

3.3) <u>Field redefinitions and flat limit</u>

The redefined fields $\quad \begin{pmatrix} e^a \\ \lambda\omega^a \end{pmatrix} = M \begin{pmatrix} e'^a \\ \lambda\omega'^a \end{pmatrix} , \quad \begin{pmatrix} E^{aa} \\ \lambda\Omega^{aa} \end{pmatrix} = N \begin{pmatrix} E'^{aa} \\ \lambda\Omega'^{aa} \end{pmatrix}$

with redefined gauge parameters $\quad \begin{pmatrix} \epsilon^a \\ \lambda\tilde{\Lambda}^a \end{pmatrix} = M \begin{pmatrix} \epsilon'^a \\ \lambda\tilde{\Lambda}'^a \end{pmatrix} , \quad \begin{pmatrix} \xi^{aa} \\ \lambda\tilde{\alpha}^{aa} \end{pmatrix} = N \begin{pmatrix} \xi'^{aa} \\ \lambda\tilde{\alpha}'^{aa} \end{pmatrix}$

where $\quad M , N \in Gl(2,\mathbb{R})$

will be related by the same transformation laws,

with $\quad A' = M^{-1} A M , \quad B' = M^{-1} B N , \quad C' = N^{-1} C M , \quad D' = N^{-1} D N$

$\hookrightarrow$ Two solutions of $\boxed{}$ related by $Gl(2,\mathbb{R}) \times Gl(2,\mathbb{R})$ are regarded as equivalent.

We showed that the general solution of the quadratic matrix-valued equations

$$\boxed{A^2 - \tfrac{5}{6}\, BC = \sigma\, \mathbb{1}_2 \,, \quad D^2 + 2\, CB = 4\sigma\, \mathbb{1}_2 \,, \quad CA - \tfrac{3}{2}\, DC = 0 \,, \quad AB - \tfrac{3}{2}\, BD = 0}$$

for the 16 parameters x_i

is in the $Gl(2,\mathbb{R}) \times Gl(2,\mathbb{R})$ orbit of the simple solution.

$$A = \begin{pmatrix} 0 & 1 \\ \frac{95}{4} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ \frac{35}{2} & 0 \end{pmatrix} = C \,, \quad D = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} .$$

$\hookrightarrow$ Unicity of the solution in $(A)dS_3$!

In order to recover the flat limit, must

1) act with some appropriate $(M, N)_{\gamma, z}$

2) send $\lambda \to 0$.

Matrices $(M, N)_{\gamma, z}$ are

$$M = \begin{pmatrix} \frac{3}{\sqrt{2}}\Delta & z \\ -\frac{9\sigma}{4}z & \frac{3}{\sqrt{2}}\Delta \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ \frac{3\sigma}{4} & 0 \end{pmatrix},$$

where Δ is the square root

$$\Delta = \sqrt{\gamma\sigma\left(2\gamma z^2 - 1\right)}.$$

Because the numerical matrices (A, B, C, D) are real and $\gamma = \pm 1$,

extensions from flat to $(A)dS_3$ depend on values of z and γ.

- If $\gamma = +1$, we have $\Delta = \sqrt{\sigma z^2 \left(2z^2 - 1\right)}$: the model can be extended to dS when $z^2 < 1/2$, to AdS for $z^2 > 1/2$, and to both when $z^2 = 1/2$. In particular, the original action of [1] corresponds to $z = -1$ and therefore can only be continued to AdS, not to dS.
- If $\gamma = -1$, we have $\Delta = \sqrt{\sigma z^2 \left(2z^2 + 1\right)}$: these models can only be deformed to AdS.

④ Relation with quivers

Use spinor notation instead of vector $so(1,2)$. The one-forms can be grouped in

$$\Phi(y,x) = \sum_{N,i} \frac{1}{N!} y_{\alpha_1} \cdots y_{\alpha_N} \Phi_i^{\alpha_1 \cdots \alpha_N}(x)$$

$(e^{\alpha\alpha}, \omega^{\alpha\alpha}, E^{\alpha(4)}, \Omega^{\alpha(4)})$ in our case.

Field equations $\qquad \nabla \Phi = Q \Phi \qquad ,$

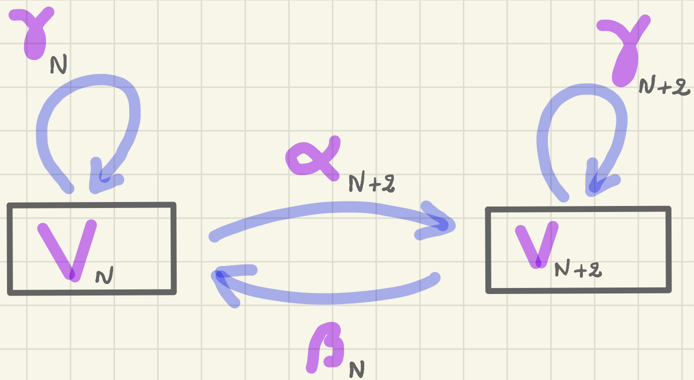
where $\qquad Q = \alpha(N)\, h^{\alpha\alpha} y_\alpha y_\alpha + \beta(N)\, h^{\alpha\alpha} \partial_\alpha \partial_\alpha + \gamma(N)\, h^{\alpha\alpha} y_\alpha \partial_\alpha$

For a topological system, $D = \nabla - Q$ is nilpotent. This imposes constraints on

$$\alpha_N : V_{N-2} \to V_N \quad , \quad \beta_N : V_{N+2} \to V_N \quad , \quad \gamma_N : V_N \to V_N$$

Moreover, redefinitions $\Phi_N \to A_N \Phi_N$ by automorphisms.

Quiver



In AdS, one can use $Gl_2 \times Gl_2$

to reach $\alpha_{N+2} = \beta_N = \gamma_N = \gamma_{N+2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In flat space, $\beta_N = \gamma_N = \gamma_{N+2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\alpha_{N+2} = q \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ 1-parameter.