Manifestly Covariant Worldline Actions from Coadjoint Orbits

Part I: Generalities and Vectorial Descriptions

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Abstract: We derive manifestly covariant actions of spinning particles starting from coadjoint orbits of isometry groups, by using Hamiltonian reductions. We show that the defining conditions of a classical Lie group can be treated as Hamiltonian constraints which generate the coadjoint orbits of another, *dual*, Lie group. In case of (inhomogeneous) orthogonal groups, the dual groups are (centrally-extended inhomogeneous) symplectic groups. This defines a symplectic dual pair correspondence between the coadjoint orbits of the isometry group and those of the dual Lie group, whose quantum version is the reductive dual pair correspondence à la Howe. We show explicitly how various particle species arise from the classification of coadjoint orbits of Poincaré and (A)dS symmetry. In the Poincaré case, we recover the data of the Wigner classification, which includes continuous spin particles, (spinning) tachyons and null particles with vanishing momenta, besides the usual massive and massless spinning particles. In (A)dS case, our classification results are not only consistent with the pattern of the corresponding unitary irreducible representations observed in the literature, but also contain novel information. In dS, we find the presence of partially massless spinning particles, but continuous spin particles, spinning tachyons and null particles are absent. The AdS case shows the largest diversity of particle species. It has all particles species of Poincaré symmetry except for the null particle, but allows in addition various exotic entities such as one parameter extension of continuous particles and conformal particles living on the boundary of AdS. Notably, we also find a large class of particles living in "bitemporal" AdS space, including ones where mass and spin play an interchanged role. We also discuss the relative inclusion structure of the corresponding orbits.

Contents

1 Introduction

A coadjoint orbit of a Lie group is equipped with a symplectic structure [1], and therefore can be viewed as the phase space of a classical mechanical system. When the Lie group is the isometry group of a spacetime, and it is large enough — typically the relativistic ones (Poincaré and (A)dS group) or their non-relativistic counterparts (such as the Galilean group) — even the dynamics (that is, the time evolution) of the system can be ascribed by the symmetry, making it integrable. Such mechanical systems can be interpreted as particles moving in the spacetime having this isometry group. Therefore, actions for relativistic particles can be derived from coadjoint orbits of their isometry group, and there have been many works in this direction, which we shall summarise shortly in one of the following paragraphs. Typically, the resulting actions are not manifestly covariant under the isometry group and heavily depend on the coordinate system of the coadjoint orbits. Since the system is integrable, too good coordinates, such as the action-angle variables, would render the system essentially trivial, obscuring the spacetime propagation. Therefore, the art is in the choice of an appropriate set of coordinates with which the mechanical system can be interpreted as a dynamical worldline particle, keeping both the spacetime motion as well as the isometries explicit. In this regard, the covariance of the system under the isometry group is crucial. However, this covariance will not be manifest unless we introduce additional degrees of freedom together with constraints.

Many relativistic spinning particle actions have been constructed as spin generalisations of the relativistic scalar particle action, without explicitly relying on coadjoint orbits. Like the scalar case, such systems have Hamiltonian constraints and involve additional variables to describe the spin degrees of freedom. Since the spin degrees of freedom are discrete,[1] the additional variables can be introduced as fermionic ones and this leads to supersymmetry. One may also persist to use bosonic variables for the spin degrees of freedom. Then, the classical system has additional continuous degrees of freedom, on top of the position and momentum variables, rather than the desired discrete ones. These continuous spin degrees of freedom should be projected, afterwards, to discrete ones in the course of a quantisation procedure. The twistor formulations for spinning particles are also obtained in a similar fashion, by employing an appropriate set of constraints. Because these works do not make use of the coadjoint orbits, or at least its role is implicit, one often needs a separate constraint analysis to check whether the system indeed describes the sought after spinning particles.

In this work, we reconsider the worldline particle actions from the vantage point of view of a manifestly covariant description of coadjoint orbits of a classical Lie group. Since the Poincaré, (A)dS as well as the Lie groups behind twistor descriptions are all classical ones, our approach is sufficiently general to cover particles in Minkowski and (A)dS spaces. Using the fact that a classical Lie group is a subgroup of the matrix group $GL(N,\mathbb{R})$ subject to a certain set of defining conditions compatible with the matrix product, we can describe a coadjoint orbit of a classical Lie group G as a reduced phase space lying inside

[1] In the sense that, upon quantisation, they yield a *finite-dimensional* Hilbert space.

a coadjoint orbit of an embedding $GL(N,\mathbb{R})$ group, where the Hamiltonian reduction is induced by the Hamiltonian constraints stemming from the defining conditions of the group G. For a given G-coadjoint orbit, the resulting constraints are given by components of the moment map for another Lie group $\tilde{G}$, with certain constant shifts. We will refer to this Lie group $\tilde{G}$ as the *dual group*. If there is no constant shift, all constraints are first class, but for a non-vanishing shift, they are a mixture of first and second class constraints. The first class constraints generate a subgroup of $\tilde{G}$, whereas the second class constraints can be associated with a $\tilde{G}$-coadjoint orbit. This establishes a correspondence between the set of G-coadjoint orbits and a set of $\tilde{G}$-coadjoint orbits. In physical terms, the information of particle species, such as mass and spin, is originally encoded in the G-coadjoint orbit. Then, our construction maps such information to a $\tilde{G}$-coadjoint orbit through the constant shifts, where the constants are given by the particle labels. As the information of the G-coadjoint orbit (the starting point) is encoded in the particle action through the data of the constant shift of the dual $\tilde{G}$-coadjoint orbit, the action always enjoys a manifest G-symmetry.

In this setting, once the starting group G is fixed, the form of the particle action is essentially universal, and only the constant shift differentiates particle species.[2] Therefore, together with the general construction of the above system, we devote a part of our work to the classification of G-coadjoint orbits as well as the identification of the corresponding $\tilde{G}$-coadjoint orbits, i.e. the identification of the corresponding constant shift. In the case of Poincaré symmetry, the classification of coadjoint orbits can be done in a very analogous manner as in Wigner classification: we classify the coadjoint orbits in terms of the representative coadjoint elements, like the way we choose the momentum in the rest frame for the representative momentum vector of a massive particle in the Wigner classification. This allows to identify the coadjoint orbits of massive, massless, tachyonic spinning particles and even those of the continuous spin particle and the null particles with vanishing momentum. The same classification scheme can be equally applied to (A)dS cases. In dS, we find the presence of partially massless spinning particles, but continuous spin particles, spinning tachyons and null particles are absent. The AdS case shows the largest diversity of particle species. It has all particles species of Poincaré symmetry except for the null particle, but allows in addition various exotic entities such as particles with entangled mass and spin, which contain a one parameter extension of continuous spin particle as a subcase, and conformal particles living on the boundary of AdS. Notably, we also find a large class of particles living in "bitemporal" AdS space, defined by $X^2 = +1$ in the ambient space with the $(-,-,+,\ldots,+)$ metric. This class includes ones where mass and spin play an interchanged role. The classification can be easily extended to mixed symmetry cases, where we find various shortening conditions consistent with the pattern of the corresponding unitary irreducible representations observed in the literature. In each of these cases, we identify the dual group $\tilde{G}$ and the dual coadjoint orbit from which the worldline particle action can be readily expressed.

[2] In this paper, we often use a very loose terminology and refer to a coadjoint orbit of an isometry group as a *particle* simply.

The general construction used in this paper has a close relation to the reductive dual pair correspondence, about which the first two authors of the current paper carried out explicit analysis in [2]. The relation works as follows. After a part of the constraints simply removes non-dynamical spectator variables, the effective embedding phase space of our model becomes a flat one, which is the minimal coadjoint orbit of $Sp(2n,\mathbb{R}) \subset GL(N,\mathbb{R})$. The pair of G- and $\tilde{G}$-coadjoint orbits is an example of symplectic dual pair [3], and it ensures the one-to-one correspondence between the coadjoint orbits of G and $\tilde{G}$ when the group G is reductive. The reductive dual pairs — pairs of subgroups $(G,\tilde{G}) \subset Sp(2n,\mathbb{R})$ which are mutual stabilisers — ensure even the existence of a one-to-one correspondence between the G-irreducible representations (irreps) and the $\tilde{G}$-irreps which arise in the restriction of the metaplectic representation of $Sp(2n,\mathbb{R})$ onto $G \times \tilde{G}$. This correspondence is known as the reductive dual pair correspondence or simply Howe duality [4, 5]. Since a suitable quantisation — such as the geometric quantisation — of G-coadjoint orbits and $\tilde{G}$-coadjoint orbits would result in G-irreps and $\tilde{G}$-irreps, respectively, the current picture can be viewed as the classical counterpart of the reductive dual pair correspondence.

Let us provide a brief overview of previous works on particle actions. As previously mentioned, one of the most common ways of describing spinning particles consists in introducing fermionic variables to the phase space.[3] The latter are used to realise supersymmetry on the worldline, with the number $\mathcal{N}$ of supercharges corresponding to a particle of spin-$\frac{\mathcal{N}}{2}$, as shown in [7, 8], drawing on earlier work on massive superparticles [9–12] (see also [13–23]). Another approach is to use (super)twistor variables in $d = 3,\ 4$, and 6 dimensions to describe spinning massive [24–30] and massless particles [31–35] in flat spacetime, as well as in AdS_{d+1} [36–40]. More recently, these techniques were also used to obtain actions for continuous spin particles [41–45].

The use of the symplectic structure on a coadjoint orbit in describing a particle dynamics has also a long history starting from the pioneering work of Souriau [46]. In the formulation with twistor variables of relativistic particles this was used starting from the early works [24, 35, 47] to a more recent one [27]. In the formulation with spacetime variables, this appeared in e.g. [48–52]. See also [53–55] for other applications to particle dynamics.

A closely related set up to derive a particle action starting from a Lie group is known as the *nonlinear realisation* method which proved particularly useful to construct actions for p-branes as well as non-relativistic particle actions, see e.g. [56–63] and references therein. See also [64] for its use in a color-extension of spacetime symmetry, [65] for particles in BMS space, and [66–71] for discussions of the path integral quantisation of this kind of model.

Let us end this brief tour of the literature by mentioning that worldline models can serve in various quantum field theory contexts [72, 73], for instance to compute heat kernel/effective action coefficients [74–78] and scattering amplitudes [79] or to probe properties of the gauge theory associated with background fields [80–85]. In the context of higher spin

[3]The first introduction of Grassmannian variables in a classical mechanics setting seems to go back to the paper [6].

gravity, coadjoint orbits play an important role, in that several higher spin algebras arise as the quantisation of particular orbits of $\mathfrak{so}(2, d-1)$. To be more precise, the simplest higher spin algebra (sometimes referred to as the type-A algebra), is the symmetry algebra of the minimal representation of $\mathfrak{so}(2, d-1)$, representation which is obtained by quantising its minimal nilpotent orbit [86, 87] (see also [88] and [89] for a discussion of the partially-massless generalisation in relation to the quantisation of coadjoint orbits). On top of that, higher spin algebras are commonly realised using the dual pair correspondence (also known as Howe duality [4, 5], see e.g. [2, 90, 91] for reviews) previously mentioned, a classical counterpart of which is recovered in this paper.

The organization of the paper is as follows: In Section 2, we start by reviewing the basics of coadjoint orbits of a Lie group and their symplectic structures. We explain how one can associate a particle action to each orbit, and discuss the conditions under which the path integral is well-defined. After detailing simple examples of three-dimensional Lie groups, we point out the issue of coordinate choice in this action and argue for the necessity of manifest covariant description of the actions using Hamiltonian constraints. In Section 3, we present several general results of a constrained Hamiltonian system where the constraints are given by constant shifts of the moment map associated with the dual Lie algebra. In particular, we demonstrate that the second and first class constraints correspond to a coadjoint orbit and its stabiliser of the dual group. In Section 4, we explain how the coadjoint action for a classical Lie group and its semi-direct product with an Abelian ideal can be reformulated as a constrained Hamiltonian system by making use of the set-up explained in Section 3. After briefly covering the general cases, we provide more details on the orthogonal and inhomogeneous orthogonal group cases, relevant to the symmetries of spacetime. In Section 5, we apply the construction of worldline action for semi-direct product groups detailed in the previous section to the Poincaré case, and (re)derive the actions for various particles in Minkowski spacetime. In Section 6, we move to the (A)dS case and derive various particle actions by using the same method. On top of the usual massive and massless particles, we spell out various other particle species. In Section 7, we discuss the inclusion structure of both nilpotent orbits — which is known to admit a convenient description in terms of Young and Hasse diagrams — and semisimple ones, which seem to have received less attention. In Section 8, we conclude this paper with a short discussion of the remaining questions that we intend to address in our follow-up paper [92]. Finally, this paper includes several appendices containing additional details and material complementing its bulk. In Appendix A, we summarise the conventions and notations used. In Appendix B, we explain how one can convert the second classes appearing in the Hamiltonian system detail in Section 3 into first class ones. Appendix C contains details on the classification of orbits of the orthogonal groups $O(n)$. We collect in Appendix D the data defining the coadjoint orbits and their duals identified in this paper, and detail in Appendix E the relation between coadjoint orbits of $SO^+(2,2)$ and of $SO^+(2,1)$. Finally, we compare our classification with the results of Metsaev [93] in Appendix F.

2 Coadjoint orbits and particles

In order to understand how a particle action can be obtained from a coadjoint orbit of the associated symmetry group, let us first consider the simple example of a relativistic scalar particle action,

$$S = \int \mathrm{d}t \left[p_\mu \, \dot{x}^\mu - e\,(p^2 - m^2)\right], \tag{2.1}$$

where $\dot{x}^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$, and the einbein e plays the role of a Lagrange multiplier which sets the mass-shell constraint $p^2 = m^2$. By solving the latter as $p_0 = \pm\sqrt{p_a\, p^a + m^2}$ and fixing x^0 to t using the reparametrisation symmetry, we find an equivalent action,

$$S = \int \mathrm{d}t \left[p_a\, \dot{x}^a \pm \sqrt{p_a\, p^a + m^2}\right]. \tag{2.2}$$

Here, the sign $\pm$ distinguishes the positive energy and negative energy solutions which can be mapped to each other by the time inversion $t \to -t$.

The same action can be obtained from a coadjoint orbit of the Poincaré group, whose Lie algebra $\mathfrak{iso}(1, d-1)$ is generated by P_μ and $J_{\mu\nu}$. A vector ϕ in the coadjoint space $\mathfrak{iso}(1, d-1)^*$ has the form $\phi = p_\mu\, \mathcal{P}^\mu + j_{\mu\nu}\, \mathcal{J}^{\mu\nu}$ where $\mathcal{P}^\mu$ and $\mathcal{J}^{\mu\nu}$ are the dual basis vectors satisfying $\langle \mathcal{P}^\mu, P_\nu\rangle = \delta^\mu_\nu$, $\langle \mathcal{J}^{\mu\nu}, J_{\rho\sigma}\rangle = \delta^{[\mu}_\rho\, \delta^{\nu]}_\sigma$ and $\langle \mathcal{P}^\mu, J_{\rho\sigma}\rangle = 0 = \langle \mathcal{J}^{\mu\nu}, P_\rho\rangle$. The orbit corresponding to a massive scalar particle is given by the representative vector $\phi = m\, \mathcal{P}^0$ whose only non-vanishing component is $p_0 = m > 0$. Under the coadjoint action of the Poincaré group on ϕ, all $j_{\mu\nu}$ components remain zero, while p_μ forms an upper hyperboloid given by $p_\mu\, p^\mu = m^2$ and $p_0 > 0$, the typical momentum orbit. Note that this orbit is embedded in the $d(d+1)/2$ dimensional space $\mathfrak{iso}(1, d-1)^*$.

The action corresponding to the orbit $\mathcal{O}_\phi$ is given by (we shall review the details later),

$$S[g] = \int \mathrm{d}t\, \langle \phi, g^{-1}\dot{g}\rangle\,, \tag{2.3}$$

where g is a generic element of the Poincaré group. Parameterising the element as[4]

$$g = e^{x^\mu\, P_\mu}\, e^{v^a\, J_{a0}}\, e^{\theta^{ab}\, J_{ab}}\,, \tag{2.4}$$

we find

$$\langle \phi, g^{-1}\dot{g}\rangle = \dot{x}^\mu \langle e^{v^a\, J_{a0}}\, m\, \mathcal{P}^0\, e^{-v^a\, J_{a0}}, P_\mu\,\rangle. \tag{2.5}$$

The boost parameters v^a parameterise the momentum orbit as

$$e^{v^a\, J_{a0}}\, m\, \mathcal{P}^0\, e^{-v^a\, J_{a0}} = -\sqrt{m^2 + p_a\, p^a}\, \mathcal{P}^0 + p_a\, \mathcal{P}^a\,, \tag{2.6}$$

where $p_a = \frac{\sinh(v)}{v}\, v_a$ and $v = \sqrt{v^a\, v_a}$. So we can reformulate the right hand side of the above equation as $p_\mu\, \mathcal{P}^\mu$ by appending the constraints $p^2 = m^2$ and $p_0 > 0$. In this way, we recover the action (2.1) of a massive scalar particle in Minkowski space. The method of using the Maurer–Cartan one-form $g^{-1}\,\mathrm{d}g$ has been well developed under the name of *nonlinear realisation* and it has been shown that this method can be applied to various

[4]In this paper, we use the convention where the Lie algebra generators are anti-Hermitian.

particles (or even branes) with different symmetries, whether relativistic, non relativistic or conformal, see e.g. [61, 64, 94–101] and references therein.

The aim of the current paper is to generalise the above procedure of obtaining the constrained action (2.1) with manifest covariance to spinning particles as well as more exotic types of particles such as continuous spin particle. For that purpose, in the current section we consider the generalisation of the unconstrained action (2.3). In the following, to be self-contained, we begin with reviewing the classical result of Kirillov–Kostant–Souriau that there exists a G-invariant symplectic structure on any coadjoint G-orbit. Then, we discuss several issues arising in interpreting the coadjoint orbit action (2.3) as a particle action. Let us stress that in this paper, we will be using the term 'particle' loosely to refer to the different types of coadjoint orbits that we will encounter. We have in mind that, when the quantisation of these coadjoint orbits is possible, it will give rise to a unitary and irreducible representation of the isometry group. Moreover, we will see that the parameters that label the different coadjoint orbits correspond, in the 'quantisable' case, to usual physical parameters such as the mass and spin of the particle.

2.1 Coadjoint orbits: generalities

Let us begin with introducing a few mathematical notions relevant to the study of coadjoint orbits. For a general introduction to the subject, one can consult e.g. [1, 102, 103].

Recall that given a Lie group G with Lie algebra $\mathfrak{g}$, the coadjoint orbit $\mathcal{O}^G_\phi$ of an element $\phi \in \mathfrak{g}^*$ is the submanifold of $\mathfrak{g}^*$ whose points are related to ϕ by the coadjoint action of G, i.e.

$$\mathcal{O}^G_\phi := \big\{ \mathrm{Ad}^*_g\, \phi\,,\ g \in G \big\} \subset \mathfrak{g}^*\,, \tag{2.7}$$

where Ad^* denotes the coadjoint action of G on $\mathfrak{g}^*$, defined by

$$\langle \mathrm{Ad}^*_g \varphi, \xi \rangle = \langle \varphi, \mathrm{Ad}_{g^{-1}} \xi \rangle\,, \tag{2.8}$$

for any $\varphi \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and $g \in G$. Here Ad is the adjoint action of G on its Lie algebra $\mathfrak{g}$, and $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. The element $\phi \in \mathfrak{g}^*$ above simply serves as a reference point for the coadjoint orbit $\mathcal{O}^G_\phi$, and can be used as a label for the latter. Of course, there is no privileged choice for this reference point as it is a representative of the equivalence class of element in $\mathfrak{g}^*$ under the coadjoint action of G. In the rest of the paper, we will use the representative ϕ to designate the corresponding coadjoint orbit. Note that when the Lie group G has disconnected parts, related by finite subgroups, their coadjoint orbits may have also disconnected parts.

One can identify a coadjoint orbit with the quotient space,

$$\mathcal{O}^G_\phi \simeq G/G_\phi = \{\, [g]\,, \forall g \in G \,|\, [g\, h] = [g]\,, \forall h \in G_\phi \,\}\,, \tag{2.9}$$

where G_ϕ is the subgroup of G which leaves ϕ invariant under its coadjoint action,

$$G_\phi = \big\{ g \in G \,|\, \mathrm{Ad}^*_g\, \phi = \phi \big\}\,, \tag{2.10}$$

and is called its stabiliser or isotropy subgroup. Therefore, the coadjoint orbit $\mathcal{O}^G_\phi$ can be viewed as the base space of the principal G_ϕ-bundle with projection map π_ϕ,

$$\begin{aligned} \pi_\phi : G \; &\to \; \mathcal{O}^G_\phi \,, \\ g \; &\mapsto \; \pi_\phi(g) = \mathrm{Ad}^*_g \, \phi \,. \end{aligned} \tag{2.11}$$

Notice that the stabilisers of any two elements of a coadjoint orbit are isomorphic.[5]

The tangent space of the quotient manifold (2.9) at a point $\varphi \in \mathcal{O}^G_\phi$ is therefore given by the quotient of the corresponding Lie algebras,

$$T_\varphi \mathcal{O}^G_\phi \;\cong\; \mathfrak{g}/\mathfrak{g}_\varphi \,, \tag{2.12}$$

with $\mathfrak{g}_\varphi$ the Lie algebra of G_φ, which can be described as

$$\mathfrak{g}_\varphi = \big\{ \xi \in \mathfrak{g} \,|\, \mathrm{ad}^*_\xi \, \varphi = 0 \big\} \,, \tag{2.13}$$

where $\mathrm{ad}^*_\xi \, \varphi := -\varphi \circ \mathrm{ad}_\xi$ denotes the coadjoint action of a Lie algebra element $\xi \in \mathfrak{g}$ on $\varphi \in \mathfrak{g}^*$. Consequently, any vector $V_\xi \in T_\varphi \mathcal{O}^G_\phi$ can be generated by an element $\xi \in \mathfrak{g}/\mathfrak{g}_\varphi$,

$$V_\xi := \mathrm{ad}^*_\xi \, \varphi \,. \tag{2.14}$$

The coadjoint orbits can be grouped into two categories: *semisimple* and *nilpotent* coadjoint orbits. If a coadjoint orbit $\mathcal{O}^G_\phi$ satisfies $\mathfrak{g}_\phi \subset \mathrm{Ker}\,\phi$, that is,

$$\langle \phi, \xi \rangle = 0 \,, \qquad \forall \xi \in \mathfrak{g}_\phi, \tag{2.15}$$

the orbit is *nilpotent*, and if not, the orbit is *semisimple* (see e.g. [104, Sec. 1.3]). For a given Lie algebra $\mathfrak{g}$, there is a continuum of semisimple orbits, and they are labelled by a set of continuous parameters. On the contrary, there is only a finite discretum of nilpotent orbits, and hence representative vectors of nilpotent orbits do not contain any parameters which label the orbits. In other words, coadjoint vectors with rescaled parameters belong to the same nilpotent coadjoint orbit. As we shall review shortly below, each coadjoint orbit is an even dimensional subspace of $\mathfrak{g}^*$ with G-invariant symplectic form.

Various properties of a coadjoint orbit can be captured by the quotient Lie algebra,

$$\mathfrak{g}^{\mathrm{Ab}}_\phi := \mathfrak{g}_\phi / [\mathfrak{g}_\phi, \mathfrak{g}_\phi] \,, \tag{2.16}$$

the Abelianisation of $\mathfrak{g}_\phi$, since the derived algebra $[\mathfrak{g}_\phi, \mathfrak{g}_\phi]$ verifies

$$[\mathfrak{g}_\phi, \mathfrak{g}_\phi] \cong \Big\{ \xi \in \mathfrak{g}_\phi \mid \langle \phi, \xi \rangle = 0 \Big\} \equiv \mathrm{Ker}\phi \cap \mathfrak{g}_\phi \,. \tag{2.17}$$

For a nilpotent orbit, $\mathfrak{g}^{\mathrm{Ab}}_\phi = \emptyset$ by definition, whereas $\mathfrak{g}^{\mathrm{Ab}}_\phi$ is non-trivial for a semisimple orbit, and it is *elliptic* if $\mathfrak{g}^{\mathrm{Ab}}_\phi$ is compact.

[5] Indeed, a simple computation shows that $G_{\mathrm{Ad}^*_g \phi} = g^{-1} G_\phi g$ for any $g \in G$, and $\mathfrak{g}_{\mathrm{Ad}^*_g \phi} = \mathrm{Ad}_{g^{-1}} \mathfrak{g}_\phi$. Note also that the projection $G \twoheadrightarrow \mathcal{O}^G_\phi$ does not depend on a choice of representative of the orbit: one can verify that $\pi_{\mathrm{Ad}^*_h \phi}(g) = \pi_\phi(gh)$, which implies that different choices of orbit representatives to define the projection explicitly lead to diffeomorphic G_ϕ-principal bundle structures.

Let us conclude this section by recalling that, when the Lie algebra $\mathfrak{g}$ is endowed with a symmetric bilinear form

$$\kappa : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}\,, \tag{2.18}$$

which is *Ad-invariant*, meaning it verifies

$$\kappa(\mathrm{Ad}_g\xi, \mathrm{Ad}_g\zeta) = \kappa(\xi,\zeta)\,, \qquad \xi,\zeta \in \mathfrak{g}\,, \tag{2.19}$$

for any Lie group element $g \in G$, then one can relate coadjoint orbits to adjoint ones — orbits of the Lie group G on its Lie algebra $\mathfrak{g}$ defined by the adjoint action. Indeed, one can define the 'musical morphism',

$$\begin{aligned} \kappa^\flat :\ \mathfrak{g} &\longrightarrow\ \mathfrak{g}^* \\ \xi &\longmapsto\ \kappa^\flat(\xi) := \kappa(\xi, -)\,, \end{aligned} \tag{2.20}$$

which, by Ad-invariance of κ, implies

$$\kappa^\flat\big(\mathcal{O}^G_\xi\big) = \mathcal{O}^G_{\kappa^\flat(\xi)}\,, \tag{2.21}$$

where on the left-hand side, one has the adjoint orbit of $\xi \in \mathfrak{g}$, and on the right hand side the coadjoint orbit of $\kappa^\flat(\xi)$. On top of that, if κ is non-degenerate, i.e.

$$\kappa(\xi,\zeta) = 0 \quad \forall \zeta \in \mathfrak{g} \qquad \Rightarrow \qquad \xi = 0\,, \tag{2.22}$$

the musical morphism $\kappa^\flat$ is an isomorphism, and therefore defines a diffeomorphism between the adjoint orbit of any $\xi \in \mathfrak{g}$ and coadjoint orbit of $\kappa^\flat(\xi) \in \mathfrak{g}^*$. In particular, for semisimple Lie groups the Killing form is non-degenerate, and hence one can equivalently study their coadjoint or adjoint orbits.

Coadjoint orbits of real semisimple Lie groups

In real semisimple Lie groups, coadjoint orbits can be bijectively identified with adjoint orbits via the Killing form. The representative element of an adjoint orbit admits a unique decomposition, the Jordan decomposition, in terms of elliptic, hyperbolic and nilpotent elements. An element $\xi \in \mathfrak{g}$ is called nilpotent if the matrix ad_ξ is a nilpotent matrix with zero eigenvalue. An element $\xi \in \mathfrak{g}$ is called semisimple if the matrix ad_ξ is diagonalisable over the complex numbers. Semisimple elements are divided into elliptic and hyperbolic ones depending on whether their non-zero eigenvalues are all pure imaginary or not (with anti-Hermitian convention for $\mathfrak{g}$). Compact semisimple Lie groups have only semisimple coadjoint orbits, which are in one-to-one correspondence with orbits of the Weyl group in the Cartan subalgebra. For classical Lie groups, that is real forms of GL_N, O_N or Sp_{2N} which can be compact or non-compact, the classification of adjoint orbits has been worked out in [105, 106].

Nilpotent orbits are of particular interest, both in mathematics and physics: see e.g. [107–111] for recent progress on complex nilpotent orbits. These orbits have been classified, and can be labeled by *signed Young diagrams* [112] (see also [113] for the classification of nilpotent orbits of the complex forms, and [104, Chap. 9] for a textbook account), which are simply Young diagrams whose boxes are filled in with plus or minus signs, in a way that encodes the real form of the Lie algebra of interest. The basic idea of this classification comes from the Jacobson–Morozov theorem which states that any nilpotent element of a semisimple Lie algebra $E \in \mathfrak{g}$ fit into a triple $\{H, E, F\}$ which span an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra in $\mathfrak{g}$, as its raising operator. The fundamental representation V of $\mathfrak{g}$ is completely reducible under the action of this $\mathfrak{sl}(2,\mathbb{R})$, as a direct sum of highest weight modules. This collection of highest weights allows one to associate a partition of the dimension of V, i.e. a Young diagram with $\dim V$ boxes, to a given nilpotent orbit. Moreover, each box of these Young diagrams should be filled in with either a $+$ or a $-$ sign, in an alternating manner in each row, according to rules that depend on the particular real form $\mathfrak{g}$. Two signed Young diagrams are equivalent if one can be related to the other by a permutation of its rows. The interested reader may find a detailed account of this classification in [104, Chap. 9].

An adjoint orbit is called *regular* if its elements are regular, which is to say that their centralisers are of minimal dimension, namely the rank of the algebra [114, Chap. II.2]. Consequently, these orbits are of maximal dimensions, and can be described as surfaces in $\mathfrak{g}^*$ defined as the common level sets of the functions dual to the Casimir operators.[6] Hence, their dimension is $\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$. Among regular orbits, there is a unique nilpotent orbit, usually called the principal nilpotent orbit, defined by the zero locus of the Casimir functions. The other nilpotent orbits have smaller dimensions, as they are defined by a larger number of polynomial equations. The nilpotent orbit with minimum dimension, apart from the trivial orbit $\{0\}$, is also unique and called the minimal orbit.

2.2 Kirillov–Kostant–Souriau symplectic two-form and symplectic potential

Coadjoint orbits form an interesting class of symplectic manifolds, as they are endowed with a symplectic form, called the Kirillov–Kostant–Souriau (KKS) symplectic form. Its value at any point $\varphi \in \mathcal{O}^G_\phi$ is defined by

$$\omega_\varphi(V_{\xi_1}, V_{\xi_2}) = \langle \varphi, [\xi_1, \xi_2] \rangle \,, \tag{2.23}$$

with $\xi_1, \xi_2 \in \mathfrak{g}$. The pullback Ω of the KKS symplectic form ω on G gives

$$\Omega_g(\xi_1, \xi_2) := (\pi^*_\phi \omega)_g(\xi_1, \xi_2) = \langle \phi, [\Theta_g(\xi_1), \Theta_g(\xi_2)] \rangle \,, \tag{2.24}$$

where Θ is the Maurer–Cartan form — the left-invariant $\mathfrak{g}$-valued one-form on G, locally given by

$$\Theta_g = g^{-1} \mathrm{d}_G g \,, \tag{2.25}$$

where d_G is the de Rham differential on the group manifold G. Since the Maurer–Cartan form satisfies the Maurer–Cartan equation,

$$\mathrm{d}_G \, \Theta + \tfrac{1}{2} [\Theta, \Theta] = 0 \,, \tag{2.26}$$

the two-form Ω is exact :

$$\Omega = -\mathrm{d}_G \langle \phi, \Theta \rangle \,. \tag{2.27}$$

[6] In the sense that the space of polynomial functions on $\mathfrak{g}^*$ is isomorphic to $S(\mathfrak{g})$, to which the Casimir operators of $\mathfrak{g}$ belong.

The orbit $\mathcal{O}_\phi^G$ can be covered by several coordinate patches $U_i \subset \mathcal{O}_\phi^G$ with local sections $\sigma_i : U_i \hookrightarrow G$. We can pullback the two-form Ω by σ_i to obtain the symplectic two-form ω and the corresponding symplectic potential θ_i in each U_i :

$$\omega = -\mathrm{d}_{\mathcal{O}}\, \theta_i \,, \qquad \theta_i = \langle \phi, \sigma_i^* \Theta \rangle = \langle \phi, {\sigma_i}^{-1}\, \mathrm{d}_{\mathcal{O}} \sigma_i \rangle \,, \tag{2.28}$$

where $\mathrm{d}_{\mathcal{O}}$ is the de Rham differential on the coadjoint orbit $\mathcal{O}_\phi^G$. Note that the two-form ω does not depend on the choice of sections but θ_i does: two sections are related by

$$\sigma_j = \sigma_i\, \tau_{ij} \,, \tag{2.29}$$

with the transition map $\tau_{ij} \;: U_i \cap U_j \to G_\phi$, and consequently the symplectic potentials are related by

$$\theta_j = \theta_i + \langle \phi, {\tau_{ij}}^{-1}\, \mathrm{d}_{\mathcal{O}} \tau_{ij} \rangle \,. \tag{2.30}$$

The fact that the second term is closed, due to $\mathrm{Ad}^*_{\tau_{ij}} \phi = \phi$, shows that the two-form ω is gauge independent, that is, independent of sections.

2.3 Worldline action and its quantisation

The worldline action is given by the integral of θ_i on a path γ lying in $U_i \subset \mathcal{O}_\phi^G$, or equivalently, the integral of Θ on the lifted path $\sigma_i(\gamma)$ lying in $\sigma_i(U_i) \subset G$,

$$S_i[\gamma] = \int_{\gamma \subset \mathcal{O}_\phi} \theta_i = \int_{\sigma_i(\gamma) \subset G} \langle \phi, \Theta \rangle \,. \tag{2.31}$$

Note that this type of action has been considered in various contexts: see e.g. [66, 115–117]. The action transforms as

$$S_j = S_i + \int_\gamma \langle \phi, {\tau_{ij}}^{-1}\, \mathrm{d}_{\mathcal{O}} \tau_{ij} \rangle \,. \tag{2.32}$$

We consider only local change of section, that is to say, the transition function τ_{ij} becomes identity at the end points of the path γ.[7] In the case the transition map τ_{ij} is connected to identity, the difference of the action vanishes. In the other case, it gives a non-trivial contribution. When we quantize the system through the path integral,[8]

$$Z = \int \mathcal{D}\gamma\, \exp\left(\tfrac{i}{\hbar}\, S[\gamma]\right) \,, \tag{2.33}$$

we can also ask the invariance of Z under a change of section by a transition map τ_{ij} which may not be connected to identity. The difference of the action by such a τ_{ij} belongs to the first de Rham cohomology group $H^1(G_\phi^{\mathrm{Ab}})$, the Lie group associated with $\mathfrak{g}_\phi^{\mathrm{Ab}}$. Since $\langle \phi, [\xi, \zeta] \rangle = 0$ for any $\xi, \zeta \in \mathfrak{g}_\phi$, it is sufficient to consider $\mathfrak{g}_\phi^{\mathrm{Ab}}$ instead of $\mathfrak{g}_\phi$. The group G_ϕ^{Ab} is Abelian, and we can parameterise an element $\tau_{ij} \in G_\phi^{\mathrm{Ab}}$ as

$$\tau_{ij} = e^{\zeta_{ij}^1\, J_1}\, e^{\zeta_{ij}^2\, J_2} \cdots e^{\zeta_{ij}^k\, J_k} \,, \qquad k = \mathrm{dim} G_\phi^{\mathrm{Ab}} \,, \tag{2.34}$$

[7] Transformations of the end points may involve issues of large gauge transformations. See [71] for related discussions.

[8] The issues of quantisation, including the path integral measure, will be addressed in the forthcoming paper [92].

where J_a's are the generators of the Lie algebra $\mathfrak{g}_\phi^{\mathrm{Ab}}$. We require that the τ_{ij} transformation leaves the path integral invariant:

$$\exp\left[\frac{1}{\hbar}\int_\gamma \langle\phi, \tau_{ij}^{-1}\mathrm{d}_{\mathcal{O}}\tau_{ij}\rangle\right] = \exp\left[\frac{1}{\hbar}\langle\phi, J_a\rangle\int_\gamma \mathrm{d}_{\mathcal{O}}\zeta_{ij}^a\right] = 1\,. \tag{2.35}$$

Recall that the transition function becomes identity (i.e. ζ_{ij}^a vanish) at the end points of the path γ. In this case, we find

$$\int_\gamma \mathrm{d}_{\mathcal{O}}\zeta_{ij}^a = \oint_{\Gamma_{ij}} \mathrm{d}_G\zeta^a\,, \tag{2.36}$$

where $\Gamma_{ij} = \sigma_i(\gamma)\cdot\sigma_j(\gamma)^{-1}$ is the closed path lying in G. The parameters ζ^a may or may not be periodic depending on the nature of the generator J_a. If a generator J_a exponentiates to a $U(1)$ so ζ^a is periodic (with period T_a), then the integral,

$$w_a = \frac{1}{T_a}\oint_{\Gamma_{ij}} \mathrm{d}_G\zeta^a \in \mathbb{Z}\,, \tag{2.37}$$

gives the number of times that the closed path Γ_{ij} winds the cycle associated with the ζ^a coordinate. Therefore, the condition (2.35) requires that each of $\frac{T_a}{2\pi\hbar}\langle\phi, J_a\rangle$ be an integer. Recall that the latter $\langle\phi, J_a\rangle$ are all zero in a nilpotent orbit. Therefore, here only semisimple orbits are concerned. We consider $\hbar$ as a fixed constant, so $\langle\phi, J_a\rangle$ is quantised. If a generator J_b exponentiates rather to an $\mathbb{R}$, then $\zeta^b \in \mathbb{R}$ and there cannot be any non-trivial winding of Γ_{ij}. Therefore, $\oint_{\Gamma_{ij}} \mathrm{d}_G\zeta^b = 0$ and no condition is imposed on $\langle\phi, J_b\rangle$. From the above discussion, we see that the quantisation selects a certain discretum of coadjoint orbits among an infinite continuum of semisimple coadjoint orbits.[9]. This selection is in fact equivalent to the prequantisation condition of the geometric quantisation: if γ is closed, we can take two disks $\Sigma_i \subset U_i$ and $\Sigma_j \subset U_j$ such that $\partial\Sigma_i = \gamma = \partial\Sigma_j$. In such cases, the difference of the action reduces to

$$\int_\gamma \langle\phi, \tau_{ij}{}^{-1}\,\mathrm{d}_{\mathcal{O}}\tau_{ij}\rangle = \oint_{\Sigma_{ji}} \omega\,, \tag{2.38}$$

where $\Sigma_{ji} = \Sigma_j \cup \bar{\Sigma}_i$ (here, $\bar{\Sigma}_i$ is the disk Σ_i with the opposite orientation) has the topology of a two sphere S^2. Remark however that the quantisation of $\langle\phi, J_a\rangle$ takes place even when there is no Σ_i or Σ_j satisfying the condition.

Clearly, the change of sections by a transition map τ_{ij} can be interpreted as a gauge transformation. The role of this gauge symmetry will become more manifest when we reformulate the action as a constrained Hamiltonian system. It is also worth noting that the condition (2.37) depends on the topology of G: it changes if we change the Lie group G by its one of covering groups. Since the coadjoint orbits of G and its various covers are all the same, the KKS symplectic form ω is also the same. However, the symplectic potential θ_i depends on the covering structure of the group.

[9] This condition is also an example of the mechanism of quantisation of coupling constants in field theory spelled out in e.g. [118].

Spin

Typically, the components $\phi_a := \langle \phi, J_a \rangle$ match the labels of particle species such as mass and spin. Due to the mechanism described above, some of these labels may be quantised: the (conventional) spin label ought to be quantised always, but sometimes the mass label is quantised as well, e.g. in AdS spacetime.[10] Then, what are the key differences between mass and spin from the viewpoint of coadjoint orbits? A key feature of the spin is that when quantised, it leads to a finite-dimensional Hilbert space. When a coadjoint orbit is compact, we will find that only a finite number of modes survive upon imposing quantisation conditions, and hence the associated Hilbert space is finite-dimensional. Let us illustrate the issue with an example. Consider a coadjoint orbit of $\mathfrak{so}(3)$ spanned by J_i. Up to $SO(3)$ rotation, there is only one type of coadjoint vector $\phi = s\,\mathcal{J}^3$ (here, $\mathcal{J}^a$ are the dual basis of $\mathfrak{so}(3)^*$ with $\langle \mathcal{J}^a, J_b \rangle = \delta^a_b$), and the corresponding orbit is S^2 with radius s. This orbit is two-dimensional, so the system has one mechanical degree of freedom. When S^2 is quantised, only integral s is allowed, and the space of phase space functions is reduced from the space of functions on S^2 to the space of spin-s spherical harmonics on S^2. Hence, the dimension of the Hilbert space is reduced from ∞ to $2s+1$, and the number of degrees of freedom — mathematically speaking twice the Gelfand–Kirillov dimension — is reduced from 2 to 0. This reduction is a generic feature of compact coadjoint orbits as they are associated with finite dimensional representations. When a coadjoint orbit is non-compact, a similar reduction of the number of modes may take place due to the presence of a compact subspace.

Let us comment here that the spinning particle action with bosonic variables should not be confused with the model of relativistic spherical top (see e.g. [119] for the classical account and also [120, 121] and reference therein for recent developments). For example, the spin degrees of freedom of the four dimensional massive spinning particle (in the sense of the current paper) are the coordinates of S^2, a $SO(3)$ coadjoint orbit, whereas the spin degrees of freedom of a spherical top are the coordinates of the cotangent bundle $T^*SO(3)$. The quantisation of the latter gives the infinite direct sum of the tensor product of two spin s representations, without any projection. See e.g. [122] for the description of a spinning particle inspired by the spherical top model. We postpone the relevant discussions to the sequel paper where we cover the issues of quantisation.

Geometric quantisation

Let us conclude this section by pointing out that the quantisation condition (2.38) also appears in the context of geometric quantisation, where it is known as the prequantisation condition (see e.g. [1, 123–126]). In this approach to quantisation, one aims at defining, from a symplectic manifold $(\mathcal{M}, \omega)$, a Hilbert space $\mathcal{H}$ and a quantisation map $\mathcal{Q} : \mathscr{C}^\infty(\mathcal{M}) \to \mathsf{Op}(\mathcal{H})$ from functions on $\mathcal{M}$ to linear operators on the Hilbert space.

[10] Let us point out that the mass label of generic massive particles is quantised in AdS_{d+1} since the time translation forms a compact subgroup $SO(2)$ of the AdS group $SO^+(2,d)$ (or its double cover). This is to be contrasted with the discrete mass level of partially-massless particles, for which the value of the mass is related to that of the spin and depth of the field. For a continuous spectrum of mass for massive particles, one can replace AdS spacetime by its infinite cover CAdS.

This map should verify a few conditions, the most constraining ones being that it defines a morphism of Lie algebra between $\mathscr{C}^\infty(\mathcal{M})$ endowed with its Poisson bracket to $\mathsf{Op}(\mathcal{H})$ endowed with the commutator, i.e.

$$[\mathcal{Q}(f),\mathcal{Q}(g)] = -i\hbar\,\mathcal{Q}(\{f,g\})\,, \qquad \forall f,g\in\mathscr{C}^\infty(\mathcal{M})\,, \tag{2.39}$$

which is usually referred to as the Dirac condition. In order for such a map to be well-defined globally, on top of obeying all conditions including Dirac's, one is lead to introducing a linear connection ∇ on a line bundle over $\mathcal{M}$ (that is, a vector bundle whose fibers are isomorphic to $\mathbb{C}$) whose curvature is proportional to the symplectic form ω. The existence of a line bundle equipped with such a connection requires that

$$\oint_\Sigma \omega \in 2\pi\hbar\,\mathbb{Z}\,, \tag{2.40}$$

for any closed 2-dimensional manifold of $\Sigma\subset\mathcal{M}$. In our case, the linear connection is simply the pullback of the Maurer–Cartan form of G on the coadjoint orbit $\mathcal{O}^G_\phi$, evaluated on ϕ, which we have seen is subject to the above condition, see (2.38). For more details, see e.g. [126, Sec. 3].

2.4 Examples: $\mathfrak{so}(3)$, $\mathfrak{so}(2,1)$, $\mathfrak{iso}(2)$ and $\mathfrak{iso}(1,1)$

For concrete examples, let us consider the coadjoint orbits of three-dimensional Lie groups $SO(3)$, $SO(2,1)$, $ISO(2)$, $ISO(1,1)$ and their simply connected counterparts as well as their double covers: Note the isomorphisms $\widetilde{SO}(3)\cong SU(2)$ and $\widetilde{SO}^+(2,1)\cong SU(1,1)\cong SL(2,\mathbb{R})\cong Sp(2,\mathbb{R})$. The example of $SL(2,\mathbb{R})$ coadjoint orbits has been treated in numerous papers, e.g. [127–129].

Let us fix the convention first. The Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ are generated by J_a $(a=1,2,3)$ obeying

$$[J_a,J_b]=\epsilon_{ab}{}^c\,J_c\,, \tag{2.41}$$

where the Levi–Civita tensor ϵ_{abc} is defined with $\epsilon_{123}=1$. The Latin indices are raised and lowered with the Euclidean metric for $\mathfrak{so}(3)$, and with the Minkowski metric $\eta=\mathrm{diag}(-1,-1,1)$ for $\mathfrak{so}(2,1)$.

The Lie algebras $\mathfrak{iso}(2)$ and $\mathfrak{iso}(1,1)$ are generated by P_a $(a=1,2)$ and J obeying

$$[P_a,P_b]=0\,, \qquad [J,P_a]=\epsilon_a{}^b\,P_b\,, \tag{2.42}$$

where the Levi–Civita tensor ϵ_{ab} is defined with $\epsilon_{12}=1$. The indices are raised and lowered with the Euclidean metric for $\mathfrak{iso}(2)$, and with the Minkowski metric $\eta=\mathrm{diag}(1,-1)$ for $\mathfrak{iso}(1,1)$. The Lie algebra $\mathfrak{iso}(2)$ can also be obtained from $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ by contracting the J_3 generator, whereas $\mathfrak{iso}(1,1)$ can be obtained from $\mathfrak{so}(2,1)$ by contracting the J_1 generator.

Geometries

An arbitrary element in $\mathfrak{so}(3)^*$ or $\mathfrak{so}(1,2)^*$ can be written as

$$\phi = j_a\,\mathcal{J}^a\,, \tag{2.43}$$

where $\mathcal{J}^a$ are the dual basis satisfying $\langle \mathcal{J}^a , J_b \rangle = \delta^a_b$.

For $\mathfrak{so}(3)^*$, any coadjoint vector ϕ can be rotated to the form,

$$\phi = \sqrt{j^2}\,\mathcal{J}^3\,, \tag{2.44}$$

and it has a stabiliser $SO(2)$ generated by J_3. The above is the representative of the coadjoint orbit $\mathcal{O}^{SO(3)}_\phi$ which is a two-sphere $S^2 \cong SO(3)/SO(2)$ with radius $\sqrt{j^2}$. The coadjoint space $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ is foliated by a continuum of spherical orbits of different radii (see Figure 1). The stabilisers of each orbit are all $\mathfrak{g}_\phi = \mathrm{span}\{J_3\} \simeq \mathfrak{u}(1)$. Since the quotient algebra $\mathfrak{g}^{\mathrm{Ab}}_\phi = \mathfrak{g}_\phi = \mathfrak{u}(1)$ is compact, the orbit is elliptic.

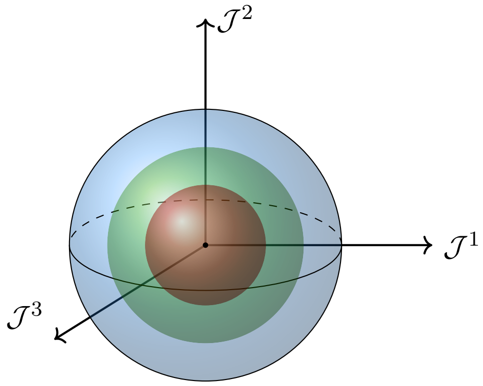

Figure 1. Examples of coadjoint orbits of $SO(3)$, which are simply two-spheres of different radii.

For $\mathfrak{so}(2,1)^*$, depending on the value of $j^2 = j_a\, j^a$,[11] a coadjoint vector ϕ can be rotated or boosted to one of the three representatives:

$$\phi = \begin{cases} \pm\sqrt{j^2}\,\mathcal{J}^3\,, & [j^2 > 0,\ \pm j_3 > 0] \\ \pm(\mathcal{J}^1 + \mathcal{J}^3), & [j^2 = 0,\ \pm j_3 > 0] \\ \sqrt{-j^2}\,\mathcal{J}^1\,, & [j^2 < 0] \\ 0\,, & [j_a = 0] \end{cases}. \tag{2.45}$$

The coadjoint vectors with $\pm$ signs belong to two distinct coadjoint orbits of $SO^+(2,1)$. These two orbits form a single disconnected coadjoint orbit of $SO(2,1)$ as they are mapped to each other by the "time reversal" transformation, forming the $\mathbb{Z}_2$ finite subgroup. The coadjoint orbits with the above representative vectors are all given by two-dimensional quadratic surfaces,

$$H^2(a) = \{(x, y, z) \in \mathbb{R}^3 \,|\, -x^2 - y^2 + z^2 = a\}\,. \tag{2.46}$$

The surface with $a < 0$ is the one-sheeted hyperbolic hyperboloid, and the surface with $a > 0$ is the two-sheeted elliptic hyperboloid. The special case $a = 0$ corresponds to the two-dimensional cone: $H^2(0) = C^2$. When $a \geq 0$, namely the two-sheeted hyperboloids

[11] Note that here, we are using the convention that the direction 3 is the time-like one (usually denoted by 0).

and the cone, contain two disconnected parts: the upper/lower hyperboloids $H^2_\pm(a>0) = \{(x,y,z)\in H^2(a)\,|\,\pm z>0\}$ and the upper/lower cones $C^2_\pm = \{(x,y,z)\in C^2\,|\,\pm z>0\}$.

The coadjoint orbit represented by the first ϕ has the stabiliser $SO(2)$ generated by J_3, and it is an elliptic orbit since $\mathfrak{g}_\phi^{\mathrm{Ab}} = \mathrm{span}\{J_3\} \simeq \mathfrak{u}(1)$. It has the geometry of the two-dimensional elliptic hyperboloid $H^2_\pm(j^2) \cong SO^+(2,1)/SO(2)$. The second case has the stabiliser $\mathbb{R}$ generated by $J_1 - J_3$, and the orbit is nilpotent since $\langle \phi, J_1 - J_3\rangle = 0$. Its geometry is a two dimensional cone $C^2_\pm \cong SO^+(2,1)/\mathbb{R}$. The third case has the stabiliser $SO^+(1,1)$ generated by J_1, and the orbit is hyperbolic since $\mathfrak{g}_\phi^{\mathrm{Ab}} = \mathrm{span}\{J_1\} \simeq \mathbb{R}$. The geometry is a two-dimensional hyperbolic hyperboloid $H^2(-j^2) \cong SO^+(2,1)/SO^+(1,1)$. The last case has the entire $SO^+(2,1)$ as its stabiliser and the orbit is the single point at the origin. The coadjoint space $\mathfrak{so}(2,1)^* \cong \mathbb{R}^3$ is foliated by a continuum of hyperboloid-type orbits $H^2_\pm(j^2)$ and $H^2(-j^2)$ with different j's, two conical orbits $C^2_\pm$ and the origin (see Figure 2).

Since the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ are semisimple, their coadjoint spaces can be identified with the adjoint spaces through the Killing forms. This allows us to view the coadjoint actions of Lie group elements as mere rotations or boosts, that is, the adjoint actions of $SO(3)$ or $SO^+(2,1)$. In other words, we may as well study their adjoint orbits. The adjoint representation of J_3 for both $SO(3)$ and $SO^+(2,1)$ is

$$\mathrm{ad}_{J_3} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \tag{2.47}$$

and it has the eigenvalues $+i, -i, 0$, confirming that the orbit is elliptic. On the other hand, the adjoint representations of J_1 and $J_1 + J_3$ for $SO^+(2,1)$ are

$$\mathrm{ad}_{J_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \qquad \mathrm{ad}_{J_1+J_3} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \tag{2.48}$$

and they have the eigenvalues $+1, -1, 0$ and $0, 0, 0$ respectively, confirming that the corresponding orbits are hyperbolic and nilpotent, respectively.

Let us move to the non-semisimple cases $ISO(2)$ and $ISO(1,1)$. An arbitrary element of $\mathfrak{iso}(2)^*$ or $\mathfrak{iso}(1,1)^*$ can be written as

$$\phi = p_a\,\mathcal{P}^a + j\,\mathcal{J}\,, \tag{2.49}$$

where $\mathcal{P}^a$ and $\mathcal{J}$ are the dual basis satisfying $\langle \mathcal{P}^a, P_b\rangle = \delta^a_b$, $\langle \mathcal{J}, J\rangle = 1$ and $\langle \mathcal{P}^a, J\rangle = 0 = \langle \mathcal{J}, P_a\rangle$. The coadjoint action of $ISO(2)$ or $ISO^+(1,1)$ on ϕ is

$$\mathrm{Ad}^*_{e^{x^a P_a}\,\Lambda}(p_a\,\mathcal{P}^a + j\,\mathcal{J}) = p_a\,\Lambda^a{}_b\,\mathcal{P}^b + (j + \epsilon^{bc}\,p_a\,\Lambda^a{}_b\,x_c)\,\mathcal{J}\,, \tag{2.50}$$

where Λ is the rotation or boost element in $SO(2)$ or $SO^+(1,1)$ generated by J.

For the $\mathfrak{iso}(2)^*$ case, any coadjoint vector ϕ can be transformed into

$$\phi = \begin{cases} \sqrt{p^2}\,\mathcal{P}^1\,, & [p^2 \neq 0] \\ j\,\mathcal{J}\,, & [p^2 = 0] \end{cases}\,. \tag{2.51}$$

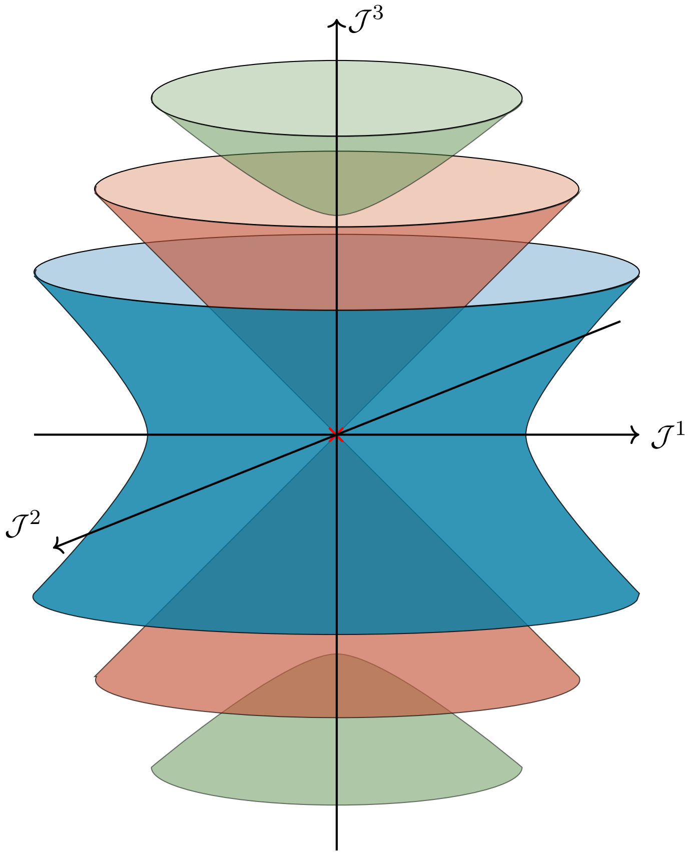

Figure 2. Example of coadjoint orbits of $SO(2,1)$: in blue the one-sheeted hyperboloid, in red the two disconnected upper and lower cones and in green two one-sheeted hyperboloids.

The first case has the stabiliser $\mathbb{R}$ generated by P_1, and the orbit is a two-dimensional cylinder $ISO(2)/\mathbb{R} \cong S^1 \times \mathbb{R}$ of radius $\sqrt{p^2}$. The stabiliser of the second case is the entire Euclidean group $ISO(2)$, and the orbit is a single point located on the j-axis. Again the coadjoint space $\mathfrak{iso}(2)^* \cong \mathbb{R}^3$ is foliated by a continuum of cylindrical orbits and a continuum of points on the j-axis, see Figure 3.

For $ISO^+(1,1)$ case, any coadjoint vector ϕ can be transformed into

$$\phi = \begin{cases} \pm\sqrt{p^2}\,\mathcal{P}^1\,, & [p^2 > 0,\ \pm p_1 > 0] \\ \pm\sqrt{-p^2}\,\mathcal{P}^2\,, & [p^2 < 0,\ \pm p_2 > 0] \\ \pm\mathcal{P}^1 \pm' \mathcal{P}^2, & [p^2 = 0,\ \pm p_1 > 0\,,\ \pm' p_2 > 0] \\ j\,\mathcal{J}, & [p_a = 0] \end{cases}, \tag{2.52}$$

where $\pm'$ means an independent sign possibilities with respect to $\pm$. The stabilisers of the first three classes of the coadjoint orbits are all $\mathbb{R}$, generated respectively by P_1, P_2 and $\pm P_1 \mp' P_2$. The corresponding orbits are hyperbolic cylinders and conical cylinder. The last case has the entire $ISO^+(1,1)$ as its stabiliser, and the orbit is a single point on the j-axis. The coadjoint space $\mathfrak{iso}(1,1)^* \cong \mathbb{R}^3$ is foliated by a continuum of hyperbolic cylinder shaped orbits and one conical cylinder (which is subdivided by four pieces of two half-planes) and a continuum of points on j-axis, see Figure 4.

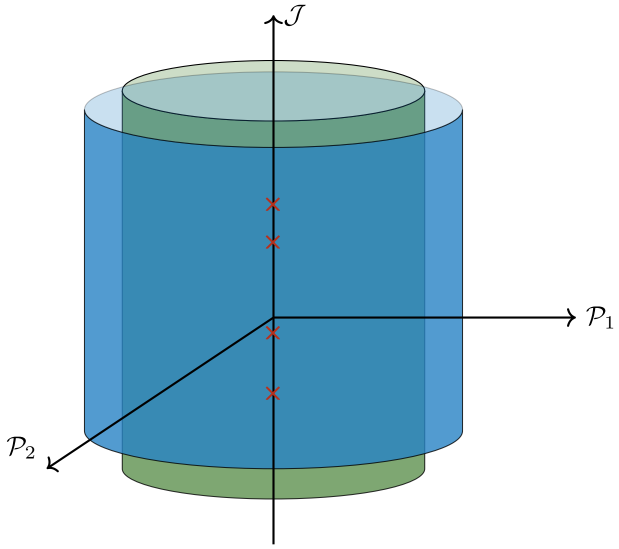

Figure 3. Examples of coadjoint orbits of $ISO(2)$: two cylinders centered around the $\mathcal{J}$ axis of different radii in blue and green, corresponding to orbits with $p^2 \neq 0$, and four isolated points on this same axis red (randomly distributed), corresponding to orbits with $p^2 = 0$.

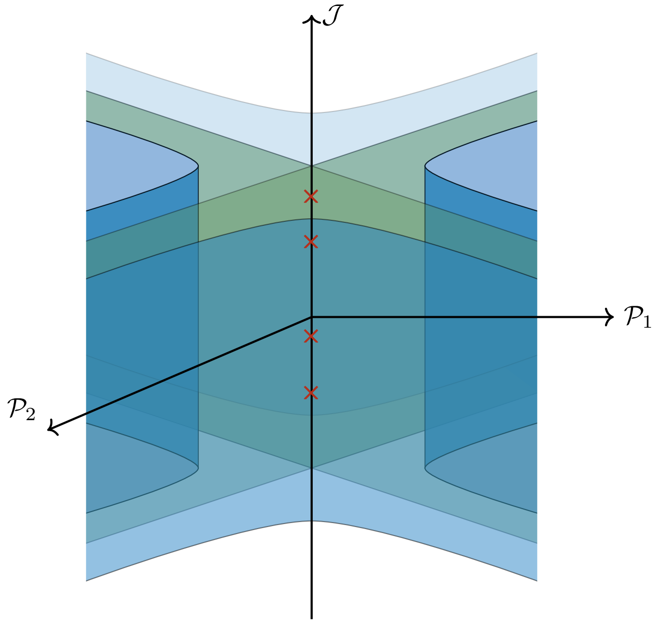

Figure 4. Examples of coadjoint orbits of $ISO(1,1)$: in blue (a two-sheet) hyperbolic cylinder, in green a conical cylinder, and in red four isolated points (randomly distributed).

Recall that $ISO(2)$ and $ISO^+(1,1)$ can be obtained by a Inönü–Wigner contraction of $SO(3)$ and $SO^+(2,1)$, respectively. In fact, $ISO(2)$ can be obtained from either $SO(3)$ or $SO^+(2,1)$, and in the latter case, the generator J_3 needs to be contracted. The spherical cylinder of $ISO(2)$ are the contractions of a sphere of $SO(3)$ as well as a one-sheeted hyperboloid of $ISO^+(1,1)$. The hyperbolic cylinder and the conical cylinder of $ISO^+(1,1)$

can be obtained from a one- or two-sheeted hyperboloid and cone of $SO^+(2,1)$, respectively. The isolated points on the $\mathcal{J}$ axis can be also obtained by assigning a suitable scaling of ϕ under the contraction. See [130, 131] for more discussions about the contraction of (A)dS orbits to Poincaré ones.

So far we have not considered the double cover of $SO(3)$ or $SO^+(2,1)$ because they define the same hypersurface in $\mathfrak{g}^*$: they have the same coadjoint representations. As far as the geometries are concerned, there is no difference. Below, we shall see that the difference arises when considering their symplectic potentials.

Symplectic structures

Let us first have a closer look at the elliptic coadjoint orbits of $SO(3)$ and $SO^+(2,1)$ and their double cover $SU(2)$ and $SU(1,1)$. These coadjoint orbits are all represented by the coadjoint vector $\phi = \sqrt{j^2}\,\mathcal{J}^3$ with the stabiliser $G_\phi = \{\exp(\gamma\, J_3)\}$, and can be described respectively by a sphere S^2 or an elliptic hyperboloid H^2_+. To proceed the analysis, let us parameterise a group element g with the Euler angles ξ, υ, ζ as

$$g(\xi,\upsilon,\zeta) = \exp(\xi\, J_3)\, \exp(\upsilon\, J_1)\, \exp(\zeta\, J_3)\,, \tag{2.53}$$

where the range of parameters ξ, υ, ζ depends on the cases. First, ξ always belongs to $[0, 2\pi)$, whereas υ belongs to $[0, 2\pi)$ for the compact case and $[0, \infty)$ for the non-compact case. Lastly, the range of ζ depends on whether the Lie groups associated with $\mathfrak{so}(3)^*$ and $\mathfrak{so}(2,1)^*$ are $SO(3)$ and $SO^+(2,1)$ or $SU(2)$ and $SU(1,1)$: $\zeta \in [0, 2\pi)$ in the former cases while $\zeta \in [0, 4\pi)$ for the latter cases. With the periodic conditions on ξ and ζ, this coordinate system is well-defined everywhere except for the region near the north pole N ($\upsilon = 0$) for both $\mathfrak{so}(3)^*$ and $\mathfrak{so}(2,1)^*$ and the south pole S ($\upsilon = \pi$) for only $\mathfrak{so}(3)^*$.

A simple computation gives

$$\langle \phi, \Theta \rangle = \langle \sqrt{j^2}\,\mathcal{J}^3, g^{-1}\mathrm{d}_G g\rangle = \sqrt{j^2} \times \begin{cases} \cos\upsilon\, \mathrm{d}_G\xi + \mathrm{d}_G\zeta\,, & [\,\mathfrak{so}(3)\,] \\ \cosh\upsilon\, \mathrm{d}_G\xi + \mathrm{d}_G\zeta\,, & [\,\mathfrak{so}(2,1)\,] \end{cases}. \tag{2.54}$$

On the coordinate chart (2.53), we choose different local sections $\sigma_i(\xi,\upsilon)$ which determine ζ_i as functions $\zeta_i(\xi,\upsilon)$ of the coadjoint orbit coordinates ξ, υ:

$$\sigma_i(\xi,\upsilon) = \exp(\xi\, J_3)\, \exp(\upsilon\, J_1)\, \exp(\zeta_i(\xi,\upsilon)\, J_3)\,. \tag{2.55}$$

Then, by pulling back $\langle \phi, \Theta \rangle$ with σ_i, we obtain the symplectic potential θ_i as

$$\theta_i(\xi,\upsilon) = \sqrt{j^2} \times \begin{cases} \cos\upsilon\, \mathrm{d}_{\mathcal{O}}\xi + \mathrm{d}_{\mathcal{O}}\zeta_i(\xi,\upsilon)\,, & [\,\mathfrak{so}(3)\,] \\ \cosh\upsilon\, \mathrm{d}_{\mathcal{O}}\xi + \mathrm{d}_{\mathcal{O}}\zeta_i(\xi,\upsilon)\,, & [\,\mathfrak{so}(2,1)\,] \end{cases}. \tag{2.56}$$

Here, ξ is the azimuthal angle, and υ is the inclination angle or rapidity of the coadjoint orbit S^2 or H^2_+, respectively. The difference between two symplectic potentials θ_i is

$$\theta_i = \theta_j + \sqrt{j^2}\, \mathrm{d}_{\mathcal{O}}\zeta_{ij}\,, \tag{2.57}$$

where $\zeta_{ij} = \zeta_i - \zeta_j$ and $\tau_{ij} = \exp(\zeta_{ij} J_3)$. The transformation of the worldline action under the change of section by ζ_{ij} is

$$S_j = S_i + \sqrt{j^2} \int_\gamma \mathrm{d}_{\mathcal{O}} \zeta_{ij} \tag{2.58}$$

and

$$\int_\gamma \mathrm{d}_{\mathcal{O}} \zeta_{ij} = \oint_{\Gamma_{ij}} \mathrm{d}_G \zeta \in \begin{cases} 2\pi\,\mathbb{Z} & [SO(3) \text{ or } SO^+(2,1)] \\ 4\pi\,\mathbb{Z} & [SU(2) \text{ or } SU(1,1)] \end{cases} , \tag{2.59}$$

where $\mathbb{Z}$ corresponds to the set of possible numbers that the trajectory Γ_{ij} winds the cycle corresponding to the ζ coordinate. The invariance of $e^{i\,S/\hbar}$ under this transformation leads to the quantisation of the orbit radius:

$$\sqrt{j^2} = \hbar\,\ell\,, \qquad \ell \in \begin{cases} \mathbb{N} & [SO(3) \text{ or } SO^+(2,1)] \\ \mathbb{N}/2 & [SU(2) \text{ or } SU(1,1)] \end{cases} . \tag{2.60}$$

In the case of the $\mathfrak{so}(3)^*$, we can cover the entire orbit S^2 with two charts $U_N = S^2 - \{\mathsf{S}\}$ and $U_S = S^2 - \{\mathsf{N}\}$. By choosing the sections as

$$\zeta_N(\xi, \upsilon) = -\xi\,, \qquad \zeta_S(\xi, \upsilon) = \xi\,, \qquad \zeta_{NS}(\xi, \upsilon) = -2\xi\,, \tag{2.61}$$

the symplectic potential in Euler angles is well-defined in each chart, that is, near the north pole N and the south pole S. For $SO(3)$ where $\zeta_{NS} \in [0, 2\pi)$, the transition map $\xi \in S^1_{2\pi} \mapsto \zeta_{NS} \in S^1_{2\pi}$ winds twice. For $SU(2)$ where $\zeta_{NS} \in [0, 4\pi)$, the transition map $\xi \in S^1_{2\pi} \mapsto \zeta_{NS} \in S^1_{4\pi}$ winds once and this fiber bundle structure corresponds to the Hopf fibration of $SU(2) \cong S^3$ over the two-sphere S^2. In both cases, the difference of the worldline action under the change of the sections (2.61) is

$$\int_\gamma \sqrt{j^2}\, \mathrm{d}_{\mathcal{O}} \zeta_{NS} = \int_0^{2\pi} \sqrt{j^2}\, (-2\, \mathrm{d}_{\mathcal{O}} \xi) = -4\pi \sqrt{j^2}\,, \tag{2.62}$$

and this can be rewritten as the integral of the symplectic two-form over the orbit S^2 :

$$\int_\gamma \sqrt{j^2}\, \mathrm{d}_{\mathcal{O}} \zeta_{NS} = \int_\gamma (\theta_N - \theta_S) = \int_{\Sigma_N \cup \bar{\Sigma}_S = S^2} \omega\,. \tag{2.63}$$

The above quantity should be $2\pi\, n$ for an integer n in order for the path integral to be invariant under such a transformation, and this is the prequantisation condition in the context of geometric quantisation. Note that the prequantisation condition is weaker than the condition of the invariance of the action under a change of section: in the latter case we find (2.60) whereas the prequantisation condition does not give any restriction on the H^2_+ orbit of $\mathfrak{so}(2,1)^*$ and it allows the half-integral radius for $SO(3)$ case. See [127] for related discussions.

Next, let us consider the nilpotent coadjoint orbit satisfying $j^2 = 0$ and $j_3 > 0$. Any such vector can be rotated to $\mathcal{J}^1 + \mathcal{J}^3$. Again to proceed the analysis, we take the Iwasawa decomposition,

$$g(\xi, \upsilon, \zeta) = \exp(\xi\, J_3)\, \exp(\upsilon\, J_2)\, \exp(\zeta\, (J_1 + J_3))\,, \tag{2.64}$$

which is well adapted to the nilpotent orbit. Here, the ranges of the parameters are $\xi \in [0, 2\pi)$ and $\upsilon, \zeta \in (-\infty, +\infty)$. The one-form $\langle \phi, \Theta \rangle$ is

$$\langle \phi, \Theta \rangle = \langle \mathcal{J}^1 + \mathcal{J}^3, g^{-1} \mathrm{d}_G g \rangle = e^{\upsilon} \, \mathrm{d}\xi + 2 \, \mathrm{d}_G \zeta \,. \tag{2.65}$$

Since ζ belongs to $\mathbb{R}$, the difference of the action always vanishes, $\oint_{\Gamma_{ij}} \mathrm{d}_G \zeta = 0$.

Lastly, let us consider the hyperbolic coadjoint orbit $H^2(j^2)$ of $\mathfrak{so}(2,1)^*$ given by $j^2 < 0$. Any vector in it can be rotated to $\sqrt{-j^2}\, \mathcal{J}^1$, and a convenient decomposition is

$$g(\xi, \upsilon, \zeta) = \exp(\xi \, J_1) \, \exp(\upsilon \, J_3) \, \exp(\zeta \, J_1) \,, \tag{2.66}$$

where both ξ and ζ belong to $(-\infty, \infty)$ and υ belongs to $[0, 2\pi)$ for $SO^+(2,1)$ and $[0, 4\pi)$ for $SU(1,1)$. The one-form $\langle \phi, \Theta \rangle$ is

$$\langle \phi, \Theta \rangle = \langle \sqrt{-j^2}\, \mathcal{J}^1, g^{-1} \mathrm{d}_G g \rangle = \sqrt{-j^2}\, (\cos \upsilon \, \mathrm{d}_G \xi + \mathrm{d}_G \zeta) \,. \tag{2.67}$$

Again ζ belongs to $\mathbb{R}$, and the difference of the action always vanishes. Therefore, no condition is imposed on $\sqrt{-j^2}$.

About the cylindrical orbits of $\mathfrak{iso}(2)^*$ and the hyperbolic cylinder orbits of $\mathfrak{iso}(1,1)^*$, we use the decomposition,

$$g(\xi, \upsilon, \zeta) = \exp(\xi \, P_2) \, \exp(\upsilon \, J) \, \exp(\zeta \, P_1) \,, \tag{2.68}$$

for a Lie group element, where ξ and ζ belongs to $\mathbb{R} = (-\infty, +\infty)$ and υ belongs to $[0, 2\pi)$ for $ISO(2)$ and $\mathbb{R}$ for $ISO(1,1)$. The one-form $\langle \phi, \Theta \rangle$ is

$$\langle \phi, \Theta \rangle = \langle \sqrt{p^2}\, \mathcal{P}^1, g^{-1} \mathrm{d}_G g \rangle = \sqrt{p^2} \times \begin{cases} \sin \upsilon \, \mathrm{d}_G \xi + \mathrm{d}_G \zeta \,, & [\, \mathfrak{iso}(2) \,] \\ \sinh \upsilon \, \mathrm{d}_G \xi + \mathrm{d}_G \zeta \,, & [\, \mathfrak{iso}(1,1) \,] \end{cases} . \tag{2.69}$$

The other $\mathfrak{iso}(1,1)^*$ orbits with $p^2 < 0$ are isomorphic to the ones with $p^2 > 0$. For the $\mathfrak{iso}(1,1)^*$ orbit which has the shape of a conical cylinder, we use the decomposition,

$$g(\xi, \upsilon, \zeta) = \exp\big(\xi \, (P_1 + P_2)\big) \, \exp(\upsilon \, J) \, \exp\big(\zeta \, (P_1 - P_2)\big) \,. \tag{2.70}$$

The one-form $\langle \phi, \Theta \rangle$ is

$$\langle \phi, \Theta \rangle = \langle \mathcal{P}^1 + \mathcal{P}^2, g^{-1} \mathrm{d}_G g \rangle = e^{\upsilon} \, \mathrm{d}_G \xi + \mathrm{d}_G \zeta \,. \tag{2.71}$$

In all cases of $\mathfrak{iso}(2)^*$ and $\mathfrak{iso}(1,1)^*$, the coordinate ζ belongs to $\mathbb{R}$. Therefore, it has no contribution to the action under a change of section.

2.5 Phase space and dynamics

The coadjoint orbit is a symplectic space, so it can serve as a phase space of a mechanical system, but it does not seem to provide a Hamiltonian at first glance. Indeed, the examples that we have treated just above did not show any Hamiltonian. On the contrary, in the introduction, we showed briefly how a relativistic scalar particle action can be obtained from a coadjoint orbit of Poincaré group. The difference between the two cases is in different parameterizations of a Lie group element. For concreteness, let us consider again the massive scalar orbit of Poincaré group with $\phi = m\,\mathcal{P}_0$.

First, let us consider the parameterization of a Lie group element given by the decomposition,

$$g = e^{y^a\,P_a}\,e^{v^a\,J_{0a}}\,e^{y^0\,P_0}\,R\,, \tag{2.72}$$

where $a = 1, 2, \ldots, d{-}1$ and $R \in SO(d{-}1)$. When $d = 2$, this choice reduces to the $\mathfrak{iso}(1,1)$ example we treated just above. Since all the stabiliser $G_\phi \cong \mathbb{R}\times SO(d-1)$ is present on the right side of g, it is well suited for the right quotient $ISO^+(1,d-1)/(\mathbb{R}\times SO(d-1))$. This choice leads to

$$\langle \phi, \Theta\rangle = p_a\,\mathrm{d}_G y^a + m\,\mathrm{d}_G y^0\,, \tag{2.73}$$

where $p_a = m\,\frac{\sinh v}{v}\,v_a$. Up to the boundary term $\mathrm{d}_G y^0$, we recover the canonical symplectic structure $p_a\,\mathrm{d}y^a$ but without any non-trivial Hamiltonian. The boundary term $m\,\mathrm{d}_G y^0$ might also be regarded as a constant Hamiltonian if we take y^0 as the proper time of the worldline. In any case, it is a static system.

Instead, if we take the group element as

$$g = e^{x^a\,P_a + x^0\,P_0}\,e^{v^a\,J_{0a}}\,R\,, \tag{2.74}$$

we would find

$$\Theta = p_a\,\mathrm{d}_G x^a - \sqrt{m^2 + p_a\,p^a}\,\mathrm{d}_G x^0\,. \tag{2.75}$$

We can set x^0 as the proper time using a reparametrization of the worldline, then we recover the familiar scalar particle Lagrangian with a non-trivial Hamiltonian.

Since different decompositions of g correspond to different coordinate systems for G, the two choices are related by a coordinate transformation,

$$(x^0, x^a) = \left(\frac{m\,y^0}{\sqrt{m^2 + p_a\,p^a}}, y^a + \frac{p^a\,y^0}{\sqrt{m^2 + p_a\,p^a}}\right), \tag{2.76}$$

which can be easily obtained by reordering (2.74) into (2.72). This coordinate transformation — which trivialize the particle dynamics — is similar to the canonical transformation resulting in action-angle variables: Hamiltonian in action-angle variables can be simply reabsorbed by shifting the angle variable ϕ_i by the frequencies w_i: $\phi_i \to \phi_i - t\,w_i$.

As we could see from the above example, the Hamiltonian action associated to a coadjoint orbit always can be written in the trivial form $p_i\,\mathrm{d}x^i$ without a Hamiltonian (up to a total derivative term), at least locally (Darboux's theorem guarantees it). Therefore, in order to interpret a coadjoint orbit action as a relativistic spinning particle action, it is crucial to choose an appropriate set of coordinates. And the appropriateness is the covariance of the system under the Lie group G. This perspective resonates with the appropriate choice of a group decomposition in nonlinear realisation where the distinction of broken symmetries and unbroken symmetries is important. A good coordinate system may make a certain part of the symmetry manifest, but it can never do so for the entire symmetry. For the full manifest covariance, we need to involve additional variables and make the system a constrained one.

3 Constrained Hamiltonian system

Before moving to the reformulation of coadjoint orbit actions as constrained systems, let us review the standard formulation of constrained Hamiltonian systems with an emphasis on its relation to coadjoint orbits. We shall see in particular how a coadjoint orbit is related to second class constraints whereas the stabiliser is related to first class constraints.

3.1 Hamiltonian reduction

When a Lie group G acts on a symplectic space $\mathcal{M}$ and the action is Hamiltonian, we can make a correspondence between coadjoint orbits of $\mathfrak{g}^*$ and a certain set of constrained surfaces (or the reduced phase spaces thereof) inside $\mathcal{M}$. For a better understanding of this perspective, let us review the relevant mathematical material. In the next subsection, we will recast the content of this subsection in terms of Hamiltonian mechanics.

A symplectic manifold $(\mathcal{M}, \Omega)$ is equipped with the Poisson bracket,

$$\{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}g)\,, \tag{3.1}$$

where the Poisson bivector Π is the inverse of the symplectic two-form Ω: in a coordinate system $\{y^\mu\}$, the symplectic two-form $\Omega = \Omega_{\mu\nu}(y)\,\mathrm{d}y^\mu \wedge \mathrm{d}y^\nu$ and Poisson bivector $\Pi = \Pi^{\mu\nu}(y)\,\frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu}$ are related by $\Omega_{\mu\nu}(y)\,\Pi^{\nu\rho}(y) = \delta^\rho_\mu$, and we have $\{y^\mu, y^\nu\} = \Pi^{\mu\nu}(y)$.

Suppose we have the vectors fields $V_a \in T\mathcal{M}$ corresponding to the generators J_a of a Lie algebra $\mathfrak{g}$ satisfying

$$[J_a, J_b] = f_{ab}{}^c\, J_c\,. \tag{3.2}$$

If these vectors fields are Hamiltonian, that is to say, there exists a set of functions $\mu_a \in \mathscr{C}^\infty(\mathcal{M})$ obeying $i_{V_a}\Omega = \mathrm{d}\mu_a$, then, these functions satisfy

$$\{\mu_a, \mu_b\} = f_{ab}{}^c\, \mu_c + \tau_{ab}\,, \tag{3.3}$$

where $\tau_{ab} \in \mathscr{C}^\infty(\mathcal{M})$ is a central function, i.e. a function whose Poisson bracket with any other function vanishes.[12] A consequence of the action of $\mathfrak{g}$ being Hamiltonian is that the symplectic form Ω is preserved by infinitesimal diffeomorphism generated by the fundamental vector fields V_a,

$$\mathcal{L}_{V_a}\Omega = 0\,, \tag{3.4}$$

as can be easily seen by using Cartan's homotopy formula. Whenever τ vanishes, the co-moment map defined as

$$\begin{aligned} \mu^* \ :\quad & \mathfrak{g} && \longrightarrow \mathscr{C}^\infty(\mathcal{M})\,, \\ & \xi = \xi^a\, J_a && \longmapsto \mu^*(\xi) = \langle \mu, \xi\rangle = \mu_a\, \xi^a\,, \end{aligned} \tag{3.5}$$

[12] The condition that $V_a = \{\mu_a, -\}$ obey $[V_a, V_b] = f_{ab}{}^c\, V_c$ is equivalent to $\{\{\mu_a, \mu_b\} - f_{ab}{}^c\, \mu_c, -\} = 0$, which in turn imply that $\{\mu_a, \mu_b\} - f_{ab}{}^c\, \mu_c = \tau_{ab}$ where $\{\tau_{ab}, f\} = 0$ for any function $f \in \mathscr{C}^\infty(\mathcal{M})$. The Jacobi identity for the Poisson bracket implies that τ_{ab} is a Chevalley–Eilenberg two-cocycle in the trivial module. If the corresponding cohomology class is non-trivial, then τ defines a central extension $\mathfrak{g}$.

is a Lie algebra morphism from $(\mathfrak{g}, [\cdot,\cdot])$ to $(\mathscr{C}^\infty(\mathcal{M}), \{\cdot,\cdot\})$. One can also assemble μ_a into the moment map μ,

$$\begin{aligned} \mu = \mu_a\, \mathcal{J}^a \;:\; \mathcal{M} &\longrightarrow \mathfrak{g}^*\,, \\ y &\longmapsto \mu(y) = \mu_a(y)\, \mathcal{J}^a\,, \end{aligned} \tag{3.6}$$

where $\mathcal{J}^a$ are the basis of $\mathfrak{g}^*$ dual to J_a: $\langle \mathcal{J}^a, J_b \rangle = \delta^a_b$. The moment and co-moment maps are related by

$$\langle \mu(y), \xi \rangle = \mu^*(\xi)(y)\,, \tag{3.7}$$

for any $\xi \in \mathfrak{g}$ and $y \in \mathcal{M}$, so that one can think of the moment map μ as the dual of the co-moment map μ^* (and vice versa).

The pre-image of the element $\phi \in \mathfrak{g}^*$ under μ,

$$\mu^{-1}(\phi) = \{y \in \mathcal{M} \,|\, \mu(y) = \phi\} \subset \mathcal{M}\,, \tag{3.8}$$

is not a symplectic submanifold of $\mathcal{M}$: using the inclusion map $\iota_\phi : \mu^{-1}(\phi) \hookrightarrow \mathcal{M}$, one can pullback Ω onto $\mu^{-1}(\phi)$ to get the two-form $\iota_\phi^*\Omega$, which is degenerate unless G_ϕ is trivial. If G_ϕ acts freely and properly[13] on $\mu^{-1}(\phi)$, and $\phi \in \mathfrak{g}^*$ is a regular value of μ,[14] the quotient space,

$$\mathcal{N}_\phi := \mu^{-1}(\phi)/G_\phi\,, \tag{3.9}$$

is the base space of the principal G_ϕ-bundle $\mu^{-1}(\phi)$ with the canonical projection $\pi : \mu^{-1}(\phi) \to \mathcal{N}_\phi$. Then, $\mathcal{N}_\phi$ has a unique symplectic two-form ω satisfying $\pi^*\omega = \iota_\phi^*\Omega$: we can use $\mu^{-1}(\phi)$ to compare the symplectic form on the G-manifold $\mathcal{M}$ and the reduced phase space $\mathcal{N}_\phi$ by pulling them back with the inclusion and projection respectively, as illustrated below.

$$\begin{array}{ccc} \mu^{-1}(\phi) & \xhookrightarrow{\iota_\phi} & \mathcal{M} \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathcal{N}_\phi & & \end{array} \tag{3.10}$$

This result is known as the Marsden–Weinstein–Meyers theorem, see e.g. [132].

Note that the moment map μ is equivariant with respect to the $\mu^*(\mathfrak{g})$ action and the $\mathrm{ad}^*_{\mathfrak{g}}$ action: for any $\xi \in \mathfrak{g}$,

$$\{\mu^*(\xi), \mu\} = \mathrm{ad}^*_\xi\, \mu\,. \tag{3.11}$$

If the vector fields V_a can be integrated, the equivariance can be promoted to the Lie group G: for any $g \in G$,

$$\mu(g \triangleright y) = \mathrm{Ad}^*_g \mu(y)\,. \tag{3.12}$$

[13] Recall that the action of a group G on $\mathcal{M}$ (denoted by $\triangleright : G \times \mathcal{M} \to \mathcal{M}$) is called *free* if the stabiliser of any point $y \in \mathcal{M}$ is trivial, meaning if $g \in G$ fixes a point y, that is $g \triangleright y = y$, then it is the group identity, $g = \mathbf{1}$, necessarily. It is called *proper* if the inverse image of compact sets under the group action are compact. These two conditions ensure that the quotient space $\mathcal{M}/G$ admits a structure of smooth manifold, and $\mathcal{M} \twoheadrightarrow \mathcal{M}/G$ is a smooth principal G-bundle.

[14] A regular value of $\mu : \mathcal{M} \to \mathfrak{g}^*$ is an element $\phi \in \mathfrak{g}^*$ such that, for any point in its pre-image $y \in \mu^{-1}(\phi)$, the pushforward $(\mu_*)_y : T_y\mathcal{M} \to T_\phi\mathfrak{g}^* \cong \mathfrak{g}^*$ is surjective. This implies that $\mu^{-1}(\phi)$ is a submanifold of $\mathcal{M}$ (see e.g. [132, Sec. 1.1.13.]).

As the moment map μ defines a G-equivariant homomorphism from $\mathcal{M}$ to $\mathfrak{g}^*$, $\mu^{-1}(\phi)$ satisfies

$$g \triangleright (\mu^{-1}(\phi)) = \mu^{-1}(\mathrm{Ad}^*_g \phi) \,. \tag{3.13}$$

This shows that $\mu^{-1}(\phi)$ is closed under the action of the stabiliser G_ϕ, and hence,

$$\mu^{-1}(\mathcal{O}^G_\phi) \cong \mathcal{O}^G_\phi \times \mu^{-1}(\phi) \,. \tag{3.14}$$

Note that the hypersurfaces $\mu^{-1}(\phi)$ and $\mu^{-1}(\mathcal{O}^G_\phi)$ in $\mathcal{M}$ have co-dimension $\dim \mathfrak{g}$ and $\dim \mathfrak{g} - \dim \mathcal{O}^G_\phi = \dim \mathfrak{g}_\phi$, respectively. The coadjoint space is foliated by the coadjoint orbits, $\mathfrak{g}^* = \bigcup_{\phi\in\Phi} \mathcal{O}^G_\phi$, with an infinite set $\Phi = \mathfrak{g}^*/G$ of representative vectors, and Φ can be further decomposed as $\Phi = \bigcup_{\varphi\in\Psi} \Phi_\varphi$ with a finite set Ψ and infinite sets Φ_φ where φ are stereotypical representative vectors. The pre-image of the entire coadjoint space $\mathfrak{g}^*$, that is nothing but $\mathcal{M}$, also admits the foliation,

$$\mathcal{M} = \mu^{-1}(\mathfrak{g}^*) \cong \bigcup_{\phi\in\Phi} \mathcal{O}^G_\phi \times \mu^{-1}(\phi) \cong \bigcup_{\varphi\in\Psi} G_\varphi \times \bigcup_{\phi\in\Phi_\varphi} \mathcal{O}^G_\phi \times \mathcal{N}_\phi \,. \tag{3.15}$$

Remark that both $\mathcal{O}^G_\phi$ and $\mathcal{N}_\phi$ are symplectic submanifold of $\mathcal{M}$, whereas G_φ is an isotropic one. When G is compact, the infinite set Φ corresponds to $\mathfrak{h}^*/W$ where $\mathfrak{h}$ is the Cartan subalgebra and W the Weyl group. In plain words, Φ is the set of orbits of W in $\mathfrak{h}^*$. Each Φ_φ corresponds to either interior, boundary, or corner regions of Φ.

3.2 Constrained Hamiltonian mechanics

Let us rephrase the above discussion in the framework of constrained Hamiltonian mechanics. The symplectic space $\mathcal{M}$ is the embedding phase space endowed with the canonical structure $\Omega(y)$, and the hypersurface $\mu^{-1}(\phi)$ is the constraint surface determined by the Hamiltonian constraints,

$$\chi_a(y) = \mu_a(y) - \phi_a \approx 0 \,, \tag{3.16}$$

and therefore has dimension $\dim \mu^{-1}(\phi) = \dim \mathcal{M} - \dim \mathfrak{g}$. The Poisson bracket of any two constraints then takes the form

$$\{\chi_a, \chi_b\} = f_{ab}{}^c \, \mu_c \approx f_{ab}{}^c \, \phi_c \,, \tag{3.17}$$

where $\approx$ denotes a weak equality, i.e. an equality on the constraint surface $\mu^{-1}(\phi)$. Recall that in a constrained Hamiltonian system, one distinguishes between first and second class constraints: the former are constraints whose Poisson brackets with any other constraint weakly vanish (i.e. they vanish on the constraint surface) while the latter are constraints whose Poisson brackets with at least one constraint does not vanish. To distinguish between first and second class constraints χ_a, it is convenient to introduce the notation $\chi^*(\xi) := \xi^a \, \chi_a$ so that each constraint can be labeled by an element $\xi = \xi^a \, J_a$ of $\mathfrak{g}$. It can also be understood as a shifted co-moment map $\chi^* = \mu^* - \phi$. In this notation, the Poisson bracket (3.17) between any two constraints can be written as

$$\{\chi^*(\xi), \chi^*(\zeta)\} \approx \langle \phi, [\xi, \zeta] \rangle = -\langle \mathrm{ad}^*_\xi \phi, \zeta \rangle \,, \tag{3.18}$$

for any $\xi, \zeta \in \mathfrak{g}$. For $\xi \in \mathfrak{g}_\phi$, the constraints $\chi^*(\xi)$ weakly commute with any other constraints, and hence they are the first-class constraints. The remaining constraints $\chi^*(\xi)$ with $\xi \notin \mathfrak{g}_\phi$ are the second-class constraints. To recapitulate, the set of constraints, $\chi^*(\mathfrak{g})$, is divided into the set of the first class constraints, $\chi^*(\mathfrak{g}_\phi)$, and the set of the second class constraints, $\chi^*(\mathfrak{g}/\mathfrak{g}_\phi) \cong \chi^*(T_\phi \mathcal{O}^G_\phi)$.

The quotient space $\mathcal{N}_\phi = \mu^{-1}(\phi)/G_\phi$ is the physical phase space, i.e. the constraint surface reduced by the action of the gauge symmetry generated by the first class constraints. The latter corresponds to the stabiliser $\mathfrak{g}_\phi$, so that the reduced phase space $\mathcal{N}_\phi$ has dimension,

$$\dim \mathcal{N}_\phi = \dim \mu^{-1}(\phi) - \dim \mathfrak{g}_\phi = \dim \mathcal{M} - \dim \mathcal{O}^G_\phi - 2 \dim \mathfrak{g}_\phi \,, \tag{3.19}$$

and one can confirm that each first and second class constraints remove respectively two and one dimension from the embedding phase space.

The action corresponding to this phase space is

$$S[y, A] = \int_I \vartheta(y) - \langle \chi(y), A \rangle \,, \tag{3.20}$$

where ϑ is the symplectic potential of Ω satisfying $\Omega = -\mathrm{d}\vartheta$. The Lagrange multiplier $A \in \Omega^1(I, \mathfrak{g})$ is a worldline one-form, valued in the Lie algebra $\mathfrak{g}$. Note that $y^\mu(t) \equiv y^\mu(\gamma(t))$ where $t \in I \subset \mathbb{R}$ is the worldline parameter and $\gamma : I \to \mathcal{M}$ is the worldline, i.e. the (phase space) trajectory of a point particle in $\mathcal{M}$. Under the transformation generated by the gauge parameter $\lambda \in \Omega^0(I, \mathfrak{g})$,

$$\delta_\lambda y^\mu = \{\chi^*(\lambda), y^\mu\} \,, \qquad \delta_\lambda A = \mathrm{d}\lambda + [A, \lambda] \,, \tag{3.21}$$

the action transforms as

$$\delta_\lambda S[y, A] = \int_I \mathrm{d}\left(i_\lambda \vartheta(y) - \langle \mu(y), \lambda \rangle\right) + \langle \phi, \mathrm{d}\lambda + [A, \lambda] \rangle \,. \tag{3.22}$$

Up to a total derivative, the above reduces to the integral of $\langle \phi, [A, \lambda] \rangle$ and it vanishes only when λ takes value in the isotropy subalgebra $\mathfrak{g}_\phi$. This shows that only the first class constraints associated with $\mathfrak{g}_\phi$ lead to gauge symmetries. Under a finite gauge transformation $h \in G_\phi$,

$$y \ \to \ y^h = h^{-1} \triangleright y \,, \qquad A \ \to \ A^h = h^{-1}(A + \mathrm{d})h \,, \tag{3.23}$$

the action changes as

$$\begin{aligned} S[y^h, A^h] - S[y, A] &= \int_I \vartheta(y^h) - \vartheta(y) - \langle \chi, \mathrm{d}h\, h^{-1} \rangle \\ &= \int_I \vartheta(y^h) - \vartheta(y) - \langle \mu(y), \mathrm{d}h\, h^{-1} \rangle + \langle \phi, h^{-1} \mathrm{d}h \rangle \,, \end{aligned} \tag{3.24}$$

where the first three terms are the finite counterpart of the total derivative $\mathrm{d}(i_\lambda \vartheta - \langle \mu(y), \lambda \rangle)$ appearing for infinitesimal gauge transformations. The invariance of $\exp(\frac{i}{\hbar} S)$ under the above transformation requires that the last term be proportional to $2\pi\, \hbar$ times an integer. This leads precisely to the same quantisation condition on ϕ as in (2.37). One can also convert all the second class constraints into first class ones by introducing additional variables: see Appendix B.

3.3 Example: Cotangent bundle of a Lie group

An important class of examples of the above discussion is the cotangent bundle $\mathcal{M} = T^*G$ of a Lie group G, which is a symplectic manifold as any cotangent bundle: Locally, the symplectic form reads

$$\Omega_{T^*G} = \mathrm{d}\vartheta\,, \qquad \vartheta = p_\mu\,\mathrm{d}x^\mu\,, \tag{3.25}$$

where $\{x^\mu\}$ are coordinates on G and $\{p_\mu\}$ are coordinates in the fiber directions of T^*G. The Lie group G acts on its algebra of functions via left- or right- invariant vector fields, which will be denoted by

$$\rho_a = \rho_a{}^\mu(x)\,\tfrac{\partial}{\partial x^\mu} \in \Gamma(TG)\,, \tag{3.26}$$

and where the Latin index a refers to a basis $\{J_a\}$ of the Lie algebra $\mathfrak{g}$. These vector fields can be lifted to functions on T^*G via

$$\mu_a(x,p) := \rho_a{}^\mu(x)\,p_\mu\,, \tag{3.27}$$

which verifies

$$\{\mu_a, \mu_b\} = f_{ab}{}^c\,\mu_c\,, \tag{3.28}$$

where $\{-,-\}$ denotes the Poisson bracket associated with the symplectic two-form (3.25). With this data, we can consider a constrained Hamiltonian system of the type described previously, whose corresponding worldline is given by

$$S[x,p,A] = \int p_\mu\,\mathrm{d}x^\mu - A^a\left(\rho_a{}^\mu(x)\,p_\mu - \phi_a\right)\,. \tag{3.29}$$

The constraints

$$\chi_a(x,p) = \rho_a{}^\mu(x)\,p_\mu - \phi_a \approx 0\,, \tag{3.30}$$

can be solved simply by

$$p_\mu = e^a_\mu(x)\,\phi_a\,, \tag{3.31}$$

where $e^a_\mu(x)$ are the components of the left-invariant Maurer–Cartan form of G,

$$g(x)^{-1}\mathrm{d}g(x) = e^a_\mu(x)\,\mathrm{d}x^\mu\,J_a \in \Omega^1(G,\mathfrak{g})\,, \tag{3.32}$$

which are the inverse of the components of the left-invariant vector fields. Inserting the solution of the constraints in the action, we recover the expression,

$$S[x] = \int \phi_a\,e^a_\mu(x)\,\mathrm{d}x^\mu = \int \langle \phi, g(x)^{-1}\mathrm{d}g(x)\rangle\,. \tag{3.33}$$

Remark that the constraint surface is

$$\mu^{-1}(\phi) = \{(g, \mathrm{Ad}^*_g\phi) \in T^*G\,|\,g \in G\} \cong G\,, \tag{3.34}$$

where we used the fact that the cotangent bundle of a Lie group is trivial, $T^*G \cong G \times \mathfrak{g}^*$. Further quotienting by the gauge symmetry generated by the first class constraints, which is given by the action of the isotropy group G_ϕ, leads to

$$\mathcal{N}_\phi = \mu^{-1}(\phi)/G_\phi \cong \mathcal{O}^G_\phi\,, \tag{3.35}$$

i.e. the reduced phase space is nothing but the coadjoint orbit of ϕ.[15] Applying the general story (3.15) to this case, we find

$$T^*G \cong \bigcup_{\varphi \in \Psi} G_\varphi \times \bigcup_{\phi \in \Phi_\varphi} \mathcal{O}^G_\phi \times \mathcal{O}^G_\phi \,. \tag{3.36}$$

Note that the quantisation of the above, in the case when the Lie group G is compact, leads to the Peter–Weyl theorem,

$$L^2(G) = \bigoplus_\lambda \pi^G_\lambda \otimes (\pi^G_\lambda)^* \,, \tag{3.37}$$

where π_λ is the unitary irreducible representations of G labelled by λ (its highest weight, since G is assumed compact here). The above decomposition of T^*G can be recovered as its orbit space under the action of $G \times G$ where the first factor acts from the left and the second from the right (see e.g. [133] for a recent discussion in that direction). In the next section, we shall see a similar pattern of decompositions, but involving coadjoint orbits of two different Lie groups.

4 Manifestly covariant formulation of coadjoint orbit action

In this section, we will explain how a mechanical system given by a coadjoint orbit $\mathcal{O}^G_\phi$ of a Lie group G can be realised as a constrained Hamiltonian system, where the constraints are associated with a different coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$ of a different Lie group $\tilde{G}$. Remark that the analysis of the previous section applies to $\tilde{G}$, associated with the *gauge* symmetry of the system, while G is the *global* symmetry. We will show that the Lie groups G and $\tilde{G}$ are dual in the sense of *symplectic dual pairs* à la Weinstein [3] (see also [134, Chap. 4] for a textbook account). The quantised picture corresponds to Howe duality [4, 5], also known as dual pair correspondence (see also e.g. [90, 91, 135–137]).

The construction below can be understood as a method of obtaining a good coordinate system for the coadjoint orbit $\mathcal{O}^G_\phi$, along the lines of the discussion in 2.5. More precisely, we want to reformulate the system in such a way that its global symmetries are manifest. In other words, we want the global symmetry to be realised linearly, as opposed to a nonlinear realisation, so that all the phase space variables carry faithful representations of the global symmetry. This can be achieved by using the definition of various matrix groups as Hamiltonian constraints. In this set up, the phase space variables carry the defining representations of the global symmetry, as well as a representation of a gauge symmetry

[15] In this simple case, we can also verify the Marsden–Meyers–Weinstein theorem explicitly. To do so, let us note that, under the trivialization provided by the Maurer–Cartan form, the tautological form on the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ reads $\vartheta|_{(g,\varphi)} = \langle \varphi, \Theta_g \rangle$ implying $\Omega_{T^*G}|_{(g,\varphi)} = \mathrm{d}\langle \varphi, \Theta_g \rangle$ at any point $(g,\varphi) \in G \times \mathfrak{g}^*$, and where $\Theta \in \Omega^1(G,\mathfrak{g})$ is the left-invariant Maurer–Cartan form of G. We can now compare the pullback of the symplectic form on the reduced phase space, which is the coadjoint orbit $\mathcal{O}^G_\phi$, by the projection $\pi_\phi : G \twoheadrightarrow \mathcal{O}^G_\phi : g \mapsto \mathrm{Ad}^*_g \phi$, with the pullback of the symplectic form on the phase space $T^*G \cong G \times \mathfrak{g}^*$ by the inclusion $\iota_\phi : G \hookrightarrow G \times \mathfrak{g}^* : g \mapsto (g,\phi)$. We already computed the first pullback in Section 2.2, while the second one simply amounts to the evaluation of Ω_{T^*G} at $\phi \in \mathfrak{g}^*$, so that we find $\pi^*_\phi \Omega_{\mathcal{O}^G_\phi} = \mathrm{d}_G \langle \phi, \Theta \rangle = \iota^*_\phi \Omega_{T^*G}$ as expected.

group. We will find an exquisite relation between the global and gauge symmetries of the system.

In the first two subsections 4.1 and 4.2, we present the construction of worldline action for all classical Lie groups and their semi-direct product with Abelian ideals. The treatment here will be rather brief as we consider the case of indefinite orthogonal group and inhomogeneous orthogonal group in detail in the subsections 4.3 and 4.4. The other cases will be detailed in the sequel paper [92], along with twistor descriptions. The readers who wish to focus on the case of (inhomogeneous) orthogonal groups may skip the first two subsections.

4.1 Classical Lie groups

Let us consider the classical Lie groups $GL(N,\mathbb{F})$, $U(p,N-p)$, $O(p,N-p)$, $O^*(2N)$, $O(N,\mathbb{C})$, $Sp(2N,\mathbb{R})$, $Sp(p,N-p)$ and $Sp(2N,\mathbb{C})$, where, $\mathbb{F}=\mathbb{R},\mathbb{C}$, or the quaternion $\mathbb{H}$. They are all reductive, and include many physically relevant cases, such as Lorentz groups and (A)dS isometry groups: the family $O(p,N-p)$ contains the AdS$_d$ symmetry $O(2,d-1)$ and the dS$_d$ symmetry $O(1,d)$. The double covers of the orthochronous Lorentz groups $SO^+(2,1)$, $SO^+(3,1)$ and $SO^+(5,1)$ are isomorphic to $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$ and $SL(2,\mathbb{H})$, the simple parts of $GL(2,\mathbb{F})$. The double covers of the conformal groups $SO^+(2,3)$, $SO^+(2,4)$ and $SO^+(2,6)$ are isomorphic to $Sp(4,\mathbb{R})$, $SU(2,2)$ and $O^*(8)$.

All classical Lie groups are subgroups of a general linear group $GL(N,\mathbb{F})$ with $\mathbb{F}=\mathbb{R},\mathbb{C}$, defined by a quadratic equation — expressing the fact that they preserve a certain bilinear form. We can introduce most of them in a unified fashion as in [35]: let us define

$$B(b_{(N)},\mathbb{F})=\{\,A\in GL(N,\mathbb{F})\mid A^\dagger\,b_{(N)}\,A=b_{(N)}\,\}\,, \tag{4.1}$$

where $b_{(N)}$ is an element of $GL(N,\mathbb{F})$, and $A^\dagger=(A^t)^*$ with t the matrix transpose and $*$ the conjugation of $\mathbb{F}$, which is the identity map for $\mathbb{R}$, the complex conjugation for $\mathbb{C}$, and the quaternionic conjugation for $\mathbb{H}$. Up to a $GL(N,\mathbb{F})$ transformation $b_{(N)}\to T^\dagger\,b_{(N)}\,T$, we have essentially two possibilities: the Hermitian ones are equivalent to $b_{(N)}=\eta^{(p,N-p)}$, the standard flat metric of $(p,N-p)$ signature. The anti-Hermitian ones are equivalent to $b_{(N)}=\Omega_{(N)}$, the canonical symplectic matrix. See e.g. [104, Prop. 9.3.2.] for relevant discussions. The group $B(b_{(N)},\mathbb{F})$ either simply coincides with one of classical Lie groups or is isomorphic to it:

$$B(\eta^{(p,N-p)},\mathbb{R})=O(p,N-p)\,,\qquad B(\Omega_{(N)},\mathbb{R})=Sp(N,\mathbb{R})\,, \tag{4.2}$$

$$B(\eta^{(p,N-p)},\mathbb{C})=U(p,N-p)\,,\qquad B(\Omega_{(N)},\mathbb{C})\simeq U(\tfrac{N}{2},\tfrac{N}{2})\,, \tag{4.3}$$

$$B(\eta^{(p,N-p)},\mathbb{H})=Sp(p,N-p)\,,\qquad B(\Omega_{(N)},\mathbb{H})\simeq O^*(2N)\,. \tag{4.4}$$

In the right column, the cases with $\mathbb{F}=\mathbb{R}$ and $\mathbb{C}$ are defined only for even N. For the case $\mathbb{F}=\mathbb{H}$, N can be both even and odd because the second element of $\mathbb{H}$ can be seen as a two-dimensional symplectic matrix so we can take $\Omega_{(N)}=\mathbf{j}\,I_{(N)}$. Note that $B(\Omega_{(N)},\mathbb{C})$ are isomorphic to unitary groups because we can diagonalize $\Omega_{(N)}$ as $i\,\eta^{(N/2,N/2)}$. On the contrary, $B(\Omega_{(N)},\mathbb{H})$ is not isomorphic to $Sp(N/2,N/2)$ but $O^*(2N)$, even though $\Omega_{(N)}$ can still be diagonalised as $\mathbf{i}\,\eta^{(N/2,N/2)}$. It is because $\mathbf{i}$, the first element of the basis of $\mathbb{H}$, does not commute with a quaternionic matrix.

We can also define another class of classical Lie groups by using the transpose t at the place of the Hermitian conjugate in the definition (4.1). Then, for $\mathbb{F}=\mathbb{R}$, this trivially coincides with $B(b_{(N)},\mathbb{R})$. For $\mathbb{F}=\mathbb{H}$, this fails to form a group. Only $\mathbb{F}=\mathbb{C}$, it defines a new classical Lie group,

$$C(b_{(N)}) = \{A \in GL(N,\mathbb{C}) \,|\, A^t\, b_{(N)}\, A = b_{(N)}\}\,. \tag{4.5}$$

Again, up to a $GL(N,\mathbb{C})$ transformation, we have two possibilities, $b_{(N)}=I_{(N)}$ and $b_{(N)}=\Omega_{(N)}$, corresponding to

$$C(I_{(N)}) = O(N,\mathbb{C})\,, \qquad C(\Omega_{(N)}) = Sp(N,\mathbb{C})\,. \tag{4.6}$$

The latter case is defined only for even N. Note that these Lie groups are not simple as a real Lie group, but semisimple.

On the manifold $GL(N,\mathbb{F})$, the components of $X^a{}_b \in \mathbb{F}$ of an element $X\in GL(N,\mathbb{F})$ serve a natural and global coordinate system, and a left-invariant vector field is given by $V = V^a{}_b\, X^c{}_a\, \frac{\partial}{\partial X^c{}_b}$. The action takes the simple form,

$$S[X,P,A] = \frac{1}{2}\int \mathrm{Tr}_{N\times N}\left[P\,\mathrm{d}X + A\,(P\,X-\phi) + (\mathrm{conj})\right], \tag{4.7}$$

where all three fields X, P and A as well as ϕ take value in $Mat_{N\times N}(\mathbb{F})$, and the symbol (conj) stands for the conjugate in $\mathbb{F}$. Remark that adding the conjugate is trivial for $\mathbb{F}=\mathbb{R}$ as it duplicates the Lagrangian and only replaces the factor $\frac{1}{2}$ by 1 in the end. For $\mathbb{F}=\mathbb{C}$ and $\mathbb{F}=\mathbb{H}$ complementing the Lagrangian with the conjugate is necessary to recover the $2(2N)^2$ and $2(4N)^2$ dimensional symplectic potentials. Note that we can solve the constraint algebraically to get

$$S[X] = \frac{1}{2}\int \mathrm{Tr}_{N\times N}\left[\phi\, X^{-1}\mathrm{d}X + (\mathrm{conj})\right], \tag{4.8}$$

which is nothing but a matrix form of (3.33).

If the coadjoint element ϕ is a rank M matrix with $M\le N$, the above action can be reduced to[16]

$$S[X,P,A] = \frac{1}{2}\int \mathrm{Tr}_{M\times M}\left[P\,\mathrm{d}X + A\,(P\,X-\tilde{\phi}) + (\mathrm{conj})\right], \tag{4.9}$$

where the fields X, P and A takes value in $Mat_{N\times M}(\mathbb{F})$, $Mat_{M\times N}(\mathbb{F})$ and $Mat_{M\times M}(\mathbb{F})$, respectively. Here $\tilde{\phi}$ also belongs to $Mat_{M\times M}(\mathbb{F})$ and it is the $M\times M$ submatrix of a triangulation of ϕ. The resulting action describes a $G=GL(N,\mathbb{F})$ coadjoint orbit $\mathcal{O}^G_\phi$ as a reduced phase space inside $\mathbb{F}^{2MN}$ where the constraints are given by the moment maps $\tilde{\mu}(X,P)=P\,X\in Mat_{M\times M}(\mathbb{F})$ generating $\tilde{G}=GL(M,\mathbb{F})$ under Poisson bracket. Note

[16] To be more precise, this reduction requires integrating out non-dynamical variables corresponding to the components of X in the subspace $Mat_{N\times(N-M)}(\mathbb{F})$.

also that the moment map $\mu(X,P) = X\,P \in Mat_{N\times N}(\mathbb{F})$ associated with the original $GL(N,\mathbb{F})$ symmetry commutes with $\tilde{\mu}$. The constraints $\tilde{\chi}(X,P) = \tilde{\mu}(X,P) - \tilde{\phi} \approx 0$ are associated with the $\tilde{G} = GL(M,\mathbb{F})$ coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$.

Now, let us move to the classical Lie groups $B(b_{(N)},\mathbb{F})$. Adding the definition (4.1) to the action (4.8) as a constraint, we start with the action

$$
\begin{aligned}
S[X,A] &= \int \mathrm{Tr}_{N\times N}\left[\tfrac{1}{2}\left(\phi\,X^{-1}\mathrm{d}X + (\mathrm{conj})\right) + A\,(X^\dagger\,b_{(N)}\,X - b_{(N)})\right] \\
&\cong \int \mathrm{Tr}_{N\times N}\left[\tfrac{1}{2}\left(\phi\,b_{(N)}^{-1}\,X^\dagger\,b_{(N)}\,\mathrm{d}X + (\mathrm{conj})\right) + A\,(X^\dagger\,b_{(N)}\,X - b_{(N)})\right]\,, \qquad (4.10)
\end{aligned}
$$

where $\cong$ means the equivalence up to a redefinition of A. For any ϕ, there exists a $T \in Mat_{N\times M}(\mathbb{F})$ and $\tilde{b}_{(M)}$ with $M \leq N$ such that

$$
\phi = T\,\tilde{b}_{(M)}\,T^\dagger\,b_{(N)}\,. \qquad (4.11)
$$

Here, $\tilde{b}_{(M)} = \Omega_{(M)}$ for $b_{(N)} = \eta^{(p,N-p)}$, and $\tilde{b}_{(M)} = \eta^{(q,M-q)}$ for $b_{(N)} = \Omega_{(N)}$. With a suitable redefinition of X and A in terms of T, we can express the action as

$$
S[X,A] = \int \mathrm{Tr}_{M\times M}\left[\frac{1}{2}\left(\tilde{b}_{(M)}\,X^\dagger\,b_{(N)}\,\mathrm{d}X + (\mathrm{conj})\right) + A\left(X^\dagger\,b_{(N)}\,X - \tilde{b}_{(M)}^{-1}\,\tilde{\phi}\right)\right], \qquad (4.12)
$$

where X takes value in $Mat_{N\times M}(\mathbb{F})$ and $\tilde{\phi} \in Mat_{M\times M}$ is given by

$$
\tilde{\phi} = \tilde{b}_{(M)}\,T^\dagger\,b_{(N)}\,T\,. \qquad (4.13)
$$

Note that the moment maps $\tilde{\mu}(X) = X^\dagger\,b_{(N)}\,X \in Mat_{M\times M}(\mathbb{F})$ generates the dual symmetry $\tilde{G} = B(\tilde{b}_{(M)},\mathbb{F})$ whereas $\mu(X) = X\,\tilde{b}_{(M)}\,X^\dagger \in Mat_{N\times N}(\mathbb{F})$ generates the original symmetry $G = B(b_{(N)},\mathbb{F})$.

The classical Lie group $C(b_{(N)})$ can be treated in a very similar manner. We find

$$
S[X,A] = \int \frac{1}{2}\,\mathrm{Tr}_{M\times M}\left[\tilde{b}_{(M)}\,X^t\,b_{(N)}\,\mathrm{d}X + A\left(X^t\,b_{(N)}\,X - \tilde{b}_{(M)}^{-1}\,\tilde{\phi}\right) + (\mathrm{conj})\right], \qquad (4.14)
$$

where the dual coadjoint vector $\tilde{\phi}$ is related to the coadjoint vector ϕ through

$$
\phi = T\,\tilde{b}_{(M)}\,T^t\,b_{(N)}, \qquad \tilde{\phi} = \tilde{b}_{(M)}\,T^t\,b_{(N)}\,T\,, \qquad (4.15)
$$

with a $T \in Mat_{N\times M}(\mathbb{C})$. Here, $\tilde{b}_{(M)} = \Omega_{(M)}$ for $b_{(N)} = I_{(N)}$, and $\tilde{b}_{(M)} = I_{(M)}$ for $b_{(N)} = \Omega_{(N)}$. The moment maps $\tilde{\mu}(X) = X^t\,b_{(N)}\,X \in Mat_{M\times M}(\mathbb{C})$ generate the dual symmetry $\tilde{G} = C(\tilde{b}_{(M)})$ whereas $\mu(X) = X\,\tilde{b}_{(M)}\,X^t \in Mat_{N\times N}(\mathbb{C})$ generates the original symmetry $G = C(b_{(N)})$.

4.2 Semi-direct product group

The above construction can be extended to a class of non-reductive Lie groups G, which are given by semi-direct product $G = I \rtimes H$ between a reductive group H treated above, and an Abelian ideal I carries a H-representation π. Any element g of G can be denoted by $g = i\,h$ with $h \in H$ and $i \in I$. The semi-direct product rule can be deduced from $h\,i = (\pi(h)\,i)\,h$. The adjoint and coadjoint actions of an element $(a,h) \in G$ read

$$\begin{aligned} &\mathrm{Ad}_{(e^a,h)}(\chi,\xi) = \big(\pi(h)(\chi + \pi(\xi)\,a),\ \mathrm{Ad}_h\xi\big)\,, \\ &\mathrm{Ad}^*_{(e^a,h)}(\phi_I,\phi_H) = \big(\pi^*(h)\phi_I,\ \langle \pi^*(h)\phi_I, \pi(-)\,a\rangle_{\mathfrak{i}} + \mathrm{Ad}^*_h\phi_H\big)\,, \end{aligned} \tag{4.16}$$

where $\langle \phi_I, \pi(-)\,a\rangle_{\mathfrak{i}} \in \mathfrak{h}^*$ is defined such that for any $\xi \in \mathfrak{h}$,

$$\big\langle \langle \phi_I, \pi(-)\,a\rangle_{\mathfrak{i}}, \xi\big\rangle_{\mathfrak{h}} = \langle \phi_I, \pi(\xi)\,a\rangle_{\mathfrak{i}}\,. \tag{4.17}$$

From the coadjoint action $\mathrm{Ad}^*_{(i,\mathrm{id})}(\phi_I,\phi_H) = (\phi_I, \langle \phi_I, \pi(-)\,a\rangle_{\mathfrak{i}} + \phi_H)$ and the property $\big\langle \langle \phi_I, \pi(-)\,a\rangle_{\mathfrak{i}} + \phi_H, \xi\big\rangle_{\mathfrak{h}} = -\langle \pi^*(\xi)\,\phi_I, a\rangle_{\mathfrak{i}} + \langle \phi_H, \xi\rangle_{\mathfrak{h}}$, we can see that, if ξ does not belong to the little group algebra $\mathfrak{h}_{\phi_I}$ of ϕ_I, we can set $\langle \phi_H, \xi\rangle_{\mathfrak{h}}$ to zero with a suitable choice of a. This means that a coadjoint vector (ϕ_I, ϕ_H) can be always rotated in a way that ϕ_H belongs to $\mathfrak{h}^*_{\phi_I}$. It is also known that a coadjoint orbit of a semi-direct product group has the structure of a fiber bundle with the 'momentum orbit', i.e. the orbit of H on ϕ_I, as the base manifold and the direct product of the cotangent space of the momentum orbit times the coadjoint orbit of ϕ_H under the 'little group' H_{ϕ_I} as the fiber, see e.g. [138, 139] or [140, 141] and references therein.

The Maurer–Cartan element reads

$$g^{-1}\,\mathrm{d}_G\, g = (e^a\, h)^{-1}\, \mathrm{d}_G\, (e^a\, h) = \pi(h^{-1})\, \mathrm{d}_I\, a + h^{-1}\mathrm{d}_H\, h\,, \tag{4.18}$$

and the coadjoint orbit action is

$$S[g] = \int \langle \phi, g^{-1}\, \mathrm{d}_G\, g\rangle_{\mathfrak{g}} = \int \big[\langle \phi_I, \pi(h^{-1})\, \mathrm{d}_I\, a\rangle_{\mathfrak{i}} + \langle \phi_H, h^{-1}\mathrm{d}_H\, h\rangle_{\mathfrak{h}}\big]\,, \tag{4.19}$$

where the coadjoint vector ϕ is split into $\phi = \phi_H + \phi_I$ with $\phi_H \in \mathfrak{h}^*$ and $\phi_I \in \mathfrak{i}^*$. We can treat the second part of the action as in the subsection 4.1.

4.3 Orthogonal groups

In this section, we reconsider the coadjoint orbit actions of the indefinite orthogonal groups $O(p, N-p)$ with more details. From the definition, the action is given by

$$\begin{aligned} S[X,A] &= \int \mathrm{Tr}_{N\times N}\left[\phi\, X^{-1}\, \mathrm{d}X + A\,(X^t\,\eta\, X - \eta)\right] \\ &\cong \int \mathrm{Tr}_{N\times N}\left[\phi\, \eta^{-1}\, X^t\, \eta\, \mathrm{d}X + A\,(X^t\,\eta\, X - \eta)\right], \end{aligned} \tag{4.20}$$

where η is the flat metric of $(p, N-p)$ signature. The matrix $\phi\,\eta^{-1}$ is antisymmetric, and suppose that its rank is $2M \leq N$. Then we can always find a rectangular matrix $T^a{}_\beta \in Mat_{N\times M}(\mathbb{R})$ such that

$$\phi^{ab} = T^a{}_\alpha\, T^b{}_\beta\, \Omega^{\alpha\beta}\,, \tag{4.21}$$

where $\Omega^{\alpha\beta}$ is the symplectic matrix of rank $2M \leq N$ and $\phi^{ab} = (\phi\,\eta^{-1})^{ab} = \phi^a{}_c\, \eta^{cb}$

Here, the indices $a, b = 1, \ldots, N$ while $\alpha, \beta = 1, \ldots, 2M$. We can append to T a $N \times (N-2M)$ matrix R so that they jointly form a matrix $(T\, R) \in GL(N, \mathbb{R})$. Introducing the indices $\bar{\alpha}, \bar{\beta} = 2M+1, \ldots, N$, we can consider the redefinition, $A = (T\,R)\,A'\,(T\,R)^t$ or in components

$$A^{ab} = A'^{\alpha\beta}\,T^a{}_\alpha\,T^b{}_\beta + A'^{\bar{\alpha}\beta}\,R^a{}_{\bar{\alpha}}\,T^b{}_\beta + A'^{\alpha\bar{\beta}}\,T^a{}_\alpha\,R^b_{\bar{\beta}} + A'^{\bar{\alpha}\bar{\beta}}\,R^a{}_{\bar{\alpha}}\,R^b_{\bar{\alpha}}\,, \tag{4.22}$$

then also

$$X'^a{}_\alpha = X^a{}_b\,T^b{}_\alpha\,, \qquad \bar{X}'^a{}_{\bar{\alpha}} = X^a{}_b\,R^b{}_{\bar{\alpha}}\,. \tag{4.23}$$

By removing the prime from the variables, the action can be written as

$$\begin{aligned} S[X,A] = \int \Omega^{\alpha\beta}\,X^a{}_\beta\,\mathrm{d}X_{a\alpha} + A^{\alpha\beta}\left(X_{c\beta}\,X^c{}_\alpha - \tilde{\phi}_{\alpha\beta}\right) \\ + 2\,A^{\alpha\bar{\beta}}\left(\bar{X}_{c\bar{\beta}}\,X^c{}_\alpha - \varphi_{\alpha\bar{\beta}}\right) + A^{\bar{\alpha}\bar{\beta}}\left(\bar{X}_{c\bar{\beta}}\,\bar{X}^c{}_{\bar{\alpha}} - \varphi_{\bar{\alpha}\bar{\beta}}\right), \end{aligned} \tag{4.24}$$

where the Latin indices are lowered by η_{ab} and $\tilde{\phi}_{\alpha\beta}$, $\varphi_{\alpha\bar{\beta}}$, and $\varphi_{\bar{\alpha}\bar{\beta}}$ are given by

$$\tilde{\phi}_{\alpha\beta} = T^a{}_\alpha\,T_{a\beta}\,, \quad \varphi_{\alpha\bar{\beta}} = T^a{}_\alpha\,R_{a\bar{\beta}}\,, \quad \varphi_{\bar{\alpha}\bar{\beta}} = R^a{}_{\bar{\alpha}}\,R_{a\bar{\beta}}\,. \tag{4.25}$$

The constraints given by $A^{\alpha\bar{\beta}}$ and $A^{\bar{\alpha}\bar{\beta}}$ can be algebraically solved for the variables $\bar{X}^a{}_{\bar{\alpha}}$, and this results in a constant factor. Discarding this factor, the final form of the action is simply

$$S[X,A] = \int \Omega^{\alpha\beta}\,X^a{}_\beta\,\mathrm{d}X_{a\alpha} + A^{\alpha\beta}\left(X_{c\beta}\,X^c{}_\alpha - \tilde{\phi}_{\alpha\beta}\right). \tag{4.26}$$

The constraints are given by the momentum maps $\tilde{\mu}_{\beta\alpha} = X_{c\beta}\,X^c{}_\alpha$ closed under the Poisson bracket as

$$\{\tilde{\mu}_{\alpha\beta}, \tilde{\mu}_{\gamma\delta}\} = \Omega_{\beta\gamma}\,\tilde{\mu}_{\alpha\delta} + \Omega_{\alpha\gamma}\,\tilde{\mu}_{\beta\delta} + \Omega_{\alpha\delta}\,\tilde{\mu}_{\beta\gamma} + \Omega_{\beta\delta}\,\tilde{\mu}_{\alpha\gamma}\,, \tag{4.27}$$

and hence defines the dual Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{sp}(2M,\mathbb{R})$. Since the constraints $\tilde{\chi}_{\alpha\beta} = \tilde{\mu}_{\alpha\beta} - \tilde{\phi}_{\alpha\beta} \approx 0$ are given with a constant shift $\tilde{\phi}_{\alpha\beta}$, they are a mixture of the first and the second class constraints. According to the general results presented in Section 3.2, the first class constraints are the linear combinations $\tilde{\chi}^*(\xi) = \xi^{\alpha\beta}\,\tilde{\chi}_{\alpha\beta}$ satisfying

$$\{\tilde{\chi}^*(\xi), \tilde{\chi}_{\gamma\delta}\} \approx 2\,\xi^{\alpha\beta}(\Omega_{\beta\gamma}\,\tilde{\phi}_{\alpha\delta} + \Omega_{\beta\delta}\,\tilde{\phi}_{\alpha\gamma}) = 0 \qquad \forall\gamma,\,\delta\,. \tag{4.28}$$

This forms a subalgebra $\tilde{\mathfrak{g}}_{\tilde{\phi}} \subset \tilde{\mathfrak{g}} = \mathfrak{sp}(2M,\mathbb{R})$, whose structure is determined by $\tilde{\phi}_{\alpha\beta}$ hence by ϕ^{ab}. The matrix $\tilde{\phi}_{\alpha\beta} = \Omega_{\alpha\gamma}\,\tilde{\phi}^\gamma{}_\beta$ then $\tilde{\phi}^\alpha{}_\beta$ corresponds to the coadjoint vector of $\tilde{\mathfrak{g}}^* = \mathfrak{sp}(2M,\mathbb{R})^*$. The remaining constraints are the second class ones corresponding to the dual coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}} = \tilde{G}/\tilde{G}_{\tilde{\phi}}$.

This construction clearly exhibits the intimate relation between $\phi^a{}_b$ and $\tilde{\phi}^\alpha{}_\beta$: they are both given by two different contractions of $T^a{}_\beta$:

$$\phi^a{}_b = T^a{}_\alpha\,\Omega^{\alpha\beta}\,T^c{}_\beta\,\eta_{cb}\,, \qquad \tilde{\phi}^\alpha{}_\beta = \Omega^{\alpha\gamma}\,T^a{}_\gamma\,\eta_{ab}\,T^b{}_\beta\,. \tag{4.29}$$

Here, it is worth emphasising that the components $\tilde{\phi}^\alpha{}_\beta$ are determined by $\phi^a{}_b$, up to a $Sp(2M,\mathbb{R})$ transformation. The choice of a coadjoint element $\phi^a{}_b$ itself is also fixed up to a $O(p,N-p)$ transformation. In other words, the choice of the matrix $T^a{}_\beta$ is determined up to a $O(p,N-p)\times Sp(2M,\mathbb{R})$ transformation. In matrix form, the relations (4.29) read

$$\phi = T\,\Omega\,T^t\,\eta, \qquad \tilde{\phi} = \Omega\,T^t\,\eta\,T\,, \tag{4.30}$$

and they satisfy the same invariant equations,

$$\mathrm{Tr}(\phi^n) = I_n = \mathrm{Tr}(\tilde{\phi}^n)\,, \tag{4.31}$$

for any natural number n. Allowing ϕ and $\tilde{\phi}$ to vary, the above are polynomial functions in $\mathfrak{g}^*$ and $\tilde{\mathfrak{g}}^*$ respectively, which commute (with respect to the Poisson bracket) with any other functions, i.e. they are Casimir functions of $\mathfrak{g}^*$ and $\tilde{\mathfrak{g}}^*$ respectively. The previous identity therefore tells us that evaluating these Casimir functions on two dual coadjoint orbits of G and $\tilde{G}$ yields the same result. This is another 'classical' counterpart of a feature present in the dual pair correspondence: the values of the Casimir operators of two dual groups, on a pair of representations which are dual to one another, are related [142]. In this last case, however, the relation between the values of the two Casimir operators involves a rank-dependant shift stemming from a quantum mechanical ordering issue.

4.4 Inhomogeneous orthogonal group

The coadjoint orbits of inhomogeneous orthogonal group $IO(p, N-p)$ can be classified as follows. Let P_a and J_{ab} the generators of the Lie algebra and $\mathcal{P}^a$ and $\mathcal{J}^{ab}$ their duals. A coadjoint vector is given by $\phi = \phi_{I\,a}\,\mathcal{P}^a + \phi_{H\,ab}\,\mathcal{J}^{ab}$. If a coadjoint orbit has $\phi_{I\,a} = 0$, then it reduces to that of the subgroup $O(p, N-p)$, which, for $p = 1$, might be interpreted as the dS group of one lower dimensions. Therefore, we focus on the coadjoint orbits with non-vanishing ϕ_I. As we have seen below (4.17), ϕ_H can be chosen in the dual space of the little group algebra associated with ϕ_I: $O(p-1, q)$, $IO(p-1, q-1)$ and $O(p, q-1)$ for $\phi_I^2 > 0$, $\phi_I^2 = 0$ and $\phi_I^2 < 0$. The classification of ϕ_H simply follows that of the coadjoint orbits of the corresponding little group algebra.

Let us apply the general method outlined previously to $IO(p, N-p)$ for which H as the indefinite orthogonal group $O(p, N-p)$ and I as the translation group, $\mathbb{R}^N$ carrying a vector representation of H. The resulting action reads

$$\begin{aligned} S[x, \Sigma, A] &= \int \Big[\phi_{I\,a}\,(\Sigma^{-1})^a{}_b\,\mathrm{d}x^b + \phi_H{}^a{}_b\,(\Sigma^{-1})^b{}_c\,\mathrm{d}\Sigma^c{}_a\Big] \\ &\cong \int \Big[\phi_{I\,a}\,\Sigma_b{}^a\,\mathrm{d}x^b + \phi_H{}^{ab}\,\Sigma_{cb}\,\mathrm{d}\Sigma^c{}_a + A^{ab}(\Sigma_{cb}\,\Sigma^c{}_a - \eta_{ba})\Big]\,, \end{aligned} \tag{4.32}$$

where we denote elements of the homogeneous group H by Σ, and elements of the Abelian ideal I by x. We first decompose the Lagrange multiplier as $A^{ab} = \phi_I{}^a\,\phi_I{}^b\,B + \phi_I{}^{(a}\,B^{b)} + B^{ab}$, then skew-diagonalise and normalize ϕ_H as $\phi_H{}^{ab} = T^a{}_\alpha\,T^b{}_\beta\,\Omega^{\alpha\beta}$. Here again, $\alpha, \beta = 1, \dots, 2M$ where $2M$ is the rank of ϕ_H. Finally, by substituting (B, B^a, B^{ab}) with (A, A^a, A^{ab}) again, and discarding non-dynamical variables, we reduce the action as

$$\begin{aligned} S[x, p, \Sigma, A] = \int \Big[& p_a\,\mathrm{d}x^a + \Omega^{\alpha\beta}\,\Sigma_{c\beta}\,\mathrm{d}\Sigma^c{}_\alpha + A\,(p_a\,p^a - \tilde{\phi}_C) \\ & + A^\alpha\,(p_a\,\Sigma^a{}_\alpha - \tilde{\phi}_{I\,\alpha}) + A^{\alpha\beta}\,(\Sigma_{c\beta}\,\Sigma^c{}_\alpha - \tilde{\phi}_{H\,\beta\alpha})\Big]\,, \end{aligned} \tag{4.33}$$

where $p_a = \Sigma_a{}^b\,\phi_{I\,b}$ and

$$\tilde{\phi}_{H\,\alpha\beta} = T^a{}_\alpha\,T^b{}_\beta\,\eta_{ab}\,, \quad \tilde{\phi}_{I\,\alpha} = \phi_{I\,a}\,T^a{}_\alpha, \quad \tilde{\phi}_C = \phi_{I\,a}\,\phi_I{}^a\,. \tag{4.34}$$

Here, the dual algebra is associated with the moment maps $\mu_{\alpha\beta} = \Sigma_{c\alpha}\, \Sigma^c{}_\beta$, $\mu_\alpha = p_a\, \Sigma^a{}_\alpha$ and $\mu = p_a\, p^a$ satisfying (4.27) and

$$\{\mu_{\alpha\beta}, \mu_\gamma\} = 2\, \mu_{(\alpha}\, \Omega_{\beta)\gamma}\,, \qquad \{\mu_\alpha, \mu_\beta\} = \Omega_{\alpha\beta}\, \mu\,, \tag{4.35}$$

where μ is the center. This Lie algebra is isomorphic to $\mathfrak{heis}_{2M} \ni \mathfrak{sp}(2M, \mathbb{R})$ with dimension $(M+1)(2M+1)$, the semi-direct sum of the Heisenberg and the symplectic algebra. To recapitulate,

$$\phi_H = T\, \Omega\, T^t\, \eta\,, \qquad \tilde{\phi}_H = \Omega\, T^t\, \eta\, T\,, \qquad \phi_I\, T = \tilde{\phi}_I\,, \tag{4.36}$$

and we find that the following three quantities

$$\mathrm{Tr}({\phi_H}^n) = I_n = \mathrm{Tr}({\tilde{\phi}_H}{}^n)\,, \qquad (\phi_I|\phi_I) = J_2 = \tilde{\phi}_C\,, \tag{4.37}$$

and

$$(\phi_I|{\phi_H}^n|\phi_I) = J_{n+2} = \langle\tilde{\phi}_I|{\tilde{\phi}_H}{}^{n-1}|\tilde{\phi}_I\rangle \qquad [n \geq 1]\,, \tag{4.38}$$

where

$$(v|A|w) = v_a\, A^a{}_b\, \eta^{bc}\, w_c\,, \qquad \langle v|A|w\rangle = v_\alpha\, A^\alpha{}_\beta\, \Omega^{\beta\gamma}\, w_\gamma\,, \tag{4.39}$$

relating the same traces of powers of $\phi = (\phi_H, \phi_I)$ and its dual $\tilde{\phi} = (\tilde{\phi}_H, \tilde{\phi}_I)$. However, I_n and J_n are invariant only under the homogeneous part of the group, G_H or $\tilde{G}_H$, except for J_2. The higher order Casimir invariant functions $C_{2(n+1)}$ for the full Poincaré algebra can be constructed using the Pauli–Lubanski tensors $W_{(n+1)}$ given by,

$$W_{(n+1)}{}^{a_1\cdots a_{d-2n-1}} = \frac{1}{2^n\, n!}\, \varepsilon^{a_1\ldots a_d}\, \phi_{H\, a_{d-2n}a_{d-2n+1}} \cdots \phi_{H\, a_{d-2}a_{d-1}}\, \phi_{I\, a_d}\,, \tag{4.40}$$

as (see e.g. [143])

$$C_{2(n+1)} = \frac{1}{(d-2n-1)!}\, W_{(n+1)}{}^{a_1\cdots a_{d-2n-1}}\, W_{(n+1)\, a_1\cdots a_{d-2n-1}}\,. \tag{4.41}$$

These invariant functions can be expressed in terms of I_n and J_n , and they are also invariant under the dual group. For our purpose, it is sufficient to consider the first two,

$$C_2 = -J_2\,, \qquad C_4 = -J_4 + \frac{1}{2}\, I_2\, J_2\,. \tag{4.42}$$

5 Vectorial description of particles in Minkowski space

In Section 4.4, we have already presented the derivation of covariant actions from coadjoint orbits of inhomogeneous Lorentz groups. Let us resume our analysis with the action (4.33),

$$\begin{aligned} S[x,p,\Sigma,A] = \int \Big[\, & p_a\, \mathrm{d}\, x^a + \Omega^{\alpha\beta}\, \Sigma_{c\beta}\, \mathrm{d}\, \Sigma^c{}_\alpha + A\, (p_a\, p^a - \tilde{\phi}_C) \\ & + A^\alpha\, (p_a\, \Sigma^a{}_\alpha - \tilde{\phi}_{I\,\alpha}) + A^{\alpha\beta}\, (\Sigma_{c\beta}\, \Sigma^c{}_\alpha - \tilde{\phi}_{H\,\beta\alpha})\Big]\,, \end{aligned} \tag{5.1}$$

where the vector indices a, b run from 0 to $d-1$, and the spin-variable indices α, β run from 1 to $2M$. Recall that $2M$ is the rank of ϕ_H. In case of the usual spinning particle,

M corresponds essentially to the number of rows of the Young diagram of the (mixed-symmetry) tensor field associated with the spinning particle under consideration. The Hamiltonian constraints correspond to a coadjoint orbit of the dual algebra

$$\tilde{\mathfrak{g}} = \mathfrak{heis}_{2M} \ni \mathfrak{sp}(2M, \mathbb{R})\,, \tag{5.2}$$

generated by

$$T = p^2\,, \qquad M_\alpha = p_a\,\Sigma^a{}_\alpha\,, \qquad S_{\alpha\beta} = \Sigma_{a\alpha}\,\Sigma^a{}_\beta\,. \tag{5.3}$$

The dual coadjoint vector $\tilde{\phi} \in \tilde{\mathfrak{g}}^*$ is given by

$$\tilde{\phi} = \tilde{\phi}_C\,\mathcal{T} + \tilde{\phi}_{I\,\alpha}\,\mathcal{M}^\alpha + \tilde{\phi}_{H\,\alpha\beta}\,\mathcal{S}^{\alpha\beta}\,. \tag{5.4}$$

We can integrate out p_a from the Hamiltonian type action (5.1) to get a Polyakov-type Lagrangian action,

$$L = -\frac{1}{2\,e}(\dot{x}^a + \lambda^\alpha\,\Sigma^a{}_\alpha)^2 - \lambda^\alpha\,\phi_{I\,\alpha} - \frac{e}{2}\,{\phi_I}^2 + \Omega^{\alpha\beta}\,\Sigma_{a\alpha}\,\dot{\Sigma}^a{}_\beta + \lambda^{\alpha\beta}\,(\Sigma_{a\beta}\,\Sigma^a{}_\alpha - \tilde{\phi}_{H\beta\alpha})\,, \tag{5.5}$$

with $A = \frac{e}{2}\,\mathrm{d}t$, $A^\alpha = \lambda^\alpha\,\mathrm{d}t$ and $A^{\alpha\beta} = \lambda^{\alpha\beta}\,\mathrm{d}t$. If the matrix $\Sigma_{a\alpha}\,\Sigma^a{}_\beta \simeq \tilde{\phi}_{H\alpha\beta}$ is invertible with the inverse $\Delta^{\alpha\beta}$, we can also remove λ^α in terms of its equation of motion to get

$$\begin{aligned} L = &-\frac{1}{2\,e}\left(\dot{x}^{a2} - \dot{x}^a\,\Sigma_{a\alpha}\,\Delta^{\alpha\beta}\,\Sigma_{b\beta}\,\dot{x}^b\right) - \frac{e}{2}\left(\tilde{\phi}_C - \tilde{\phi}_{I\alpha}\,\Delta^{\alpha\beta}\,\tilde{\phi}_{I\beta}\right) \\ &+ \Sigma_{a\beta}\left(\Omega^{\alpha\beta}\,\dot{\Sigma}^a{}_\alpha + \dot{x}^a\,\Delta^{\alpha\beta}\,\tilde{\phi}_{I\alpha}\right) + \lambda^{\alpha\beta}\,(\Sigma_{a\beta}\,\Sigma^a{}_\alpha - \tilde{\phi}_{H\beta\alpha})\,. \end{aligned} \tag{5.6}$$

In the case where λ^α cannot be solved, it is associated with a first class constraint. Finally, when $\tilde{\phi}_C - \tilde{\phi}_{I\alpha}\,\Delta^{\alpha\beta}\,\tilde{\phi}_{I\beta} \neq 0$, we can solve e out to find the Nambu-type action,

$$\begin{aligned} L = &-\sqrt{\left(\tilde{\phi}_C - \tilde{\phi}_{I\alpha}\,\Delta^{\alpha\beta}\,\tilde{\phi}_{I\beta}\right)(\dot{x}^{a2} - \dot{x}^a\,\Sigma_{a\alpha}\,\Delta^{\alpha\beta}\,\Sigma_{b\beta}\,\dot{x}^b)} \\ &+ \Sigma_{a\beta}\left(\Omega^{\alpha\beta}\,\dot{\Sigma}^a{}_\alpha + \dot{x}^a\,\Delta^{\alpha\beta}\,\tilde{\phi}_{I\alpha}\right) + \lambda^{\alpha\beta}\,(\Sigma_{a\beta}\,\Sigma^a{}_\alpha - \tilde{\phi}_{H\beta\alpha})\,. \end{aligned} \tag{5.7}$$

Even though the above action contains all the parameters $\tilde{\phi}_C, \tilde{\phi}_{I\alpha}$ and $\tilde{\phi}_{H\alpha\beta}$, the dependence on $\tilde{\phi}_{I\alpha}$ is in fact irrelevant because $\tilde{\phi}_{I\alpha}$ are non-trivial only when $\Sigma_{a\alpha}\,\Sigma^a{}_\beta$ is not invertible. The above type of the action has been derived in [50] for massive spinning particles. One may even convert the spin variables into Lagrangian [48, 49] ending up with a double square root type action.

Below, we present more details of the scalar particles ($M = 0$) and the spinning particles ($M = 1$), along with the classification of coadjoint orbits of the Poincaré group. See [144, 145] for explicit characterisations of the Poincaré group orbits. See also [146] for a discussion of the coadjoint orbits of the Carroll group.

5.1 Scalar particles

In the scalar particle case, we have $\phi_{H\,ab} = 0$, that is $\tilde{\phi}_{H\,\alpha\beta} = 0$, and the action gets simplified to the familiar form,

$$S[x, p, A] = \int p_a\,\mathrm{d}x^a + A\,(p_a\,p^a - \tilde{\phi}_C)\,. \tag{5.8}$$

The dual algebra in this case is merely $\tilde{\mathfrak{g}} = \mathbb{R}$, generated by $T = p^2$. The dual coadjoint orbit is given by $\tilde{\phi} = \tilde{\phi}_C \, \mathcal{T}$ and the stabiliser is the dual algebra itself $\tilde{\mathfrak{g}}_{\tilde{\phi}} = \tilde{\mathfrak{g}} = \mathbb{R}$. Depending on the signature of the vector $\tilde{\phi}_{I\,a}$, we have massive, massless and tachyonic particles with mass squared given by ${\phi_I}^2$. The corresponding stabilisers are

$$\mathfrak{iso}(1,d-1)_{m\,\mathcal{P}^0} = \mathbb{R}_{P_0} \oplus \mathfrak{so}(d-1)_{[1,\dots,d-1]}\,, \tag{5.9a}$$

$$\mathfrak{iso}(1,d-1)_{E\,\mathcal{P}^+} = \mathbb{R}_{P_-} \oplus \mathfrak{iso}(d-2)_{[-;1,\dots,d-2]}\,, \tag{5.9b}$$

$$\mathfrak{iso}(1,d-1)_{\mu\,\mathcal{P}^{d-1}} = \mathbb{R}_{P_{d-1}} \oplus \mathfrak{so}(1,d-2)_{[0,1,\dots,d-2]}\,, \tag{5.9c}$$

where the subscripts refer to a basis for the stabilisers of the representative considered here, see Appendix A for details on this notation. The dimensions of these coadjoint orbits are all $2(d-1)$ implying that they describe d-dimensional particles. Indeed, the Hilbert space corresponding to these particles will consist of wave functions on a $(d-1)$-dimensional Cauchy surface, which is a Lagrangian submanifold of the phase space. The dual coadjoint orbits are all zero-dimensional as each of them is a single point. Remark that the massless coadjoint orbit is nilpotent and it is dual to the trivial orbit. In the dimension counting (3.19), we have dim $\mathcal{M} = 2d$ (here $\mathcal{M} = T^*\mathbb{R}^d$), dim $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}} = 0$ and dim $\tilde{\mathfrak{g}}_{\tilde{\phi}} = 1$, and hence dim $\mathcal{N}_{\tilde{\phi}} =$ dim $\mathcal{O}^G_\phi = 2(d-1)$.

5.2 Spinning particles

In the spinning particle case with $M = 1$, we relabel once again $({\Sigma^a}_\chi, {\Sigma^a}_\pi)$, the two non-trivial vectors in ${\Sigma^a}_\alpha$, as (χ^a, π^a) (note here we use χ, π to denote both the indices and the vectors). The resulting action reads

$$\begin{aligned} S[x,p,\chi,\pi,A] = \int \Big[& p \cdot \mathrm{d}x + \pi \cdot \mathrm{d}\chi + A\,(p^2 - \tilde{\phi}_C) + A^\chi\,(p \cdot \chi - \tilde{\phi}_{I\chi}) + A^\pi\,(p \cdot \pi - \tilde{\phi}_{I\pi}) \\ & + A^{\chi\chi}\,(\chi^2 - \tilde{\phi}_{H\chi\chi}) + A^{\chi\pi}\,(\chi \cdot \pi - \tilde{\phi}_{H\chi\pi}) + A^{\pi\pi}\,(\pi^2 - \tilde{\phi}_{H\pi\pi}) \Big]\,, \end{aligned} \tag{5.10}$$

where we used the shorthand notation $v \cdot w = v^a\,w_a$ for contraction of Lorentz indices. Below, we shall provide the classification of the coadjoint orbits $\mathcal{O}^G_\phi$ of Poincaré algebra with $M = 1$ and the corresponding coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$ of the dual algebra $\tilde{\mathfrak{g}} = \mathfrak{heis}_2 \ni \mathfrak{sp}(2,\mathbb{R})$. Note that we will always assume that the labels of the representative vectors are generic: they are non-vanishing and different unless stated otherwise. From the vector,

$$\tilde{\phi} = \tilde{\phi}_C\;\mathcal{T} + \tilde{\phi}_{I\,\chi}\,\mathcal{M}^\chi + \tilde{\phi}_{I\,\pi}\,\mathcal{M}^\pi + \tilde{\phi}_{H\,\chi\chi}\,\mathcal{S}^{\chi\chi} + 2\,\tilde{\phi}_{H\,\chi\pi}\,\mathcal{S}^{\chi\pi} + \tilde{\phi}_{H\,\pi\pi}\,\mathcal{S}^{\pi\pi}\,, \tag{5.11}$$

one can read off the parameters which appear in the particle action. The dimensions of $\mathcal{O}^G_\phi$ is $\dim \mathfrak{iso}(1,d-1) - \dim\,\mathfrak{g}_\phi = \frac{d(d+1)}{2} - \dim\,\mathfrak{g}_\phi$, whereas in the dimension counting (3.19) we find

$$\begin{aligned} \dim \mathcal{O}^G_\phi = \dim \mathcal{N}_{\tilde{\phi}} &= \dim \mathcal{M} - \dim \mathcal{O}^{\tilde{G}}_{\tilde{\phi}} - 2\dim \tilde{\mathfrak{g}}_{\tilde{\phi}} \\ &= \dim \mathcal{M} - \dim \tilde{\mathfrak{g}} - \dim \tilde{\mathfrak{g}}_{\tilde{\phi}} = 2(2d-3) - \dim \tilde{\mathfrak{g}}_{\tilde{\phi}}\,. \end{aligned} \tag{5.12}$$

As we shall see below, dim $\tilde{\mathfrak{g}}_{\tilde{\phi}}$ is either 2 or 4 in the $M = 1$ case. Therefore, the corresponding phase spaces have dimensions either $2(2d-4)$ or $2(2d-5)$. Comparing these with the phase space dimensions of scalar particles, $2(d-1)$, they are greater by $2(d-3)$ or $2(d-4)$. As a mechanical system, one may interpret these dimensions directly as the number of degrees of freedom but here lies a subtlety. As we have discussed with the example of the compact coadjoint orbit S^2 of $SU(2)$, a compact phase space does not contribute to the continuous degrees of freedom but only discrete labels. In the usual spinning particle case, the additional dimensions can be understood as 'spin-orbits', which often correspond to the compact part of the fiber of the coadjoint orbit viewed as a fiber bundle over the momentum orbit [138, 139], contributing again only some discrete labels.

In the following, we present the classification of coadjoint orbits with $M = 1$.

- A massive spinning particle corresponds to the coadjoint orbit $\mathcal{O}^G_\phi$ with representative and stabiliser given by

$$\phi = m\,\mathcal{P}^0 + s\,\mathcal{J}^{12}\,, \qquad \mathfrak{g}_\phi = \mathbb{R}_{P_0} \oplus \mathfrak{u}(1)_{J_{12}} \oplus \mathfrak{so}(d-3)_{[3,\ldots,d-1]}\,. \tag{5.13}$$

  The coadjoint orbit $\mathcal{O}^G_\phi \simeq \frac{IO(1,d-1)}{\mathbb{R}\times O(2)\times O(d-3)}$ can be viewed as a fiber bundle with the massive scalar coadjoint orbit $\frac{IO(1,d-1)}{\mathbb{R}\times O(d-1)}$ (which is the cotangent bundle of the momentum orbit) as the base space and the spin-orbit $\frac{O(d-1)}{O(2)\times O(d-3)}$ as the fiber. The latter, the real Grassmannian $\mathrm{Gr}_{\mathbb{R}}(2, d-1)$, is a compact manifold, and hence contributes only to discrete degrees of freedom, when quantised. With the choice of the indices $\chi = 1, \pi = 2$, the dual coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$ is characterised by

$$\tilde{\phi} = -m^2\,\mathcal{T} + \mathcal{S}^{\chi\chi} + s^2\,\mathcal{S}^{\pi\pi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_T \oplus \mathfrak{u}(1)_{S_{\pi\pi}+s^2\,S_{\chi\chi}}\,. \tag{5.14}$$

  The quadratic and quartic Casimir functions of this orbit are given by

$$C_2 = m^2\,, \qquad C_4 = m^2\,s^2\,, \tag{5.15}$$

  and, up to a shift (that should originate from an ordering issue when quantising) reproduces the value of the Casimir operators of the Poincaré group on the irrep corresponding to a massive spinning particle. See [50] for the derivation of a related worldline action for a massive spinning particle in flat spacetime.

- A massless spinning particle corresponds to the coadjoint orbit $\mathcal{O}^G_\phi$ with representative and stabiliser

$$\phi = E\,\mathcal{P}^+ + s\,\mathcal{J}^{12}\,, \qquad \mathfrak{g}_\phi = \big(\mathfrak{heis}_2 \niplus \mathfrak{u}(1)_{J_{12}}\big) \oplus \mathfrak{iso}(d-4)_{[-;3,\ldots,d-2]}\,, \tag{5.16}$$

  and where the Heisenberg algebra is generated by

$$-E\,J_{-2} + s\,P_1, \qquad E\,J_{-1} + s\,P_2 \qquad \text{and} \qquad P_-\,. \tag{5.17}$$

  With the indices $\chi = 1, \pi = 2$, we find the dual coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$, characterised by

$$\tilde{\phi} = \mathcal{S}^{\chi\chi} + s^2\,\mathcal{S}^{\pi\pi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{heis}_2 \niplus \mathfrak{u}(1)\,. \tag{5.18}$$

with stabiliser generated by T, M_π, M_χ, and $S_{\pi\pi} + s^2\, S_{\chi\chi}$. Note that E is not a proper label for the orbit $\mathcal{O}^G_\phi$ because rescaling of E does not change the orbit. We can verify this in the dual coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$: the coadjoint vector $\tilde{\phi}$ does not depend on E. The Casimir functions of this orbit vanish: $C_2 = 0 = C_4$.

- A continuous spinning particle corresponds to the coadjoint orbit $\mathcal{O}^G_\phi$ with representative and stabiliser

$$\phi = E\, \mathcal{P}^+ + \epsilon\, \mathcal{J}^{-1}\,, \qquad \mathfrak{g}_\phi = \mathbb{R}_{P_-} \oplus \mathbb{R}_{2\,\epsilon\, P_+ + E J_{-1}} \oplus \mathfrak{so}(d-3)_{[2,\dots,d-2]}\,. \tag{5.19}$$

With the indices $\chi = 1, \pi = -$, we find the dual coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$ is characterised by

$$\tilde{\phi} = -E\,\epsilon\, \mathcal{M}^\pi + \mathcal{S}^{\chi\chi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_T \oplus \mathbb{R}_{2\,M_\pi - E\,\epsilon\, S_{\chi\chi}}\,. \tag{5.20}$$

Here again E and ϵ are not proper labels for $\mathcal{O}^G_\phi$, but only the combination $\varepsilon^2 = E\,\epsilon$ is (note that the sign of $\mathcal{M}^\pi$ term can be changed by a coadjoint action, and that the square symbol should not be understood literally, i.e. ε^2 may be either positive or negative). Hence, the particle action involves only one parameter, ε. The Casimir functions of this orbit take the values

$$C_2 = 0\,, \qquad C_4 = 4\,\varepsilon^4. \tag{5.21}$$

Remark that the massive and massless spinning particles share the same spin part $s\,\mathcal{J}^{12}$ and $\mathcal{S}^{\chi\chi} + s^2 \mathcal{S}^{\pi\pi}$ in the coadjoint vectors ϕ and $\tilde{\phi}$. We may refer to this as space-like spin. In the continuous spin case, the spin part of the coadjoint orbits are $\epsilon\,\mathcal{J}^{-1}$ and $\mathcal{S}^{\chi\chi}$ are null-vectors, so we may refer this as light-like spin. Note that a particle action for continuous spin fields was discussed in [41, 147, 148], which involves 4 first class constraints corresponding to Wigner's equations [149], whereas our system involves 2 first and 4 second class constraints, which can be viewed as a partially gauge fixed version of the former. See e.g. [150–154] for related works.

- There are three sub-categories for a tachyonic spinning particle. The first case is the space-like spin coadjoint orbit $\mathcal{O}^G_\phi$ with representative and stabiliser given by

$$\phi = \mu\, \mathcal{P}^{d-1} + s\, \mathcal{J}^{12}\,, \qquad \mathfrak{g}_\phi = \mathbb{R}_{P_{d-1}} \oplus \mathfrak{u}(1)_{J_{12}} \oplus \mathfrak{so}(1, d-4)_{[0,3,\dots,d-2]}\,. \tag{5.22}$$

With the indices $\chi = 1, \pi = 2$, we find the dual coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$ is characterised by

$$\tilde{\phi} = \mu^2\, \mathcal{T} + \mathcal{S}^{\chi\chi} + s^2\, \mathcal{S}^{\pi\pi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_T \oplus \mathfrak{u}(1)_{S_{\pi\pi} + s^2\, S_{\chi\chi}}\,. \tag{5.23}$$

Here, one can also note that the spin part shares the same structure as the massive and massless spinning case. The Casimir functions of this orbits are

$$C_2 = -\mu^2\,, \qquad C_4 = -\mu^2 s^2. \tag{5.24}$$

and are related to those of a massive spinning orbit (5.13) by setting $m = i\mu$, in accordance with our interpretation as a tachyonic spinning orbit.

- The second case is the time-like spin coadjoint orbit $\mathcal{O}^G_\phi$, with representative and stabiliser

$$\phi = \mu\,\mathcal{P}^{d-1} + \nu\,\mathcal{J}^{01}\,, \qquad \mathfrak{g}_\phi = \mathbb{R}_{P_{d-1}} \oplus \mathbb{R}_{J_{02}} \oplus \mathfrak{so}(d-3)_{[2,\ldots,d-2]}\,. \tag{5.25}$$

With the indices $\chi = 0, \pi = 1$, we find the dual coadjoint orbit $\mathcal{O}^{\tilde{G}}_{\tilde{\phi}}$ is characterised by

$$\tilde{\phi} = \mu^2\,\mathcal{T} + \mathcal{S}^{\chi\chi} - \nu^2 \mathcal{S}^{\pi\pi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_T \oplus \mathbb{R}_{S_{\pi\pi} - \nu^2\,S_{\chi\chi}}\,. \tag{5.26}$$

The Casimir functions of this orbit are given by

$$C_2 = -\mu^2\,, \qquad C_4 = \mu^2\nu^2\,, \tag{5.27}$$

and one can notice that they are related to those of the tachyonic spinning orbit with space-like (5.24) by setting $s = i\,\nu$.

- The last case is the light-like spin coadjoint orbit $\mathcal{O}^G_\phi$ with representative and stabiliser

$$\phi = \mu\,\mathcal{P}^{d-1} + \epsilon\,\mathcal{J}^{-2}\,, \qquad \mathfrak{g}_\phi = \mathbb{R}_{P_{d-1}} \oplus \mathbb{R}_{J_{+2}} \oplus \mathfrak{iso}(d-4)_{[+;3,\ldots,d-2]}\,. \tag{5.28}$$

With the indices $\chi = 2, \pi = -$, we find that the dual coadjoint orbit $\tilde{\mathcal{O}}^{\tilde{G}}_{\tilde{\phi}}$ is characterised by

$$\tilde{\phi} = \mu^2\,\mathcal{T} + \mathcal{S}^{\chi\chi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_T \oplus \mathbb{R}_{S_{\pi\pi}}\,. \tag{5.29}$$

The Casimir functions of this orbit read

$$C_2 = -\mu^2\,, \qquad C_4 = 0\,. \tag{5.30}$$

Remark that except for the continuous spin particles, all other particles are described by the action with two parameters $\mathcal{C}_M$ and $\mathcal{C}_S$ in the end:

$$\begin{aligned} S[x,p,\chi,\pi,A] = \int \Big[& p_a\,\mathrm{d}\,x^a + \pi_a\,\mathrm{d}\,\chi^a + A\,(p^2 + \mathcal{C}_M) + A^\chi\,p\cdot\chi + A^\pi\,p\cdot\pi \\ & + A^{\chi\chi}\,(\chi^2 - 1) + A^{\chi\pi}\,\chi\cdot\pi + A^{\pi\pi}\,(\pi^2 - \mathcal{C}_S)\Big]\,. \end{aligned} \tag{5.31}$$

Massive, massless and tachyonic particles of spin s are described by $\mathcal{C}_S = s^2$ and positive, zero and negative values of $\mathcal{C}_M$, respectively. The tachyonic particles of time-like and light-like spins are described by a negative and zero $\mathcal{C}_S$, respectively, and a negative $\mathcal{C}_M$.

5.3 Spinning particles with mixed symmetry

The coadjoint orbits with higher M correspond typically to spinning particles with mixed symmetry, characterised by a M-row Young diagram. In the following, we shall provide the representative vectors ϕ of the coadjoint orbits with higher M and their stabilisers $\mathfrak{g}_\phi$. The stabilisers of the dual algebra $\tilde{\mathfrak{g}}_{\tilde{\phi}}$ are always isomorphic to the d-independent part of $\mathfrak{g}_\phi$.

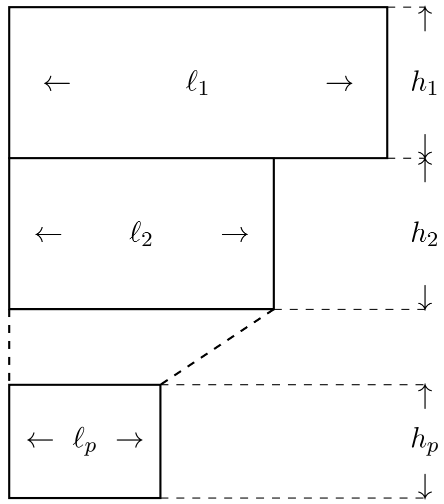

Figure 5. Young diagram presented in block form.

The coadjoint orbit of a mixed symmetry spinning particle is described by a representative vector where the space-like spin $s\,\mathcal{J}^{12}$ is replaced by

$$s_1\,\mathcal{J}^{12} + s_2\,\mathcal{J}^{34} + \cdots + s_M\,\mathcal{J}^{2M-1\,2M}\,, \qquad M \leq [\tfrac{d}{2}]\,, \tag{5.32}$$

where $[x]$ denotes the integer part of x, and where we can also assume that

$$s_1 \geq s_2 \geq \cdots \geq s_M\,, \tag{5.33}$$

without loss of generality. If $s_k \in \mathbb{N}$, then this defines a Young diagram. In order to take into account the possibility that several consecutive rows of the diagram have the same length, i.e.

$$\ell_1 := s_1 = \cdots = s_{h_1}\,, \qquad \ell_2 := s_{h_1+1} = \cdots = s_{h_1+h_2}\,, \tag{5.34}$$

and so on, it is convenient to describe the diagram in terms of blocks of width ℓ_k and height h_k, as illustrated in Figure 5 below. The stabilisers of such spinning particles are given by

$$\mathfrak{g}_\phi = \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \begin{cases} \mathfrak{so}(d-1-2M) & \text{[massive]}, \\ \mathfrak{so}(1,d-2-2M) & \text{[tachyonic]}, \end{cases} \tag{5.35}$$

and in the massless case,

$$\mathfrak{g}_\phi = \mathfrak{heis}_{2M} \ni \Big(\mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p)\Big) \oplus \mathfrak{iso}(d-2-2M)\,, \tag{5.36}$$

where $h_1 + \cdots + h_p = M$.

In the case of the coadjoint orbits with light-like or time-like spin, simply the $M-1$ space-like spins are added on top of the former. We therefore find the stabilisers

$$\mathfrak{g}_\phi = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \begin{cases} \mathfrak{so}(d-1-2M) & \text{[continuous spin} \\ & \quad \text{and time-like tachyonic]}, \\ \mathfrak{iso}(d-2-2M) & \text{[light-like tachyonic]}, \end{cases} \tag{5.37}$$

where $h_1 + \cdots + h_p = M - 1$.

5.4 Null particles

The last class of particles with Poincaré symmetry are what we refer to as 'null' particles, corresponding to the coadjoint orbit with $\phi_I = 0$. Clearly, the condition $\phi_I = 0$ trivializes the ideal – translational – part of the Poincaré algebra, and hence such orbits are simply identical to Lorentz coadjoint orbits. Upon quantisation, these coadjoint orbits would correspond to the unfaithful representations of Poincaré with $p_\mu = 0$ and the little group $SO(1, d-1)$, hence again reduces to unitary irreducible representations of Lorentz group. Since the Lorentz group can be viewed as dS group of one lower dimensions, the classification of null coadjoint orbits are the same as the classification of dS coadjoint orbits. The only differences are in the interpretation. Even though the null coadjoint orbits seem somewhat dull in their defining nature, they may capture some important peculiarities of Poincaré symmetry because analogous dull orbits are not present for (A)dS symmetry. In fact, the null particles can be interpreted as the 'soft limit' of massless particles. We shall come back to this point in Section 7.2. Let us conclude this section by remarking that, in a sense, these null particles can be viewed as a kind of flat space analogue of the AdS singleton. See [155–157] for more serious proposals concerning this issue.

6 Vectorial description of particles in (A)dS space

For the covariant description of various particles in dS and AdS spaces, we begin with the covariant action (4.26) of the orthogonal groups $O(p, N-p)$: $O(1,d)$ for dS and $O(2, d-1)$ for AdS. The indices A, B take values $0, 1, \ldots, d-1$ and $\bullet$ ($\bullet = d$ for dS and $\bullet = 0'$ for AdS). The metric $\eta_{\mathrm{AB}}$is $\mathrm{diag}(-1, +1, \ldots, +1, \sigma)$ where $\sigma = +1$ for dS and -1 for AdS. Relabeling the variables as

$$X^{\mathrm{A}}{}_{\bullet} = X^{\mathrm{A}}\,, \qquad X^{\mathrm{A}}{}_{\beta}\, \phi_P{}^{\beta} = \tfrac{1}{2}\, P^{\mathrm{A}}\,, \qquad X^{\mathrm{A}}{}_{\alpha} = \Sigma^{\mathrm{A}}{}_{\alpha}\,, \qquad [\alpha \geq 3]\,, \tag{6.1}$$

the action can be expressed in a more familiar form,

$$\begin{aligned} S[X, P, \Sigma, A] = \int \Big[& P \cdot \mathrm{d}X + \Omega^{\alpha\beta}\, \Sigma_\beta \cdot \mathrm{d}\Sigma_\alpha \\ & + A^{XX}(X^2 - \tilde{\phi}_{XX}) + A^{PP}(P^2 - \tilde{\phi}_{PP}) + A^{XP}(X \cdot P - \tilde{\phi}_{XP}) \\ & + A^{X\alpha}(X \cdot \Sigma_\alpha - \tilde{\phi}_{X\alpha}) + A^{P\alpha}(P \cdot \Sigma_\alpha - \tilde{\phi}_{P\alpha}) + A^{\alpha\beta}\big(\Sigma_\beta \cdot \Sigma_\alpha - \tilde{\phi}_{\alpha\beta}\big) \Big]\,, \end{aligned} \tag{6.2}$$

where X^{A} and P_{A} will play the role of the ambient space position and momentum. Here again we used the notation $v \cdot w = v^{\mathrm{A}}\, w_{\mathrm{A}}$ for contraction of ambient indices. The Hamiltonian constraints are associated with the dual algebra,[17]

$$\tilde{\mathfrak{g}} = \mathfrak{sp}(2(M+1), \mathbb{R})\,, \tag{6.3}$$

generated by

$$T := P^2\,, \quad U := X^2\,, \quad V := X \cdot P\,, \tag{6.4a}$$

[17] Note that in this section, we use the convention that the rank of the dual group is $M+1$, different from the convention used in Section 4.3 where the rank was M.

$$M_\alpha := P \cdot \Sigma_\alpha\,, \quad N_\alpha := X \cdot \Sigma_\alpha\,, \quad S_{\alpha\beta} := \Sigma_\alpha \cdot \Sigma_\beta\,. \tag{6.4b}$$

Note that the dimensions of $\tilde{\mathfrak{g}}$ is $(M+1)(2M+3)$, and differs from the dimension of the dual algebra of Poincaré $\mathfrak{heis}_{2M} \ni \mathfrak{sp}(2M,\mathbb{R})$, which is $(M+1)(2M+1)$, by $2(M+1)$.[18] This corresponds to the number of additional constraints necessary to bring the ambient space to the intrinsic (A)dS. In the following, the parameters in the action can be read off from the dual coadjoint vector $\tilde{\phi} \in \tilde{\mathfrak{g}}^*$,

$$\tilde{\phi} = \tilde{\phi}_{PP}\,\mathcal{T} + \tilde{\phi}_{XX}\,\mathcal{U} + \tilde{\phi}_{XP}\,\mathcal{V} + \tilde{\phi}_{P\alpha}\,\mathcal{M}^\alpha + \tilde{\phi}_{X\alpha}\,\mathcal{N}^\alpha + \tilde{\phi}_{\alpha\beta}\,\mathcal{S}^{\alpha\beta}\,. \tag{6.5}$$

Similarly to the Minkowski case, we can integrate out $P_{\rm A}$ from the Hamiltonian type action (6.3) to get a Polyakov-type Lagrangian action,

$$\begin{aligned} L =& -\frac{1}{2\,e}\,(\mathcal{D}_t\,X^{\rm A} + \lambda^\alpha\,\Sigma^{\rm A}{}_\alpha)^2 - \lambda^\alpha\,\tilde{\phi}_{P\alpha} - \frac{e}{2}\,\tilde{\phi}_{PP} + \Omega^{\alpha\beta}\,\Sigma_\alpha \cdot \dot{\Sigma}_\beta \\ &+ \lambda^{\alpha\beta}\,(\Sigma_\alpha\cdot\Sigma_\beta - \tilde{\phi}_{H\beta\alpha}) + \rho\,(X^2 - \tilde{\phi}_{XX}) - \tau\,\tilde{\phi}_{XP} + \tau^\alpha\,(X\cdot\Sigma_\alpha - \tilde{\phi}_{X\alpha})\,, \end{aligned} \tag{6.6}$$

where $\mathcal{D}_t\,X^{\rm A} = \dot{X}^{\rm A} + \tau\,X^{\rm A}$ and the components of the gauge fields are $A^{XX} = \rho\,{\rm d}t$, $A^{PP} = \frac{e}{2}\,{\rm d}t$, $A^{XP} = \tau\,{\rm d}t$, $A^{X\alpha} = \tau^\alpha\,{\rm d}t$, $A^{P\alpha} = \lambda^\alpha\,{\rm d}t$ and $A^{\alpha\beta} = \lambda^{\alpha\beta}\,{\rm d}t$. As we shall see, we can always choose a representative vector with $\tilde{\phi}_{XP} = 0$. In such a case, the equation for τ simply reduces to $\tau\,X^2 + X\cdot\dot{X} + \lambda^\alpha\,\Sigma_\alpha\cdot X = 0$. If $\Sigma_\alpha\cdot X \simeq \tilde{\phi}_{X\alpha} = 0$ and $X^2 \simeq \tilde{\phi}_{XX} \neq 0$, we can remove τ to get

$$\mathcal{D}_t X^{\rm A} = \bar{\mathcal{D}}_t X^{\rm A} = \dot{X}^{\rm A} - \frac{X\cdot\dot{X}}{X^2}\,X^{\rm A}\,. \tag{6.7}$$

Note that this expression is the pullback to the worldline of the ambient lift of an (A)dS covariant derivative. If the matrix $\Sigma_\alpha\cdot\Sigma_\beta \simeq \tilde{\phi}_{\alpha\beta}$ is invertible with the inverse $\Delta^{\alpha\beta}$, we can remove λ^α to get

$$\begin{aligned} L =& -\frac{1}{2\,e}\left(\bar{\mathcal{D}}_t X^2 - \bar{\mathcal{D}}_t X\cdot\Sigma_\alpha\,\Delta^{\alpha\beta}\,\Sigma_\beta\cdot\bar{\mathcal{D}}_t X\right) - \frac{e}{2}\left(\tilde{\phi}_{PP} - \tilde{\phi}_{P\alpha}\,\Delta^{\alpha\beta}\,\tilde{\phi}_{P\beta}\right) \\ &+ \Sigma_\beta\cdot\left(\Omega^{\alpha\beta}\,\dot{\Sigma}_\alpha + \bar{\mathcal{D}}_t X\,\Delta^{\alpha\beta}\,\tilde{\phi}_{P\beta}\right) + \lambda^{\alpha\beta}\,(\Sigma_\alpha\cdot\Sigma_\beta - \tilde{\phi}_{\beta\alpha}) \\ &+ \rho\,(X^2 - \tilde{\phi}_{XX}) + \tau^\alpha\,X\cdot\Sigma_\alpha\,, \end{aligned} \tag{6.8}$$

and for $\tilde{\phi}_{PP} - \tilde{\phi}_{P\alpha}\,\Delta^{\alpha\beta}\,\tilde{\phi}_{P\beta} \neq 0$, we can further remove e to get the Nambu-type action,

$$\begin{aligned} L =& -\sqrt{\left(\tilde{\phi}_{PP} - \tilde{\phi}_{P\alpha}\,\Delta^{\alpha\beta}\,\tilde{\phi}_{P\beta}\right)\left(\bar{\mathcal{D}}_t X^2 - \bar{\mathcal{D}}_t X\cdot\Sigma_\alpha\,\Delta^{\alpha\beta}\,\Sigma_\beta\cdot\bar{\mathcal{D}}_t X\right)} \\ &+ \Sigma_\beta\cdot\left(\Omega^{\alpha\beta}\,\dot{\Sigma}_\alpha + \bar{\mathcal{D}}_t X\,\Delta^{\alpha\beta}\,\phi_{I\beta}\right) + \lambda^{\alpha\beta}\,(\Sigma_\alpha\cdot\Sigma_\beta - \tilde{\phi}_{\beta\alpha}) \\ &+ \rho\,(X^2 - \tilde{\phi}_{XX}) + \tau^\alpha\,X\cdot\Sigma_\alpha\,. \end{aligned} \tag{6.9}$$

[18] Let us remark that the dual of the Poincaré algebra appears as a subalgebra of the Inönü–Wigner contraction of $\mathfrak{sp}\big(2(M+1),\mathbb{R}\big)$ that preserves an $\mathfrak{sp}(2M,\mathbb{R})$ subalgebra. More precisely, this contraction yields a semi-direct sum $\mathfrak{sp}(2M,\mathbb{R}) \Subset \mathfrak{n}_{2M}$ where $\mathfrak{n}_{2M}$ is a nilpotent Lie algebra, made out of two copies of $\mathfrak{heis}_{2M}$ and a central term, and these two Heisenberg algebras only commute with one another up to this central term.

For the usual spinning particle type, a similar type of action has been derived in [26, 49].

In the following, we present the coadjoint vectors of (A)dS algebra using a basis which singles out the Lorentz subalgebra and with remaining, *transvection*, generators defined as

$$P_a = \tfrac{1}{\ell}\, J_{\bullet a}\,, \qquad [P_a, P_b] = \tfrac{\sigma}{\ell^2}\, J_{ab}\,, \qquad [a, b = 0, 1, \dots, d-1]\,, \tag{6.10}$$

where ℓ is the (A)dS radius and its dual $\mathcal{P}^a = \ell\, \mathcal{J}^{\bullet a}$, in order to make the analogy with the Minkowski case manifest. From now on, we set $\ell = 1$ for simplicity. As we shall see below, many cases can be viewed as the (A)dS counterparts of the Poincaré orbits, but there are also several cases which *do not* have a Poincaré analogue. For the purpose of comparison between them, it will be convenient to parameterise the Casimir functions in terms of I_2 and I_4 (defined in eq. (4.31) previously) as

$$C_2 = \sigma\, \tfrac{1}{2}\, I_2\,, \qquad C_4 = \tfrac{1}{8}\,(I_2)^2 - \tfrac{1}{4}\, I_4\,. \tag{6.11}$$

This will also be useful to compare with the results in literature. For explicit characterisation of some (A)dS coadjoint orbits, see e.g. [130, 131, 158].

6.1 Scalar particles

In the scalar particle case, we can always set $\tilde{\phi}_{XX} = \sigma$ and $\tilde{\phi}_{XP} = 0$, so that the action depends only on $\tilde{\phi}_{PP}$:

$$S[X, P, A] = \int \Big[P \cdot \mathrm{d}X + A^{XX}(X^2 - \sigma) + A^{PP}(P^2 - \tilde{\phi}_{PP}) + A^{XP}\, X \cdot P\Big]\,, \tag{6.12}$$

where the Hamiltonian constraints are associated with the dual algebra $\mathfrak{sp}(2, \mathbb{R})$. From the constraint $X^2 = \sigma$, we can naturally interpret X^{A} as the ambient space coordinate for (A)dS spacetime. The condition $X \cdot P = 0$ can be understood as a fixed homogeneity, and finally $P^2 = \tilde{\phi}_{PP}$ is the mass-shell constraint. See e.g. [159] for related analysis and discussions. The coadjoint orbits of the dual $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 1)$ are given by the two-dimensional surfaces $H^2(\sigma\, \tilde{\phi}_{PP})$. For more details, let us introduce a σ-dependent notation for the one-dimensional Lie group $I(\sigma)$,

$$I(+1) = U(1)\,, \qquad I(-1) = \mathbb{R}\,, \tag{6.13}$$

and the associated Lie algebra $\mathfrak{i}(\sigma)$: $\mathfrak{i}(+1) = \mathfrak{u}(1)$ and $\mathfrak{i}(-1) = \mathbb{R}$. In the following, we match each one of the three types of (A)dS scalar orbits — massive, massless and tachyonic — with one of the three types of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 1)$ orbits $H^2(a)$ defined in (2.46).

- The (A)dS orbit of the massive particle is given by

$$\phi = m\, \mathcal{P}^0\,, \qquad \mathfrak{g}_\phi = \mathfrak{i}(-\sigma)_{P_0} \oplus \mathfrak{so}(d-1)_{[1,\dots,d-1]}\,, \tag{6.14}$$

  while the dual $\mathfrak{sp}(2, \mathbb{R})$ orbit is given by

$$\tilde{\phi} = -m^2\, \mathcal{T} + \sigma\, \mathcal{U}\,, \qquad \tilde{G}_{\tilde{\phi}} = I(-\sigma)\,, \tag{6.15}$$

  has the geometry of hyperboloid $H^2(-\sigma\, m^2)$.

- The massless particle orbit has representative and stabiliser,

$$\phi = E\,\mathcal{P}^{+}\,, \qquad \mathfrak{g}_\phi = \mathbb{R}_{P_-} \oplus \mathfrak{iso}(d-2)_{[-;1,\dots,d-2]}\,. \tag{6.16}$$

  The dual orbit is characterised by

$$\tilde{\phi} = \sigma\,\mathcal{U}\,, \qquad \tilde{G}_{\tilde{\phi}} = \mathbb{R} \tag{6.17}$$

  and corresponds to the cone C^2.

- Lastly, the tachyonic particle orbit is given by

$$\phi = \mu\,\mathcal{P}^{1}\,, \qquad \mathfrak{g}_\phi = \mathfrak{i}(\sigma)_{P_1} \oplus \mathfrak{so}(1,d-2)_{[0,2,\dots,d-1]}\,. \tag{6.18}$$

  The dual orbit has

$$\tilde{\phi} = \mu^2\,\mathcal{T} + \sigma\,\mathcal{U}\,, \qquad \tilde{G}_{\tilde{\phi}} = I(\sigma)\,, \tag{6.19}$$

  and corresponds to the hyperboloid $H^2(\sigma\,\mu^2)$.

Remark that the map of massive and tachyonic coadjoint orbits of spacetime symmetry to the one-sheet and two-sheet hyperbolic coadjoint orbits of $\mathfrak{sp}(2,\mathbb{R})$ works oppositely for AdS and dS. Note that $C_2 = \tilde{\phi}_{XP}^2 - \tilde{\phi}_{XX}\,\tilde{\phi}_{PP}$ is a constant on these orbits. Massive scalar orbits in dS and tachyonic scalar orbits in AdS have positive C_2, and they can be given by different representative vectors with $\tilde{\phi}_{PP} = 0$: $(\tilde{\phi}_{PP}, \tilde{\phi}_{XX}, \tilde{\phi}_{XP}) = (0,1,m)$ or $(0,-1,\mu)$. Massive scalar orbits in AdS and tachyonic scalar orbits in dS have negative C_2, and they do not contain a vector with $\tilde{\phi}_{PP} = 0$. However, if we insist on it naively, they could be given by the complex vectors $(\tilde{\phi}_{PP}, \tilde{\phi}_{XX}, \tilde{\phi}_{XP}) = (0,-1,i\,m)$ or $(0,1,i\,\mu)$. This seemingly ill-defined choice of coadjoint vector makes sense after quantisation: since $P_{\rm A} = -i\,\hbar\,\partial/\partial X^{\rm A}$, it defines a homogeneity condition with a real degree of homogeneity and hence corresponds to a more standard way to describe a AdS field using ambient space. The two choices are related by a complexified global transformation — a $Sp(2,\mathbb{C})$ rotation.

6.2 Spinning particles

Let us move to the spinning case with $M = 1$. Relabelling the non-trivial elements $(\Sigma^{\rm A}{}_\chi, \Sigma^{\rm A}{}_\pi) = (\chi^{\rm A}, \pi^{\rm A})$, we find

$$\begin{aligned} S[X,P,\chi,\pi,A] = \int \Big[& P\cdot {\rm d}X + \pi\cdot{\rm d}\chi + A^{XX}(X^2-\sigma) + A^{PP}(P^2 - \tilde{\phi}_{PP}) + A^{XP}\,X\cdot P \\ & + A^{X\pi}(X\cdot\pi - \tilde{\phi}_{X\pi}) + A^{X\chi}(X\cdot\chi - \tilde{\phi}_{X\chi}) \\ & + A^{P\pi}(P\cdot\pi - \tilde{\phi}_{P\pi}) + A^{P\chi}(P\cdot\chi - \tilde{\phi}_{P\chi}) \\ & + A^{\pi\pi}\big(\pi^2 - \tilde{\phi}_{\pi\pi}\big) + A^{\chi\pi}\big(\chi\cdot\pi - \tilde{\phi}_{\chi\pi}\big) + A^{\chi\chi}\big(\chi^2 - \tilde{\phi}_{\chi\chi}\big)\Big]\,, \end{aligned} \tag{6.20}$$

where the Hamiltonian constraints are associated with the dual algebra $\mathfrak{sp}(4,\mathbb{R})$. Note that here $\chi^{\rm A}$ and $\pi_{\rm A}$ are $(d+1)$-dimensional vectors. Comparing the dimension counting (5.12), we find the same result as in the Minkowski case:

$$\dim\,\mathcal{O}_\phi = \dim\,\mathcal{M} - \dim\,\tilde{\mathfrak{g}} - \dim\,\tilde{\mathfrak{g}}_\phi = 2(2d-3) - \dim\,\tilde{\mathfrak{g}}_{\tilde{\phi}}\,, \tag{6.21}$$

where the increase of dimension in $\mathcal{M}$ is compensated by that of $\tilde{\mathfrak{g}}$. Compared to the Minkowski particles and also to the (A)dS scalar particles, the association of (A)dS coadjoint orbits with the spinning particles in (A)dS is more subtle. Therefore, we first provide the classification using the simple terminologies that distinguish the causal properties of the momenta and spins.

- The coadjoint orbit with time-like momenta and space-like spins has representative and stabiliser

$$\phi = m\,\mathcal{P}^0 + s\,\mathcal{J}^{12}\,, \qquad \mathfrak{g}_\phi = \mathfrak{i}(-\sigma)_{P_0} \oplus \mathfrak{u}(1)_{J_{12}} \oplus \mathfrak{so}(d-3)_{[3,\dots,d-1]}\,. \tag{6.22}$$

Here, we take the non-trivial indices as $\chi = 1, \pi = 2$, and find

$$\tilde{\phi} = -\,m^2\,\mathcal{T} + \sigma\,\mathcal{U} + \mathcal{S}^{\chi\chi} + s^2\,\mathcal{S}^{\pi\pi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{i}(-\sigma) \oplus \mathfrak{u}(1)\,, \tag{6.23}$$

where the stabiliser is generated by $T - \sigma\,m^2\,U$ and $s^2\,S_{\chi\chi} + S_{\pi\pi}$. The Casimir functions are given by

$$C_2 = m^2 - \sigma\,s^2\,, \qquad C_4 = m^2 s^2\,, \tag{6.24}$$

which, again up to a dimension-dependent shift, agree with those of the $\mathfrak{so}(2, d-1)$-irreps corresponding to massive spinning particles. See [48, 49] for the derivation of a related worldline action of massive spinning particle in AdS. Note also that the above set of constraints is the same as the ones identified and used in the treatment of massive and (partially-)massless mixed-symmetry fields in AdS_d using BRST techniques and the ambient space approach [160, 161].

For dS, the mass parameter is a positive real $m > 0$, and all these orbits correspond to massive spinning particles.

For AdS, the quantisation condition requires $m \in \mathbb{N}$, and the value $m = s$ is singular because in that case we have different stabilisers,

$$\mathfrak{g}_\phi = \mathfrak{u}(1,1) \oplus \mathfrak{so}(d-3)_{[3,\dots,d-1]}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{u}(1,1)\,, \tag{6.25}$$

where the $\mathfrak{u}(1,1)$ subalgebras are generated by

$$P_1 - J_{02}\,, \qquad P_2 + J_{01}\,, \qquad P_0 + J_{12}\,, \qquad P_0 - J_{12}\,, \tag{6.26}$$

and

$$T + s^2\,U\,, \qquad S_{\chi\chi} + s^2\,S_{\pi\pi}\,, \qquad M_\chi + N_\pi\,, \qquad M_\pi - s^2\,N_\chi\,, \tag{6.27}$$

respectively. Therefore, in AdS, we can interpret the case with $m > s$ as massive spinning particles, and $m = s$ as massless spinning particle.

One may expect that the coadjoint orbits with $m = s-1, s-2, \dots, 1$ correspond to the partially massless representations in AdS, with conformal weights $\Delta = s+d-4, s+d-5, \dots, d-2$. Up to the quantum shift, $\Delta = m+d-3$, the labels of these representations seem to match those of the coadjoint orbits with $m = s-1, s-2, \dots, 1$. However, these representations are not unitary — they are unitary only in dS — and one may conclude that this class of coadjoint orbits do lead to non-unitary representations upon quantisations. We believe that this is not the case for the following reasons.

The coadjoint orbits with $m < s$ rather lead to an unfamiliar class of *unitary* representations which are not of the lowest energy type: see our forthcoming paper [92] for explicit construction of such representations. Luckily, the $d = 3$ case can give us good lessons about this issue: using the isomorphism $\mathfrak{so}(2,2) \simeq \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$, we can decompose the $\mathfrak{so}(2,2)$ orbit as a product of two $\mathfrak{so}(2,1)$ orbits (see Appendix E for a dictionary). The massive spinning particle orbits with $m > s$ correspond to the products of two elliptic hyperboloids $H^2_\pm$ of radius $j_{\rm L} = \frac{m+s}{2} > 0$ and $j_{\rm R} = \frac{m-s}{2} > 0$ (see (2.46) for the definition of $H^2_\pm$). Since the $H^2_\pm(j^2)$ orbit corresponds the the lowest/highest weight representation $\mathcal{D}^\pm_j$ with the lowest/highest weight $\pm j$, the massive spinning orbits correspond to the representations $(\mathcal{D}^\pm_{j_{\rm L}} \otimes \mathcal{D}^\pm_{j_{\rm R}}) \oplus (\mathcal{D}^\pm_{j_{\rm R}} \otimes \mathcal{D}^\pm_{j_{\rm L}})$. At the level of representations, we find the decomposition,

$$\mathcal{D}^\pm_{-\frac{t-1}{2}} = \mathcal{D}_{\frac{t-1}{2}} \oplus \mathcal{D}^\pm_{\frac{t+1}{2}}\,, \tag{6.28}$$

when $2j$ becomes a non-positive integer $-(t-1)$ with $t \geq 1$. Here, $\mathcal{D}_{\frac{t-1}{2}}$ is the t-dimensional representation. And, for $m = s - t$, the quotient representations

$$(\mathcal{D}^\pm_{s-\frac{t-1}{2}} \otimes \mathcal{D}_{\frac{t-1}{2}}) \oplus (\mathcal{D}_{\frac{t-1}{2}} \otimes \mathcal{D}^\pm_{s-\frac{t-1}{2}})\,, \tag{6.29}$$

correspond to partially massless representations of depth t, which are non-unitary for $t > 1$. Here, the non-unitarity is due to the finite-dimensional representation $\mathcal{D}_{\frac{t-1}{2}}$ of $\mathfrak{so}(1,2)$ algebra. Therefore, one might confirm once again that the orbits with $m = 0, \ldots, s-2$ lead to non-unitary representations. However, this is not correct because the non-unitary representation $\mathcal{D}_{\frac{t-1}{2}}$ should arise from S^2, while the orbits with $m = s - t + 1$ is given by the product space,

$$\big[H^2_\pm\big((s - \tfrac{t-1}{2})^2\big) \times H^2_\mp\big((\tfrac{t-1}{2})^2\big)\big] \cup \big[H^2_\mp\big((\tfrac{t-1}{2})^2\big) \times H^2_\pm\big((s - \tfrac{t-1}{2})^2\big)\big]\,. \tag{6.30}$$

This coadjoint orbit would correspond to the *unitary* representation,

$$(\mathcal{D}^\pm_{s-\frac{t-1}{2}} \otimes \mathcal{D}^\mp_{\frac{t-1}{2}}) \oplus (\mathcal{D}^\mp_{\frac{t-1}{2}} \otimes \mathcal{D}^\pm_{s-\frac{t-1}{2}})\,, \tag{6.31}$$

whose particle interpretation is unclear for the moment. In Section 6.4, we propose an interpretation for this type of orbits.

We may understand this issue from a different angle: the $O(2,2)$ group has two discrete symmetries, the time reversal sending $m \to -m$ and the parity sending $s \to -s$. In terms of $O(1,2) \times O(1,2)$ it corresponds to $(j_{\rm L}, j_{\rm R}) \to (-j_{\rm R}, -j_{\rm L})$ and $(j_{\rm L}, j_{\rm R}) \to (j_{\rm R}, j_{\rm L})$. We can consider yet another automorphism sending $(j_{\rm L}, j_{\rm R}) \to (j_{\rm L}, -j_{\rm R})$ or equivalently $(m, s) \to (s, m)$, which is not an element of $O(2,2)$. Note that this "inversion" — up to a dimension related shift which would arise upon quantisation — has been used within the context of conformal field theory [162].

Therefore, the coadjoint orbits with $m < s$ in any dimensions would also correspond to unitary representations, which are somehow mixed with the usual massive spinning particle through the inversion.

- The coadjoint orbits with light-like momenta and space-like spins are given by

$$\phi = E\,\mathcal{P}^{+} + s\,\mathcal{J}^{12}\,, \qquad \mathfrak{g}_\phi = \mathbb{R}_{P_-} \oplus \mathfrak{u}(1)_{J_{12}} \oplus \mathfrak{iso}(d-4)_{[-;3,\dots,d-2]}\,, \tag{6.32}$$

while, with the indices $\chi = 1, \pi = 2$, we find that the dual orbit is characterised by

$$\tilde{\phi} = \sigma\,\mathcal{U} + \mathcal{S}^{\chi\chi} + s^2\,\mathcal{S}^{\pi\pi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_T \oplus \mathfrak{u}(1)_{S_{\pi\pi}+s^2 S_{\chi\chi}}\,. \tag{6.33}$$

The Casimir functions of this orbit read

$$C_2 = -\sigma\, s^2\,, \qquad C_4 = 0\,. \tag{6.34}$$

In comparison with the Minkowski case, these (A)dS orbits seem to be related to the massless spinning particles, but we have already seen that for AdS, the massless spinning particle is associated with $\phi = s\,\mathcal{P}^0 + s\,\mathcal{J}^{12}$. In fact, we see that the stabiliser $\tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R} \oplus \mathfrak{u}(1)$ is smaller than that of Minkowski, $\mathfrak{heis}_2 \ni \mathfrak{u}(1)$: the former has dimension 2 and the latter has 4. Therefore, these orbits are too big for a massless spinning particle, and they just correspond to the end point of the spectrum of the massive and tachyonic spinning particles in dS and AdS, respectively.

- The coadjoint orbits with light-like momenta and spins are given by

$$\phi = E\,\mathcal{P}^{+} + \epsilon\,\mathcal{J}^{-1}\,. \tag{6.35}$$

Here again, E and ϵ are not separately good parameters but the combination $\varepsilon^2 = E\,\epsilon$ is (we can always set $E\,\epsilon > 0$ by a suitable rotation). By analogy with the Minkowski case, the corresponding action can be interpreted as the action for continuous spin particles in (A)dS.

In dS, the coadjoint vector (6.39) actually belongs to a massive spinning orbits with $\varepsilon \in \sqrt{2}\,\mathbb{N}$. Rescaling ϕ with J_{+-} we can set

$$\phi = \varepsilon\,\left(\mathcal{J}^{\bullet +} + \mathcal{J}^{-1}\right) = \tfrac{\varepsilon}{2}\left(\mathcal{J}^{\bullet 0} - \mathcal{J}^{10} + \mathcal{J}^{\bullet d-1} + \mathcal{J}^{1\,d-1}\right). \tag{6.36}$$

Note here that only when the components $\bullet$ and 1 have the same signature, that is only in dS, we can perform a $\pi/2$-rotation in the $\bullet$–1 plane to get

$$\begin{aligned}\phi &= \frac{\varepsilon}{\sqrt{2}}\left(\mathcal{J}^{\bullet 0} + \mathcal{J}^{1\,d-1}\right)\\ &\simeq \frac{\varepsilon}{\sqrt{2}}\left(\mathcal{P}^{0} + \mathcal{J}^{12}\right),\end{aligned} \tag{6.37}$$

where we interchanged the coordinate $d-1$ with the coordinate 2 by a rotation to get a canonical form. In AdS, this cannot be done, so the coadjoint orbit given by (6.39) is a genuinely new one.

Lagrangians for continuous particles have been constructed by Metsaev in [163] where only the case of AdS is shown to be unitary. Our orbit classification is consistent with this result. Below, we shall see that the continuous spin particle in AdS belongs to a larger class of particle species with two labels, which are also consistent with the result of Metsaev. We shall come back to this point shortly below.

Now, focusing on the AdS case with $\sigma = -1$, we find that the stabiliser of ϕ is

$$\mathfrak{g}_\phi = \mathfrak{u}(1)_{P_0+J_{1\,d-1}} \oplus \mathbb{R}_{P_{d-1}+J_{01}} \oplus \mathfrak{so}(d-3)_{[2,\dots,d-2]}\,, \tag{6.38}$$

with the indices $\chi = 1, \pi = -$, we find that the dual coadjoint orbit is characterised by

$$\tilde{\phi} = -\mathcal{U} + \mathcal{S}^{\chi\chi} - \varepsilon^2\,\mathcal{M}^\pi\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{u}(1) \oplus \mathbb{R}\,, \tag{6.39}$$

with stabiliser generated by $T - S_{\pi\pi} + 2\,\varepsilon^2\,N_\chi$ and $2\,M_\chi + \varepsilon^2\,(U - S_{\chi\chi})$. Note that the sign of $\mathcal{M}^\pi$ term is not important as it can be changed by a conjugation, and the stabiliser contains a $\mathfrak{u}(1)$ subalgebra leading to the quantisation of $\varepsilon \in \mathbb{N}$. The Casimir functions on this orbit take the values

$$C_2 = 0\,, \qquad C_4 = 4\,\varepsilon^4\,, \tag{6.40}$$

which is identical (up to a multiplicative factor) to that of the continuous spin orbit identified in the Poincaré case — in accordance with our interpretation as the orbit corresponding to continuous spin particle as defined by Metsaev. The dual coadjoint vector $\tilde{\phi}$ provides the worldline action for the continuous spin particle in AdS, which is a simple ambient space generalisation of the Minkowski one.

- Coadjoint orbits with space-like momenta have three subcases. First, the coadjoint orbit with space-like spin is given by

$$\phi = \mu\,\mathcal{P}^{d-1} + s\,\mathcal{J}^{12}\,, \qquad \mathfrak{g}_\phi = \mathfrak{i}(\sigma)_{P_{d-1}} \oplus \mathfrak{u}(1)_{J_{12}} \oplus \mathfrak{so}(1,d-4)_{[0,3,\dots,d-2]}\,. \tag{6.41}$$

Here we take the non-trivial indices as $\chi = 1$ and $\pi = 2$, to find for the dual orbit

$$\tilde{\phi} = \mu^2\,\mathcal{T} + \sigma\,\mathcal{U} + \mathcal{S}^{\chi\chi} + s^2\,\mathcal{S}^{\pi\pi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{i}(\sigma)_{T+\sigma\,\mu^2\,U} \oplus \mathfrak{u}(1)_{s^2\,S_{\chi\chi}-S_{\pi\pi}}\,. \tag{6.42}$$

The Casimir functions of this orbit are given by

$$C_2 = -\mu^2 - \sigma\,s^2\,, \qquad C_4 = \mu^2 s^2\,, \tag{6.43}$$

and one can notice that they agree with those of a massive spinning orbit upon setting $m = i\mu$. Since $\mathfrak{i}(+1) = \mathfrak{u}(1)$ for dS, the value of μ should be quantised: $\mu \in \mathbb{N}$. When $\mu = s$, the situation becomes singular and the stabilisers of the pair of dual orbits are respectively enhanced to

$$\mathfrak{g}_\phi = \mathfrak{u}(2) \oplus \mathfrak{so}(1,d-4)_{[0,3,\dots,d-2]}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{u}(2)\,, \tag{6.44}$$

where the $\mathfrak{u}(2)$ subalgebras are generated by

$$P_2 + J_{1d-1}\,, \qquad P_1 - J_{2d-1}\,, \qquad P_{d-1} + J_{12}\,, \qquad P_{d-1} - J_{12}\,, \tag{6.45}$$

and

$$T + s^2\, U\,, \qquad s^2\, S_{\chi\chi} + S_{\pi\pi}\,, \qquad M_\chi - N_\pi\,, \qquad M_\pi + s^2\, N_\chi\,, \tag{6.46}$$

respectively. This special case actually corresponds to the massless spinning particles in dS. The other lower values of $\mu = 1, 2, \ldots, s-1$ correspond to the partially massless spinning particles. The remaining values $\mu = s+1, s+2, \ldots$ might correspond to the spinning tachyons, but there is a subtlety here: Since $\mathcal{P}^{d-1} = \mathcal{J}^{d\,d-1}$ and $\mathcal{J}^{12}$ can be interchanged by a finite rotation, there is no genuine difference between the parameters μ and s. For this reason, we can simply assume that the smaller one among two is μ and the greater one is s: the equal case $\mu = s$ corresponds to the massless case. In this interpretation, there is no coadjoint action for tachyonic spinning particle in dS. This would mean in turn that there is no unitary irrep of spinning tachyons in dS. In AdS, μ is a real parameter and all of them correspond to spinning tachyons.

- Second, the coadjoint orbit with space-like momenta and time-like spins is given by

$$\phi = \mu\, \mathcal{P}^{d-1} + \nu\, \mathcal{J}^{01}\,, \qquad \mathfrak{g}_\phi = \mathfrak{i}(\sigma)_{P_{d-1}} \oplus \mathbb{R}_{J_{01}} \oplus \mathfrak{so}(d-3)_{[2,\ldots,d-2]}\,. \tag{6.47}$$

Taking the non-trivial indices as $\chi = 0$ and $\pi = 1$, we find for the dual orbit

$$\tilde{\phi} = \mu^2\, \mathcal{T} - \mathcal{U} + \mathcal{S}^{\chi\chi} - \nu^2\, \mathcal{S}^{\pi\pi}\,. \tag{6.48}$$

Again, the above case corresponds to a new case only in AdS, because in dS it is the same as the massive spinning case with $m = \nu$ and $s = \mu$. The stabiliser is given by

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_{T-\mu^2\, U} \oplus \mathbb{R}_{\nu^2\, S_{\chi\chi} - S_{\pi\pi}}\,. \tag{6.49}$$

This is the AdS analogue of the tachyonic particle with time-like spin (5.25) in Minkowski, in accordance with the fact that the Casimir functions of this orbit are given by

$$C_2 = -\mu^2 - \nu^2\,, \qquad C_4 = \mu^2\, \nu^2\,. \tag{6.50}$$

and hence obtained from the massive spinning orbit (6.22) by setting $m = i\mu$ and $s = i\nu$. Since $\mathcal{P}^{d-1}$ and $\mathcal{J}^{01}$ are in the same conjugacy class, we can assume $\mu \geq \nu$. When $\mu = \nu$, we find yet another enhancement of the stabilisers,

$$\mathfrak{g}_\phi = \mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{so}(d-3)_{[2,\ldots,d-2]}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{gl}(2, \mathbb{R})\,, \tag{6.51}$$

where the $\mathfrak{gl}(2, \mathbb{R})$ subalgebras are generated by

$$P_0 + J_{1d-1}\,, \qquad P_1 + J_{0d-1}\,, \qquad P_{d-1} - J_{01}\,, \qquad P_{d-1} + J_{01}\,, \tag{6.52}$$

and

$$T - \nu^2\, U\,, \qquad \nu^2\, S_{\chi\chi} - S_{\pi\pi}\,, \qquad M_\chi + N_\pi\,, \qquad M_\chi + \nu^2\, N_\pi\,, \tag{6.53}$$

respectively. We may refer to this case as *short tachyon*.[19] This exotic case can be better understood in terms of $\mathfrak{so}(2,2)$, again. A generic coadjoint orbit with space-like momenta and time-like spins is mapped to the $\mathfrak{so}(2,1)\oplus\mathfrak{so}(2,1)$ coadjoint orbit, $H^2(-(\mu+\nu)^2)\times H^2(-(\mu-\nu)^2)$. For $\mu=\nu$, remark that the last factor becomes a point and not the cone, as the latter correspond to another orbit to be discussed below.

- Finally, the coadjoint orbit with space-like momenta and light-like spins is given by

$$\phi=\mu\,\mathcal{P}^{d-1}+\epsilon\,\mathcal{J}^{-2}\,, \tag{6.54}$$

which again gives a new orbit only in AdS: in dS, it is equivalent to the light-like momenta and space-like spins. The stabiliser is

$$\mathfrak{g}_\phi=\mathbb{R}_{P_{d-1}}\oplus\mathbb{R}_{J_{+2}}\oplus\mathfrak{iso}(d-4)_{[+;3,\dots,d-2]}\,, \tag{6.55}$$

and taking the non-trivial indices as $\chi=2$ and $\pi=-$, we find for the dual orbit

$$\tilde\phi=\mu^2\,\mathcal{T}-\mathcal{U}+\mathcal{S}^{\chi\chi}\,,\qquad \tilde{\mathfrak{g}}_{\tilde\phi}=\mathbb{R}_{T-\mu^2\,U}\oplus\mathbb{R}_{S_{\pi\pi}}\,. \tag{6.56}$$

The Casimir functions of this orbit,

$$C_2=-\mu^2\,,\qquad C_4=0\,. \tag{6.57}$$

take the same values as those of the the tachyonic particle with light-like spin (5.28) in Minkowski, and hence can be considered as its anti-de Sitter analogue.

Remark once again that apart from the continuous spin particles, all other particles are described by the action with two parameters $\mathcal{C}_M$ and $\mathcal{C}_S$ as

$$\begin{aligned} S[X,P,\pi,\chi,A] = \int \Big[& P\cdot \mathrm{d}X+\pi\cdot\mathrm{d}\chi+A^{XX}\,(X^2-\sigma)+A^{PP}\,(P^2+\mathcal{C}_M) \\ & +A^{XP}\,X\cdot P+A^{X\pi}\,X\cdot\pi+A^{X\chi}\,X\cdot\chi+A^{P\pi}\,P\cdot\pi+A^{P\chi}\,P\cdot\chi \\ & +A^{\pi\pi}\left(\pi^2-\mathcal{C}_S\right)+A^{\chi\chi}\left(\chi^2-1\right)+A^{\chi\pi}\,\chi\cdot\pi\Big]\,. \end{aligned} \tag{6.58}$$

Massive, massless and tachyonic particles of spin s are described by $\mathcal{C}_S=s^2$ and positive, zero and negative values of $\mathcal{C}_M+\sigma\,\mathcal{C}_S$, respectively. Note that we have a spin-dependent shift and this quantity is different from the Casimir invariant $C_2=\mathcal{C}_M-\sigma\,\mathcal{C}_S$. For $\mathcal{C}_M=-\sigma\,\mathcal{C}_S$, the gauge symmetry is enhanced for $\mathcal{C}_S>0$ in both AdS and dS but for $\mathcal{C}_S<0$ only the AdS case shows this gauge symmetry enhancement. The tachyonic particles of time-like and light-like spins are described by a negative and zero $\mathcal{C}_S$, respectively, and a negative $\mathcal{C}_M+\sigma\,\mathcal{C}_S$. However, in dS, seemingly tachyonic particles are all equivalent to the massive cases, except for the scalar case.

In AdS spacetime, besides the coadjoint orbits associated with spinning particles, we have three additional classes of coadjoint orbits.

[19] We will refer to the representations having a relatively smaller/larger size as short/long representations. On the other hand, when we refer to the orbits, we will use more often the geometric adjectives, small/large.

6.3 Particles with entangled mass and spin

In the previous section, we have seen three special points where coadjoint orbits become small: the AdS massless particle given by $\phi = s\,(\mathcal{P}^0 + \mathcal{J}^{12})$, the AdS *short* tachyon with time-like spin given by $\phi = \nu\,(\mathcal{P}^{d-1} + \mathcal{J}^{02})$, and dS massless particle (or *short* tachyon with space-like spin) given by $\phi = s\,(\mathcal{P}^{d-1} + \mathcal{J}^{12})$. Their stabilisers are $\mathfrak{u}(1,1) \oplus \mathfrak{so}(d-3)$, $\mathfrak{gl}(2,\mathbb{R}) \oplus \mathfrak{so}(d-3)$ and $\mathfrak{u}(2) \oplus \mathfrak{so}(1,d-4)$ respectively. We can add to this representative vector ϕ a new 'spin' vector taken from the dual of the stabiliser algebra. We may limit ourselves to take this vector from the first part of the stabiliser (meaning the d-independent part), because taking other spin components from the latter part will be interpreted as mixed symmetry ones. It turns out the dS short tachyon (or equivalently the massless spinning particle) becomes either a non-short tachyon or it changes the spin depending on whether the 'spin' vector is taken from the dual of $\mathfrak{su}(2)$ or the central $\mathfrak{u}(1)$ in $\mathfrak{u}(2) \subset \mathfrak{g}_\phi$. Also in the AdS cases, if we add an elliptic vector of $\mathfrak{su}(1,1)^*$ or a hyperbolic vector of $\mathfrak{sl}(2,\mathbb{R})^*$, we do not find new coadjoint orbits but the ones equivalent to non-small coadjoint orbits which we already considered. Similarly, taking the 'spin' vector from the center will end up changing the label of the small orbits.

A new coadjoint orbit with AdS symmetry can be obtained either from a massless one, which is elliptic, by adding a hyperbolic vector $\nu\,(\mathcal{P}^1 - \mathcal{J}^{02}) \in \mathfrak{u}(1,1)^*$ or from a short tachyon with time-like spin, which is hyperbolic, by adding an elliptic vector $s\,(\mathcal{P}^0 + \mathcal{J}^{12}) \in \mathfrak{gl}(2,\mathbb{R})^*$. In either ways, the resulting orbit is given by

$$\phi = s\,(\mathcal{P}^0 + \mathcal{J}^{12}) + \nu\,(\mathcal{P}^1 - \mathcal{J}^{02})\,, \tag{6.59}$$

and has the stabiliser,

$$\mathfrak{g}_\phi = \mathfrak{u}(1)_{P_0+J_{12}} \oplus \mathbb{R}_{P_1-J_{02}} \oplus \mathfrak{so}(d-3)_{[3,\dots,d-1]}\,. \tag{6.60}$$

The dual coadjoint orbit is given by

$$\tilde{\phi} = -\mathcal{U} + \mathcal{S}^{\chi\chi} + 2\,s\,\nu\,\mathcal{M}^\pi + (s^2-\nu^2)\left(\mathcal{T} + \mathcal{S}^{\pi\pi}\right), \tag{6.61}$$

with stabiliser,

$$\tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{u}(1) \oplus \mathbb{R}\,, \tag{6.62}$$

generated by

$$\begin{aligned} &T - S_{\pi\pi} - 2\,s\,\nu\,N_\chi - (s^2-\nu^2)\,(U + S_{\chi\chi})\,,\\ &\qquad \text{and} \qquad M_\pi - \tfrac{s\,\nu}{2}\,U - (s^2-\nu^2)\,N_\chi - \tfrac{s^2-\nu^2}{2\,s\,\nu}\,S_{\pi\pi} - \tfrac{(2\,s^2-\nu^2)(s^2-2\,\nu^2)}{2\,s\,\nu}\,S_{\chi\chi}\,. \end{aligned} \tag{6.63}$$

The Casimir functions of this orbit are given by

$$C_2 = 2(s^2-\nu^2)\,, \qquad C_4 = (s^2+\nu^2)^2\,. \tag{6.64}$$

Depending on the sign of $s-\nu$, the corresponding particle could be interpreted either massive ($s>\nu$), massless ($s=\nu$) or tachyonic ($s<\nu$), but with a rather strange spin. In fact, it reduces to the continuous spin particle with $\phi = 2s\,(\mathcal{P}^+ + \mathcal{J}^{-2})$ in the massless case.

We may interpret these particles as massive, massless and tachyonic particles of continuous spin.

Two other orbits can be obtained in a similar fashion by adding a nilpotent vector proportional to ϵ, taken from $\mathfrak{u}(1,1)^*$ and $\mathfrak{gl}(2,\mathbb{R})^*$, respectively. Firstly, the coadjoint orbit given by

$$\phi = s\,(\mathcal{P}^0 + \mathcal{J}^{12}) + \epsilon\,(\mathcal{P}^0 + \mathcal{P}^1 - \mathcal{J}^{12} - \mathcal{J}^{02})\,, \tag{6.65}$$

with stabiliser

$$\mathfrak{g}_\phi = \mathbb{R}_{P_1 - J_{02} + 2J_{12}} \oplus \mathfrak{u}(1)_{P_0 + J_{12}} \oplus \mathfrak{so}(d-3)_{[3,\dots,d-1]}\,, \tag{6.66}$$

has the dual orbit given by the representative,

$$\tilde{\phi} = -\mathcal{U} - \mathcal{N}^\chi - 4\,s^2\,(\mathcal{S}^{\pi\pi} + \mathcal{M}^\pi)\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_{M_\pi + 4\,s^2\,N_\chi} \oplus \mathfrak{u}(1)_{T + 4\,s^2\,S_{\chi\chi}}\,. \tag{6.67}$$

The Casimir functions of this orbit are

$$C_2 = 2\,s^2\,, \qquad C_4 = s^4\,, \tag{6.68}$$

and they coincide with those of massless spin s orbit. This orbit can be understood as follows. When the mass value m of the massive orbit (6.22) tends to the shortening point $m = s$, the $2(2d-4)$ dimensional massive orbit splits into two: the massless orbit of dimension $2(2d-5)$ and a $2(2d-4)$-dimensional remnant orbit, corresponding to the one given by (6.65). Let us contemplate this issue in terms of representations. The massive spinning orbit would correspond to the irrep $\mathcal{D}(m+d-3,s)$, which in the massless limit splits into the massless irrep $\mathcal{D}(s+d-3,s)$ and a massive one of one lower spin $\mathcal{D}(s+d-2,s-1)$ (see e.g. [164] and also [165] for a proposal wherein this splitting could lead massless higher spin fields to become massive). The Casimir operator eigenvalues of these two irreps are identical. In this reasoning, quantisation of the orbit (6.65) may give rise to $\mathcal{D}(s+d-2,s-1)$. At the same time the latter irrep can certainly arise from the massive spinning orbit of mass $s+1$ and spin $s-1$, with $C_2 = 2(s^2+1)$ and $C_4 = (s^2-1)^2$, which are slightly different from (6.68). This reflects the fact that the quantisation of the orbit (6.65) is rather peculiar. We expect that this is a common feature of the orbits which contains a nilpotent part in it.

The above phenomenon can be better understood from the $d=3$ case, where the orbit (6.65) corresponds to $(H^2_+(s^2) \times C^2_+) \cup (C^2_+ \times H^2_+(s^2))$. On the other hand, the orbit of mass $s+1$ and spin $s-1$ corresponds to $(H^2_+(s^2) \times H^2_+(1)) \cup (H^2_+(1) \times H^2_+(s^2))$. The $O(2,1)$ orbit $H^2_+(1)$ can be quantised to result in the irrep $\mathcal{D}^+_1$ with vanishing Casimir. The nilpotent orbit C^2_+ admits a one-parameter family of quantisation [166], and gives $\mathcal{D}^+_\lambda$ with $\lambda > 0$.[20] Therefore, the orbit (6.65) can be quantised to $\mathcal{D}(s+\lambda, s-\lambda)$ with a continuous spin label $s-\lambda$. In $3d$, all spin eigenstates are one-dimensional, and the spin number is quantised only for the global consistency of $O(2,2)$, i.e. as a result of requiring to have a UIR of the *group*. In higher dimensions, for a non-(half-)integral spin, the number of spin states cannot be finite and the corresponding fields will have infinitely many components.

[20] Here, we consider the Fock model of deformed oscillator, i.e. the representation space is the space of excited oscillator states of the Fock vacuum.

In other words, no 'spin projection' takes place. Note that for $0 < \lambda < 1$, the discrete series representation $\mathcal{D}^+_\lambda$ mixes with the complementary series representation which could arise by quantising $C^2_+ \cup \{0\} \cup C^2_-$.[21] Here, the inclusion of the origin $\{0\}$ indicates that the massless spin s orbit is also contained in it. The Metsaev's infinite-component field [93] seems to provide the first quantised description of the above case. See Appendix F for related discussions. Among the one-parameter possibility of quantisation of the remnant orbit, the discrete series representation with $\lambda = 1$, i.e. $\mathcal{D}(s+1, s-1)$ in 3d, is consistent with the splitting phenomenon of the long massive spin s representation into a massless spin s and a massive spin $s-1$ representations.

Secondly, the coadjoint orbit given by

$$\phi = \nu\,(\mathcal{P}^2 + \mathcal{J}^{01}) + \epsilon\,(\mathcal{P}^0 + \mathcal{P}^2 - \mathcal{J}^{21} - \mathcal{J}^{01}), \tag{6.69}$$

with stabiliser

$$\mathfrak{g}_\phi = \mathbb{R}_{P_0+2J_{01}+J_{12}} \oplus \mathbb{R}_{P_2+J_{01}} \oplus \mathfrak{so}(d-3)_{[3,\dots,d-1]}\,, \tag{6.70}$$

has the dual orbit given by

$$\tilde\phi = -\mathcal{U} - \mathcal{N}^\chi + 4\,\nu^2\,(\mathcal{S}^{\pi\pi} + \mathcal{M}^\pi)\,, \qquad \tilde{\mathfrak{g}}_{\tilde\phi} = \mathbb{R}_{M_\pi - 4\,\nu^2\,N_\chi} \oplus \mathbb{R}_{T-4\,\nu^2\,S_{\chi\chi}}\,. \tag{6.71}$$

The Casimir functions of this orbit are

$$C_2 = -2\,\nu^2\,, \qquad C_4 = \nu^4\,, \tag{6.72}$$

which coincides with those of the short tachyon orbit. This orbit corresponds again to the $2(2d-4)$ dimensional remnant of the shortening phenomenon.

As briefly mentioned above, Metsaev constructed a Lagrangian for infinite component fields having independent quadratic and quartic Casimir values [93] (see also [163, 167–170] for further developments). The model contains two constants parameterising the Casimir values and is divided into several subcases depending on the regions of these constants. For all these subcases, the Lagrangian was generally referred to as continuous spin in AdS. Comparing this work of Metsaev with our classification, the various subcases of AdS continuous spin in [93] corresponds to various coadjoint orbits identified in this paper. See Appendix F for more details.

6.4 Particles in bitemporal AdS space

The coadjoint orbits with vanishing momenta, but space-like spins are given by

$$\phi = m\,\mathcal{J}^{12}\,, \qquad \mathfrak{g}_\phi = \mathfrak{u}(1)_{J_{12}} \oplus \mathfrak{so}(2, d-3)_{[0,0',3,\dots,d-1]}\,, \tag{6.73}$$

and the dual coadjoint orbit is characterised by

$$\tilde\phi = m^2\,\mathcal{T} + \mathcal{U}\,, \qquad \tilde{\mathfrak{g}}_{\tilde\phi} = \mathfrak{u}(1)_{T+m^2\,U}\,. \tag{6.74}$$

[21] The complementary series representation might be obtained from the deformed oscillators [166] by considering a Segal–Bargmann model instead of the Fock model.

The corresponding action has the form,

$$S[X,P,A]=\int P\cdot \mathrm{d}X+A^{XX}(X^2-1)+A^{PP}(P^2-m^2)+A^{XP}\,X\cdot P\,, \tag{6.75}$$

where the ambient space condition is given with the opposite sign $X^2=+1$. This means that the corresponding particle lives in a spacetime with two temporal directions. Let us refer to this spacetime as Bitemporal Anti de Sitter in short BdS. Note that this case exists only for AdS because the analogue in dS is essentially the same as the tachyonic scalar. Moreover, the analogue orbit with time-like spins is equivalent to the tachyonic scalar in AdS and the massive scalar in dS.

In the regard of BdS physics, let us consider the coadjoint orbit given by

$$\phi=E\,\mathcal{J}^{1+}\,,\qquad \mathfrak{g}_\phi=\mathbb{R}_{J_{-1}}\oplus\mathfrak{iso}(1,d-3)_{[-;0',2,\dots,d-2]}\,, \tag{6.76}$$

which is dual to the orbit characterised by

$$\tilde\phi=\mathcal{U}\,,\qquad \tilde{\mathfrak{g}}_{\tilde\phi}=\mathbb{R}_T\,. \tag{6.77}$$

This orbit can be interpreted as "massless" scalar in BdS, while the previous one as massive scalar (with $C_2=m^2>0$) in BdS. Remark that the scalar tachyon in AdS with $\phi=\mu\,\mathcal{P}^1$ can be equally interpreted as a scalar tachyon in BdS, and hence it will be more useful to group particles with $\mathfrak{so}(2,d-1)$ symmetry into AdS particles, BdS particles and tachyons. We shall comment more on the intriguing relations between these three species in the next section. We admit that there is no concrete physical context for the BdS particles (even tachyons). However, we find useful to use these concepts with physics flavor in analysing the mathematical objects that are coadjoint orbits.

We may add space-like spins to the massive or massless scalars in BdS. The orbit of the massive space-like spin particle in BdS is determined by the coadjoint vector,

$$\phi=m\,\mathcal{J}^{12}+s\,\mathcal{J}^{34}\,. \tag{6.78}$$

For $m>s$, the stabiliser is

$$\mathfrak{g}_\phi=\mathfrak{u}(1)_{J_{12}}\oplus\mathfrak{u}(1)_{J_{34}}\oplus\mathfrak{so}(2,d-5)_{[0',0,5,\dots,d-1]}\,, \tag{6.79}$$

and the dual coadjoint orbit is characterised by

$$\tilde\phi=m^2\,\mathcal{T}+\mathcal{U}+s^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}\,,\qquad \tilde{\mathfrak{g}}_{\tilde\phi}=\mathfrak{u}(1)_{T+m^2\,U}\oplus\mathfrak{u}(1)_{S_{\pi\pi}+s^2\,S_{\chi\chi}}\,. \tag{6.80}$$

The Casimir functions of this orbit coincide with those of the massive spinning orbit (6.24).

For $m=s$, the stabiliser and the dual stabiliser are enhanced to

$$\mathfrak{g}_\phi=\mathfrak{u}(2)\oplus\mathfrak{so}(2,d-5)_{[0',0,5,\dots,d-1]}\,,\qquad\text{and}\qquad \tilde{\mathfrak{g}}_{\tilde\phi}=\mathfrak{u}(2)\,, \tag{6.81}$$

where the $\mathfrak{u}(2)$ subalgebras are generated by

$$J_{13}+J_{24}\,,\qquad J_{14}-J_{23}\,,\qquad J_{12}+J_{34}\,,\qquad J_{12}-J_{34}\,, \tag{6.82}$$

and

$$T+s^2\,U\,,\qquad S_{\pi\pi}+s^2\,S_{\chi\chi}\,,\qquad M_\pi+s^2\,N_\chi\,,\qquad M_\chi-N_\pi\,,\tag{6.83}$$

respectively, and hence we can interpret this as a massless spinning BdS particle. We can also consider a BdS particle with light-like momentum and space-like spin determined by the coadjoint vector,

$$\phi=E\,\mathcal{J}^{1+}+s\,\mathcal{J}^{23}\,,\qquad \mathfrak{g}_\phi=\mathbb{R}_{J_{-1}}\oplus\mathfrak{u}(1)_{J_{23}}\oplus\mathfrak{iso}(1,d-5)_{[-;0',4,\dots,d-2]}\tag{6.84}$$

with its dual coadjoint orbit characterised by

$$\tilde\phi=\mathcal{U}+s^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}\,,\qquad \tilde{\mathfrak{g}}_{\tilde\phi}=\mathfrak{u}(1)_{S_{\pi\pi}+s^2\,S_{\chi\chi}}\oplus\mathbb{R}_{M_\pi+s^2\,N_\chi}\,.\tag{6.85}$$

The Casimir functions of this orbit are given by

$$C_2=s^2\,,\qquad C_4=0\,.\tag{6.86}$$

In fact, the coadjoint orbits of time-like momenta and space-like spins with $m<s$, the ones having the risk of confusion with the partially massless particles, can also be regarded as spinning BdS particles. Changing the role of (X,P) and (χ,π) and m and s, the action becomes

$$\begin{aligned}S[X,P,\chi,\pi,A]=\int\Big[&P_a\,\mathrm{d}X^a+\pi_a\,\mathrm{d}\chi^a+A^{XX}(X^2-1)+A^{PP}(P^2-\,m^2)\\&+A^{XP}\,X\cdot P+\,A^{X\pi}\,X\cdot\pi+A^{X\chi}\,X\cdot\chi+A^{P\pi}\,P\cdot\pi\\&+A^{P\chi}\,P\cdot\chi+\,A^{\pi\pi}\left(\pi^2-s^2\right)+A^{\chi\chi}\left(\chi^2+1\right)+A^{\chi\pi}\,\chi\cdot\pi\Big]\,.\end{aligned}\tag{6.87}$$

The above action can be interpreted as the action for a particle of mass m and space-like spin s in BdS. Note here that the space-like spin is given by $s\,\mathcal{J}^{0'0}$ which is inequivalent to the space-like spin considered in the previous two cases. In fact, we have more types of spins in BdS. We can exclude time-like spins in BdS because they can be interpreted as tachyonic particles. Otherwise, light-like or doubly-light-like spins in BdS give us a new class of orbits.

Firstly, let us consider the light-like spin given by $\epsilon\,\mathcal{J}^{1+}$. In the case of massive BdS particles, the light-like spin is simply equivalent to the massless BdS particles with space-like spin. On the other hand, the massless BdS particles with light-like spin are new ones. The orbit is given by

$$\phi=E\,\mathcal{J}^{1+}+\epsilon\,\mathcal{J}^{2+'}\,,\tag{6.88}$$

with the stabiliser (for the choice $E=\epsilon=1$)

$$\mathfrak{g}_\phi=\mathbb{R}_{J_{1-}-J_{2-'}}\oplus\mathfrak{so}(2)_{J_{12}-2J_{-+'}-2J_{+-'}}\in\mathfrak{heis}_{2(d-4)}\ni\mathfrak{so}(d-5)_{[3,\dots,d-3]}\,,\tag{6.89}$$

where $\mathfrak{heis}_{2(d-4)}$ is generated by

$$J_{1-}+J_{2-'},\quad J_{1-'}+J_{2-},\quad J_{i-},\quad J_{i-'},\quad J_{--'}\,,\qquad [i=3,\dots,d-3]\,.\tag{6.90}$$

Here, we take the non-trivial indices as $\chi = 2$, $\pi = +$ to find that the dual coadjoint orbit is given by

$$\tilde{\phi} = \mathcal{U} + \mathcal{S}^{\chi\chi}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_{T+S_{\pi\pi}} \oplus \mathfrak{iso}(2)_{T-S_{\pi\pi},M_\pi,N_\pi-M_\chi}\,. \tag{6.91}$$

Note that in this case E and ϵ can be rescaled independently leaving no label for this orbit, and hence it is a nilpotent orbit, with vanishing Casimir functions: $C_2 = 0 = C_4$.

Secondly, we can consider doubly-light-like spin. In the massive case, the orbit is characterised by the following representative and its stabiliser,

$$\phi = m\,\mathcal{J}^{12} + \epsilon\,\mathcal{J}^{++'}\,, \qquad \mathfrak{g}_\phi = \mathfrak{u}(1)_{J_{12}} \oplus [\mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{so}(d-5)_{[3,\dots,d-3]}] \ni \mathfrak{heis}_{2(d-5)}\,, \tag{6.92}$$

where the $\mathfrak{sp}(2,\mathbb{R})$ subalgebra is generated by

$$J_{+-} - J_{+'-'}\,, \qquad J_{+-'}\,, \qquad J_{-+'}\,, \tag{6.93}$$

and the Heisenberg subalgebra by

$$J_{-i}\,, \qquad J_{-'i}\,, \qquad J_{--'}\,, \qquad [i = 3,\dots,d-3]\,. \tag{6.94}$$

The dual orbit is given by

$$\tilde{\phi} = m^2\,\mathcal{T} + \mathcal{U}\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathfrak{u}(1)_{U+m^2\,T} \oplus \mathfrak{sp}(2,\mathbb{R})_{S_{\pi\pi},S_{\pi\chi},S_{\chi\chi}}\,. \tag{6.95}$$

The Casimir functions of this orbit read

$$C_2 = m^2\,, \qquad C_4 = 0\,. \tag{6.96}$$

Finally, the massless doubly-light-like spinning BdS particle is given by

$$\phi = E\,\mathcal{J}^{1+} + \epsilon\,\mathcal{J}^{-+'}\,, \tag{6.97}$$

with the stabiliser (for the choice $E = \epsilon = 1$),

$$\mathfrak{g}_\phi = \mathbb{R}_{J_{-'-}} \oplus \mathbb{R}_{J_{1-}-2\,J_{+-'}} \oplus \mathfrak{iso}(d-4)_{[-';2,\dots,d-3]}\,. \tag{6.98}$$

The dual orbit and dual stabiliser are given by

$$\tilde{\phi} = \mathcal{U} + \mathcal{M}^\pi\,, \qquad \tilde{\mathfrak{g}}_{\tilde{\phi}} = \mathbb{R}_{T-N_\chi} \oplus \mathbb{R}_{S_{\chi\chi}}\,. \tag{6.99}$$

Again, the above orbit is nilpotent as can be seen from the fact that E and ϵ can be rescaled independently, and the Casimir functions vanish: $C_2 = 0 = C_4$.

6.5 Conformal particles on the boundary

For the AdS algebra $\mathfrak{so}(2,d-1)$, we have yet another class of coadjoint orbits, which are very *small* compared to others. Consider the coadjoint orbit given by

$$\phi = \epsilon\,\mathcal{J}^{++'}\,, \tag{6.100}$$

where $\pm'$ is the lightcone coordinate from $0'$ and $d-2$. The stabiliser is

$$\mathfrak{g}_\phi = \big(\mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{so}(d-3)_{[1,\dots,d-3]}\big) \inplus \mathfrak{heis}_{2(d-3)}\,, \tag{6.101}$$

where the $\mathfrak{sp}(2,\mathbb{R})$ subalgebra is generated by

$$J_{+-} - J_{+'-'}\,, \qquad J_{+-'}\,, \qquad J_{-+'}\,, \tag{6.102}$$

and the Heisenberg subalgebra by

$$J_{-i}\,, \qquad J_{-'i}\,, \qquad J_{--'}\,, \qquad [i=1,\dots,d-3]\,. \tag{6.103}$$

Interestingly, the dual coadjoint orbit is trivial:

$$\tilde\phi = 0\,, \qquad \tilde{\mathfrak{g}}_{\tilde\phi} = \mathfrak{sp}(2,\mathbb{R})_{T,U,V}\,. \tag{6.104}$$

Here, we take $X^{\rm A}{}_{+} = X^{\rm A}$ and $X^{\rm A}{}_{+'} = P^{\rm A}$ to find the action

$$S[X,P,A] = \int \big[P\cdot {\rm d}X + A^{XX}\,X^2 + A^{PP}\,P^2 + A^{XP}\,X\cdot P\big]\,. \tag{6.105}$$

The constraint $X^2=0$ and the homogeneity condition $X\cdot P=0$ tells that the particle leaves on a $(d-1)$-dimensional section of the cone $X^2=0$, and it corresponds to the conformal particle in $(d-1)$-dimensions. Indeed, we can see that the dimension of the coadjoint orbit is $2(d-2)$. See e.g. [159] for related analysis and discussions. When quantised, this leads to the scalar singleton representation. See [87, 171–173] for the references. As we shall see below, there are also conformal spinning particles on the boundary. In order to understand them, we need to discuss about mixed symmetry cases, first.

6.6 Spinning particles with mixed symmetry

Similarly to the Poincaré case, the coadjoint orbits of (A)dS algebra with higher M correspond typically to spinning particles with mixed symmetry of M-row Young diagram. Interestingly, we find the classical analogues of various subtleties of mixed symmetry representations in (A)dS algebra. In the following, we present the representative vectors ϕ of the coadjoint orbits with higher M and their stabilisers $\mathfrak{g}_\phi$. The dual stabilisers $\tilde{\mathfrak{g}}_{\tilde\phi}$ are isomorphic to the d-independent part of $\mathfrak{g}_\phi$.

In AdS$_d$, the massive and massless spinning particles are given by

$$\phi = m\,\mathcal{P}^0 + s_1\,\mathcal{J}^{12} + \cdots + s_M\,\mathcal{J}^{2M-1\,2M}\,, \tag{6.106}$$

where $m\in\mathbb{N}$, $s_1 \geq s_2 \geq \cdots \geq s_M$, and $M \leq [\frac{d-1}{2}]$. The massless point corresponds to $m=s_1$ and hence it is massive if $m>s_1$. For $m<s_1$, we interpret the coadjoint orbits as mixed-symmetry spinning particles in BdS, rather than partially-massless ones for a similar reason we explained in the symmetric spinning case. We will use the block notation introduced in the previous section, where h_k denotes the height of the kth block, of width $\ell_k = s_{h_1+\cdots+h_{k-1}+1}$, and $h_1+\cdots+h_p = M$. When $m=\ell_n$, the stabiliser becomes

$$\mathfrak{g}_\phi = \mathfrak{u}(h_1)\oplus\cdots\oplus\mathfrak{u}(1,h_n)\oplus\cdots\oplus\mathfrak{u}(h_p)\oplus\mathfrak{so}(d-1-2M)\,, \tag{6.107}$$

for $1 \leq n \leq p$. Therefore, we do have a rich variety of exotic class of particles living in BdS. The tachyonic particles with space-like spins are generalised to

$$\phi = \mu\, \mathcal{P}^{d-1} + s_1\, \mathcal{J}^{12} + \cdots + s_M\, \mathcal{J}^{2M-1\,2M}\,, \tag{6.108}$$

with

$$\mathfrak{g}_\phi = \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \mathfrak{so}(d-1-2M)\,, \tag{6.109}$$

and the coadjoint orbit given by

$$\phi = E\, \mathcal{P}^{+} + s_1\, \mathcal{J}^{12} + \cdots + s_M\, \mathcal{J}^{2M-1\,2M}\,, \tag{6.110}$$

with

$$\mathfrak{g}_\phi = \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \mathfrak{iso}(d-2-2M)\,, \tag{6.111}$$

corresponds to the end point of the tachyonic spectrum.

In dS, the massive spinning particle is given again by (6.106) but with $m \in \mathbb{R}$ and the stabiliser is

$$\mathfrak{g}_\phi = \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \mathfrak{so}(d-1-2M)\,. \tag{6.112}$$

The coadjoint orbit given by (6.110) with stabiliser

$$\mathfrak{g}_\phi = \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \mathfrak{iso}(d-2-2M)\,, \tag{6.113}$$

corresponds to the end point of the massive spectrum. The coadjoint orbits given by the representative vector (6.108) with $\mu \in \mathbb{N}$ contain (partially-)massless spinning particles of mixed symmetry. As discussed in the $M = 1$ case, the generator $\mathcal{P}^{d-1} = \mathcal{J}^{d\,d-1}$ is not different from any of $\mathcal{J}^{2k\,2k+1}$ and we assume that $\mu \leq s_M$. The equality $\mu = s_M$ corresponds to the massless case whereas other cases with $\mu < s_M$ correspond to the partially-massless cases. Note that there are no spinning tachyons in dS.

Let us compare our results with the pattern of massless mixed-symmetry representations in (A)dS. In AdS, such representations are known to be unitary only when the gauge parameter has the symmetry of the gauge field Young diagram with one box removed at the bottom of the *first* block [174, 175] (see also [176–179] for more details on mixed-symmetry fields). In dS, unitarity requires the gauge parameter to have the symmetry of the gauge field Young diagram where one removes t boxes from the *last* block: here t is the depth of the partially-massless field. The mass parameter of these fields will depend on the length of the block affected by the gauge symmetry, but not the other blocks. This distinction seem to be reflected in the classes of coadjoint orbits corresponding to (partially-)massless fields in AdS or dS that we have identified here. In this comparison, it is important to take into account the coadjoint orbits of BdS particles, which would lead to unfamiliar classes of unitary representations.

In the case of the coadjoint orbits with light-like or time-like spin, simply the $M-1$ space-like spins are added on top of the light-like or time-like spin. The continuous spin particle exists only in AdS and has the stabiliser $\mathfrak{g}_\phi = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \mathfrak{so}(d-1-2M)$. The tachyon with time-like spin has the stabiliser

$$\mathfrak{g}_\phi = \mathfrak{i}(\sigma) \oplus \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \mathfrak{so}(d-1-2M)\,. \tag{6.114}$$

The tachyon with light-like spin has the stabiliser

$$\mathfrak{g}_\phi = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus \mathfrak{iso}(d-2-2M)\,. \tag{6.115}$$

In AdS, there are yet another class of coadjoint orbits given by

$$\phi = \epsilon\,\mathcal{J}^{++'} + s_1\,\mathcal{J}^{12} + \cdots + s_M\,\mathcal{J}^{2M-1\,2M}\,, \tag{6.116}$$

and the stabilisers

$$\mathfrak{g}_\phi = \mathfrak{u}(h_1) \oplus \cdots \oplus \mathfrak{u}(h_p) \oplus [\mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{so}(d-3-2M)] \Subset \mathfrak{heis}_{2(d-3-2M)}\,. \tag{6.117}$$

In AdS, besides the above orbits, we have also mixed-symmetry extension of the coadjoint orbits with entangled labels. To the 'mass' vector $\phi = s(\mathcal{P}^0 + \mathcal{J}^{12} + \cdots + \mathcal{J}^{2h-1\,2h})$ with the stabiliser $\mathfrak{u}(1,h)$, we can append a 'spin' vector taken from $\mathfrak{u}(1,h)^*$. Like in the $h = 1$ case, any elliptic vector would not lead to a new orbit, but hyperbolic or nilpotent ones will result in new coadjoint orbits.

Finally, let us comment about the particular case of maximal rank massless spinning particle in an odd D-dimensional AdS spacetime with

$$\phi = s(\mathcal{P}^0 + \mathcal{J}^{12} + \cdots + \mathcal{J}^{D-2\,D-1})\,. \tag{6.118}$$

It stabiliser in $\mathfrak{so}(2, D-1)$ is $\mathfrak{g}_\phi = \mathfrak{u}(1, \frac{D-1}{2})$, and the dimension of the coadjoint orbit is $\frac{(D-1)(D+1)}{4}$. This is to be compared with the maximal rank massless spinning particle in an even d dimensional Minkowski space with $\phi = E\,\mathcal{P}^0 + s(\mathcal{J}^{12} + \cdots + \mathcal{J}^{d-3\,d-2})$. The stabiliser in $\mathfrak{iso}(1, d-1)$ is $\mathfrak{g}_\phi = \mathfrak{heis}_{d-2} \oplus \mathfrak{u}(\frac{d-2}{2})$, and the dimension of the coadjoint orbit is $\frac{d(d+2)}{4}$. By matching $D = d+1$, we find that the two orbits have the same dimensions. The phenomena can be understood as the classical counterpart of the peculiar branching rule of spinning singletons [180–182].

Let us conclude this section with the figures which summarise the spectra of scalar (Figures 6 and 7) and spinning (symmetric space-like) particles in (A)dS (Figures 8 and 9), both in terms of the representatives and the 'mass squared' $\mathcal{C}_M$. We also indicated the regions excluded by the quantisation condition which nevertheless should be associated to a class of unitary and irreducible representations usually referred to as the complementary series.

7 Inclusion structure and soft limit

A coadjoint orbit may be contained in the closure of a larger coadjoint orbit. The simplest example is that the inclusion of the origin, the trivial orbit, in the closure of the conical nilpotent orbit of $O(2,1)$ (see Figures 11 and 12 below). The structure of inclusion is well understood for nilpotent orbits. In the following, we briefly review some of the well-known results about the inclusion structure of nilpotent orbits, and associate them with the classifications carried out in this paper. Physically the included smaller orbit can be understood as the soft or boundary limit of the larger orbit: the former can be obtained from the latter by taking a limit sending a point in the phase space to its boundary. We also discuss the analogous phenomena in semisimple orbits.

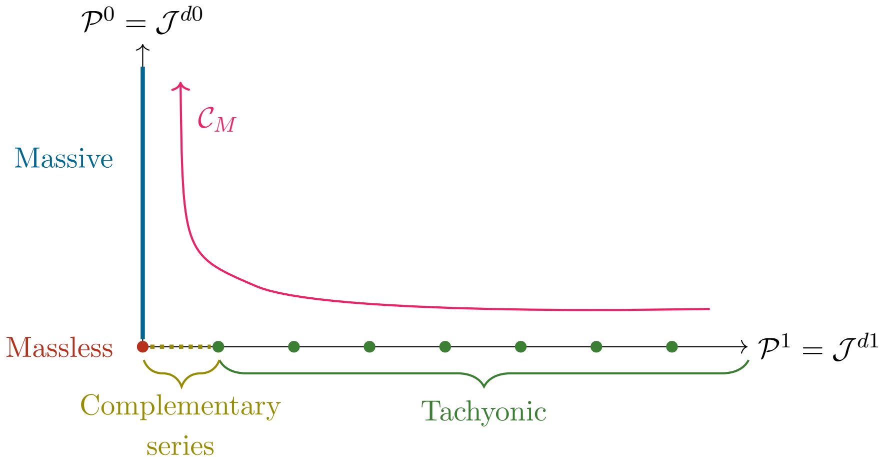

Figure 6. Scalar particles in dS

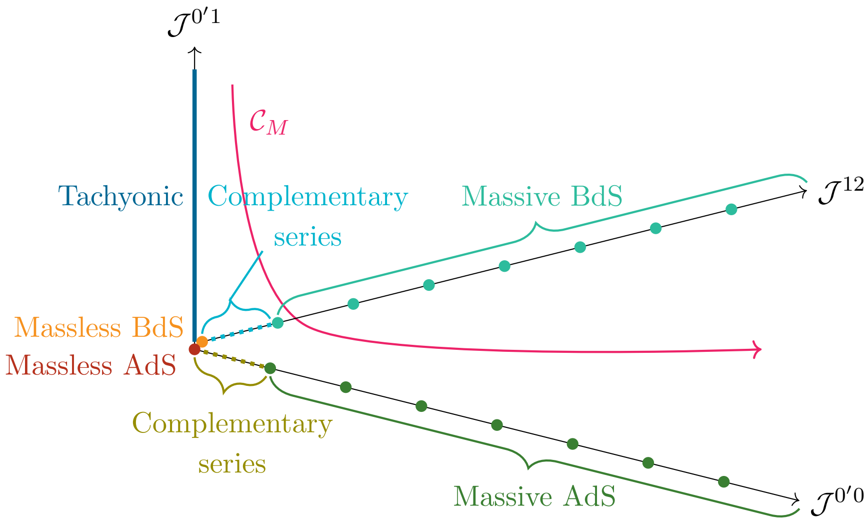

Figure 7. Scalar particles in AdS

7.1 Inclusion structure of nilpotent orbits

Nilpotent orbits of a complex Lie algebra have a rich inclusion structure, which can be described by a Hasse diagram. See e.g. [107–111] for recent progress in the study of supersymmetric moduli spaces, using nilpotent orbits. Let us limit the scope of our discussion to the $\mathfrak{so}_{\mathbb{C}}(n)$ case. Its nilpotent orbits are in one-to-one correspondence with Young diagrams of n boxes, where rows of even lengths appear with even multiplicities (see e.g. [104, Chap. 5.1]), and an orbit corresponding to the Young diagram $\mathbb{Y}_1$ contains an orbit corresponding to $\mathbb{Y}_2$ in its closure if and only if $\mathbb{Y}_2$ can be obtained from $\mathbb{Y}_1$ by repeatedly moving a box from the right edge of one row to a lower row. See the example of $n = 8$ cases depicted in Figure 10.

For the real form $\mathfrak{so}(p, n-p)$, the possible signed Young diagrams are composed of n

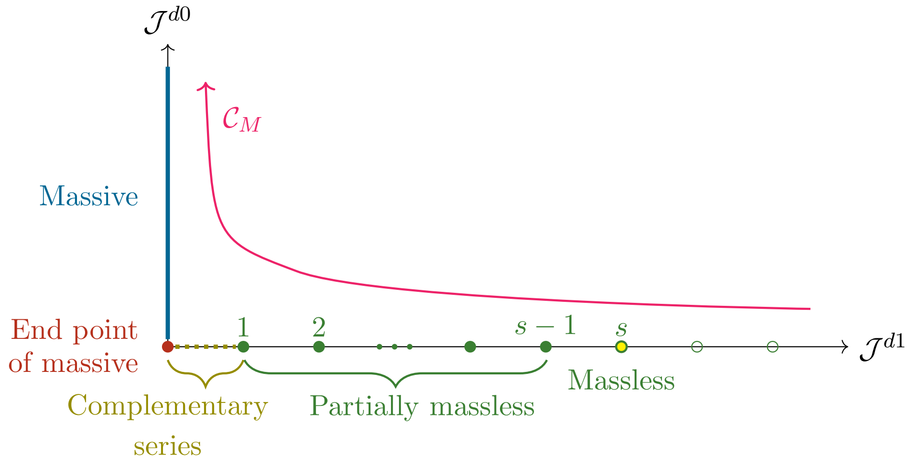

Figure 8. Spin s particles in dS

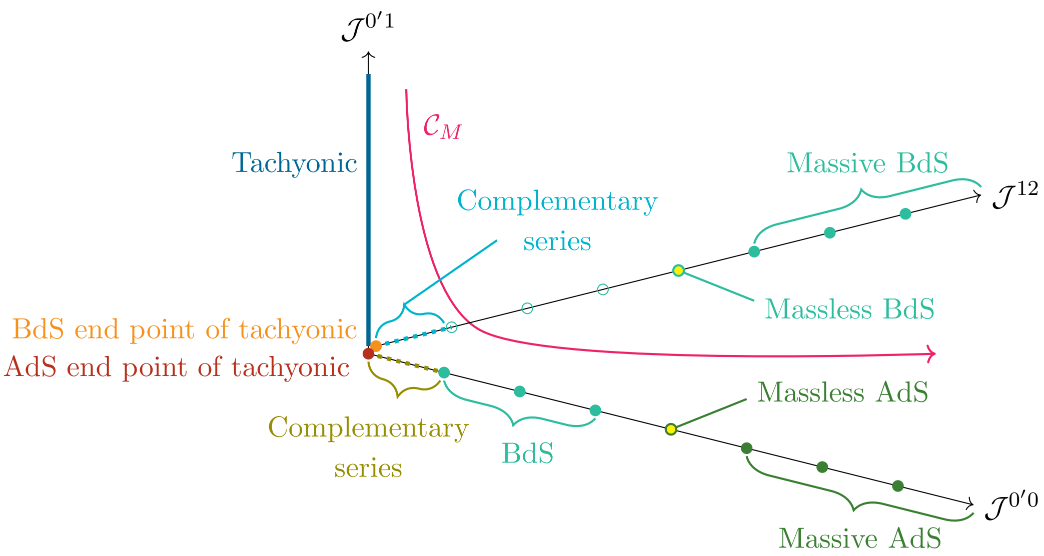

Figure 9. Spin s particles in AdS

boxes and the distribution of + and − signs corresponds to the signature $(p, n-p)$, such that the first box of even-length rows are labelled by a plus sign. The inclusion structure of real coadjoint orbits is also given by the same rule as the complex case but in terms of the signed Young diagrams. These (signed) Young diagrams can also be used to compute the dimension of the associated orbit. The dimension formula is given by

$$\dim \mathcal{O}_{[h_1,\ldots,h_k]} = \dim \mathfrak{so}(p, n-p) - \tfrac{1}{2} \sum_{i=1}^{k} h_i \big(h_i + (-1)^i\big)\,, \tag{7.1}$$

where h_i denote the height of the i-th column of the signed Young diagram.

Let us enumerate the possible signed Young diagrams for $\mathfrak{so}(1,d)$ and $\mathfrak{so}(2,d-1)$, and show their inclusion structures. For the dS algebra, only two signed Young diagram are

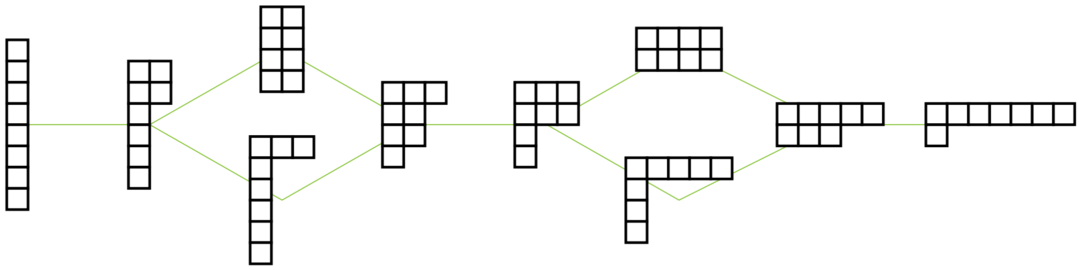

Figure 10. Nilpotent orbits of $\mathfrak{so}_{\mathbb{C}}(8)$ and its inclusion structure: orbits on the left are contained in the closure of orbits on the right, to which they are related by a green line

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possible: see Figure 11. Here, the first one is the trivial orbit and the second one is the

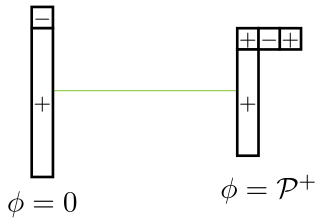


Figure 11. Inclusion structure of dS nilpotent orbits

massless scalar orbit with the representative vector $\phi = \mathcal{P}^+$.

The nilpotent coadjoint orbits of the Poincaré algebra and their inclusion structure is the same as the dS case: there are only two nilpotent orbits, the trivial one given by $\phi = 0$ and the massless scalar given by $\phi = \mathcal{P}^+$. The former is included in the closure of the latter.

For the AdS algebra, we find six possible signed Young diagrams: see Figure 12. We also provided the representative vectors of the corresponding orbits. Let us explain this inclusion structure in words. The trivial orbit is contained in the closure of conformal scalar orbit given by $\mathcal{J}^{++'}$, which is the minimal nilpotent orbit. The conformal scalar is included both in the closure of massless scalar in AdS with $\mathcal{J}^{0'+}$ and massless scalar in BdS with $\mathcal{J}^{1+}$. The former is not contained anywhere, whereas the latter is included both in the closure of massless light-like spin particle in BdS with $\mathcal{J}^{1+} + \mathcal{J}^{2+'}$ and in the massless doubly-light-like spin particle in BdS with $\mathcal{J}^{1+} + \mathcal{J}^{-+'}$.

The inclusion structure can be intuitively understood by the action of Lorentz boosts on the representative vector. Two different Lorentz boosts act on the $\pm$ and $\pm'$ components of the vectors, and they can scale the vector down (or up). In this way, one can easily understand $\mathcal{P}^+$ can be scaled down to 0 under the infinite boost along the $\pm$ directions in the dS and Poincaré cases. In AdS, $\mathcal{J}^{++'}$ can be scaled down to 0 by either boosts $\pm$ or $\pm'$. Also, $\mathcal{J}^{1+} + \mathcal{J}^{2+'}$ and $\mathcal{J}^{1+} + \mathcal{J}^{-+'}$ can be scaled down to $\mathcal{J}^{1+}$ under the $\pm'$ boost. In order to get $\mathcal{J}^{++'}$ from $\mathcal{J}^{1+}$, we need to boost in the $0' - 1$ plane to get $+'$ while renormalising the vector with the $\pm$ boost. We can do the same for $\mathcal{J}^{0'+}$ to get $\mathcal{J}^{++'}$. See Figure 13,

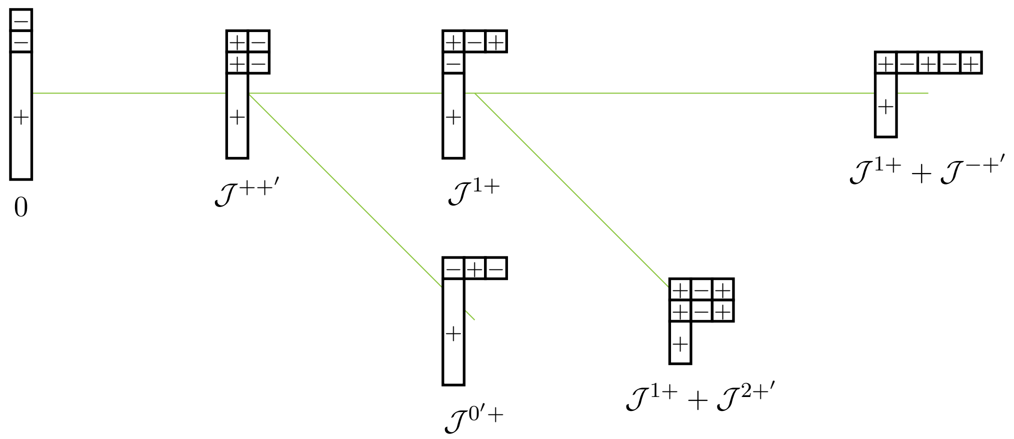


Figure 12. Inclusion structure of AdS nilpotent orbits

where we used the same colors as in Figure 7, for a cartoon picture of the inclusion of the conformal scalar orbit $\mathcal{J}^{+'+}$ in the intersection of two massless scalar orbits $\mathcal{J}^{0'+}$ and $\mathcal{J}^{1+}$. The closure of the latter two nilpotent orbits correspond to the massless limit $m \to 0$ of the semisimple orbits $m\,\mathcal{J}^{0'0}$ and $m\,\mathcal{J}^{12}$. The union of these two nilpotent orbit closures is the $\mu \to 0$ limit of the semisimple orbit $\mu\,\mathcal{J}^{0'1}$.

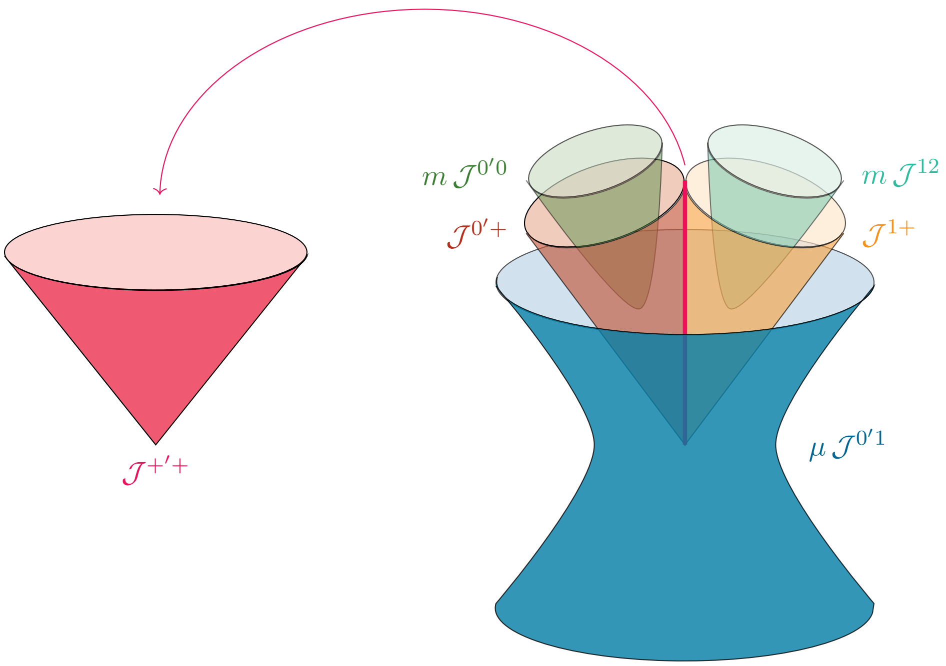


Figure 13. Scalar nilpotent orbits of AdS group and their adjacent semisimple orbits

Reasoning in terms of boosts in lightcone coordinates is also useful in understanding the nilpotent nature of the above orbits. Nilpotent orbits should not have any labels. Therefore, any coefficients in a representative vector of nilpotent orbit should be adjustable by a suitable boost. One can convince oneself that the above are all possible such vectors up to rotations.

Remark that we had initially introduced the representative vector of massive scalar as $\phi = E\,\mathcal{P}^+$ while we removed the E dependence in this section as it can be rescaled to any number. And the rescaling E to 0 is how we understand that the trivial orbit is included in the massless scalar orbit. In this context, the rescaling $E \to 0$ can be interpreted as the soft limit, which unfortunately leaves nothing (trivial orbit) in the scalar particle case (see more discussion in the following section). In the case of the inclusion of the conformal scalar with $\phi = \mathcal{J}^{++'}$ in the massless scalar in AdS with $\phi = \mathcal{J}^{0'+}$, the mutual boosts in the $0' - 1$, and $\pm$ planes can be understood as the AdS boundary limit. It is interesting to note that the boundary limit of massless scalar in BdS also leads to the conformal scalar. Thinking in terms of the ambient space, these boundary limits can be understood from the fact that the infinite regions of both AdS ($X^2 = -1$) and BdS ($X^2 = 1$) approach to the cone ($X^2 = 0$), whose section can be viewed as the conformal boundary.

7.2 Inclusion structure of semisimple orbits

There also exist a class of semisimple orbits which contain smaller orbits in their closure. These semisimple orbits are closely related to nilpotent ones: their representative vectors can be obtained from that of a nilpotent orbit $\mathcal{O}^G_{\phi_{\rm N}}$ by adding a representative vector of a semisimple orbit lying in the stabiliser algebra $\mathfrak{g}_{\phi_{\rm N}}$.[22] In this way, the inclusion nature is completely controlled by the nilpotent part while the semisimple part is simply a spectator. To be more concrete, let us consider a nilpotent orbit given by $\phi_{\rm N}$, which includes m sub-nilpotent orbits

$$\phi_{\rm N}^{(0)} = 0, \quad \phi_{\rm N}^{(1)}, \quad \dots \quad , \quad \phi_{\rm N}^{(m-1)}, \quad \phi_{\rm N}^{(m)} = \phi_{\rm N}\,, \tag{7.2}$$

with an inclusion structure, say,[23]

$$\mathcal{O}_{\phi_{\rm N}^{(0)}} \subset \mathcal{O}_{\phi_{\rm N}^{(1)}} \subset \cdots \subset \mathcal{O}_{\phi_{\rm N}^{(m)}}\,. \tag{7.3}$$

Then for any semisimple orbit of $\phi_{\rm S} \in \mathfrak{g}^*_{\phi_{\rm N}}$, we have the inclusion structures,

$$\mathcal{O}_{\phi_{\rm S}^{(0)}} \subset \mathcal{O}_{\phi_{\rm S}^{(1)}} \subset \cdots \subset \mathcal{O}_{\phi_{\rm S}^{(m)}}\,, \tag{7.4}$$

where $\phi_{\rm S}^{(i)}$ are given by

$$\phi_{\rm S}^{(i)} = \phi_{\rm N}^{(i)} + \phi_{\rm S}\,, \qquad [i = 0, 1, \dots, m]\,. \tag{7.5}$$

In (7.3), we have considered the simplest inclusion structure which can be depicted by a diagram of a simple line, but the same should hold for any more non-trivial inclusion structures.

Let us consider the example of $\phi_{\rm N} = \mathcal{P}^+$. In dS and Poincaré, the closure of $\mathcal{O}_{\mathcal{P}^+}$ contains only the trivial orbit, but in AdS, it also contains the orbit of the conformal scalar

[22] This corresponds to adding a vector in the normal directions of the orbit.

[23] Note that here, it should be understood that an orbit $\mathcal{O}_{\phi_{\rm N}^{(k)}}$ is included in the *Zariski closure* of the next orbit $\mathcal{O}_{\phi_{\rm N}^{(k+1)}}$, see e.g. [104] for more details.

$\mathcal{O}_{\mathcal{J}^{++'}}$. We can add a spin $\phi_{\rm S} = s\,\mathcal{J}^{12} \in \mathfrak{g}^*_{\mathcal{P}^+}$ to these orbits to find semisimple orbits with representative,

$$\phi_{\rm S}^{(0)} = s\,\mathcal{J}^{12}\,, \qquad \phi_{\rm S}^{(1)} = \mathcal{J}^{++'} + s\,\mathcal{J}^{12}\,, \qquad \phi_{\rm S}^{(2)} = \mathcal{P}^+ + s\,\mathcal{J}^{12}\,, \tag{7.6}$$

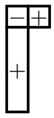
where $\phi_{\rm S}^{(1)}$ is present only for AdS. Let us first focus on the Poincaré case, where $\phi_{\rm S}^{(2)} = \mathcal{P}^+ + s\,\mathcal{J}^{12}$ corresponds to the massless spin s particle. We find that its soft limit gives the null particle with $\phi_{\rm S}^{(0)} = s\,\mathcal{J}^{12}$. The orbit of the latter has dimensions $2(d-2)$ which could be understood as a particle in one lower dimension, similar to the conformal scalar on the boundary. Hopefully, this simple observation might give a new insight on the issues of soft particles, BMS, and celestial CFT etc. Note that the representations of the BMS group in three dimensions have been constructed using the orbit method [183, 184].

In the case of dS and AdS, the soft limit of the orbit $\phi_{\rm S}^{(1)} = \mathcal{P}^+ + s\,\mathcal{J}^{12}$ still leads to the orbit $\phi_{\rm S}^{(0)} = s\,\mathcal{J}^{12}$, but we need to interpret differently. First of all, the latter orbit has dimensions $2(d-1)$, greater than that of Poincaré. And the starting orbit $\phi_{\rm S}^{(0)} = s\,\mathcal{J}^{12}$ cannot be interpreted as a massless one, but the end point of massive/tachyonic spin s in dS/AdS, respectively. As we shall comment in the discussion section, this end point will be even shielded by complementary series representation, so would become an interior point of massive/tachyonic spectrum (see Figure 9). Moreover, $\phi_{\rm S}^{(0)} = s\,\mathcal{J}^{12}$ can be interpreted as a tachyonic scalar in dS for $\mathfrak{so}(1,d)$ and a massive scalar in BdS for $\mathfrak{so}(2,d-1)$. It is intriguing that in dS, a massive spin s particle with a specially tuned mass would contain a tachyonic scalar in the boundary of its phase space. It is also intriguing that in AdS, a tachyonic spin s contains a BdS scalar, though we have already seen that in AdS tachyonic particles can live both AdS and BdS. A possibility is that due to the quantum shift, the quantisation of this end point corresponds to the opposite bound of the spectrum window associated with the complementary series representations. Then, the inclusion of the scalar orbit may be interpreted as the development of the scalar gauge symmetry in the zero mass limit of massive/tachyonic fields in dS/AdS. In dS case, this corresponds to the maximal depth partially massless spin s field which appears at the lightest mass end point of massive spectrum.

Similarly, the soft limit of the orbit $\phi_{\rm S}^{(1)} = s\,(\mathcal{P}^0 + \mathcal{J}^{12}) + \epsilon\,(\mathcal{P}^0 + \mathcal{P}^1 - \mathcal{J}^{12} - \mathcal{J}^{02})$ leads to the orbit of massless spin s given by $\phi_{\rm S}^{(0)} = s\,(\mathcal{P}^0 + \mathcal{J}^{12})$. As we discussed previously below (6.65), the former orbit can be quantised with one free parameter, and gives rise to a field with infinitely many components (that is, no spin projection). In the limit where this parameter goes to a special value, massive spin $s-1$ and massless spin s field appear besides the remaining infinite-component field (see [93] for an explicit description). Therefore, the inclusion of the small massless spin s orbit in the large orbit (6.65) can be interpreted again as the splitting of a long representation into short ones. The analogous discussions can be made also for the counterpart orbits of dS as well as for the BdS orbits.

We can also consider the mixed-symmetry analogues, the massless AdS orbits given by $\phi_{\rm S}^{(0)} = \ell_1(\mathcal{P}^0 + \mathcal{J}^{12} + \cdots + \mathcal{J}^{2h_1-1\,2h_1}) + \cdots$ with the stabiliser $\mathfrak{g}_\phi = \mathfrak{u}(1,h_1) \oplus \cdots$. Here, $\cdots$ denotes the part which depends on additional spin components. The algebra $\mathfrak{u}(1,h_1)$

has only one non-trivial nilpotent orbit $\phi_{\rm N}^{(1)}$ associated with the Young diagram



(7.7)

and by adding it to the original coadjoint vector $\phi_{\rm S}^{(0)}$, we obtain an orbit $\phi_{\rm S}^{(1)} = \phi_{\rm N}^{(1)} + \phi_{\rm S}^{(0)}$ having the same dimensions as the massive orbit given by $m\,\mathcal{P}^0 + \ell_1\,(\mathcal{J}^{12} + \cdots + \mathcal{J}^{2h_1-1\,2h_1}) + \cdots$ with the stabiliser $\mathfrak{u}(1) \oplus \mathfrak{u}(h_1) \oplus \cdots$. Classically, the large orbit $\phi_{\rm S}^{(1)}$ includes the small one $\phi_{\rm S}^{(0)}$. The quantisation of the orbit $\phi_{\rm S}^{(1)}$ would again involve a free parameter and a phenomenon analogous to the symmetric case would take a place. Remark that when $d = 2h_1 + 1$, the short massless representation is described by a field living on the $d-1$ dimensional boundary of AdS_d.

8 Conclusion

8.1 Summary and Discussions

In this paper, we studied the construction of worldline particle actions starting from a coadjoint orbit of the isometry group. The construction is based on the KKS symplectic structure and the action is given by the associated symplectic potential. In order for the path integral quantisation of this action to be well-defined, a part of the labels of particles, such as the spin labels, are quantised. Focusing on the classical Lie groups, we reformulate the coadjoint orbit actions into constrained Hamiltonian ones, where the definition of the group gives rise to a mixture of first and second class constraints. The constraints appear as moment maps — for the dual Lie algebra — shifted by some constants. As such, coadjoint orbits of the dual group are defined with the labels given by the aforementioned constants. In this way, we find pairs of coadjoint orbits $(\mathcal{O}^G_\phi, \mathcal{O}^{\tilde G}_{\tilde\phi})$, where for a given G, a choice of coadjoint orbit $\mathcal{O}^G_\phi$ defines the dual group $\tilde G$ together with its coadjoint orbit $\mathcal{O}^{\tilde G}_{\tilde\phi}$.

The mathematical structures underlying the above pairs of coadjoint orbits fall within the set-up of the *symplectic dual pair* [3].[24] Relevant to us is the case of a symplectic manifold $(\mathcal{M}, \Omega)$ equipped with the Hamiltonian actions of two Lie groups, say G and $\tilde G$,

$$\begin{array}{ccc} & (\mathcal{M},\Omega) & \\ {}^{\mu}\swarrow & & \searrow^{\tilde\mu} \\ \mathfrak{g}^* & & \tilde{\mathfrak{g}}^* \end{array} \tag{8.1}$$

such that the actions commute with one another,

$$\{\mu^*(f), \tilde\mu^*(g)\}_{\mathcal{M}} = 0\,, \qquad \forall\, f \in \mathscr{C}^\infty(\mathfrak{g}^*)\,, \quad \forall\, g \in \mathscr{C}^\infty(\tilde{\mathfrak{g}}^*)\,, \tag{8.2}$$

[24] A symplectic dual pair consists in a pair of Poisson manifolds, say $\mathcal{P}_1$ and $\mathcal{P}_2$, each equipped with a Poisson map $\pi_i : \mathcal{M} \to \mathcal{P}_i$ from the same symplectic manifold $(\mathcal{M}, \Omega)$, such that $\{\pi_1^*(f_1), \pi_2^*(f_2)\}_{\mathcal{M}} = 0$, for any $f_i \in \mathscr{C}^\infty(\mathcal{P}_i)$, where $\{-,-\}_{\mathcal{M}}$ denotes the Poisson bracket on $\mathcal{M}$ induced by the symplectic form Ω. In other words, the pullback of the algebra of functions on the Poisson manifolds $\mathcal{P}_1$ and $\mathcal{P}_2$ commute with one another in $\mathcal{M}$. For more details, the interested reader may consult [134, Chap. 9] or [132, Chap. 11].

and such that the pre-image $\mu^{-1}(\phi)$ for a fixed $\phi \in \mathfrak{g}^*$ is a single $\tilde{G}$-orbit, and vice-versa. Such actions are called 'mutually transitive' [185], and establish a one-to-one correspondence between G-orbits in $\mu(\mathcal{M})$ and $\tilde{G}$-orbits in $\tilde{\mu}(\mathcal{M})$. More precisely, the reduced phase space at $\phi \in \mathfrak{g}^*$ is symplectomorphic to a coadjoint orbit of $\tilde{G}$,

$$\mu^{-1}(\phi)/G_\phi \cong \mathcal{O}^{\tilde{G}}_{\tilde{\phi}}\,, \tag{8.3}$$

where the representative $\tilde{\phi} \in \tilde{\mathfrak{g}}^*$ is simply the image of a point $x \in \mu^{-1}(\phi)$ under the moment map for $\tilde{G}$, i.e. $\tilde{\phi} = \tilde{\mu}(x)$.

The correspondence spelled out in this work admits a description in terms of symplectic dual pairs, wherein one considers the cotangent bundle $T^*Mat_{N\times M}(\mathbb{F}) \cong \mathbb{F}^{2(N\times M)}$ of $N \times M$ matrices with coefficients in $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ as the embedding symplectic manifolds, equipped with two commuting actions of reductive groups G and $\tilde{G}$ (as suggested by the examples worked out in [185], which seem to fall in the class of models considered in this paper). The foliation of $T^*Mat_{N\times M}(\mathbb{F})$ under the action of $G \times \tilde{G}$ gives

$$T^*Mat_{N\times M}(\mathbb{F}) \cong \bigcup_\phi \mathcal{O}^G_\phi \times \mathcal{O}^{\tilde{G}}_{\tilde{\phi}(\phi)}\,, \tag{8.4}$$

where the summation for ϕ is over all coadjoint orbits present in this decomposition and the dual coadjoint orbit with the representative element $\tilde{\phi}(\phi)$ is uniquely specified by ϕ. The above is reminiscent of the foliation of the cotangent bundle T^*G under the left and right action $G \times G$, which leads to the Peter–Weyl theorem upon quantisation:

$$T^*G \cong \bigcup_\phi \mathcal{O}^G_\phi \times \mathcal{O}^G_\phi \quad \xrightarrow{\text{quantisation}} \quad L^2(G) \cong \bigoplus_\lambda \pi^G_\lambda \otimes (\pi^G_\lambda)^*\,. \tag{8.5}$$

Note that the summation of ϕ over all distinct coadjoint orbits is transmuted into the summation of λ over all distinct unitary irreducible representations. By analogy with (8.5), one can understand that the geometrical correspondence (8.4) is the classical analogue of the reductive dual pair correspondence [4, 5],

$$W_{NM} \cong \bigoplus_\lambda \pi^G_\lambda \otimes \pi^{\tilde{G}}_{\tilde{\lambda}(\lambda)}\,, \tag{8.6}$$

which consists in a bijection between irreducible representations λ and $\tilde{\lambda}$ of G and $\tilde{G}$, appearing in the decomposition of the oscillator representation W_{NM} (i.e. the metaplectic representation) of $Sp(2NM,\mathbb{R})$. The representation W_n is known to arise as the quantisation of the minimal nilpotent orbit of $Sp(2n,\mathbb{R})$, which is simply the flat symplectic manifold $(\mathbb{R}^{2n}\backslash\{0\})/\mathbb{Z}_2$ [171]. Therefore, the irreps appearing in the dual pair correspondence should arise from the quantisation of pairs of coadjoint orbits of $G \times \tilde{G} \subset Sp(2n,\mathbb{R})$ embedded in the minimal orbit of $Sp(2n,\mathbb{R})$. See [186–190] and references therein for works in this direction.

Let us come back to the content of this paper: by focusing our attention to the Poincaré and (A)dS groups, we derived manifestly covariant actions for various particle species in Minkowski and (A)dS spacetime. In (A)dS case, the manifest covariance is realised using ambient space coordinates. In the Poincaré case, the classification of coadjoint orbits is essentially the same as the Wigner classification. In the (A)dS cases, it turned out that there are far less types of coadjoint orbits of $O(1,d)$ compared to $O(2,d-1)$. Limiting our attention to the cases of no more than one spin label, we could survey all possibilities. Here, by 'spin', we mean any additional label besides the mass, the principal label. Compared to the classification of unitary irreducible representations, the same task for coadjoint orbits is so much easier, and it allowed us to see the landscape of all the available particle species with Poincaré and (A)dS symmetry, assuming a one-to-one correspondence between quantisable coadjoint orbits and unitary irreducible representations. We found that in dS there is no tachyonic particle except for the scalar. On the other hand, in AdS, not only tachyons but also more exotic entities appear. We interpret them as the particles living in bitemproal AdS (BdS) as its worldline action involves the constraint $X^2 = +1$. It is interesting to note that tachyons can be interpreted as either AdS or BdS particles, hence bridging the spectrum of the ordinary AdS particles and the BdS ones. Even though BdS particles are exotic, they may play certain roles in CFT, such as in the inversion formula. The existence of the scalar and spinning conformal particles is another peculiarity of AdS, whose closest counterpart might be the presence of null particles in Poincaré with vanishing momenta. In Poincaré and AdS, we also identified coadjoint orbits that should correspond to continuous spin particles, which is part of a two-parameter family of orbits in the AdS case. On top of that, we found many more interesting classes of coadjoint orbits whose particle/field interpretation either consistent with the existing literature or yet to be described.

8.2 Outlook

In this paper, we focused on the (inhomogeneous) orthogonal group, but our construction can be equally applied to other classical Lie groups. In the sequel paper, we shall cover these other cases with their applications to the twistor formulation of worldline particles.

Various issues related to quantisation need to be better understood. First, we need to define a proper measure for the path integral, but once this is done we expect that the constrained Hamiltonian system can be easily quantised because all the constraints are quadratic. As mentioned earlier, the quantisation of these dual pairs of coadjoint orbits should lead to the dual pair of unitary and irreducible representations appearing in Howe duality. The natural setting for the latter is the Fock space generated by several families of bosonic oscillators, with which one can realise the action of a dual pair of reductive groups [2, 91]. This is consistent with the fact that the constraints considered in our Hamiltonian systems are obtained from moment maps for the group actions of dual pairs (in the above sense), and that these moment maps are all quadratic.

However, there still remain a few important issues to be clarified. One important such issue is how the degrees of freedom associated to the spin are projected to a finite dimensional space. As we discussed with the example of $SU(2)$, this can happen for a compact coadjoint orbit, but understanding the same mechanism for a non-compact orbit will be an important task. In certain cases, such as massive spinning orbits where the coadjoint orbits have a bundle structure where the fiber is a compact coadjoint orbit, it is easy to understand the mechanism. However, in other cases, such as massless coadjoint orbits, it is not easy to grasp the precise mechanism. Especially in the dS partially massless cases, the “spin orbits” are non-compact while we still expect the projection to takes place.

Another important issue is how the classical equations of motion for the first-quantised fields arise from the Hamiltonian constraints that we derived. We already see that the constraints ought to be associated with a set of equations similar to that of Fierz equations. By properly quantising the system with BRST symmetry, we may get a gauge invariant equation from the BRST charge (see e.g. [160, 161] and references therein). When the spin degrees of freedom are to be projected, we should be able to get the equations which define tensor fields of a finite rank.

Finally, we would like to remark about the unitary irreducible representations which would arise by quantising the coadjoint orbits. We have seen that some labels need to be quantised for a well-defined path integral. These labels will get some d-dependent constant shift due to the ordering of the quantised variables: e.g. the quadratic Casimir of $SU(2)$ gets shifted from s^2 to $s(s+1)$. In the end of the quantisation procedure, we expect to recover most of unitary irreducible representations, with notable exception of the complementary series ones. The latter seems to arise generically from a coadjoint orbit containing a non-trivial nilpotent part. Quantisation of such an orbit involves deformation parameters which are associated with the labels of the complementary series representations. This type of orbits generically appear at the end point of a continuous family of orbits (such as massive dS and tachyonic AdS), or more generally whenever certain class of orbits becomes small in a limit that one of its label goes to a shortening point as we have seen in the massless spin s example.

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A Conventions and notations

- Our convention for the Lie algebras $\mathfrak{so}(1,d)$ and $\mathfrak{so}(2,d-1)$ is that they are generated by antisymmetric generators $J_{\rm AB} = -J_{\rm BA}$ with A,B $= 0,1,\ldots,d-1,\bullet$, subject to

$$[J_{\rm AB}, J_{\rm CD}] = \eta_{\rm AC}\, J_{\rm BD} - \eta_{\rm BC}\, J_{\rm AD} - \eta_{\rm AD}\, J_{\rm BC} + \eta_{\rm BD}\, J_{\rm AC}\,, \tag{A.1}$$

where $\eta_{\rm AB} = {\rm diag}(-1,1,\ldots,1,\sigma)$ and $\sigma = +1$ for $\mathfrak{so}(1,d)$ or $\sigma = -1$ for $\mathfrak{so}(2,d-1)$.

- Defining $P_a := \frac{1}{\ell} J_{\bullet a}$, with $a = 0, 1, \dots, d-1$, the above Lie bracket reads

$$[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} - \eta_{ad} J_{bc} + \eta_{bd} J_{ac} \,, \tag{A.2a}$$

$$[J_{ab}, P_c] = \eta_{cb} P_a - \eta_{ca} P_b \,, \qquad [P_a, P_b] = \frac{\sigma}{\ell^2} J_{ab} \,, \tag{A.2b}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, \dots, 1)$ is the Minkowski metric and ℓ the (A)dS radius. In plain words, the generators J_{ab} span the Lorentz subalgebra $\mathfrak{so}(1, d-1)$ common to $\mathfrak{so}(1, d)$ and $\mathfrak{so}(2, d-1)$, while P_a are the (A)dS 'transvection' generators, the non-commutative counterpart of translations in flat space.

- Sending the (A)dS radius to infinity, $\ell \to \infty$, implements the Inönü–Wigner contraction of $\mathfrak{so}(1, d)$ or $\mathfrak{so}(2, d-1)$ to the Poincaré algebra, $\mathfrak{iso}(1, d-1)$, whose Lie bracket is almost the same as above except for the fact that P_a are now genuine generators of translation and hence Abelian.

- The convention for the lightcone indices $\pm$ is as follows. For any generator V_a with a vector index and its dual $\mathcal{V}^a$, we set $V_\pm = V_0 \pm V_{d-1}$ and $\mathcal{V}^\pm = \frac{1}{2}(\mathcal{V}^0 \pm \mathcal{V}^{d-1})$.

- In order to compactly encode the information about the stabilisers of the various representatives, we use the following notation. First, given an indefinite orthogonal algebra $\mathfrak{so}(p, q)$, we denote a subalgebra $\mathfrak{so}(m, n)$ with $m \leq p$ and $n \leq q$, by

$$\mathfrak{so}(m, n)_{[I]} := \mathrm{span}\{J_{ij} \,,\ i, j \in I\} \subset \mathfrak{so}(p, q) \,, \tag{A.3}$$

where I denotes a subset of values for the indices carried by the generators of $\mathfrak{so}(p, q)$. Similarly, we denote a subalgebra $\mathfrak{iso}(m, n)$ by

$$\mathfrak{iso}(m, n)_{[a\pm b; I]} := \mathrm{span}\{J_{ai} \pm J_{bi}, J_{ij} \,,\ i, j \in I\} \subset \mathfrak{so}(p, q) \,, \tag{A.4}$$

i.e. the indices $a \pm b$ specifies which combination of generators form the Abelian ideal of translation for the subalgebra $\mathfrak{so}(m, n)_{[I]}$. Finally, for low-dimensional subalgebras $\mathfrak{h} \subset \mathfrak{g}$ (typically, one-dimensional), we write

$$\mathfrak{h}_{\mathtt{t}_1, \dots, \mathtt{t}_k} := \mathrm{span}\{\mathtt{t}_1, \dots, \mathtt{t}_k\} \subset \mathfrak{g} \,, \tag{A.5}$$

where $\mathtt{t}_i$ denotes a basis of $\mathfrak{g}$.

- We often omit the word "coadjoint" and simply say "orbit" to refer to a coadjoint orbit.

B Conversion of second class constraints

In this appendix, we spell out a simple way of converting the second class constraints appearing in the Hamiltonian action considered in Section 3 into first class ones. First, let us point out that the dimension of the reduced phase space, given in (3.19), can be also expressed as

$$\dim \mathcal{N}_\phi = \dim \mathcal{M} + \dim \mathcal{O}^G_\phi - 2 \dim \mathfrak{g} \,, \tag{B.1}$$

which suggests a way of converting the second class constraints appearing for $\phi \neq 0$ into first class ones. Doing so would amount to extending the phase space of our worldline model from $\mathcal{M}$ to $\mathcal{M} \times \mathcal{O}^G_\phi$ with the symplectic structure $\Omega(y) - \omega(\varphi)$, and potential

$$\vartheta(y) + \langle \phi, g_\varphi^{-1}\, \mathrm{d}g_\varphi \rangle \,, \tag{B.2}$$

where $\varphi = \mathrm{Ad}^*_{g_\varphi}\phi$ is a point on $\mathcal{O}^G_\phi$ and g_φ is a section $\mathcal{O}^G_\phi \hookrightarrow G$. In this extended phase space $\mathcal{M} \times \mathcal{O}^G_\phi$, we impose the constraints,

$$\chi_a(y,\varphi) = \mu_a(y) - \varphi_a \approx 0 \,, \tag{B.3}$$

where $\varphi_a = \langle \varphi, J_a \rangle$ is a function on $\mathcal{O}^G_\phi$. The Poisson bracket between two such constraints is

$$\{\chi_a(y,\varphi), \chi_b(y,\varphi)\}_{\mathcal{M}\times\mathcal{O}^G_\phi} = \{\chi_a(y,\varphi), \chi_b(y,\varphi)\}_{\mathcal{M}} - \{\chi_a(y,\varphi), \chi_b(y,\varphi)\}_{\mathcal{O}^G_\phi} \,. \tag{B.4}$$

The first Poisson brackets gives

$$\{\chi_a(y,\varphi), \chi_b(y,\varphi)\}_{\mathcal{M}} = \{\mu_a(y), \mu_b(y)\}_{\mathcal{M}} = f_{ab}{}^c\, \mu_a(y) \,, \tag{B.5}$$

whereas the second Poisson bracket reduces to

$$\{\chi_a(y,\varphi), \chi_b(y,\varphi)\}_{\mathcal{O}^G_\phi} = \{\varphi_a, \varphi_b\}_{\mathcal{O}^G_\phi} \,. \tag{B.6}$$

The Poisson bracket on $\mathcal{O}^G_\phi$ can be obtained from the symplectic structure ω, but also deduced directly from that of $\mathfrak{g}^*$. The entire coadjoint space $\mathfrak{g}^*$ is not a symplectic manifold but is endowed with a Poisson structure: for any two functions f, g on $\mathfrak{g}^*$,

$$\{f, g\}_{\mathfrak{g}^*} = f_{ab}{}^c\, x_c\, \frac{\partial f}{\partial x_a}\, \frac{\partial g}{\partial x_b} \,, \tag{B.7}$$

Here, $x = x_a\, \mathcal{J}^a$ is a vector in $\mathfrak{g}^*$. The pullback of the above by the inclusion $\iota_\phi : \mathcal{O}^G_\phi \hookrightarrow \mathfrak{g}^*$[25] gives the Poisson structure on $\mathcal{O}^G_\phi$ as

$$\iota^*_\phi(\{f, g\}_{\mathfrak{g}^*}) = \{\iota^*_\phi f, \iota^*_\phi g\}_{\mathcal{O}^G_\phi} \,. \tag{B.8}$$

Since the pullback of the coordinate function x_a is φ_a: $\iota^*_\phi x_a = \varphi_a$, we find

$$\{\varphi_a, \varphi_b\}_{\mathcal{O}^G_\phi} = f_{ab}{}^c\, \varphi_c \,. \tag{B.9}$$

Combining the two Poisson brackets, we find the constraints $\chi_a(y,\varphi)$ are all of first class type as $\{\chi_a, \chi_b\}_{\mathcal{M}\times\mathcal{O}^G_\phi} \approx 0$. The particle action corresponding to the extended constrained phase space is

$$S[y,\varphi,A] = \int \vartheta(y) + \langle \phi, g_\varphi^{-1}\mathrm{d}g_\varphi \rangle - \langle A, \chi(y,\varphi) \rangle \,, \tag{B.10}$$

which can be rewritten as

$$S[y,\varphi,A] = \int \vartheta_A(y) + \langle \phi, g_\varphi^{-1} D_A g_\varphi \rangle \,, \tag{B.11}$$

[25] Note that the inclusion of $\mathcal{O}^G_\phi$ in $\mathfrak{g}^*$ is a moment map for the coadjoint action of G.

in terms of

$$\vartheta_A = \vartheta - \langle \mu(y), A \rangle \,, \qquad g_\varphi^{-1}\, D_A g_\varphi = g_\varphi^{-1}(\mathrm{d} + A) g_\varphi \,, \tag{B.12}$$

which are separately invariant under

$$\delta_\lambda y^\mu = \{\mu^*(\lambda), y^\mu\} \,, \qquad \delta_\lambda A = \mathrm{d}\lambda + [A, \lambda] \,, \qquad \delta_\lambda g_\varphi = -\lambda\, g_\varphi \,, \tag{B.13}$$

up to a total derivative term. Here, the gauge parameter $\lambda \in \Omega^0(I, \mathfrak{g})$ takes values in the full Lie algebra $\mathfrak{g}$. Under a change of section $g_\varphi \to g_\varphi\, h_\varphi$, the action changes exactly as in the coadjoint orbit action. Therefore, we find a consistent result of quantisation condition for ϕ.

Note that the above procedure to convert second class constraints to first class ones (for the type of constrained Hamiltonian systems discussed here), is an application of what is known as the 'shifting trick' in the context of symplectic reduction, see e.g. [132, Chap. 6.5] or [191, Chap. 6.3].

C Classification of the $O(n)$ coadjoint orbits

As explained in Section 2, the classification of coadjoint orbits of $O(n)$ can be obtained using the bijection between adjoint and coadjoint orbits: the conjugacy classes of $\mathfrak{so}(n)$ correspond to the elements of the Cartan subalgebra up to Weyl reflections. The classification of orbits of the indefinite orthogonal groups is much more involved, but has been carried out in [105, 106]. To give an intuitive picture of this classification problem, let us review some details about the $O(n)$ case.

Let J_{ab} the generators of $\mathfrak{so}(n)$ and $\mathcal{J}^{ab}$ their duals. Any representative coadjoint vector can be written as $\phi = \phi_{ab}\, \mathcal{J}^{ab}$. Since ϕ_{ab} is an antisymmetric matrix, we can skew-block-diagonalize it (that is, bring it into the form $\phi = \sum_{k=1}^{[\frac{n}{2}]} \phi_{2k-1\, 2k}\, \mathcal{J}^{2k-1\, 2k} = \phi_{12}\, \mathcal{J}^{12} + \phi_{34}\, \mathcal{J}^{34} + \cdots$) by an orthogonal transformation, which is the same as the adjoint action. This is one of the key differences of the $O(n)$ case from the $O(p, n-p)$ ones because the latter cannot be skew-block-diagonalized by an adjoint action.

In the $O(n)$ case, we can furthermore set $|\phi_{12}| \geq |\phi_{34}| \geq \ldots$ by $\pi/2$-rotations. Finally, we can perform a π-rotation in the $(2k)$–$(2k{+}1)$ plane to flip the sign of $\phi_{2k-1\, 2k}$. Continuing this procedure, we can set $\phi_{2k-1\, 2k} \geq 0$ for $k = 1, \ldots, [(n{+}1)/2] - 1$: only the sign of $\phi_{n-1\, n}$ for even n cannot be adjusted in this way. In summary, for $n = 2r$ or $n = 2r + 1$, we can always set a representative coadjoint vector as

$$\phi = \sum_{k=1}^{r} \ell_k\, \mathcal{J}^{2k-1\, 2k} \,, \qquad \ell_1 \geq \cdots \geq \ell_{r-1} \geq |\ell_r| \,, \tag{C.1}$$

where $\ell_r \geq 0$ for $n = 2r + 1$. Note that this standard representative is an element of the Cartan subalgebra of $\mathfrak{so}(n)$, in accordance with the previous remark that (co)adjoint orbits of compact Lie groups are in correspondence with (equivalence classes of) Cartan subalgebra elements. For ℓ_k's satisfying

$$\ell_1 = \cdots = \ell_{h_1} > \ell_{h_1+1} = \cdots = \ell_{h_1+h_2} > \cdots > \ell_{h_1+\cdots+h_{p-1}+1} = \cdots = \ell_{h_1+\cdots+h_p} \,, \tag{C.2}$$

the stabiliser G_ϕ is isomorphic to

$$U(h_1) \times U(h_2) \times \cdots \times U(h_{p-1}) \times O\big(n-2M\big)\,, \qquad M := h_1 + \cdots + h_p\,. \tag{C.3}$$

The derived algebra of $\mathfrak{g}_\phi$ in this case is

$$[\mathfrak{g}_\phi, \mathfrak{g}_\phi] = \mathfrak{su}(h_1) \oplus \ldots \mathfrak{su}(h_p) \oplus \mathfrak{so}(n-2M)\,, \tag{C.4}$$

and therefore

$$\mathfrak{g}_\phi^{\rm Ab} = \underbrace{\mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)}_{M \text{ times}}\,, \tag{C.5}$$

is compact. According to the discussion in Section 2.3, quantisable coadjoint orbits will correspond to those having $\ell_k \in \mathbb{N}$. One can easily imagine that after quantising these orbits, the label $(\ell_1, \ldots, \ell_r)$ for coadjoint orbits becomes the typical Young diagram label for the finite dimensional representations: ℓ_i is the number of boxes in the i-th row.

D Summary of coadjoint orbits of the Poincaré and (A)dS groups

In this section, we present the summary of the coadjoint orbit data we have obtained in Section 5 and 6. In the case of Poincaré, we left out the null particles as they coincide with those of dS in one lower dimensions: in terms of particle actions, we need to interpret X and P as a part of spin variables. In the case of AdS, we omitted for simplicity the coadjoint orbits where the spin and mass are entangled. The symbols M_+, M_- and M_0 indicate the orbits of time-like, space-like and light-like momenta. Similarly, S_+, S_-, S_0 and S_{00} indicate the orbits of time-like, space-like, light-like and doubly-light-like spin.

	ϕ	$\mathfrak{g}_\phi$	$\tilde{\phi}$	$\tilde{\mathfrak{g}}_{\tilde{\phi}}$
M_+	$m\,\mathcal{P}^0$	$\mathbb{R}\oplus\mathfrak{so}(d-1)$	$-m^2\,\mathcal{T}$	$\mathbb{R}$
M_0	$\mathcal{P}^+$	$\mathbb{R}\oplus\mathfrak{iso}(d-2)$	0	$\mathbb{R}$
M_-	$\mu\,\mathcal{P}^{d-1}$	$\mathbb{R}\oplus\mathfrak{so}(1,d-2)$	$\mu^2\,\mathcal{T}$	$\mathbb{R}$
$\mathrm{M}_+\mathrm{S}_+$	$m\,\mathcal{P}^0+s\,\mathcal{J}^{12}$	$\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(d-3)$	$-m^2\,\mathcal{T}+s^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{m}_0\mathrm{s}_+$	$\mathcal{P}^++s\,\mathcal{J}^{12}$	$\big(\mathfrak{heis}_2\ni\mathfrak{u}(1)\big)\oplus\mathfrak{iso}(d-4)$	$s^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}$	$\mathfrak{heis}_2\ni\mathfrak{u}(1)$
$\mathrm{M}_0\mathrm{S}_0$	$\varepsilon\left(\mathcal{P}^++\mathcal{J}^{-1}\right)$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathfrak{so}(d-3)$	$-\varepsilon^2\,\mathcal{M}^{\pi}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathbb{R}$
$\mathrm{M}_-\mathrm{S}_+$	$\mu\,\mathcal{P}^{d-1}+s\,\mathcal{J}^{12}$	$\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(1,d-4)$	$\mu^2\,\mathcal{T}+s^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{M}_-\mathrm{S}_0$	$\mu\,\mathcal{P}^{d-1}+\mathcal{J}^{-2}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathfrak{iso}(d-4)$	$\mu^2\,\mathcal{T}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathbb{R}$
$\mathrm{M}_-\mathrm{S}_-$	$\mu\,\mathcal{P}^{d-1}+\nu\,\mathcal{J}^{01}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathfrak{so}(d-3)$	$\mu^2\,\mathcal{T}-\nu^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathbb{R}$

Table 1. Summary of the data defining a coadjoint orbit of the Poincaré group and its dual except for the null particles.

	ϕ	$\mathfrak{g}_\phi$	$\tilde{\phi}$	$\tilde{\mathfrak{g}}_{\tilde{\phi}}$
M_+	$m\,\mathcal{P}^0$	$\mathbb{R}\oplus\mathfrak{so}(d-1)$	$\mathcal{U}-m^2\,\mathcal{T}$	$\mathbb{R}$
M_0	$\mathcal{P}^+$	$\mathbb{R}\oplus\mathfrak{iso}(d-2)$	$\mathcal{U}$	$\mathbb{R}$
M_-	$\mu\,\mathcal{P}^1$	$\mathfrak{u}(1)\oplus\mathfrak{so}(1,d-2)$	$\mathcal{U}+\mu^2\,\mathcal{T}$	$\mathfrak{u}(1)$
$\mathrm{M}_+\mathrm{S}_+$	$m\,\mathcal{P}^0+s\,\mathcal{J}^{12}$	$\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(d-3)$	$\mathcal{U}+\mathcal{S}^{\chi\chi}$ $-m^2\,\mathcal{T}+s^2\,\mathcal{S}^{\pi\pi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{M}_0\mathrm{S}_+$	$\mathcal{P}^++s\,\mathcal{J}^{12}$	$\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{iso}(d-4)$	$\mathcal{U}+\mathcal{S}^{\chi\chi}+s^2\,\mathcal{S}^{\pi\pi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{M}_-\mathrm{S}_+$	$\mu\,\mathcal{P}^{d-1}+s\,\mathcal{J}^{12}$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(1,d-4)$	$\mathcal{U}+\mathcal{S}^{\chi\chi}$ $+\mu^2\,\mathcal{T}+s^2\,\mathcal{S}^{\pi\pi}$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)$
$\mathrm{m}_-\mathrm{s}_+$	$s\,(\mathcal{P}^{d-1}+\mathcal{J}^{12})$	$\mathfrak{u}(2)\oplus\mathfrak{so}(1,d-4)$	$\mathcal{U}+\mathcal{S}^{\chi\chi}$ $+s^2\,(\mathcal{T}+\mathcal{S}^{\pi\pi})$	$\mathfrak{u}(2)$

Table 2. Summary of the data defining a coadjoint orbit of the dS group and its dual.

	ϕ	$\mathfrak{g}_\phi$	$\tilde{\phi}$	$\tilde{\mathfrak{g}}_{\tilde{\phi}}$
M_+	$m\,\mathcal{P}^0$	$\mathfrak{u}(1)\oplus\mathfrak{so}(d-1)$	$-\mathcal{U}-m^2\,\mathcal{T}$	$\mathfrak{u}(1)$
M_0	$\mathcal{P}^+$	$\mathbb{R}\oplus\mathfrak{iso}(d-2)$	$-\mathcal{U}$	$\mathbb{R}$
M_-	$\mu\,\mathcal{P}^1$	$\mathbb{R}\oplus\mathfrak{so}(1,d-2)$	$-\mathcal{U}+\mu^2\,\mathcal{T}$	$\mathbb{R}$
S_+	$m\,\mathcal{J}^{12}$	$\mathfrak{u}(1)\oplus\mathfrak{so}(2,d-3)$	$\mathcal{U}+m^2\,\mathcal{T}$	$\mathfrak{u}(1)$
S_0	$\mathcal{J}^{1+}$	$\mathbb{R}\oplus\mathfrak{iso}(1,d-3)$	$\mathcal{U}$	$\mathbb{R}$
S_{00}	$\mathcal{J}^{++'}$	$[\mathfrak{sp}(2,\mathbb{R})\oplus\mathfrak{so}(d-3)]$ $\Subset\mathfrak{heis}_{2(d-3)}$	0	$\mathfrak{sp}(2,\mathbb{R})$
$\mathrm{M}_+\mathrm{S}_+$	$m\,\mathcal{P}^0+s\,\mathcal{J}^{12}$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(d-3)$	$-\mathcal{U}+\mathcal{S}^{\chi\chi}$ $-m^2\,\mathcal{T}+s^2\,\mathcal{S}^{\pi\pi}$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)$
$\mathrm{m}_+\mathrm{s}_+$	$s\,(\mathcal{P}^0+\mathcal{J}^{12})$	$\mathfrak{u}(1,1)\oplus\mathfrak{so}(d-3)$	$-\mathcal{U}+\mathcal{S}^{\chi\chi}$ $+s^2\,(-\mathcal{T}+\mathcal{S}^{\pi\pi})$	$\mathfrak{u}(1,1)$
$\mathrm{M}_0\mathrm{S}_+$	$\mathcal{P}^++s\,\mathcal{J}^{12}$	$\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{iso}(d-4)$	$-\mathcal{U}+s^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{M}_0\mathrm{S}_0$	$\varepsilon\,(\mathcal{P}^++\mathcal{J}^{-1})$	$\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(d-3)$	$-\mathcal{U}-\varepsilon^2\,\mathcal{M}^\pi+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{M}_-\mathrm{S}_+$	$\mu\,\mathcal{P}^{d-1}+s\,\mathcal{J}^{12}$	$\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(1,d-4)$	$-\mathcal{U}+\mathcal{S}^{\chi\chi}$ $+\mu^2\,\mathcal{T}+s^2\,\mathcal{S}^{\pi\pi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{M}_-\mathrm{S}_0$	$\mu\,\mathcal{P}^{d-1}+\mathcal{J}^{-2}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathfrak{iso}(d-4)$	$-\mathcal{U}+\mu^2\,\mathcal{T}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathbb{R}$
$\mathrm{M}_-\mathrm{S}_-$	$\mu\,\mathcal{P}^{d-1}+\nu\,\mathcal{J}^{01}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathfrak{so}(d-3)$	$-\mathcal{U}+\mathcal{S}^{\chi\chi}$ $+\mu^2\,\mathcal{T}-\nu^2\,\mathcal{S}^{\pi\pi}$	$\mathbb{R}\oplus\mathbb{R}$
$\mathrm{m}_-\mathrm{s}_-$	$\mu\,(\mathcal{P}^{d-1}+\mathcal{J}^{01})$	$\mathfrak{gl}(2,\mathbb{R})\oplus\mathfrak{so}(d-3)$	$-\mathcal{U}+\mathcal{S}^{\chi\chi}$ $+\mu^2\,(\mathcal{T}-\mathcal{S}^{\pi\pi})$	$\mathfrak{gl}(2,\mathbb{R})$
$\mathrm{S}_+\mathrm{S}_+$	$m\,\mathcal{J}^{12}+s\,\mathcal{J}^{34}$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{so}(2,d-5)$	$\mathcal{U}+\mathcal{S}^{\chi\chi}$ $+m^2\,\mathcal{T}+s^2\,\mathcal{S}^{\pi\pi}$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)$
$\mathrm{s}_+\mathrm{s}_+$	$s(\mathcal{J}^{12}+\mathcal{J}^{34})$	$\mathfrak{u}(2)\oplus\mathfrak{so}(2,d-5)$	$\mathcal{U}+\mathcal{S}^{\chi\chi}+s^2\,(\mathcal{T}+\mathcal{S}^{\pi\pi})$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)$
$\mathrm{S}_0\mathrm{S}_+$	$\mathcal{J}^{1+}+s\mathcal{J}^{34}$	$\mathbb{R}\oplus\mathfrak{iso}(1,d-5)$	$\mathcal{U}+s^2\,\mathcal{S}^{\pi\pi}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathfrak{u}(1)$
$\mathrm{S}_0\mathrm{S}_0$	$\mathcal{J}^{1+}+\mathcal{J}^{2+'}$	$\mathbb{R}\oplus\mathfrak{so}(2)\Subset\mathfrak{heis}_{2(d-4)}$ $\Supset\mathfrak{so}(d-5)$	$\mathcal{U}+\mathcal{S}^{\chi\chi}$	$\mathbb{R}\oplus\mathfrak{iso}(2)$
$\mathrm{S}_+\mathrm{S}_{00}$	$m\,\mathcal{J}^{12}+\mathcal{J}^{++'}$	$[\mathfrak{sp}(2,\mathbb{R})\oplus\mathfrak{so}(d-5)]$ $\Subset\mathfrak{heis}_{2(d-5)}\oplus\mathfrak{u}(1)$	$\mathcal{U}+m^2\,\mathcal{T}$	$\mathfrak{u}(1)\oplus\mathfrak{u}(1)$
$\mathrm{S}_0\mathrm{S}_{00}$	$\mathcal{J}^{1+}+\mathcal{J}^{-+'}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathfrak{iso}(d-4)$	$\mathcal{U}+\mathcal{M}^\pi$	$\mathbb{R}\oplus\mathbb{R}$

Table 3. Summary of the data defining a coadjoint orbit of the AdS group and its dual except for the cases with entangled mass and spin.

E Particles with $\mathfrak{so}(2,2)$ symmetry

In this appendix, we take advantage of the semisimple nature of the AdS_3 isometry algebra $\mathfrak{so}(2,2) \simeq \mathfrak{so}(2,1)_{\mathrm{L}} \oplus \mathfrak{so}(2,1)_{\mathrm{R}}$ to describe the (co)adjoint orbits of $SO(2,2)$ as a product of $SO(2,1)$ orbits, for which we have a clear geometrical picture. The generators of $\mathfrak{so}(2,1)_{\mathrm{L}}$ and $\mathfrak{so}(2,1)_{\mathrm{R}}$ are given by

$$J^{\mathrm{L}}_a = \frac{\frac{1}{2}\,\epsilon_a{}^{bc}\,J_{bc} + P_a}{2}, \qquad J^{\mathrm{R}}_a = \frac{\frac{1}{2}\,\epsilon_a{}^{bc}\,J_{bc} - P_a}{2}\,, \tag{E.1}$$

where $J^{\mathrm{L/R}}_a$ satisfies (2.41) with the metric $\mathrm{diag}(-,+,+)$ and the Levi–Civita symbol with $\epsilon_{012} = 1$. The corresponding dual generators are

$$\mathcal{J}^a_{\mathrm{L}} = \frac{1}{2}\,\epsilon^a{}_{bc}\,\mathcal{J}^{bc} + \mathcal{P}^a, \qquad \mathcal{J}^a_{\mathrm{R}} = \frac{1}{2}\,\epsilon^a{}_{bc}\,\mathcal{J}^{bc} - \mathcal{P}^a\,, \tag{E.2}$$

and we take the representatives for each coadjoint orbits as

$$\phi_{\mathrm{L}} = \begin{cases} \pm j_{\mathrm{L}}\,\mathcal{J}^0_{\mathrm{L}} \\ \pm\mathcal{J}^+_{\mathrm{L}} \\ k_{\mathrm{L}}\,\mathcal{J}^2_{\mathrm{L}} \end{cases}, \qquad \phi_{\mathrm{R}} = \begin{cases} \mp j_{\mathrm{R}}\,\mathcal{J}^0_{\mathrm{R}} \\ \mp\mathcal{J}^+_{\mathrm{R}} \\ -k_{\mathrm{R}}\,\mathcal{J}^2_{\mathrm{R}} \end{cases}. \tag{E.3}$$

Note that for a more intuitive picture, we take an inverted picture for $\mathfrak{so}(2,1)_{\mathrm{R}}$: the upper/lower elliptic two-sheeted hyperboloids are associated with $\pm j_{\mathrm{L}}\,\mathcal{J}^0_{\mathrm{L}}$ and $\mp j_{\mathrm{R}}\,\mathcal{J}^0_{\mathrm{R}}$, and the upper/lower cones are associated with $\pm\mathcal{J}^+_{\mathrm{L}}$ and $\mp\mathcal{J}^+_{\mathrm{R}}$. The representative in $\mathfrak{so}(2,2)^*$ basis is simply given by $\phi = \phi_{\mathrm{L}} + \phi_{\mathrm{R}}$ with (E.2). In the usual massive spinning case, the mass and spin labels are related to the labels j_{L} and j_{R} of the elliptic two-sheeted hyperboloids as

$$m = j_{\mathrm{L}} + j_{\mathrm{R}}\,, \qquad s = j_{\mathrm{L}} - j_{\mathrm{R}}\,, \tag{E.4}$$

and we have similar relations in the tachyonic case,

$$\mu = k_{\mathrm{L}} + k_{\mathrm{R}}\,, \qquad -\nu = k_{\mathrm{L}} - k_{\mathrm{R}}\,. \tag{E.5}$$

In Table 4, we have collected representatives of the coadjoint orbits of $SO^+(2,2)$ and arranged them in a 'multiplication table' to highlight the product structure of the corresponding orbits, in terms of coadjoint orbits of $SO^+(2,1)$.

For the $\mathfrak{so}(2,1)$ coadjoint orbits, we have a good understanding on their quantisations. The elliptic and hyperbolic orbits of $\mathfrak{so}(2,1)$ (as classical phase spaces) give rise to the discrete and principal series representations of $\mathfrak{so}(2,1)$ (as Hilbert spaces), respectively (see e.g. [127, Sec. 2(b)]). The nilpotent orbit gives the minimal representation which lies at the end point of the principal representation. From these data, we can also understand the quantisation the $\mathfrak{so}(2,2)$ coadjoint orbits: tensor products of two $\mathfrak{so}(2,1)$ representations give the representations of $\mathfrak{so}(2,2)$ (see e.g. [89, 192] for relevant discussions). For instance, the tensor products of two discrete series of representations describe the familiar massive and massless spinning particles in AdS_3, together with the unfamiliar spinning particles in BdS. Tensor products of a principal series representation, whose spectrum is unbounded (it is not of lowest weight type), with any other representation give other exotic types of particles with $\mathfrak{so}(2,2)$ symmetry.

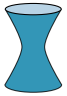
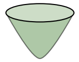
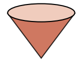
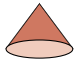
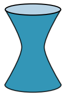
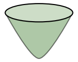
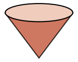
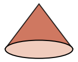
L\R						×
	$\mu\,\mathcal{P}^2+\nu\,\mathcal{J}^{01}$	$k_{\rm L}\,(\mathcal{P}^2-\mathcal{J}^{01})$ $+\,j_{\rm R}\,(\mathcal{P}^0-\mathcal{J}^{12})$	$k_{\rm L}\,(\mathcal{P}^2-\mathcal{J}^{01})$ $-\,j_{\rm R}\,(\mathcal{P}^0-\mathcal{J}^{12})$	$k_{\rm L}\,(\mathcal{P}^2-\mathcal{J}^{01})$ $+(\mathcal{P}^+-\mathcal{J}^{+2})$	$k_{\rm L}\,(\mathcal{P}^2-\mathcal{J}^{01})$ $-(\mathcal{P}^+-\mathcal{J}^{+2})$	$k_{\rm L}\,(\mathcal{P}^2-\mathcal{J}^{01})$
	$j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$ $+\,k_{\rm R}\,(\mathcal{P}^2+\mathcal{J}^{01})$	$m\,\mathcal{P}^0+s\,\mathcal{J}^{12}$	$s\,\mathcal{P}^0+m\,\mathcal{J}^{12}$	$j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$ $+(\mathcal{P}^+-\mathcal{J}^{+2})$	$j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$ $-(\mathcal{P}^+-\mathcal{J}^{+2})$	$j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$
	$-j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$ $+\,k_{\rm R}\,(\mathcal{P}^2+\mathcal{J}^{01})$	$-\,s\,\mathcal{P}^0-m\,\mathcal{J}^{12}$	$-m\,\mathcal{P}^0-s\,\mathcal{J}^{12}$	$-\,j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$ $+(\mathcal{P}^+-\mathcal{J}^{+2})$	$-\,j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$ $-(\mathcal{P}^+-\mathcal{J}^{+2})$	$-\,j_{\rm L}\,(\mathcal{P}^0+\mathcal{J}^{12})$
	$(\mathcal{P}^++\mathcal{J}^{+2})$ $+\,k_{\rm R}(\mathcal{P}^2+\mathcal{J}^{01})$	$(\mathcal{P}^++\mathcal{J}^{+2})$ $+\,j_{\rm R}(\mathcal{P}^0-\mathcal{J}^{12})$	$(\mathcal{P}^++\mathcal{J}^{+2})$ $-\,j_{\rm R}(\mathcal{P}^0-\mathcal{J}^{12})$	$2\,\mathcal{P}^+$	$2\,\mathcal{J}^{+2}$	$\mathcal{P}^++\mathcal{J}^{+2}$
	$-(\mathcal{P}^++\mathcal{J}^{+2})$ $+\,k_{\rm R}(\mathcal{P}^2+\mathcal{J}^{01})$	$-(\mathcal{P}^++\mathcal{J}^{+2})$ $+\,j_{\rm R}(\mathcal{P}^0-\mathcal{J}^{12})$	$-(\mathcal{P}^++\mathcal{J}^{+2})$ $-\,j_{\rm R}(\mathcal{P}^0-\mathcal{J}^{12})$	$-2\,\mathcal{J}^{+2}$	$-2\,\mathcal{P}^+$	$-(\mathcal{P}^++\mathcal{J}^{+2})$
×	$k_{\rm R}\,(\mathcal{P}^2+\mathcal{J}^{01})$	$j_{\rm R}\,(\mathcal{P}^0-\mathcal{J}^{12})$	$-j_{\rm R}\,(\mathcal{P}^0-\mathcal{J}^{12})$	$\mathcal{P}^+-\mathcal{J}^{+2}$	$-(\mathcal{P}^+-\mathcal{J}^{+2})$	0

Table 4. Multiplication table for orbits of $SO^+(2,1)$ and $SO^+(2,2)$

F Comparison with Metsaev's work

Through a series of work [93, 163, 167–170], Metsaev constructed and studied an infinite-component field theory Lagrangian, in terms of which he could identify many novel elementary fields. In this section, we attempt to make a correspondence between such fields and the coadjoint orbits classified in this paper. A proper quantisation of each of these orbits can make this correspondence precise eventually, but for the moment we only make a preliminary assessment.

In [93], Metsaev classified different fields in AdS according to their quadratic and quartic Casimir values. These Casimir values should coincide with those of coadjoint orbits up to a quantum shift, which would arise from an ordering issue and depends on dimensions. Discarding the shift, the Metsaev's parameterisation of C_2 and C_4 are

$$C_2 = p^2 + q^2\,, \qquad C_4 = p^2\,q^2\,, \tag{F.1}$$

where p and q are complex numbers. Basically, p and q are related to the mass m and spin s, or their analogues (see below). Imposing a unitarity of field theory Lagrangian, possible values of p and q are further restricted and there are six classes. Using the same enumeration symbol as in [93] for different classes, we have

i. $\Re p = 0\,,\, \Re q = 0\ (p = i\,\mu\,,\, q = i\,\nu)$:
This case corresponds to the orbit $\phi = \mu\,\mathcal{P}^{d-1} + \nu\,\mathcal{J}^{01}$ with Casimirs (6.50). In our classification, we found a shortening condition $\mu = \nu$ where a small orbit $\phi = \mu\,(\mathcal{P}^{d-1} + \mathcal{J}^{01})$ (6.51), together with a large remnant orbit $\phi = \mu\,(\mathcal{P}^{d-1} + \mathcal{J}^{01}) + \epsilon\,(\mathcal{P}^0 + \mathcal{P}^{d-1} - \mathcal{J}^{d-1\,1} - \mathcal{J}^{01})$ appear, but there is no analogue of this in Metsaev's result.

ii. $p^* = q\ (p = s + i\,\nu\,,\, q = s - i\,\nu)$:
This case corresponds to the orbit $\phi = s\,(\mathcal{P}^0 + \mathcal{J}^{12}) + \nu\,(\mathcal{P}^{d-1} - \mathcal{J}^{02})$ with Casimir (6.64).

iii. $p^* = -q\ (p = s + i\,\nu\,,\, q = -s + i\,\nu)$:
This case corresponds to the orbit $\phi = s\,(\mathcal{P}^0 - \mathcal{J}^{12}) + \nu\,(\mathcal{P}^{d-1} + \mathcal{J}^{02})$ which can be obtained from the previous case by a π-rotation in (2–3) plane. Hence, according to our classification, this case is equivalent to the previous one for $d \geq 4$. For $d = 3$, they are different but related by the parity map.

iv. $\Re p = 0\,,\, \Im q = 0\ (p = i\,\mu\,,\, q = s)$:
This case corresponds to the orbit $\phi = \mu\,\mathcal{P}^{d-1} + s\,\mathcal{J}^{12}$ with Casimir (6.43). In our classification, any integer values are allowed for s, but in Metsaev's result only a small interval near 0 is allowed for q. When q is on the boundary of the interval, the field becomes reducible. This reducible point seems related to the $\nu \to 0$ limit of $\phi = \mu\,\mathcal{P}^{d-1} + \nu\,\mathcal{J}^{01}$ where a short scalar tachyon orbit $\phi = \mu\,\mathcal{P}^{d-1}$ appear together with a large remnant orbit $\phi = \mu\,\mathcal{P}^{d-1} + \epsilon\,\mathcal{J}^{-1}$. The small interval may correspond to the complementary series representation arising from a quantisation of the orbit $\phi = \mu\,\mathcal{P}^{d-1} + \epsilon\,\mathcal{J}^{-1}$ containing a singularity with a deformation parameter.

v. $\Im p = 0\,,\, \Re q = 0\ (p = m\,,\, q = i\,\nu)$:
This case is also further restricted such that only a small interval near 0 is allowed for p and the field becomes reducible when p takes the boundary value of the interval. This seems again related to the $\mu \to 0$ limit of $\phi = \mu\,\mathcal{P}^{d-1} + \nu\,\mathcal{J}^{01}$ where a short scalar tachyon orbit $\phi = \nu\,\mathcal{J}^{01} \simeq \nu\,\mathcal{P}^{d-1}$ appear together with a large remnant orbit $\phi = E\,\mathcal{P}^{+} + \nu\,\mathcal{J}^{01}$ (which is of a different class from the orbit $\phi = \mu\,\mathcal{P}^{d-1} + \epsilon\,\mathcal{J}^{-1}$ appeared in the previous case). Again, the small interval may correspond to the complementary series representation arising from a quantisation of singular orbit $\phi = E\,\mathcal{P}^{+} + \nu\,\mathcal{J}^{01}$ with a deformation parameter.

vi. $\Im p = 0\,,\, \Im q = 0\ (p = m\,,\, q = s)$:
This case is divided into several sub cases, and all such cases seem related to the shortening condition $m = s$ where a small massless orbit $\phi = s\,(\mathcal{P}^0 + \mathcal{J}^{12})$ appears together with a large remnant one $\phi = s\,(\mathcal{P}^0 + \mathcal{J}^{12}) + \epsilon\,(\mathcal{P}^0 + \mathcal{P}^1 - \mathcal{J}^{02} - \mathcal{J}^{12})$. The latter orbit again contains a singularity and its quantisation may involve a deformation parameter. In such a case, the spin projection does not take place, and s does not need to be quantised either. Therefore, this will lead to a small interval either only one among p and q or for both of p and q near the shortening point given by an integer $m = s$. The Metsaev's results treat the cases with p and q exchanged as different. This may correspond again to the orbits related by a parity map.

In this section, we discussed a possible link between the results of Metsaev and our coadjoint orbit classification. Metsaev's results cover a part of coadjoint orbits and they are often related to a deformation quantisation of the orbit with a non-trivial nilpotent part. Let us conclude this section with a disclaimer that the above discussion is rather a speculation for the moment. We will revisit this issue in the sequel paper, and hopefully provide more evidences for the statements we made in this section.

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