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Microarticle

Equivalent period for a stationary quantum system

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ABSTRACT

A generalization of the Kepler's third law has been proposed for classical and quantum N-body systems in a Newtonian gravitation field. This implies the definition of the equivalent of a period for a stationary quantum system. In this paper, it is shown that a significant quantum definition for the equivalent of a period is possible and coincides with the quantities defined phenomenologically for the generalization of the Kepler's third law.

The Kepler's third law has certainly a great historical significance, but this relation between the period and the size of the orbit, or the period and the energy of the orbit, applies only for classical two-body systems. Nevertheless, a generalization for classical and quantum N-body systems in gravitational interaction has been proposed recently [1–3]. A problem is the necessity to define the equivalent of a period for a stationary quantum system. In [3], a formula has been proposed for the two-body system with a combination of some mean values of observables, and another one for the N-body systems with semiclassical considerations. It seems thus desirable to have a clear and unique definition for a quantum period. The idea is to build from a classical system a formula which can be computed for the equivalent quantum system.

Let us consider a particle of mass m moving nonrelativistically along a bounded trajectory C (this is also valid for a relative two-body motion with a reduced mass m). In one dimension, the motion is periodic, while in three dimensions, the periodicity is only guaranteed for a harmonic oscillator or a Coulomb system. If the motion has a period τ, the action I is computed by

$$I = \frac{1}{2\pi}\int_C \boldsymbol{p}\cdot d\boldsymbol{r} = \frac{1}{2\pi}\int_{t_0}^{t_0+\tau} m\,\dot{\boldsymbol{r}}^2 dt. \tag{1}$$

Defining the classical mean value of a quantity A by the integral

$$\langle A\rangle = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} A\,dt, \tag{2}$$

the period is given by

$$\tau = \frac{\pi I}{\langle T\rangle}, \tag{3}$$

where T is the kinetic energy. This formula can be used to compute the equivalent of a period for a stationary quantum system, provided $\langle T\rangle$ and I can be independently computed.

In one dimension, the action can be approximately computed in the framework of the WKB method [4]:

$$I \approx \left(n - \frac{\gamma}{4}\right)\hbar \quad \text{with} \quad n = 1, 2\ldots, \tag{4}$$

where $\gamma = 0$, 1 or 2, following the boundary conditions at the turning points. Let us look at two systems for which the WKB approximation gives the exact result. For an infinite square well of length a, we have $I = n\,\hbar$ and $\langle T\rangle = E = (n\,\pi\,\hbar)^2/(2\,m\,a^2)$[4]. Formula (3) gives

$$\tau = \frac{2\,m\,a^2}{n\,\pi\,\hbar}. \tag{5}$$

In this case, the modulus of the momentum is well defined by $|p| = n\,\pi\,\hbar/a$, and (5) reduces to

$$\tau = \frac{2a}{v}, \tag{6}$$

where $v = |p|/m$. This corresponds to the classical period for a motion between the two turning points. For the harmonic oscillator, $I = (n - 1/2)\hbar$ and $\langle T\rangle = E/2 = (n - 1/2)\hbar\omega/2$[4]. Formula (3) gives

$$\tau = \frac{2\pi}{\omega} \tag{7}$$

which is the classical result.

For the three dimensional case, let us look at the value of I from the knowledge of the period for classical periodic systems. For a harmonic oscillator, $\langle T\rangle = E/2 = (2n + \ell + 3/2)\hbar\omega/2$, and (3) gives immediately

$$I = (2n + \ell + 3/2)\hbar. \tag{8}$$

For the Coulomb potential, the classical period can be computed from the Kepler's third law

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{m\,k^2}{8\,|E|^3} \tag{9}$$

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for the attractive potential $V = -k/r$. With the corresponding quantum energy $E = -m\,k^2/(2(n+\ell+1)^2\hbar^2)$, (3) gives

$$I = (n+\ell+1)\hbar. \tag{10}$$

Let us note that τ is computed by $2\pi\,\langle 1/|\boldsymbol{r}|\rangle^{-1}/(\langle \boldsymbol{p}^2\rangle^{1/2}/m)$ in [3]. In both cases, I is simply the characteristic action of the particle, as expected from the results in one dimension.

We look now at the quantum spherical rigid rotor, which is generally associated with a many-body system (molecule, nucleus…). Its quantum energy E is given by $E = \langle T\rangle = j(j+1)\hbar^2/(2\mathscr{I})$, where $\mathscr{I}$ is the moment of inertia. Following the results obtained above, we can assume that

$$I = \sqrt{j(j+1)}\,\hbar. \tag{11}$$

Then, (3) gives

$$\tau = \frac{2\pi\,\mathscr{I}}{\sqrt{j(j+1)}\,\hbar}. \tag{12}$$

If $\boldsymbol{L}$ is the classical angular momentum, we can write $\mathscr{I}\,\omega = |\boldsymbol{L}| \approx \sqrt{j(j+1)}\,\hbar$. Then, (12) gives $\tau \approx 2\pi/\omega$, which is the expected result.

Now, we assume that (3) is also valid for general N-body quantum systems (as suggested by the study of the rigid rotor). For the N identical self-gravitating particles considered in [3], the results obtained by the envelope theory [5–7] are $I = Q\,\hbar$ and $\langle T\rangle = N\,p_0^2/(2m)$ where Q is a global quantum number and p_0 is the mean momentum of the particles. In this case, (3) gives the same result for the period as the one computed in [3] by a semiclassical treatment (a circular motion at the same speed for N particles at the vertices of a regular N-gon).

In this paper, the period for a two-body system and the one for a N-body system are recovered by a unique formula with better foundation. It is clear that formula (3) and the generalization of the Kepler's third law for classical and quantum self-gravitating systems deserve more studies to establish clearly their relevance. We think that the results obtained here could shed some light on this problematic.

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