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Generalized Freudenthal duality for rotating extremal black holes

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Abstract: Freudenthal duality (FD) is a non-linear symmetry of the Bekenstein-Hawking entropy of extremal dyonic black holes (BHs) in Maxwell-Einstein-scalar theories in four space-time dimensions realized as an anti-involutive map in the symplectic space of electric-magnetic BH charges. In this paper, we generalize FD to the class of rotating (stationary) extremal BHs, both in the under- and over-rotating regime, defining a (generalized) rotating FD (generally, non-anti-involutive) map (RFD), which also acts on the BH angular momentum. We prove that the RFD map is unique, and we compute the explicit expression of its non-linear action on the angular momentum itself. Interestingly, in the non-rotating limit, RFD bifurcates into the usual, non-rotating FD branch and into a spurious branch, named "golden" branch, mapping a non-rotating (static) extremal BH to an under-rotating (stationary) extremal BH, in which the ratio between the angular momentum and the non-rotating entropy is the square root of the golden ratio. Finally, we investigate the possibility of inducing transitions between the under- and over- rotating regimes by means of RFD, obtaining a *no-go* result.

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Contents

1 Introduction

Extremal black holes (BHs) are objects of great interest to the string theorists, though they concern about a pretty ideal situation, as they lack temperature T. On the other hand, astrophysical BHs are never exactly extremal, but, for instance, the BH GRS1915+105 observed through X-ray and a radio telescope is likely within 1% of the extremal value of its angular momentum [1]; in other words, this BH is *near-extremal*, i.e. not far from saturating the extremality bound.

For asymptotically flat and static solutions, *Freudenthal duality* (FD) is an intrinsically non-linear and anti-involutive map between the BH dyonic charges, which, unexpectedly, keeps the Bekenstein-Hawking BH entropy invariant [2–4]. Interestingly, under FD, the attractor configurations of the scalar fields at the electric-magnetic (e.m.) dyonic BH event horizon also remain invariant [3]. A suitable extension of FD has been formulated for Maxwell-Einstein theories in the presence of gaugings of the isometries of the vector multiplets' or

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hypermultiplets' scalar manifolds (*at least* as far as Abelian gaugings are concerned [5]). Furthermore, in recent years FD has turned out to characterize a number of directions of investigation within the Maxwell-Einstein (super)gravity theories coupled to non-linear sigma models of scalar fields [6–11]. A crucial point to stress is that FD is inherently different from the electro-magnetic duality (*aka* U-duality in string theory[1]), since the later act *linearly* on dyonic BH charges, contrary to the former [15–17].

Recently, in [18] and [19], FD has been studied and further extended to the class of *near-extremal* (non-rotating) BHs. As mentioned above, for extremal, (asymptotically flat) non-rotating BHs, FD is an anti-involutive non-linear map acting on the e.m. BH charges, and it is a symmetry of the Bekenstein-Hawking BH entropy, which is a homogeneous function of degree two in the charges themselves. Since the T-dependent entropy of a near-extremal BH is no longer a degree-two homogeneous function of charges [18], a consistent generalization of FD has been achieved in [19] only at the price of transforming the temperature T. The resulting map, which is no longer anti-involutive (but nevertheless is analytical and unique), has been termed *near-extremal (generalized[2]) FD*: two near-extremal BHs, whose charges and (small) temperatures are connected by the near-extremal FD, have the same entropy.

In this paper, we aim to further extend the notion of FD to the class of extremal (stationary) *rotating* BHs, within the same spirit as [19]. In section 2, we show that in general the definition of the Freudenthal dual for an extremal (asymptotically flat) over-(or under-)rotating BH is *not* consistent while keeping its angular momentum fixed. To overcome this issue, in section 3 we define a — generally, non-anti-involutive — *(generalized) rotating FD* (RFD) map: interestingly, within our formulation, RFD uniquely maps the dyonic e.m. charges and the angular momentum J of an extremal rotating BH to the ones of another extremal rotating BH, while keeping the Bekenstein-Hawking BH entropy fixed.

The plan of the paper is as follows. After a first, naïve generalizing approach and some preliminary considerations in section 2, we present the generalization of FD to RFD in section 3, analyzing in detail its non-rotating limit in section 3.1. Then, after presenting numerical evidence in section 4, we prove the uniqueness of the RFD map in section 5, and determine its analytic form in section 6. Finally, in section 7 we investigate the possibility of inducing transitions between the (stationary) under- and over-rotating regimes by means of the RFD map, obtaining a *no-go* result. Final remarks and hints for further developments are given in the concluding section 8. Appendices A and B, containing details concerning the treatment given in section 7, conclude the paper.

2 Freudenthal duality (FD) and rotating BHs

In this section, we analyse the prospect of having an F-dual of an extremal, over-(or under)-rotating BH, such that both of them have the same angular momentum and entropy whereas their e.m. charges are related by FD.

Before proceeding further, we first briefly review the Freudenthal duality for extremal, non-rotating black holes. For such black holes with e.m. charges collected into the symplectic

[1] Here U-duality is referred to as the "continuous" symmetries of [12, 13]. Their discrete versions are the U-duality of the non-perturbative string theory symmetries introduced by Hull and Townsend in [14].

[2] Not to be confused with the *general Freudenthal transformations* (GFT's) treated in [11].

vector Q and entropy $S_0(Q)$, the Freudenthal duality acts non-trivially on Q as follows

$$Q \longmapsto \hat{Q}(Q) := \Omega \frac{\partial S_0(Q)}{\partial Q} \tag{2.1}$$

such that

$$S_0(Q) = S_0\left(\Omega \frac{\partial S_0\left(Q\right)}{\partial Q}\right). \tag{2.2}$$

Ω is a symplectic matrix with $\Omega^T = -\Omega$ and $\Omega^2 = -\mathbb{I}$. In other words, two non-rotating, extremal black holes with e.m. charge vectors Q and $\hat{Q}$ related to each other by (2.1) have the same value for their entropy. The non-rotating, extremal entropy is a degree-two homogeneous function in the e.m. charge Q i.e.

$$S_0\left(\lambda Q\right) = \lambda^2 S_0(Q), \ \forall \lambda \in \mathbb{R}, \tag{2.3}$$

and this property plays a fundamental role for the invariance of the entropy under (2.1).

It is worth noticing that, in this case, the transformation (2.1) is anti-involutive, i.e. $\hat{\hat{Q}} = -Q$.

The study of the Freudenthal duality was further extended in a non-anti-involutive way for the near-extremal, non-rotating black holes in [18, 19]. Due to the presence of the temperature T in the form, the entropy of a near-extremal, non-rotating black hole is no more a degree-two homogeneous function of e.m. charges. Hence, the obvious way of charge transformation following (2.1), i.e. $\hat{Q}(Q;T) := \Omega \frac{\partial S_{NE}(Q;T)}{\partial Q}$, called as “on-shell” FD does not keep the near-extremal entropy invariant. In spite of this, it has been shown that two different near-extremal, non-rotating black holes with charge and temperature respectively (Q, T) and $(\hat{Q}, T + \delta T)$ have the same entropy when

$$Q \longmapsto \hat{Q}(Q; T + \delta T) := \Omega \frac{\partial S_{NE}(Q; T + \delta T)}{\partial Q} \tag{2.4}$$

with a unique solution for δT.

With this brief recap, we now establish the Freudenthal duality for extremal, rotating black holes.

After [20, 21], the Bekenstein-Hawking entropy for an extremal, (asymptotically flat) under-rotating Kerr-Newman BH with angular momentum J and e.m. charge Q is given by

$$S_{\text{under}}\left(Q, J\right) = \sqrt{S_0^2\left(Q\right) - J^2}; \tag{2.5}$$

$$S_0^2\left(Q\right) - J^2 > 0, \tag{2.6}$$

whereas the entropy for an extremal, (asymptotically flat) over-rotating Kerr-Newman BH is[3]

$$S_{\text{over}}\left(Q, J\right) = \sqrt{J^2 - S_0^2\left(Q\right)}; \tag{2.7}$$

$$J^2 - S_0^2\left(Q\right) > 0. \tag{2.8}$$

[3]Clearly, only the under-rotating class (2.5)–(2.6) of solutions have a well-defined non-rotating limit.

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In general, within the semi-classical supergravity approximation, the e.m. charges take real values, and so do the angular momentum J and the non-rotating entropy $S_0(Q)$, which are further constrained to be non-negative and strictly positive, respectively: $J \in \mathbb{R}^+$, $S_0 \in \mathbb{R}_0^+$.

Following the usual formulation of FD [3], the action of the "on-shell" FD map on Q reads

$$Q \longmapsto \hat{Q}(Q,J) := \Omega \frac{\partial S_{\text{under(over)}}(Q,J)}{\partial Q} = \pm \frac{S_0(Q)}{\sqrt{|S_0^2(Q) - J^2|}} \Omega \frac{\partial S_0(Q)}{\partial Q}, \tag{2.9}$$

with Ω denoting the symplectic invariant structure of the e.m. charge representation space ($\Omega^T = -\Omega$ and $\Omega^2 = -\mathbb{I}$); the "+" and "−" signs correspond to the under-rotating and over-rotating cases respectively. Thus, a naïve generalization of FD (leaving J fixed), in these cases, can be formulated as a map acting on Q as (2.9), such that

$$S_{\text{under(over)}}(Q,J) = S_{\text{under(over)}}\left(\Omega \frac{\partial S_{\text{under(over)}}(Q,J)}{\partial Q}, J\right). \tag{2.10}$$

Using (2.5) and (2.7), one can show that for both the under-rotating and over-rotating extremal Kerr-Newman BHs, the conditions for the existence of a Freudenthal dual boils down to solving a very simple algebraic equation, namely[4]

$$S_0^2 J^2 \left(J^2 - 2S_0^2\right) = 0. \tag{2.11}$$

2.1 Under-rotating

For an under-rotating BH, $J^2 - S_0^2(Q) < 0$, and therefore $J^2 - 2S_0^2(Q) < 0$ always. In this case, the only possible solution of (2.11) is $J = 0$. This is a trivial solution, since in such a limit (2.10) reduces to the usual (and anti-involutive) notion of FD for a non-rotating, extremal BH [3].

2.2 Over-rotating

For an over-rotating BH, $J^2 - S_0^2(Q) > 0$, and this rules out the possibility of the solution $J = 0$ to (2.11). Still, one could define the F-dual of an over-rotating extremal Kerr-Newman BH, when the condition (2.11) is met as

$$J^2 = 2S_0^2(Q) \Rightarrow S_{\text{over}}(Q,J) = S_0(Q). \tag{2.12}$$

Thus, one can conceive the starting extremal BH as a non-rotating BH, for which having a F-dual is obvious [3].

3 Generalized rotating Freudenthal duality (RFD)

We have just observed that, by the usual notion of FD, one cannot map two extremal rotating BHs to each other. In this section, we go beyond the usual definition of FD and try to map two extremal rotating BHs (both under- or over-rotating) with *different* angular momenta.

[4]Note that $S_0 \equiv S_0(Q)$.

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Namely, motivated by the approach carried out in [19] for a near-extremal BH with the temperature T, here we define the transformation of the dyonic e.m. BH charges as follows:

$$\begin{aligned} Q \rightarrow \hat{Q}(Q, J+\delta J) &:= \Omega \frac{\partial S_{\text{under(over)}}(Q, J+\delta J)}{\partial Q} \\ &= \pm \frac{S_0(Q)}{\sqrt{\left|S_0^2(Q) - (J+\delta J)^2\right|}} \Omega \frac{\partial S_0(Q)}{\partial Q}. \end{aligned} \tag{3.1}$$

Where the branch "+" and "−" represents the under-rotating ($S_0^2(Q) > (J+\delta J)^2$) and over-rotating ($S_0^2(Q) < (J+\delta J)^2$) cases respectively, such that the Bekenstein-Hawking BH entropy remains invariant, i.e.

$$S_{\text{under(over)}}(Q, J) = S_{\text{under(over)}}\left(\hat{Q}(Q, J+\delta J), J+\delta J\right). \tag{3.2}$$

Therefore, the set of transformations

$$\text{RFD}: \begin{cases} Q \rightarrow \hat{Q}(Q, J+\delta J); \\ \\ J \rightarrow J+\delta J; \end{cases} \tag{3.3}$$

define the *(generalized) rotating FD* (RFD) map. By using (2.5), (2.7), for both under- and over- rotating BHs, the condition (3.2) boils down to the following sextic algebraic inhomogeneous equation in the variation δJ of the angular momentum:

$$a_1(\delta J)^6 + a_2(\delta J)^5 + a_3(\delta J)^4 + a_4(\delta J)^3 + a_5(\delta J)^2 + a_6\delta J + a_7 = 0, \tag{3.4}$$

where each coefficients is a function of J and $S_0 = S_0(Q)$,

$$\begin{aligned} a_1 &= 1; \\ a_2 &= 6J; \\ a_3 &= 14J^2 - S_0^2; \\ a_4 &= 4\left(4J^2 - S_0^2\right)J; \\ a_5 &= -S_0^4 + 9J^4 - 4S_0^2J^2; \\ a_6 &= -2\left(S_0^4 - J^4\right)J; \\ a_7 &= \left(J^2 - 2S_0^2\right)S_0^2J^2. \end{aligned} \tag{3.5}$$

In order to find out two rotating extremal BHs with different angular momenta but the same entropy, having their dyonic charges related by RFD (3.3), we need to look for a real solution of (3.4), namely for $\delta J = \delta J(J, S_0) \in \mathbb{R}$ such that the transformed angular momentum reads $\hat{J} := J + \delta J \in \mathbb{R}^+$.

3.1 Non-rotating limit of RFD: the spurious, "golden" branch

Before dealing with a detailed analysis of the roots of the algebraic equation (3.4), let us investigate the limit $J \rightarrow 0^{(+)}$ of the set of solutions to (3.4). In this limit, which is well-defined only for the under-rotating class of solutions, $\hat{J} \rightarrow \delta J^{(+)} \in \mathbb{R}^+$, and eq. (3.4) reduces to

$$(\delta J)^6 - S_0^2(\delta J)^4 - S_0^4(\delta J)^2 = 0, \tag{3.6}$$

which consistently admits the solution $\delta J = 0$. This is no surprise, since in the $J \to 0^{(+)}$ limit, RFD simplifies down to its usual, non-rotating definition (namely, to FD, [3]).

Interestingly, other two real solutions to the cubic algebraic homogeneous equation (in $(\delta J)^2$) (3.6) exist. In fact, eq. (3.6) can be solved by two more real roots[5]

$$\delta J_{\pm} := \pm\sqrt{\phi}\, S_0\left(Q\right), \tag{3.7}$$

with $\phi := \frac{1+\sqrt{5}}{2}$ being the so-called *golden ratio* (see e.g. [22]). Since $\hat{J} \to \delta J^{(+)} \in \mathbb{R}^+$, only δJ_+ in (3.7) has a sensible physical meaning. Correspondingly, within the $J \to 0^+$ (i.e., non-rotating) limit, the unique non-vanishing, physically sensible solution for the angular momentum transformation reads

$$\hat{J}_{\text{golden}} := \delta J_+ = \sqrt{\phi} S_0\left(Q\right) = \sqrt{\frac{1+\sqrt{5}}{2}} S_0(Q). \tag{3.8}$$

This analysis shows the existence, in the limit $J \to 0^+$, of a spurious, "golden" branch of RFD, which in the non-rotating limit does not reduce to the usual FD, but rather it allows to map a non-rotating extremal BH to an under-rotating (stationary) one; this can be depicted as

$$\underset{\text{static extremal BH}}{\begin{cases} S = S_0\left(Q\right) \\ J = 0 \end{cases}} \xrightarrow{\text{RFD}_{J\to 0^+,\ \text{golden}}} \underset{\text{under-rotating extremal BH}}{\begin{cases} S_{\text{under}} = S_{\text{under}}\left(\hat{Q}_{\text{golden}}, \hat{J}_{\text{golden}}\right) \\ J = \hat{J}_{\text{golden}} \end{cases}}, \tag{3.9}$$

where $\hat{J}_{\text{golden}}$ is defined by (3.8), $\hat{Q}_{\text{golden}}$ is defined by

$$\begin{aligned} \hat{Q}_{\text{golden}} :=& \Omega \frac{\partial S_{\text{over}}\left(Q, \hat{J}_{\text{golden}}\right)}{\partial Q} = -\frac{S_0\left(Q\right)}{\sqrt{\hat{J}^2_{\text{golden}} - S_0^2\left(Q\right)}} \Omega \frac{\partial S_0\left(Q\right)}{\partial Q} \\ =& -\frac{S_0\left(Q\right)}{\sqrt{\phi S_0^2(Q) - S_0^2\left(Q\right)}} \Omega \frac{\partial S_0\left(Q\right)}{\partial Q} = -\frac{1}{\sqrt{\phi - 1}} \Omega \frac{\partial S_0\left(Q\right)}{\partial Q} \\ =& -\sqrt{\phi} \Omega \frac{\partial S_0\left(Q\right)}{\partial Q}, \end{aligned} \tag{3.10}$$

and $\text{RFD}_{J\to 0^+,\ \text{golden}}$ denotes such a spurious, "golden" branch of the non-rotating limit of RFD (3.3):

$$\text{RFD}_{J\to 0^+,\ \text{golden}} : \begin{cases} Q \to \hat{Q}_{\text{golden}} = -\sqrt{\phi}\Omega \frac{\partial S_0(Q)}{\partial Q}; \\ \\ J = 0 \to \hat{J}_{\text{golden}}; \end{cases} \tag{3.11}$$

Note that in the last step of (3.10) we used the crucial property of the golden ratio ϕ, namely

$$\phi - 1 = \frac{1}{\phi}. \tag{3.12}$$

[5] There exist also two purely imaginary roots with $\delta J^2 = S_0^2 \left(\frac{1-\sqrt{5}}{2}\right)$, which we ignore.

Consistently, the total entropy is preserved by the map $(\mathrm{RFD})_{J\to 0^+,\ \text{golden}}$ (3.11), because

$$\begin{aligned} S_{\text{under}}\left(\hat{Q}_{\text{golden}},\hat{J}_{\text{golden}}\right) &= \sqrt{S_0^2\left(\hat{Q}_{\text{golden}}\right)-\hat{J}^2_{\text{golden}}} \\ &= \sqrt{S_0^2\left(-\sqrt{\phi}\Omega\frac{\partial S_0\left(Q\right)}{\partial Q}\right)-\phi S_0^2\left(Q\right)} \\ &= \sqrt{\phi^2 S_0^2\left(\Omega\frac{\partial S_0\left(Q\right)}{\partial Q}\right)-\phi S_0^2\left(Q\right)} \\ &= \sqrt{\phi^2-\phi}S_0\left(Q\right)=S_0\left(Q\right), \end{aligned} \tag{3.13}$$

where in the last step the crucial property (3.12) has been used again. Moreover, in achieving (3.13), we have also exploited two crucial properties of the static extremal BH entropy $S_0(Q)$, namely its homogeneity of degree two in the e.m. charges, and its invariance under the Freudenthal duality for static extremal BHs, as given by (2.3) and (2.2) respectively.

Thus, the two extremal BHs in the l.h.s. and r.h.s. of (3.9) have the same Bekenstein-Hawking entropy, preserved by the map $(\mathrm{RFD})_{J\to 0^+,\ \text{golden}}$ (3.11).

Remark 1. It is worth remarking that the stationary extremal BH in the r.h.s. of (3.9), namely the image of an extremal, static BH under the map $\mathrm{RFD}_{J\to 0^+,\ \text{golden}}$ (3.11), is necessarily *under-rotating*, because the non-rotating limit $J\to 0^+$ is well-defined *only* in the under-rotating case. On the other hand, the BH entropy entering the definition of $\hat{Q}_{\text{golden}}$ (3.10) is necessarily of the *over-rotating* type, since

$$\phi>1\Rightarrow \hat{J}_{\text{golden}}=\sqrt{\phi}S_0\left(Q\right)>S_0\left(Q\right)\Leftrightarrow \hat{J}^2_{\text{golden}}-S_0^2\left(Q\right)>0, \tag{3.14}$$

which pertains to the over-rotating case.

Remark 2. In a sense, the use of the spurious, "golden" branch $(\mathrm{RFD})_{J\to 0^+,\ \text{golden}}$ (3.11) of the map RFD (3.3) can be regarded as a kind of solution-generating technique, which generates an under-rotating, stationary extremal BH from a non-rotating, static BH, while keeping the Bekenstein-Hawking entropy fixed. Indeed, while both such BHs are asymptotically flat, their near-horizon geometry changes under $(\mathrm{RFD})_{J\to 0^+,\ \text{golden}}$:

$$\underset{\text{static extremal BH [23]}}{AdS_2\otimes S^2}\ \overset{(\mathrm{RFD})_{J\to 0^+,\ \text{golden}}}{\longrightarrow}\ \underset{\text{under-rotating extremal BH [24]}}{AdS_2\otimes S^1}. \tag{3.15}$$

Remark 3. The expression of the Bekenstein-Hawking entropy of under-rotating resp. over-rotating stationary extremal BHs, respectively given by (2.5)–(2.6) and (2.7)–(2.8), is suggestive of a representation of the two fundamental quantities S_0 and J characterizing such BHs in terms of elements of the *split complex* (also named *hypercomplex*) numbers $\mathbb{C}_s$ (see e.g. [25]), i.e. respectively as

$$Z:=S_0+\mathbf{i}J\in\mathbb{C}_s\ \Rightarrow S_{\text{under}}=|Z|:=\sqrt{Z\bar{Z}}=\sqrt{S_0^2-J^2}; \tag{3.16}$$

$$Y:=\mathbf{i}Z=J+\mathbf{i}S_0\in\mathbb{C}_s\ \Rightarrow S_{\text{over}}=|Y|:=\sqrt{-Z\bar{Z}}=\sqrt{J^2-S_0^2}, \tag{3.17}$$

where $\mathbf{i}$ denotes the split imaginary unit ($\mathbf{i}^2=1$, not to be confused with ± 1), and the bar stands for the hypercomplex conjugation, defined as $\bar{Z}:=S_0-\mathbf{i}J$.

Thus, any map acting on S_0 and J, as the RFD map (3.3) itself, can be represented as acting on the hypercomplex (analogue of the) Argand-Gauss plane, denoted by $\mathbb{C}_s[S_0, J]$ resp. $\mathbb{C}_s[J, S_0]$ for under- resp. over-rotating BHs. Since $\mathbb{C}_s$ is not a field (because it contains zero divisors due to its split nature), there are no split analogues of the "wild" automorphisms [26] in $\mathbb{C}_s$, but rather only four discrete fundamental automorphisms, namely $\mathbb{I}$, $-\mathbb{I}$, $\mathbf{C}$ and $-\mathbf{C}$, where $\mathbb{I}$ and $\mathbf{C}$ respectively denote the identity map and the aforementioned conjugation map on $\mathbb{C}_s[S_0, J]$ or $\mathbb{C}_s[J, S_0]$; interestingly, since $S_0 \in \mathbb{R}_0^+$ and $J \in \mathbb{R}^+$, none (but, trivially, $\mathbb{I}$) of such discrete automorphisms are physically allowed, and the physically sensible quadrant of $\mathbb{C}_s[S_0, J]$ resp. $\mathbb{C}_s[J, S_0]$ is only the first one (with the J axis excluded).

In particular, the RFD map acts on any $Z \in \mathbb{C}_s[S_0, J]$ resp. $Y \in \mathbb{C}_s[J, S_0]$ as a norm-preserving transformation, because, by definition, it preserves the Bekenstein-Hawking entropy S, as given by the defining condition (3.2). Thus, considering a under- resp. over-rotating (stationary, asymptotically flat) extremal BH with Bekenstein-Hawking entropy $S = \mathcal{S} \in \mathbb{R}^+$, its RFD-dual extremal BH will belong to the arc of the hyperbola $|Z|^2 = S_0^2 - J^2 = \mathcal{S}$ resp. $|Y|^2 = J^2 - S_0^2 = \mathcal{S}$ within the first quadrant (with the J axis excluded) of the hypercomplex Argand-Gauss plane.

In the non-rotating limit, in light of the treatment of section 3.1, the action of the RFD map (3.3) on a non-rotating extremal BH (represented as a point of the S_0 axis — with its origin excluded — in the hyper-Argand-Gauss plane $\mathbb{C}_s[S_0, J]$, thus with coordinates $(S_0, 0)$) may be nothing but the identity $\mathbb{I}$ (in the usual, non-rotating branch of $\mathrm{RFD}_{J\to 0^+}$) or a point with coordinates $(\phi S_0, \sqrt{\phi} S_0)$ along the arc of hyperbola defined as $\mathcal{H} := \left\{ Z \in \mathbb{C}_s[S_0, J] : |Z|^2 = S_0^2 \right\}$ (in the spurious, "golden" branch $\mathrm{RFD}_{J\to 0^+,\ \mathrm{golden}}$ of $\mathrm{RFD}_{\mathrm{J}\to 0^+}$, discussed above).

4 Numerical interlude

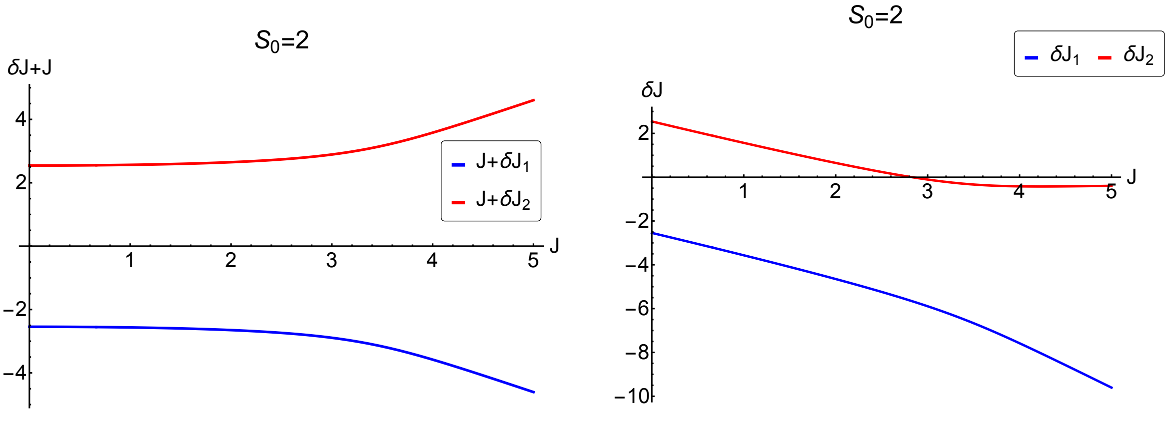
Before carrying out an analytical study of the roots of the algebraic eq. (3.4), we want to present numerical evidence that for $J > 0$ (thus going beyond the non-rotating limit), only two roots, out of the six roots of the sextic inhomogeneous algebraic eq. (3.4) (which exist in the algebraically closed field of complex numbers $\mathbb{C}$, by the fundamental theorem of algebra), are real roots. Moreover, as shown in the plots of figure 1, out of such two real roots, named δJ_1 and δJ_2, only one, say δJ_2, is consistent with the requirement of physical soundness, namely with the condition that $\hat{J} := J + \delta J \geqslant 0$. figure 1(a) shows the plot[6]of $J + \delta J$ vs. J for fixed entropy, whereas figure 1(b) shows the plot of δJ vs. J, for fixed entropy for the two aforementioned real roots of eq. (3.4). It is worth here remarking that δJ_2 actually goes into (3.8) in the limit $J \to 0^+$, as expected.

5 Uniqueness of RFD

Recalling the definition $\hat{J} := J + \delta J \in \mathbb{R}^+$ of the (physically sensible) RFD-dual angular momentum as, the sextic algebraic eq. (3.4)–(3.5) can be rewritten as

$$\hat{J}^6 - \hat{J}^4(J^2 + S_0^2) - \hat{J}^2 S_0^2(S_0^2 - 2J^2) - J^2 S_0^4 = 0. \tag{5.1}$$

[6]In the same plot, the consideration of the under-rotating or of the over-rotating case just affects the range of the plot, while not affecting the uniqueness of the real (and physically sensible) root.



(a) (b)

Figure 1. The two real roots of (3.4).

By setting $x := \hat{J}^2$, (5.1) becomes an inhomogeneous cubic algebraic equation,

$$x^3 - x^2(J^2 + S_0^2) - x(S_0^2 - 2J^2)S_0^2 - J^2 S_0^4 = 0. \tag{5.2}$$

In order to analyse the roots of (5.2), we consider the following algebraic curve[7]

$$f(x) := x^3 - x^2(J^2 + S_0^2) - x(S_0^2 - 2J^2)S_0^2 - J^2 S_0^4, \tag{5.3}$$

and look for the *turning points* on x vs. $y = f(x)$, indeed, counting the number of times $f(x)$ crosses the x-axis provides the number of real roots.

In general, a cubic equation has either one or three real roots, and *at most* two turning points. The collective information about the locations of the turning points and the x-intercept of $f(x)$ or the position of $f(0)$ provides us with a clear picture of the situation. In order to compute the number of turning points or the points where the slope goes to zero, one has to solve

$$\frac{df(x)}{dx} = (S_0^2 - x)(2J^2 - S_0^2 - 3x) = 0,$$

which generates the following turning points $T_i := (x_i, f(x_i))$, $i = 1, 2$:

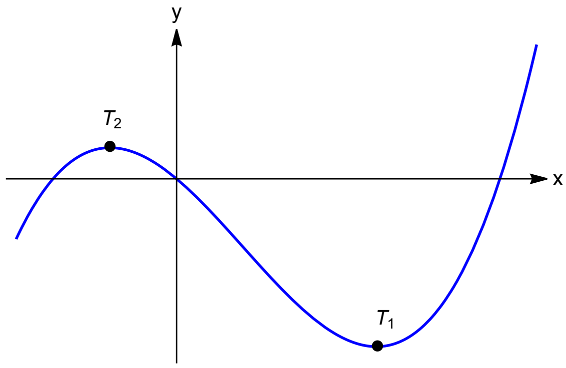
$$T_1 := \left(S_0^2, -S_0^6\right), \qquad T_2 := \left(\frac{1}{3}(2J^2 - S_0^2), \frac{\mathcal{A}}{27}\right), \tag{5.4}$$

$$\mathcal{A} := -4(J^2 - 2S_0^2)^3 - 27S_0^6. \tag{5.5}$$

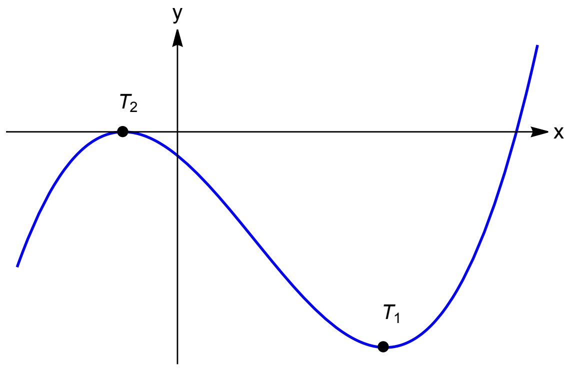
As $S_0 > 0$, T_1 always lies in the fourth quadrant of the $x - y$ plane. From the values of $f(x)$ defined in (5.3) at both the turning points T_1 and T_2, one can conclude the number of real roots for the cubic equation $f(x) = 0$ (5.2). For further insight, one can look at the second derivatives at the turning points,

$$\mathcal{S}_1 := \left.\frac{d^2 f(x)}{dx^2}\right|_{x=S_0^2} = -2(J^2 - 2S_0^2), \qquad \mathcal{S}_2 := \left.\frac{d^2 f}{dx^2}\right|_{x=1/3(2J^2 - S_0^2)} = 2(J^2 - 2S_0^2), \tag{5.6}$$

[7]The y intercept, $f(0) = -J^2 S_0^4$, is always negative, or zero only at $J = 0$.



(a) $J = 0$.



(b) $J \neq 0$, $\mathcal{A} = 0$.

Figure 2. Typical plots for sections 5.1 and 5.2.

implying that the convexity of the function $y = f(x)$ always remains opposite at T_1 and T_2. The case $J^2 = 2S_0^2$, which can only arise for over-rotating BHs, will be discussed further below.

The same information can be derived from the discriminant of the cubic equation (5.3), which can be computed to read

$$\Delta = S_0^6\,\mathcal{A}. \tag{5.7}$$

For any cubic equation, if the discriminant $\Delta > 0$, there are three real roots, whereas if $\Delta < 0$, there is only one real root. To determine the number of real roots of $f(x) = 0$, one then needs to check the sign of either the discriminant or the value of $f(x)$ at the turning points. Below, we will consider all possible cases.

5.1 $J = 0$ (non-rotating)

We start and consider the simplest, i.e. the non-rotating, case: $J = 0$. From (5.4), we find the following turning points:

$$T_1 = \left\{S_0^2, -S_0^6\right\}, \qquad T_2 = \left\{-\frac{S_0^2}{3}, \frac{5\,S_0^6}{27}\right\}. \tag{5.8}$$

Thus, T_2 always belongs to the second quadrant, with $\mathcal{S}_1 > 0$ and $\mathcal{S}_2 < 0$. Consequently, there exist 3 real roots, as shown in figure 2(a). This also can be understood by computing that $\Delta = 5S_0^{12} > 0$. Out of the resulting three real roots, one is zero, as $f(0) = 0$, hence corresponding to the usual (and anti-involutive) notion of FD for a non-rotating, extremal BH [3]. We also observe that T_2 has a negative x coordinate, whereas T_1 has a positive one: this implies that among the two non-zero real roots, one is positive and the other is negative. Since $x := \hat{J}^2 = (J + \delta J)^2$, only the positive real root is a physically sensible choice: as discussed in section 3.1, it corresponds to RFD mapping a non-rotating, static extremal BH to an under-rotating, stationary extremal BH. In order to further clarify the locations of the turning point, as an example in figure 2(a) we plot $y = f(x)$ when $J = 0$ and $S_0 = 5$.

5.2 $J \neq 0$, $\mathcal{A} = 0$ (under-rotating)

Next, we consider $J > 0$, but $\mathcal{A} = 0$. In this case, $\Delta = 0$, and therefore there are multiple real roots; since all the coefficients of the cubic equation (5.2) are real, then all roots are real. From (5.5), we find that $\mathcal{A} = 0 \Rightarrow J^2 = S_0^2 \mathbf{g}$, where $\mathbf{g} := \left(2 - \frac{3}{2^{\frac{2}{3}}}\right) \approx 0.110118$. Thus, $S_0^2 - J^2 = (1 - \mathbf{g})\, S_0^2 > 0$, implying that this is an under-rotating case. The turning points read

$$T_1 = \left\{ S_0^2, -S_0^6 \right\}, \quad T_2 = \left\{ \frac{1}{3} S_0^2 (2\mathbf{g} - 1), 0 \right\}. \tag{5.9}$$

Since there are two different turning points, again with $\mathcal{S}_1 > 0$ and $\mathcal{S}_2 < 0$, there exist two real roots, as depicted in figure 2(b) in the case in which $S_0 = 5$. Inserting the actual value of $\mathbf{g}$, we observe that T_2 has a vanishing $y = f(x)$ coordinate, but it has a negative x coordinate, and thus it must be discarded. On the other hand, T_1 lies on the positive x axis, implying the existence of only one strictly positive real root, which is the physically acceptable one. By setting $J^2 = S_0^2 \mathbf{g}$ and using $\mathbf{g} \approx 0.110118$, one can solve for the $f(x) = 0$ and find that the multiplicity arises at the negative real root, leaving a single positive real root.

5.3 $J \neq 0$, $\mathcal{A} > 0$ (under-rotating)

In this case, T_1 still belongs to the fourth quadrant, and $\Delta > 0$ implies the existence of three distinct real roots. As $\mathcal{A} > 0 \Rightarrow J^2 < S_0^2\, \mathbf{g}$, T_2 belongs to the second quadrant and this case is under-rotating, too. Again, since $f(0) = -J^2 S_0^4$ and observing that $\mathcal{S}_1 > 0$ and $\mathcal{S}_2 < 0$, one can realize that only one of the real roots lies on the positive x axis, and is therefore physically acceptable. A prototypical plot for this case is shown in figure 3 with $J = 2$ and $S_0 = 7$.

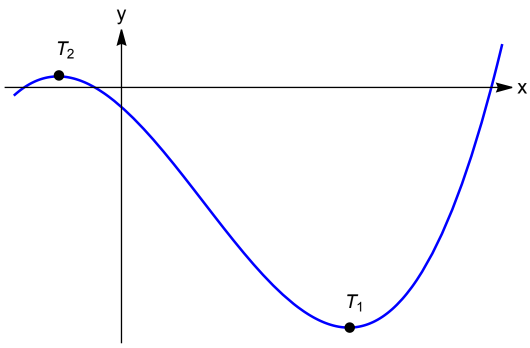


Figure 3. Typical plot for section 5.3 with $J \neq 0$ and $\mathcal{A} > 0$.

5.4 $J \neq 0$, $\mathcal{A} < 0$

As evident from the previous analysis, $\mathcal{A} < 0 \Leftrightarrow J^2 > S_0^2\, \mathbf{g}$. In this case, $\Delta < 0$, hence there exists a single real root. The turning point T_1 is still located in the fourth quadrant, whereas T_2 can be either in the third or fourth quadrant. This situation is rather delicate, as it covers both the under- and over- rotating class of rotating, stationary, extremal BHs. Depending on the various possible values of J^2 there are several situations as shown in figure 4(a) and figure 4(b). However, although the location of T_1 and T_2 changes, the curve $y = f(x)$ (5.3) always looks the same, and furthermore, it implies the existence of a unique strictly positive real root of the equation $f(x) = 0$, i.e. of the inhomogeneous cubic equation (5.2), corresponding to the physically sensible solution.

5.4.1 Under-rotating

Since $J^2 < S_0^2 \Rightarrow S_0^2 \mathbf{g} < J^2 < S_0^2$, both cases depicted in figure 4(a) and figure 4(b) are possible, again with $\mathcal{S}_1 > 0$ and $\mathcal{S}_2 < 0$. The existence of a unique and physically sensible root is evident.

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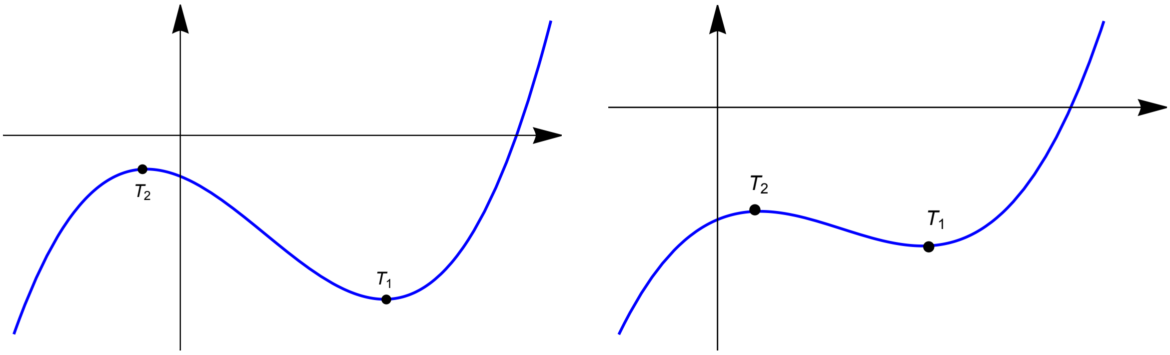
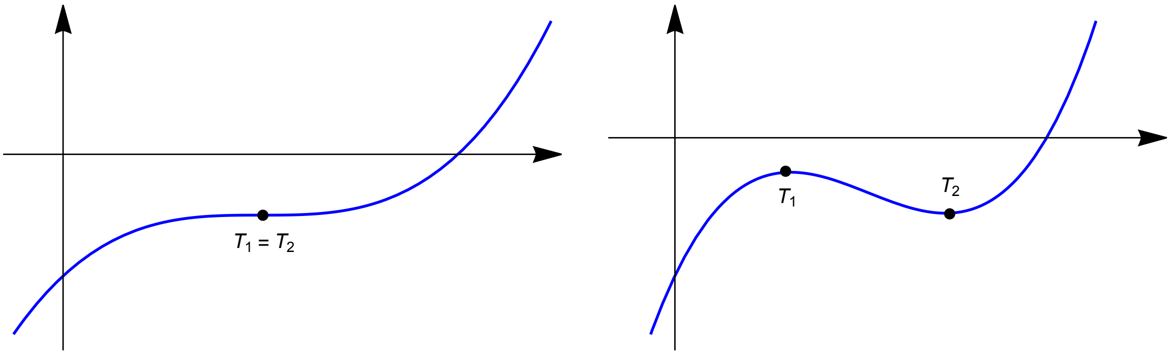


(a) Under-rotating as $S_0^2\mathbf{g} < J^2 < \frac{1}{2}S_0^2$. (b) Over(under)-rotating as $\frac{1}{2}S_0^2 < J^2 < 2S_0^2$.

Figure 4. Typical plots for section 5.4 with $S_0^2\mathbf{g} < J^2 < 2S_0^2$.



(a) Over-rotating with $J^2 = 2S_0^2$. (b) Over-rotating with $J^2 > 2S_0^2$.

Figure 5. Typical plots for section 5.4 with $J^2 \geq 2S_0^2$.

5.4.2 Over-rotating

Besides the case depicted in figure 4(b) with $S_0^2 < J^2 < 2S_0^2$, when $J^2 \geqslant 2S_0^2$ there exist two other interesting possibilities:

1. The case $J^2 = 2S_0^2$, which has been already considered in section 2.2, has $T_1 = T_2$ and $\mathcal{S}_1 = \mathcal{S}_2 = 0$, implying that $y = 2S_0^2$ is the unique (strictly) positive real root. This (over-rotating) case trivializes the RFD map, which coincides with the identity, since it leaves both J and S_0 (and thus S_{over} (2.7)) invariant.

2. The case $J^2 > 2S_0^2$ has, interestingly, $\mathcal{S}_1 < 0$ and $\mathcal{S}_2 > 0$. However, since both T_1 and T_2 swap their order along the x axis, the equation $f(x) = 0$ ends up having a unique (strictly positive, and thus) physically sensible solution.

To recap. The above-detailed analysis proves that, for any value of J and S_0 physically allowed for stationary (Kerr-Newman, asymptotically flat) extremal BHs (i.e., $J \in \mathbb{R}^+$ and $S_0 \in \mathbb{R}_0^+$), the (generalized) rotating Freudenthal duality (RFD) map defined in (3.3) allows for a *unique* RFD-dual BH. The above analysis is pictorially summarized in figure 6, in which we plotted all the real roots of (5.2) by varying $J \in \mathbb{R}^+$ (for $S_0 = 2$).

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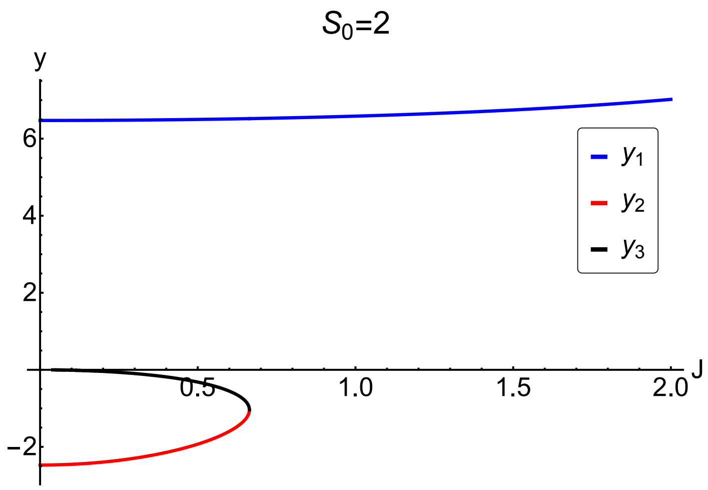

Figure 6. Real roots $y_1(J)$, $y_2(J)$ and $y_3(J)$ of the inhomogeneous cubic equation (5.2) (for $S_0 = 2$).

6 Analytic form of RFD

After the qualitative analysis of the previous section, in which we have proved that RFD (3.3) defines a one-to-one map between two rotating (stationary, Kerr-Newman, asymptotically flat) extremal BHs with the same Bekenstein-Hawking entropy, we now discuss the analytical form of the RFD itself, namely the explicit, analytical form of the solutions to the inhomogeneous cubic equation (5.2), i.e. of $f(x) = 0$, where the curve $y = f(x)$ is defined in (5.3).

From the general theory of cubic equations, the *depressed form* of the equation (5.2) reads

$$t^3 + p\,t + q = 0, \;\text{ with} \begin{cases} t := x - \frac{1}{3}(J^2 + S_0^2)\,; \\[2ex] p := -\frac{1}{3}(J^2 - 2S_0^2)^2; \\[2ex] q := \frac{1}{54}\mathcal{A} - \frac{1}{2}S_0^6, \end{cases} \tag{6.1}$$

and the discriminant can be computed to be (cf. (5.7))

$$\Delta = -(4p^3 + 27q^2) = S_0^6\mathcal{A}. \tag{6.2}$$

We have the following case study.

6.1 $\Delta < 0$

In this case, $\mathcal{A} < 0 \Leftrightarrow J^2 > S_0^2\,\mathbf{g}$. Since $\mathbf{g} < 1$, the extremal BH can be under-rotating or over-rotating, depending on whether $S_0^2\,\mathbf{g} < J^2 < S_0^2$ or $J^2 > S_0^2$, respectively. Moreover, $q < 0$, and the equation (5.2) has one real root and two non-real complex conjugate roots. By further defining

$$\mathcal{C} := \frac{q^2}{4} + \frac{p^3}{27} = -\frac{S_0^6}{108}\mathcal{A} = -\frac{\Delta}{108} > 0; \tag{6.3}$$

$$u := -\frac{q}{2} + \sqrt{\mathcal{C}} > 0; \tag{6.4}$$

$$v = -\frac{q}{2} - \sqrt{\mathcal{C}} > 0, \tag{6.5}$$

by Cardano's method, the unique real root reads

$$t_1 = \sqrt[3]{u} + \sqrt[3]{v} > 0. \tag{6.6}$$

For completeness, we mention that the other two non-real complex conjugate roots of (6.1) can be written as

$$\begin{aligned} t_2 &= \omega \sqrt[3]{u} + \bar{\omega} \sqrt[3]{v}; \\ t_3 &= \bar{\omega} \sqrt[3]{u} + \omega \sqrt[3]{v}, \end{aligned} \tag{6.7}$$

with ω and $\bar{\omega}$ being the non-real cube roots of unity,

$$\omega := \frac{-1 + i\sqrt{3}}{2}. \tag{6.8}$$

Thus, by recalling that $x := \hat{J}^2$, the unique physically sensible solution for the RFD-transformed (*aka* RFD-dual) angular momentum reads

$$\begin{gathered} \hat{J}^2 := (J + \delta J)^2 = t_1 + \frac{1}{3}\left(J^2 + S_0^2\right) > 0; \\ \Updownarrow \\ \delta J = -J + \sqrt{t_1 + \frac{1}{3}\left(J^2 + S_0^2\right)} \in \mathbb{R}, \end{gathered} \tag{6.9}$$

with t_1 given by (6.6).

6.2 $\boldsymbol{\Delta = 0}$ (under-rotating)

In this case, $\mathcal{A} = 0 \Leftrightarrow J^2 = S_0^2 \mathbf{g} < S_0^2$ (and thus the extremal BH is under-rotating). Then, since its coefficients are all real, the equation (5.2) has three real roots, with non-trivial multiplicity. Since $q < 0$ and $p < 0$, in this case the roots indeed read

$$t_1 = 3\frac{q}{p} > 0; \tag{6.10}$$

$$t_2 = t_3 = -3\frac{q}{p} < 0. \tag{6.11}$$

Therefore, the unique physically sensible solution for the RFD-dual angular momentum still formally reads

$$\begin{gathered} \hat{J}^2 := t_1 + \frac{1}{3}\left(J^2 + S_0^2\right) > 0; \\ \Updownarrow \\ \delta J = -J + \sqrt{t_1 + \frac{1}{3}\left(J^2 + S_0^2\right)} \in \mathbb{R}, \end{gathered} \tag{6.12}$$

but with t_1 given by (6.10).

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6.3 $\Delta > 0$ (under-rotating)

This particular situation goes under the name of *casus irreducibilis*, since at Cardano's time the complex roots were not known to exist. In this case, $\mathcal{A} > 0 \Rightarrow J^2 < S_0^2\,\mathbf{g}$, and therefore the extremal BH is under-rotating; from the general theory, there are three distinct real roots, but, as discussed in section 5.3, only one of them is positive. Remarkably, the same three roots t_1, t_2 and t_3 given by Cardano's method (i.e., (6.6) and (6.7)) work, *provided one takes the principal value* for the cube roots in (6.6) and (6.7), as well as for the square root in (6.9). In fact, in this case t_1, given by (6.6), still is the unique physically consistent solution, which can now more conveniently be written as

$$
\begin{aligned}
t_1 &= \frac{1}{3\cdot 2^{2/3}}\left(3^{3/2}S_0^3+\sqrt{-\mathcal{A}}\right)^{2/3}+\frac{1}{3\cdot 2^{2/3}}\left(3^{3/2}S_0^3-\sqrt{-\mathcal{A}}\right)^{2/3}\\
&= \frac{1}{3\cdot 2^{2/3}}\left(z^{2/3}+\bar{z}^{2/3}\right) > 0; &(6.13)\\
z &:= 3^{3/2}S_0^3 + i\,\sqrt{\mathcal{A}}. &(6.14)
\end{aligned}
$$

Since we are taking the principal value here, t_1 is real and positive, as the imaginary parts would cancel out in (6.13).

To recap. Therefore, in all cases, there exists a unique physically sensible root of (5.2), which can be cast in the same form, namely

$$\hat{J}^2 := t_1 + \frac{1}{3}\left(J^2+S_0^2\right) > 0, \tag{6.15}$$

$$\text{or } \delta J = -J + \sqrt{t_1 + \frac{1}{3}\left(J^2+S_0^2\right)} \in \mathbb{R}, \tag{6.16}$$

with t_1 which can generally be written as follows:

$$t_1 = \left(\frac{S_0^3}{2}+\sqrt{-\frac{\mathcal{A}}{108}}\right)^{\frac{2}{3}} + \left(\frac{S_0^3}{2}-\sqrt{-\frac{\mathcal{A}}{108}}\right)^{\frac{2}{3}}. \tag{6.17}$$

7 Under- $\rightleftarrows$ over- rotating transitions through RFD? *No-Go*

In the previous sections, we have proved the uniqueness and presented the explicit analytic form, of the RFD map introduced in (3.3). One might wonder whether RFD may be responsible for a transition from the under-rotating regime to the over-rotating one, or *vice versa*. In this section, we will answer such a question.

We start by recalling (2.5), (2.7) and (3.1), implying that, under RFD (3.3), the overall entropy *formally*[8] transforms as follows:

$$S_{\text{under(over)}}(Q,J) \mapsto S_{\text{under(over)}}\left(\hat{Q}(Q,J+\delta J), J+\delta J\right), \tag{7.1}$$

where (cf. (2.5) and (2.7))

$$S_{\text{under(over)}}(Q,J) := \sqrt{\left|S_0^2(Q)-J^2\right|}, \tag{7.2}$$

[8]We stress that the transformation (7.1) is only formal, because (3.2) crucially holds: the overall Bekenstein-Hawking entropy is *invariant* under RFD.

and

$$
\begin{aligned}
&S_{\text{under(over)}}\left(\hat{Q}(Q,J+\delta J),J+\delta J\right)\\
&=\sqrt{\left|S_0^2\left(\hat{Q}(Q,J+\delta J)\right)-(J+\delta J)^2\right|}\\
&=\sqrt{\left|S_0^2\left(\pm\frac{S_0(Q)}{\sqrt{\left|S_0^2(Q)-(J+\delta J)^2\right|}}\Omega\frac{\partial S_0(Q)}{\partial Q}\right)-(J+\delta J)^2\right|}\\
&=\sqrt{\left|\frac{S_0^4(Q)}{\left[S_0^2(Q)-(J+\delta J)^2\right]^2}S_0^2\left(\Omega\frac{\partial S_0(Q)}{\partial Q}\right)-(J+\delta J)^2\right|}\\
&=\sqrt{\left|\frac{S_0^6(Q)}{\left[S_0^2(Q)-(J+\delta J)^2\right]^2}-(J+\delta J)^2\right|}\\
&=\sqrt{\left|\tilde{S_0}^2\left(Q,\hat{J}\right)-\hat{J}^2\right|}
\end{aligned}
\tag{7.3}
$$

Where we have used the properties (2.3) and (2.2) of S_0 to obtain (7.3). Moreover, by recalling that $\hat{J}:=J+\delta J$, we have introduced

$$
\tilde{S_0}\left(Q,\hat{J}\right)\equiv\tilde{S_0}(Q,J,\delta J):=\frac{S_0^3(Q)}{\left|S_0^2(Q)-\hat{J}^2\right|}\in\mathbb{R}_0^+. \tag{7.4}
$$

Thus, depending on the sign of $\tilde{S_0}^2\left(Q,\hat{J}\right)-\hat{J}^2$, one can establish whether the *formal* transformation (7.1)–(7.3) of the overall Bekenstein-Hawking BH entropy pertains to an under- or over- rotating extremal BH, i.e.

$$
\tilde{S_0}^2\left(Q,\hat{J}\right)-\hat{J}^2>0\Rightarrow\text{the RFD-dual BH is under-rotating;} \tag{7.5}
$$

$$
\tilde{S_0}^2\left(Q,\hat{J}\right)-\hat{J}^2<0\Rightarrow\text{the RFD-dual BH is over-rotating.} \tag{7.6}
$$

In order to study this sign problem, we introduce the real, non-negative, dimensionless parameters

$$
\alpha(Q,J):=\frac{J^2}{S_0^2(Q)}\in\mathbb{R}^+:\begin{cases}0\leqslant\alpha<1: & \text{the starting BH is under-rotating;}\\ \alpha>1: & \text{the starting BH is over-rotating.}\end{cases} \tag{7.7}
$$

$$
\hat{\alpha}\left(Q,\hat{J}\right)\equiv\hat{\alpha}(Q,J,\delta J):=\frac{\hat{J}^2}{\tilde{S_0}^2\left(Q,\hat{J}\right)}\in\mathbb{R}^+:\begin{cases}0\leqslant\hat{\alpha}<1: & \text{the RFD-dual BH is under-rotating;}\\ \hat{\alpha}>1: & \text{the RFD-dual BH is over-rotating.}\end{cases} \tag{7.8}
$$

Such parameters are connected by the RFD map (3.3),

$$
\alpha(Q,J)\overset{\text{RFD (3.3)}}{\longrightarrow}\hat{\alpha}\left(Q,\hat{J}\right). \tag{7.9}
$$

There are two limit cases:

1. $\alpha = 0 \Leftrightarrow J = 0$ (non-rotating limit): the starting extremal BH is non-rotating (i.e., static); as discussed in section 3.1, the RFD map (3.3) can yield to $\hat{\alpha} = 0$:

$$\alpha\,(Q, J=0) = 0 \overset{\mathrm{RFD}_{J\to 0^+}\ (3.3)}{\longrightarrow} \hat{\alpha}\left(Q, \hat{J}=0\right) = 0, \tag{7.10}$$

but it also admits another spurious, "golden" branch, which maps α to (cf. (3.8), (3.12), (7.4) and (7.8)):

$$\alpha\,(Q, J=0) = 0 \overset{\mathrm{RFD}_{J\to 0^+,\ \mathrm{golden}}\ (3.3)}{\longrightarrow} \hat{\alpha}_{\mathrm{golden}}, \tag{7.11}$$

$$\begin{aligned}\text{where } \hat{\alpha}_{\mathrm{golden}} := \hat{\alpha}\left(Q, \hat{J}_{\mathrm{golden}}\right) &= \frac{\hat{J}^2_{\mathrm{golden}}}{\tilde{S}_0{}^2\left(Q, \hat{J}_{\mathrm{golden}}\right)} = \phi\,(1-\phi)^2 \\ &= \frac{1}{\phi} = \phi - 1 < 1,\end{aligned} \tag{7.12}$$

implying that the image of the non-rotating (static) extremal BH under the "golden" branch of $\mathrm{RFD}_{J\to 0^+}$, defined as $\mathrm{RFD}_{J\to 0^+,\ \mathrm{golden}}$ (3.11) in section 3.1, is an *under-rotating* (stationary) extremal BH; see the discussion, and in particular the Remark 1, in section 3.1.

2. $\alpha = 1 \Leftrightarrow J = S_0(Q)$: from (2.5) and (2.7), we find $S_{\mathrm{under}}\,(Q, J=S_0) = 0 = S_{\mathrm{over}}\,(Q, J=S_0))$. Hence, the starting BH is "small", i.e. it has a vanishing Bekenstein-Hawking entropy/area of the (unique) event horizon, *at least* within the two-derivative (Einsteinian) treatment understood in the present treatment. The RFD map (3.3) yields to (cf. (7.4) and (7.8))

$$\alpha\,(Q, J=S_0) = 1 \overset{\mathrm{RFD}\ (3.3)}{\longrightarrow} \check{\alpha}\left(\frac{\delta J}{J}\right), \tag{7.13}$$

$$\begin{aligned}\text{where } \check{\alpha}\left(\frac{\delta J}{J}\right) &:= \hat{\alpha}\left(Q, \hat{J} = S_0 + \delta J\right) \\ &= \frac{\left[S_0\,(Q) + \delta J\right]^2 \left[S_0^2\,(Q) - \left(S_0\,(Q) + \delta J\right)^2\right]^2}{S_0^6(Q)} \\ &= \frac{(\delta J)^2 \left[S_0\,(Q) + \delta J\right]^2 \left[2S_0(Q) + \delta J\right]^2}{S_0^6(Q)} \\ &= \left(\frac{\delta J}{J}\right)^2 \left(1 + \frac{\delta J}{J}\right)^2 \left(2 + \frac{\delta J}{J}\right)^2.\end{aligned} \tag{7.14}$$

However, since the RFD map (3.3) preserves (by definition) the overall Bekenstein-Hawking BH entropy, the RFD-dual extremal BH is also "small", with

$$S_{\mathrm{under}}\,(Q, J=S_0) = 0 = S_{\mathrm{over}}\,(Q, J=S_0) \mapsto S\left(\hat{Q}(J+\delta J), J+\delta J\right) = 0, \tag{7.15}$$

implying that (cf. (7.14))

$$\check{\alpha}\left(\frac{\delta J}{J}\right)=1 \tag{7.16}$$

$$\Updownarrow$$

$$\left(\frac{\delta J}{J}\right)^2\left(1+\frac{\delta J}{J}\right)^2\left(2+\frac{\delta J}{J}\right)^2=1 \tag{7.17}$$

$$\Updownarrow$$

$$(\delta J)^2\,(J+\delta J)^2\,(2J+\delta J)^2=J^6. \tag{7.18}$$

This is an inhomogeneous algebraic equation of degree 6 in δJ, equivalently obtained by plugging $S_0 = J$ in (3.4) and (3.5):

$$0=a_1\,(\delta J)^6+a_2\,(\delta J)^5+a_3\,(\delta J)^4+a_4\,(\delta J)^3+a_5\,(\delta J)^2+a_6\delta J+a_7, \tag{7.19}$$

$$\begin{aligned}
&a_1=1;\\
&a_2=6J;\\
&a_3=13J^2;\\
&a_4=12J^3;\\
&a_5=4J^4;\\
&a_6=0;\\
&a_7=-J^6.
\end{aligned} \tag{7.20}$$

Solving (7.19), the unique real and positive (and thus physically meaningful) solution reads $\delta J \simeq 0.324718J$; the details are provided in appendix A.[9] However, since both the starting and the RFD-dual extremal BHs are "small", this limit case is not interesting, *at least* at two-derivative level.

Reconsidering the general treatment, by using the explicit form of the physically sensible root of (3.4), namely (6.15) and (6.17), and recalling the definitions (5.5) of $\mathcal{A}$, (7.7) of α, and (7.4) of $\tilde{S}_0$, one can compute[10]

$$\begin{aligned}
\hat{J}^2&=t_1+\frac{1}{3}\left(J^2+S_0^2\right)\\
&=\left(\frac{S_0^3}{2}+\sqrt{-\frac{\mathcal{A}}{108}}\right)^{\frac{2}{3}}+\left(\frac{S_0^3}{2}-\sqrt{-\frac{\mathcal{A}}{108}}\right)^{\frac{2}{3}}+\frac{1}{3}\left(J^2+S_0^2\right)\\
&=\frac{S_0^2}{2^{2/3}}\left(\left(1+\sqrt{1+\frac{4}{27}(\alpha-2)^3}\right)^{\frac{2}{3}}+\left(1-\sqrt{1+\frac{4}{27}(\alpha-2)^3}\right)^{\frac{2}{3}}\right)+\frac{S_0^2}{3}\,(\alpha+1)\\
&=S_0^2\,f(\alpha),
\end{aligned} \tag{7.21}$$

and

$$\tilde{S_0}^2=S_0^2\,(1-f(\alpha))^{-2}\,, \tag{7.22}$$

[9]With (7.23) and appendix A one can check the relation $\left(\frac{\hat{J}}{J}\right)^2=f(1)$.

[10]$S_0 \equiv S_0(Q)$ throughout.

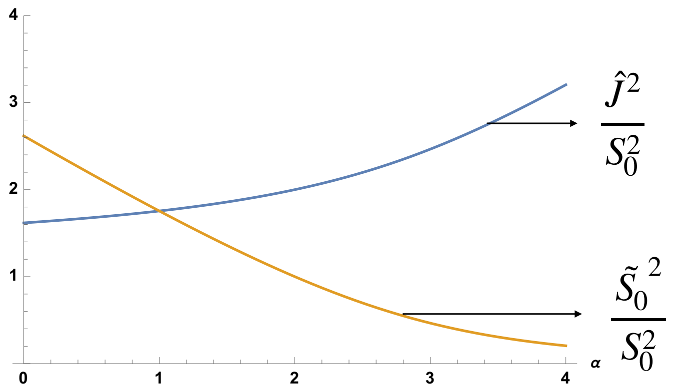

Figure 7. The yellow curve is for $\frac{\tilde{S}_0{}^2}{S_0^2}$ and the blue one is for $\frac{\hat{J}^2}{S_0^2}$.

where we have defined the following non-negative function of α:

$$f(\alpha) := \frac{1}{2^{2/3}}\left(1+\sqrt{1+\frac{4}{27}(\alpha-2)^3}\right)^{2/3} + \frac{1}{2^{2/3}}\left(1-\sqrt{1+\frac{4}{27}(\alpha-2)^3}\right)^{2/3} + \frac{1}{3}(\alpha+1), \tag{7.23}$$

We can now discuss the possibility of under $\rightleftarrows$ over -rotating transition by means of the RFD map (3.3), by conveniently plotting $\hat{J}^2/S_0^2 = f(\alpha)$ and $\tilde{S}_0^2/S_0^2 = (1-f(\alpha))^{-2}$ vs. α in figure 7. Indeed, we have to study the sign of

$$\tilde{S}_0^2 - \hat{J}^2 = \left[(1-f(\alpha))^{-2} - f(\alpha)\right] S_0^2, \tag{7.24}$$

depending on the value of $\alpha = J^2/S_0^2 \geqslant 0$ (by recalling that for $0 \leqslant \alpha < 1$ the extremal BH is under-rotating, while for $\alpha > 1$ it is over-rotating). By defining $z := f(\alpha)$, it holds that

$$\tilde{S}_0^2 - \hat{J}^2 = 0 \Leftrightarrow (1-z)^{-2} = z \Leftrightarrow z(1-z)^2 = 1. \tag{7.25}$$

This peculiar cubic equation has been solved in appendix A (cf. eq. (A.2)), giving as unique[11] real non-negative solution[12] (cf. eq. (A.4))

$$z = f(1) \Leftrightarrow f(\alpha) = f(1). \tag{7.26}$$

In appendix B we prove that $f(\alpha)$ is monotone increasing for $\alpha \geqslant 0$; thus, the result (7.26) holds iff $\alpha = 1$, and the following *no-go* result is achieved:

$$0 \leqslant \alpha < 1 \Rightarrow \tilde{S}_0^2 - \hat{J}^2 > 0; \tag{7.27}$$

$$\alpha > 1 \Rightarrow \tilde{S}_0^2 - \hat{J}^2 < 0. \tag{7.28}$$

In other words, no under $\rightleftarrows$ over -rotating transition can be achieved by means of the RFD map (3.3), starting from an under-rotating resp. over-rotating stationary extremal BH.

[11]Under the condition that $f(\alpha) \neq 1$. This condition is however always satisfied, since, as we will prove in appendix B, $f(\alpha)$ is monotone increasing for $\alpha \geqslant 0$, and $f(0) = 4/3$, thus implying that $f(\alpha) \geqslant 4/3 > 1$ $\forall \alpha \geqslant 0$.

[12]From the treatment given in appendix A, the value of $f(1)$ can be computed to be $f(1) = \left.\frac{\hat{J}^2}{S_0^2}\right|_{J=S_0} = \frac{\left(S_0 + \delta J|_{J=S_0}\right)^2}{S_0^2} \simeq (1.324718)^2 \simeq 1.754877$.

8 Conclusion

In this work, we focused on extremal (stationary, asymptotically flat) rotating BHs in four space-time dimensions, and we have shown that there exists a (generally non-anti-involutive) non-linear symmetry of their Bekenstein-Hawking entropy, both in the under-rotating and in the over-rotating regime. We have named such a non-linear symmetry (generalized) rotating Freudenthal duality (RFD): this map generalizes, in the presence of non-vanishing (constant) angular momentum J, the usual Freudenthal duality (FD) for non-rotating (static) extremal BHs introduced in [3], and it is part of the program, started in [18] and [19] dealing with near-extremal non-rotating BHs, to extend FD to various classes of BH solutions beyond the extremal static one. The RFD map has been defined as an intrinsically non-linear map acting both on the e.m. dyonic charges of the BH (collectively denoted by the symplectic vector Q) and on its angular momentum J, and keeping the Bekenstein-Hawking BH entropy S invariant. We have proved the uniqueness of the RFD map, and we also explicitly computed the analytical expression of the transformation of the angular momentum J under such a map, which generally is a quite involved function of the charges of the starting extremal BH (through the non-rotating BH entropy $S_0(Q)$) as well as of its angular momentum J itself.

In the non-rotating limit, RFD reduces to the non-rotating, usual (anti-involutive) definition of FD (introduced, with the name of "on-shell" FD, in [3]). However, there also exists a spurious, *"golden"* branch of the RFD map when $J \to 0^+$, which maps a non-rotating (static) extremal BH (with near-horizon geometry $AdS_2 \otimes S^2$, and Bekenstain-Hawking entropy $S_0(Q)$) to an under-rotating (stationary) extremal BH (with near-horizon geometry $AdS_2 \otimes S^1$), with a (constant) angular momentum given by $\sqrt{\phi} S_0(Q)$, where ϕ is the *golden ratio*; it is then noteworthy that both such BHs have the same Bekenstein-Hawking entropy! Furthermore, one can state that the space of (generalized) FD functionals undergoes a sort of phase transition in the non-rotating limit $J \to 0^+$, such that $J = 0$ appears to be a bifurcation point between the non-rotating usual FD branch and the aforementioned *"golden"* branch. It is suggestive to remark that the duality between a non-rotating extremal BH and an under-rotating extremal BH could be traced back to a common parent five-dimensional extremal BH [27, 28]. This observation needs further investigation, and we leave that as a future endeavour.

It will also be interesting to investigate the effect of RFD on the dual CFT_2 of an extremal Kerr-Newman BH, by exploiting the so-called Kerr/CFT correspondence. On the other hand, we should recall that so far we have only formulated the generalization of the Freudenthal duality map for the semi-classical Bekenstein-Hawking entropy of various classes of BHs; the invariance of the quantum-corrected BH entropy under a suitable generalization of Freudenthal duality still stands as a crucial issue, which we hope to address in future investigation.

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A Solution for $J = S_0$

We will start by simplifying (7.19) as

$$(\hat{J}^2 - J^2)^2 \hat{J}^2 = J^6, \tag{A.1}$$

with $\hat{J} = J + \delta J$. This can further be simplified as

$$(y-1)^2 y = 1, \tag{A.2}$$

by defining $y = \frac{\hat{J}^2}{J^2}$. The depressed form of the equation is then

$$\tilde{t}^3 - \frac{\tilde{t}}{3} - \frac{25}{27} = 0, \tag{A.3}$$

with $\tilde{t} = y - \frac{2}{3}$. Evidently, (A.3) is the same as (6.1) with the identification that $\tilde{t} \equiv \frac{t}{J^2}$ and p and q evaluated at $\alpha = 1$. Therefore we have the relation that

$$\frac{\hat{J}^2}{J^2} = \frac{\hat{J}^2}{S_0^2} = y = \tilde{t} + \frac{2}{3} = f(1), \tag{A.4}$$

with $f(\alpha)$ being defined in (7.23).

B Proof of monotonicity of $f(\alpha)$

We will here prove that both $f(\alpha)$ and $(f(\alpha)-1)^{-2}$ are monotone (increasing resp. decreasing) functions $\forall \alpha \geqslant 0$.

We start and recall eq. (6.1),

$$t^3 + p\,t + q = 0, \text{ with } \begin{cases} t := \hat{J}^2 - \frac{1}{3}(J^2 + S_0^2) = g(\alpha) S_0^2; \\ p := -\frac{1}{3}(J^2 - 2S_0^2)^2 = S_0^4 \tilde{p}\,(\alpha)\,; \\ q := -\frac{2}{27}\left(J^2 - 2S_0^2\right)^3 - S_0^6 = S_0^6 \tilde{q}\,(\alpha)\,, \end{cases} \tag{B.1}$$

where the relations $J^2 = \alpha S_0^2$ and $\hat{J}^2 = f(\alpha) S_0^2$ have been used, and the following definitions have been introduced:

$$g\,(\alpha) := f\,(\alpha) - \frac{(\alpha+1)}{3} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{\alpha-2}{3}\right)^3}\right)^{2/3} + \left(\frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{\alpha-2}{3}\right)^3}\right)^{2/3}; \tag{B.2}$$

$$\tilde{p}\,(\alpha) := -\frac{1}{3}(\alpha-2)^2; \tag{B.3}$$

$$\tilde{q}\,(\alpha) := -\frac{1}{27}\left[2\,(\alpha-2)^3 - 1\right]. \tag{B.4}$$

Therefore, by understanding the dependence on α, the depressed cubic (6.1) can be written as ($S_0 \equiv S_0(Q) > 0$ throughout)

$$g^3 + \tilde{p} g + \tilde{q} = 0, \tag{B.5}$$

whose first derivative then reads

$$3g^2\frac{dg}{d\alpha}+\frac{d\tilde{p}}{d\alpha}g+\tilde{p}\frac{dg}{d\alpha}+\frac{d\tilde{q}}{d\alpha}=0; \tag{B.6}$$

$$\Updownarrow$$

$$\frac{dg}{d\alpha}=-\frac{\left(\frac{d\tilde{p}}{d\alpha}g+\frac{d\tilde{q}}{d\alpha}\right)}{3g^2+\tilde{p}}=\frac{2\left(\alpha-2\right)}{3\left(3g-\alpha+2\right)}, \tag{B.7}$$

where the definitions (B.3) and (B.4) have been exploited, respectively implying

$$\frac{d\tilde{p}}{d\alpha}=-\frac{2\left(\alpha-2\right)}{3}; \tag{B.8}$$

$$\frac{d\tilde{q}}{d\alpha}=-\frac{2\left(\alpha-2\right)^2}{9}. \tag{B.9}$$

Thus, since $f\left(\alpha\right)=g\left(\alpha\right)+\frac{(\alpha+1)}{3}$, one obtains

$$\frac{df}{d\alpha}=\frac{dg}{d\alpha}+\frac{1}{3}=\frac{3g+\alpha-2}{3\left(3g-\alpha+2\right)}=\frac{f-1}{3f-2\alpha+1}. \tag{B.10}$$

On the other hand, by defining

$$k\left(\alpha\right):=\left(1-f\left(\alpha\right)\right)^{-2}, \tag{B.11}$$

one can compute that

$$\frac{dk}{d\alpha}=\frac{2}{\left(1-f\right)^3}\frac{df}{d\alpha}=-\frac{2}{\left(f-1\right)^2\left(3g-\alpha+2\right)}. \tag{B.12}$$

Thus, we obtain that

$$f\equiv f\left(\alpha\right)\text{ monotone } \textit{increasing} \Leftrightarrow \frac{df}{d\alpha}>0\Leftrightarrow\begin{cases}3g+\alpha-2\gtrless 0,\\ 3g-\alpha+2\gtrless 0,\end{cases}\forall\alpha\geqslant 0, \tag{B.13}$$

and

$$k\equiv k\left(\alpha\right)\text{ monotone } \textit{decreasing} \Leftrightarrow \frac{dk}{d\alpha}<0\Leftrightarrow 3g-\alpha+2>0,\ \forall\alpha\geqslant 0. \tag{B.14}$$

Thence, in order to prove that both $f(\alpha)$ and $(f(\alpha)-1)^{-2}$ are monotone (increasing resp. decreasing) functions $\forall\alpha\geqslant 0$, we have to show that

$$3g+\alpha-2>0,\quad\forall\alpha\geqslant 0; \tag{B.15}$$

$$3g-\alpha+2>0,\quad\forall\alpha\geqslant 0. \tag{B.16}$$

(B.16) can be proven by observing that, since

$$\mathcal{G}\left(x;\gamma\right):=\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\gamma x^3}\right)^{2/3}+\left(\frac{1}{2}-\frac{1}{2}\sqrt{1+\gamma x^3}\right)^{2/3}>x,\quad\forall\gamma\in\left(1,\infty\right), \tag{B.17}$$

it holds that

$$g\left(\alpha\right)=\mathcal{G}\left(\frac{\alpha-2}{3};4\right)>\frac{\alpha-2}{3}\Leftrightarrow 3g-\alpha+2>0,\quad\forall\alpha\geqslant 0. \tag{B.18}$$

On the other hand, (B.7) implies that $g\left(\alpha\right)$ is minimized at $\alpha=2$, at which it acquires the value $g\left(2\right)=1$. Thus, (B.16) and (B.7) imply (B.15):

$$3g+\alpha-2>1>0,\quad\forall\alpha\geqslant 0.\ \square \tag{B.19}$$

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