Geometry of Super Null Infinity

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Ongoing work with N. Boulanger and Y. Herfray

Workshop Beyond Lorentzian Geometry II

Table of Contents

Table of Contents

Context

- Null Infinity is the boundary of asymptotically flat spacetimes
- Asymptotically flat spacetimes are relevant
- Asymptotic group of symmetries of these space is BMS [BMS 62]
- More recently, it was shown that these symmetries are equivalent to the conformal Carrolian symmetries (symmetries of Null Infinity) [Duval, Gibbons, Horvathy]

Goal of the work

Generalize this in a superspace formulation, with a Carrollian approach :

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Generalize this in a superspace formulation, with a Carrollian approach : Several works already investigated the subject [AGS 86][Henneaux et al.]

- We would like to propose a definition for asymptotically flat superspaces, for which the boundary would carry a *superconformal Carrollian geometry*
- This would give a geometrical realization of the super BMS group, as the conformal Carrollian symmetry group of super Null Infinity
- Starting point : super Minkowski space, through the study of homogeneous spaces

Klein Geometry and homogeneous spaces

Homogeneous space

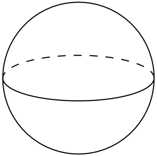
Space M with a transitive action of a Lie group G

"All points look the same"

$\longrightarrow M \simeq G/H$, where H is the stabilizer of one point $x \in G$

Examples :

- $\mathbb{S}^2 \simeq \frac{\mathrm{SO(3)}}{\mathrm{SO(2)}}$
- $\mathbb{M}^{1,3} \simeq \frac{\mathrm{ISO(1,3)}}{\mathrm{SO(1,3)}}$



Klein pair

The pair (G, H) is a Klein geometry

Example : Conformally compactified Minkowski

Conformally compactified Minkowski $\overline{\mathbb{M}}^{1,3}$ is a homogeneous space for the conformal group

$$\overline{\mathbb{M}}^{1,3} = \frac{\mathrm{SO}(2,4)}{\mathbb{R}^4 \rtimes (\mathbb{R} \times \mathrm{SO}(1,3))}$$

We can choose $\mathrm{ISO}(1,3) \subset \mathrm{SO}(2,4)$ and break the conformal invariance by imposing to stabilize the preferred degenerate direction called null infinity tractor $I^I = \begin{bmatrix} 1 \\ 0^{AA'} \\ 0 \end{bmatrix}$

→ Split of $\overline{\mathbb{M}}^{1,3}$ into orbits of Poincaré

Orbit decomposition

Three orbits (subspaces invariant under the action of Poincaré)

$$\overline{\mathbb{M}}^{1,3} = \mathbb{M}^{1,3} \ \sqcup \mathscr{I} \ \sqcup \{I\}$$

Because Poincaré acts transitively on each of these subspaces, they are homogeneous spaces for ISO$(1,3)$:

$$\overline{\mathbb{M}}^{1,3} = \frac{\mathsf{ISO}(1,3)}{\mathsf{SO}(1,3)} \sqcup \ \frac{\mathsf{ISO}(1,3)}{\mathbb{R}^3 \rtimes (\mathbb{R} \times \mathsf{ISO}(2))} \sqcup \frac{\mathsf{ISO}(1,3)}{\mathsf{ISO}(1,3)}$$

Conformal Carrollian geometry

$$\mathscr{I} = \frac{\mathrm{ISO}(1,3)}{\mathbb{R}^3 \rtimes (\mathbb{R} \times \mathrm{ISO}(2))} \simeq \mathbb{R} \times S^2$$

Conformal boundary of asymptotically flat spacetime is null, n^μ normal vector at $\mathscr{I}$ such that

$$n^b h_{ab} = 0$$

where the degenerate metric h_{ab} is the induced conformal metric from the conformal manifold $\overline{M}$.

$\Rightarrow (n^a, h_{ab})$ constitues a conformal Carrollian geometry

Table of Contents

Super Minkowski space

Goal : generalize this to the supersymmetric case (naive generalization)

$$\begin{array}{ccc} \overline{\mathbb{M}}^{4} & \rightarrow & \text{super compactified Minkowski space } \overline{\mathbb{M}}^{4|2\mathcal{N}} \\ \mathrm{SU}(2,2) \twoheadrightarrow \mathrm{SO}(2,4) & \rightarrow & \text{super conformal group } \mathrm{SU}(2,2|\mathcal{N}) \\ \mathrm{SU}(2,2) \circlearrowleft \overline{\mathbb{M}}^{4} & \rightarrow & \mathrm{SU}(2,2|\mathcal{N}) \circlearrowleft \overline{\mathbb{M}}^{4|2\mathcal{N}} \end{array}$$

$\Rightarrow \overline{\mathbb{M}}^{4|2\mathcal{N}}$ (real) is an homogeneous space for the superconformal group

e.g. [?]

Question : Orbit decomposition of $\overline{\mathbb{M}}^{4|2\mathcal{N}}$ for super Poincaré group ?

Grassmannian definition of compactified Minkowski

Definition that generalizes to the susy case : (complexified)

$$\begin{aligned}\overline{M}^4 &:= \mathrm{Gr}(2, \mathbb{C}^4) \\ &= \{\mathrm{span}(Z^{\alpha 1}, Z^{\alpha 2}) \mid Z^{\alpha b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}{}^{b} \end{bmatrix} \in \mathbb{C}^4 \text{ for } b = 1, 2\}\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\overline{M}_{\ell}^{4|2\mathcal{N}} &:= \mathrm{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}}) \\ &= \{\mathrm{span}(Z^{\hat{\alpha} 1}, Z^{\hat{\alpha} 2}) \mid Z^{\hat{\alpha} b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}{}^{b} \\ \theta^{Ib} \end{bmatrix} \in \mathbb{C}^{4|\mathcal{N}} \text{ for } b = 1, 2\}\end{aligned}$$

We can change the basis of the plane :

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^{01} & \omega^{02} \\ \omega^{11} & \omega^{12} \\ {\pi_{0'}}^{1} & {\pi_{0'}}^{2} \\ {\pi_{1'}}^{1} & {\pi_{1'}}^{2} \\ \theta^{I1} & \theta^{I2} \end{bmatrix} \sim \begin{bmatrix} \omega^{01} & \omega^{02} \\ \omega^{11} & \omega^{12} \\ {\pi_{0'}}^{1} & {\pi_{0'}}^{2} \\ {\pi_{1'}}^{1} & {\pi_{1'}}^{2} \\ \theta^{I1} & \theta^{I2} \end{bmatrix} \cdot M$$

where $M \in \mathrm{GL}(2, \mathbb{C})$.

$+$ Natural action of $\mathrm{SL}(4|\mathcal{N}, \mathbb{C})$ (complexified super conformal group)

Orbit decomposition

Choice of a preferred super null direction $I^{\alpha b} = \begin{bmatrix} 1^{Ab} \\ 0_{A'}{}^{b} \\ 0^{Ib} \end{bmatrix}$

$\iff$ Choice of $\mathrm{ISO}(1,3|\mathcal{N})_{\mathbb{C}} \subset \mathrm{SU}(2,2|\mathcal{N})_{\mathbb{C}}$

Result of the decomposition : more orbits !

$$\overline{M}_{\ell}^{4|2\mathcal{N}} = M_{\ell}^{4|2\mathcal{N}} \sqcup \mathscr{I}_{\ell}^{(3|\mathcal{N})} \sqcup \mathcal{O}_{\ell} \sqcup \mathcal{H} \sqcup \{I\}$$

$\hookrightarrow$ Each of these is an homogeneous space for super Poincaré

Coordinates

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}{}^{b} \\ \theta^{Ib} \end{bmatrix} \in \overline{M}_{\ell}^{4|2\mathcal{N}}$$

Coordinates

$$Z^{\hat{\alpha}b} = \begin{bmatrix}\omega^{Ab}\\ \pi_{A'}{}^{b}\\ \theta^{Ib}\end{bmatrix} \in \overline{M}_{\ell}^{4|2\mathcal{N}}$$

$\det \pi \neq 0$

$\det \pi = 0$

Coordinates

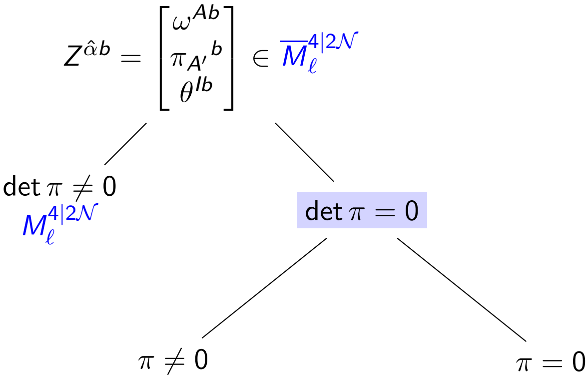
$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^{Ab} \\ {\pi_{A'}}^{b} \\ \theta^{Ib} \end{bmatrix} \in \overline{M}_{\ell}^{4|2\mathcal{N}}$$

$\det \pi \neq 0$

$$Z^{\hat{\alpha}b} \sim \begin{bmatrix} iX_{\ell}^{AA'} \\ {\delta_{A'}}^{b} \\ \theta^{Ib} \end{bmatrix}$$

$\det \pi = 0$

Coordinates



$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}{}^{b} \\ \theta^{Ib} \end{bmatrix} \in \overline{M}_\ell^{4|2\mathcal{N}}$$

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$M_\ell^{4|2\mathcal{N}}$

$\det \pi = 0$

$\pi \neq 0$

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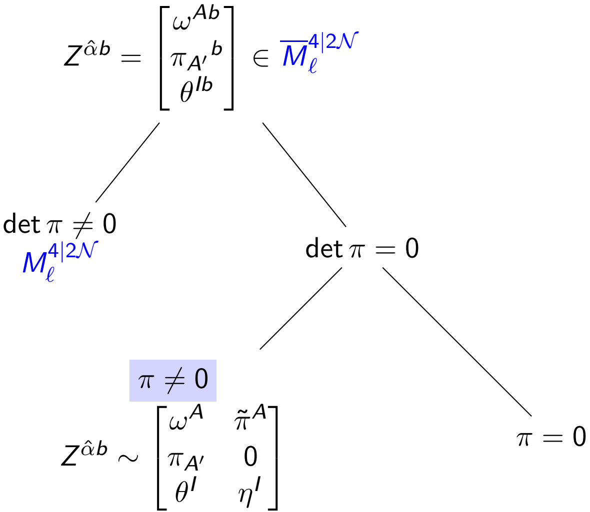
Coordinates



$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}{}^{b} \\ \theta^{Ib} \end{bmatrix} \in \overline{M}_{\ell}^{4|2\mathcal{N}}$$

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$M_{\ell}^{4|2\mathcal{N}}$

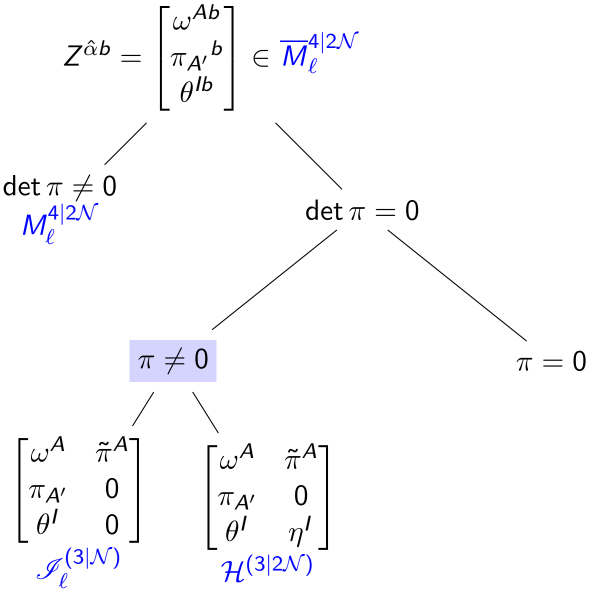
$\det \pi = 0$

$\pi \neq 0$

$$Z^{\hat{\alpha}b} \sim \begin{bmatrix} \omega^{A} & \tilde{\pi}^{A} \\ \pi_{A'} & 0 \\ \theta^{I} & \eta^{I} \end{bmatrix}$$

$\pi = 0$

Coordinates



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$M_{\ell}^{4|2\mathcal{N}}$

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$\pi \neq 0$

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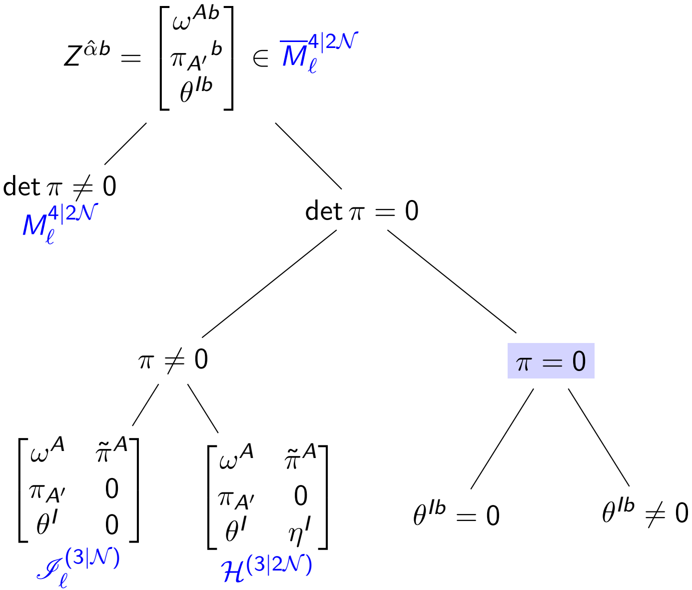
$$\begin{bmatrix} \omega^{A} & \tilde{\pi}^{A} \\ \pi_{A'} & 0 \\ \theta^{I} & 0 \end{bmatrix}$$

$\mathscr{I}_{\ell}^{(3|\mathcal{N})}$

$$\begin{bmatrix} \omega^{A} & \tilde{\pi}^{A} \\ \pi_{A'} & 0 \\ \theta^{I} & \eta^{I} \end{bmatrix}$$

$\mathcal{H}^{(3|2\mathcal{N})}$

Coordinates



$$Z^{\hat{\alpha} b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}{}^{b} \\ \theta^{Ib} \end{bmatrix} \in \overline{M}_{\ell}^{4|2\mathcal{N}}$$

$\det \pi \neq 0$

$M_{\ell}^{4|2\mathcal{N}}$

$\det \pi = 0$

$\pi \neq 0$

$\pi = 0$

$$\begin{bmatrix} \omega^{A} & \tilde{\pi}^{A} \\ \pi_{A'} & 0 \\ \theta^{I} & 0 \end{bmatrix}$$

$\mathscr{I}_{\ell}^{(3|\mathcal{N})}$

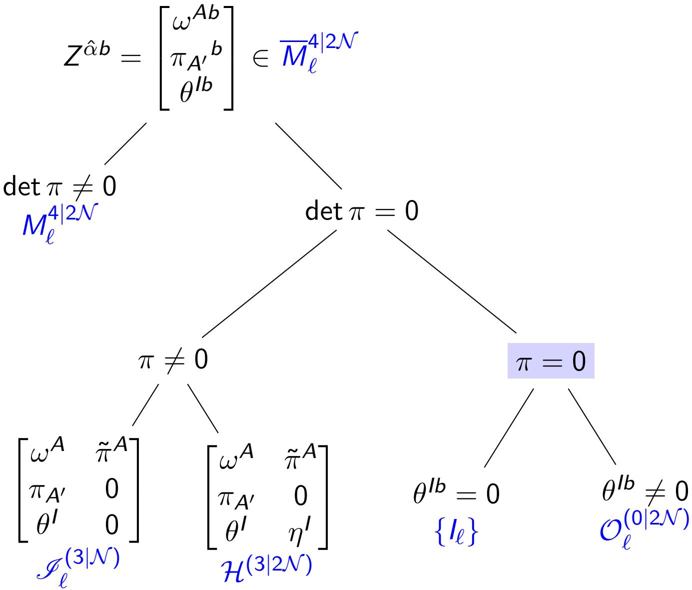
$$\begin{bmatrix} \omega^{A} & \tilde{\pi}^{A} \\ \pi_{A'} & 0 \\ \theta^{I} & \eta^{I} \end{bmatrix}$$

$\mathcal{H}^{(3|2\mathcal{N})}$

$\theta^{Ib} = 0$

$\theta^{Ib} \neq 0$

Coordinates



$$Z^{\hat{\alpha}b} = \begin{bmatrix}\omega^{Ab}\\ \pi_{A'}{}^{b}\\ \theta^{Ib}\end{bmatrix} \in \overline{M}_{\ell}^{4|2\mathcal{N}}$$

$\det\pi \neq 0$

$M_{\ell}^{4|2\mathcal{N}}$

$\det\pi = 0$

$\pi \neq 0$

$\pi = 0$

$$\begin{bmatrix}\omega^{A} & \tilde{\pi}^{A}\\ \pi_{A'} & 0\\ \theta^{I} & 0\end{bmatrix}$$

$\mathscr{I}_{\ell}^{(3|\mathcal{N})}$

$$\begin{bmatrix}\omega^{A} & \tilde{\pi}^{A}\\ \pi_{A'} & 0\\ \theta^{I} & \eta^{I}\end{bmatrix}$$

$\mathcal{H}^{(3|2\mathcal{N})}$

$\theta^{Ib} = 0$

$\{I_{\ell}\}$

$\theta^{Ib} \neq 0$

$\mathcal{O}_{\ell}^{(0|2\mathcal{N})}$

Coordinates on these orbits ? (reality conditions)

- On $M_{\ell}^{1,3|2\mathcal{N}} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{\mathrm{SO}(1,3)\times \mathrm{SU}(\mathcal{N})}$ we find coordinates $(X_{\ell}^{AA'}, \theta^{A'}{}_{I})$ such that one can write

$$X_{\ell}^{AA'} = X^{AA'} + \frac{i}{2}\theta^{A'}{}_{I}\bar{\theta}^{A}{}_{\bar{J}}\,\delta^{I\bar{J}},$$

for a real $X^{AA'}$

Chiral left coordinates appear naturally !

Coordinates on these orbits ? (reality conditions)

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for a real $X^{AA'}$

Chiral left coordinates appear naturally !

- On $\mathscr{I}_\ell^{(3|\mathcal{N})} \simeq \frac{\mathrm{ISO}(1,3|\mathcal{N})}{\mathbb{R}^3 \rtimes \left(\mathbb{R}^{0|\mathcal{N}} \rtimes (\mathrm{ISO}(2)\times\mathbb{R}\times\mathrm{SU}(\mathcal{N}))\right)}$ we find coordinates $([\pi^A], u_\ell = -i\omega^A\tilde{\pi}_A, \theta_I)$ such that one can write

$$u_\ell = u + \frac{i}{2}\theta_I \bar{\theta}_{\bar{J}} \delta^{I\bar{J}}$$

for a real u

Table of Contents

Chiral structure

So far, we have been looking only at one half of the history ! Complexify and [?] :

$$\begin{tikzcd}
& \overline{M} = F(2|0,2|\mathcal{N},\mathbb{C}^{4|\mathcal{N}}) \arrow[dl, dash, dashed, "\pi_\ell"'] \arrow[dr, dash, dashed, "\pi_r"] & \\
\overline{M}_\ell = \mathrm{Gr}(2|0,\mathbb{C}^{4|\mathcal{N}}) & & \overline{M}_r
\end{tikzcd}$$

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega_\ell^{Ab} \\ \pi_{A'\ell}{}^{b} \\ \theta_\ell^{Ib} \end{bmatrix}$$

Chiral right

Chiral left

Flag variety

Let $0 \leq d_0 < d_1 < ... < d_k \leq \dim V$

$$F(d_0, d_1, ..., d_k, V) := \{V_0 \subset V_1 \subset ... \subset V_k \text{ sub v.s. of } V \mid \dim(V_i) = d_i\}$$

Flag variety

Let $0 \leq d_0 < d_1 < ... < d_k \leq \dim V$

$F(d_0, d_1, ..., d_k, V) := \{V_0 \subset V_1 \subset ... \subset V_k \text{ sub v.s. of } V \mid \dim(V_i) = d_i\}$

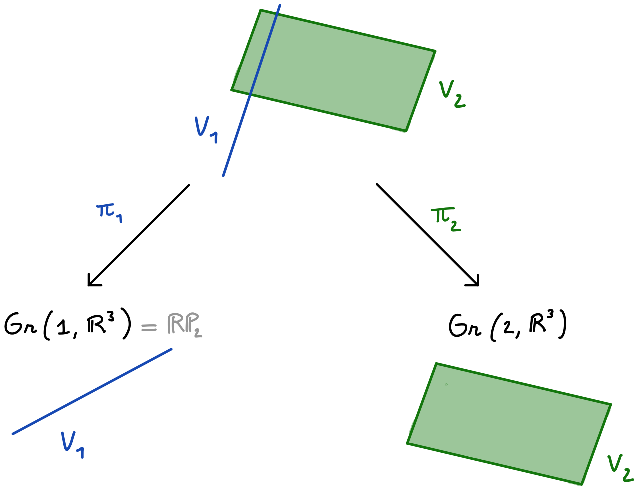
in particular

$$F(k, V) = \mathrm{Gr}(k, V)$$

$$F(1, V) = \mathbb{P}_{\dim V - 1}(V)$$

Example

$$F(1,2,\mathbb{R}^3) := \{V_1 \subset V_2 \text{ sub v.s. of } \mathbb{R}^3 \mid \dim(V_i) = i\}$$



Chiral structure

$$\begin{tikzcd}
 & \overline{M} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N})} \arrow[dl, dash, "\pi_\ell"'] \arrow[dr, dash, "\pi_r"] & \\
\overline{M}_\ell = \mathrm{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}}) & & \overline{M}_r = \mathrm{Gr}(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})
\end{tikzcd}$$

[?]

Chiral structure

Full complexified compactified super Minkowski space

$$\begin{tikzcd}
 & \overline{M} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \arrow[dl, dash, "\pi_\ell"'] \arrow[dr, dash, "\pi_r"] & \\
\overline{M}_\ell = \mathrm{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}}) & & \begin{array}{c} \overline{M}_r = \mathrm{Gr}(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \\ \simeq \mathrm{Gr}(2|0, (\mathbb{C}^{4|\mathcal{N}})^*) \end{array}
\end{tikzcd}$$

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega_\ell^{Ab} \\ \pi_{A'\ell}{}^b \\ \theta_\ell^{Ib} \end{bmatrix}$$

$$\tilde{Z}_{\hat{\alpha}}{}^b = \begin{bmatrix} \pi^b_{Ar} \\ \omega_r^{A'b} \\ \theta_{Ir}{}^b \end{bmatrix}^t$$

Chiral left

Chiral right

[?]

Chiral structure

Full complexified compactified super Minkowski space

$$\begin{tikzcd}
& \overline{M} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \arrow[dl, dash, "\pi_\ell"'] \arrow[dr, dash, "\pi_r"] & \\
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\end{tikzcd}$$

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega_\ell^{Ab} \\ \pi_{A'\ell}{}^b \\ \theta_\ell^{Ib} \end{bmatrix}$$

$$\tilde{Z}_{\hat{\alpha}}{}^b = \begin{bmatrix} \pi_{Ar}^b \\ \omega_r^{A'b} \\ \theta_{Ir}{}^b \end{bmatrix}^t$$

Chiral left

Chiral right

[?] Flag condition : $\forall a, b, Z^{\hat{\alpha}a}\tilde{Z}_{\hat{\alpha}}{}^b = 0$

Chiral right

Two importants orbits :

- In chiral right super Minkowski space M_R, the subspace of $\overline{M}_R$ where $\det \pi \neq 0$, it exists $X^{A'A} \in \mathbb{R}$ such that we can write

$$X_r^{A'A} = X^{A'A} - \frac{i}{2}\theta^{IA'}\delta_{IJ'}\bar{\theta}^{J'B}.$$

Chiral right

Two importants orbits :

- In chiral right super Minkowski space M_R, the subspace of $\overline{M}_R$ where $\det \pi \neq 0$, it exists $X^{A'A} \in \mathbb{R}$ such that we can write

$$X_r^{A'A} = X^{A'A} - \frac{i}{2}\theta^{IA'}\delta_{IJ'}\bar{\theta}^{J'B}.$$

- On the chiral right super Null Infinity $\mathscr{I}_r$ it exists $u \in \mathbb{R}$ such that we can write

$$u_- = u - \frac{i}{2}\theta_I\overline{\theta}_{\bar{J}}\delta^{I\bar{J}}.$$

Non chiral

- Non chiral super Minkowski space is defined as

$$\{(Z^{\hat{\alpha}b}, \tilde{Z}_{\hat{\alpha}b}) \mid Z^{\hat{\alpha}b} \in M_L \text{ and } \tilde{Z}_{\hat{\alpha}b} \in M_R\}$$

→ Flag condition imposes $x_{\ell}^{AA'} - x_r^{AA'} = 2i\theta_{\ell}^{AI}\theta_{rI}^{A'}$

Non chiral

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$$\{(Z^{\hat{\alpha}b}, \tilde{Z}_{\hat{\alpha}b}) \mid Z^{\hat{\alpha}b} \in M_L \text{ and } \tilde{Z}_{\hat{\alpha}b} \in M_R\}$$

→ Flag condition imposes $x_\ell^{AA'} - x_r^{AA'} = 2i\theta_\ell^{AI}\theta_{rI}^{A'}$

- The obvious candidate for non chiral super null infinity is

$$\{(Z^{\hat{\alpha}b}, \tilde{Z}_{\hat{\alpha}b}) \mid Z^{\hat{\alpha}b} \in \mathscr{I}_L^{(\mathcal{N})} \text{ and } \tilde{Z}_{\hat{\alpha}b} \in \mathscr{I}_R^{(\mathcal{N})}\}$$

→ Flag condition imposes $\tilde{\pi}_\ell^A \propto \pi_r^A$, $\tilde{\pi}_r^A \propto \pi_\ell^A$ and $u_\ell - u_r = i\delta_{IJ}\theta_I^I\theta_r^J$

Non chiral super Minkowski (big cell)

Super Minkowski

$$\begin{tikzcd}
& \begin{matrix} M \\ (x^{AA'} := \frac{1}{2}(x_\ell^{AA'} + x_r^{A'A}),\ \theta_\ell^{Ib},\ \theta_{Ir}{}^{b}) \end{matrix} \arrow[dl, dash, "\pi_\ell"'] \arrow[dr, dash, "\pi_r"] & \\
\begin{matrix} M_\ell \\ (x_\ell^{AA'} = x^{AA'} + \frac{i}{2}\theta_\ell^{IA}\theta_{Ir}^{A'},\ \theta_\ell^{Ib}) \end{matrix} & & \begin{matrix} M_r \\ (x_r^{A'A} = x^{AA'} - \frac{i}{2}\theta_\ell^{IA}\theta_{Ir}^{A'},\ \theta_{Ir}{}^{b}) \end{matrix}
\end{tikzcd}$$

Chiral left

Chiral right

Non chiral super Null Infinity

Super Null Infinity

$$\begin{tikzcd}
& \overset{\mathscr{I}}{\left(u := \frac{1}{2}(u_\ell + u_r), \theta_\ell, \theta_r, [\pi_\ell^A], [\pi_r^A]\right)} \arrow[dl, dash, "\pi_\ell"] \arrow[dr, dash, "\pi_r"] & \\
\overset{\mathscr{I}_\ell}{\left(u_\ell = u + \frac{i}{2}\theta_\ell\theta_r,\ \theta_\ell,\ [\pi_\ell^A],\ [\pi_r^A]\right)} & & \overset{\mathscr{I}_r}{\left(u_r = u - \frac{i}{2}\theta_\ell\theta_r,\ \theta_r,\ [\pi_{A'\ell}], [\pi_r^A]\right)}
\end{tikzcd}$$

Chiral left

Chiral right

Table of Contents

Distributions

For super Minkowski :

$$\begin{tikzcd}
 & \begin{array}{c} M \\ (x^{AA'} := \frac{1}{2}(x_\ell^{AA'} + x_r^{A'A}),\ \theta_\ell^{Ib},\ \theta_{Ir}{}^{b}) \end{array} \arrow[dl, dash, "\pi_\ell"'] \arrow[dr, dash, "\pi_r"] & \\
\begin{array}{c} M_\ell \\ (x_\ell^{AA'} = x^{AA'} + \frac{i}{2}\theta_\ell^{IA}\theta_{Ir}^{A'},\ \theta_\ell^{Ib}) \end{array} & & \begin{array}{c} M_r \\ (x_r^{A'A} = x^{AA'} - \frac{i}{2}\theta_\ell^{IA}\theta_{Ir}^{A'},\ \theta_{Ir}{}^{b}) \end{array}
\end{tikzcd}$$

Distributions

For super Minkowski :

Two integrable distributions $D_\ell, D_r \subset TM$ associated to the two fibrations
Basis of the sections given by :

$$
\begin{aligned}
D_{\ell Ib} &= \partial_{\theta_\ell^{Ib}} + i\theta_{rI}^{b}\partial_{x^{AA'}} && \longrightarrow \text{ s.t. } D_\ell(x_r, \theta_r) = 0 \\
D^I_{rb} &= \partial_{\theta^b_{Ir}} + i\theta_I^{Ib}\partial_{x^{AA'}} && \longrightarrow \text{ s.t. } D_r(x_\ell, \theta_\ell) = 0
\end{aligned}
$$

$\Rightarrow$ Covariant derivatives

Distributions

$$\begin{tikzcd}
 & \overline{M} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \arrow[dl, dash, "\pi_\ell"'] \arrow[dr, dash, "\pi_r"] & \\
\overline{M}_\ell = \mathrm{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}}) & & \overline{M}_r = \mathrm{Gr}(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})
\end{tikzcd}$$

Distributions (”covariant derivatives”) already there in the homogeneous superconformal model !

Superconformal geometry

In fact, they are part of the superconformal geometry :

Superconformal geometry [**?**, **?**]

A superconformal geometry on a complex supermanifold $M^{4|4}$ is a pair of integrable $(0|2)$-dimensional distributions $D_\ell, D_r \subset TM$ s.t.

- Their sum is direct in TM
- $\varphi : D_\ell \otimes D_r \to TM/(D_\ell \oplus D_r) : (X \otimes Y) \mapsto [X, Y] \mathrm{mod}(D_\ell \oplus D_r)$ is an isomorphism

Distributions

For super Null Infinity : works in the same way

$$\begin{tikzcd}[column sep=small]
 & \begin{matrix}\mathscr{I}\\ (u := \frac{1}{2}(u_\ell + u_r), \theta_\ell, \theta_r, [\pi_\ell^A], [\pi_r^A])\end{matrix} \arrow[dl, dash, "\pi_\ell"'] \arrow[dr, dash, "\pi_r"] & \\
\begin{matrix}\mathscr{I}_\ell\\ (u_\ell = u + \frac{i}{2}\theta_\ell\theta_r,\ \theta_\ell,\ [\pi_\ell^A],\ [\pi_r^A])\end{matrix} & & \begin{matrix}\mathscr{I}_r\\ (u_r = u - \frac{i}{2}\theta_\ell\theta_r,\ \theta_r,\ [\pi_{A'\ell}], [\pi_r^A])\end{matrix}
\end{tikzcd}$$

Distributions

For super Null Infinity : works in the same way

Two integrable distributions $D_\ell, D_r \subset T\mathscr{I}$ associated to the two fibrations

$$\begin{aligned} D_\ell &= \partial_{\theta_\ell} - i\theta_r\partial_u && \rightarrow \text{ s.t. } D_\ell(u_r, \theta_r) = 0 \\ D_r &= \partial_{\theta_r} - i\theta_\ell\partial_u && \rightarrow \text{ s.t. } D_r(u_\ell, \theta_\ell) = 0 \end{aligned}$$

Super conformal Carrollian : curved setup

Based on the flat model, we propose to extend this definition for $\mathscr{I}^{3|1}$, as a model for a *super conformal Carrollian geometry* : the data of (D_ℓ, D_r, n), with some compatibility conditions

Results :

Super conformal Carrollian : curved setup

Based on the flat model, we propose to extend this definition for $\mathscr{I}^{3|1}$, as a model for a *super conformal Carrollian geometry* : the data of (D_ℓ, D_r, n), with some compatibility conditions

Results :

- We can always write :
  - $D_\ell = \partial_{\theta_\ell} - i\theta_r\partial_u$
  - $D_r = \partial_{\theta_r} - i\theta_\ell\partial_u$
  - $n = \partial_u$
- The symmetry group of these is the super BMS group of Awada-Gibbons-Shaw [AGS 86]

Table of Contents

Conclusion

Short summary

- Mostly in the flat case
- Chirality : the non chiral super Minkowski space is a flag variety
- Identify two remarkable orbits inside the compactification : super Minkowski and super $\mathscr{I}$
- For non chiral super Minkowski and non chiral super $\mathscr{I}$ we have the same structure of double fibration, characterized by two distributions
- This structure is the basis of the definition of superconformal geometry

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Still in progress ...

- Give precise definitions to treat the general asymptotically flat case
- Application to self dual supergravity

Thank you for your attention !

References