Higher spin symmetry/gravity and $3d$ bosonization duality

Recent developments in CFTs

Evgeny Skvortsov, UMONS

February 1st, 2023

European Research Council
Established by the European Commission

FREEDOM TO RESEARCH

Main Messages/Teaser/Spoiler

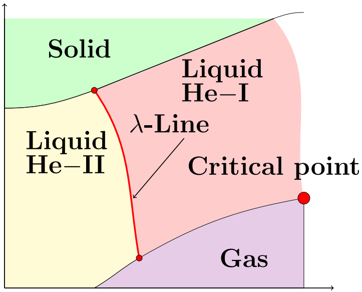
Main message: higher spin symmetry is $3d$ Virasoro 

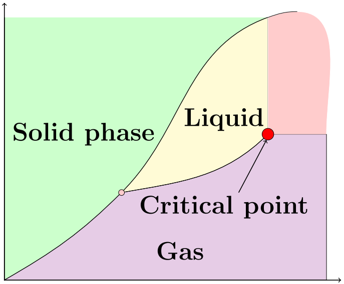
- Infinite-dimensional symmetry is usually useful: Virasoro, Yangian, ... Is there any symmetry behind (Chern-Simons) vector models like Ising model? Slightly-broken higher spin symmetry (Maldacena, Zhiboedov) is clearly not a usual Lie-type symmetry ...
- The structure behind is L_∞ strong homotopy algebra. Invariants = correlators and are uniquely fixed by the symmetry, which implies the $3d$ bosonization duality
- There is a closed subsector of vector models (including the Ising?), which has a local UV-complete AdS_4 description — Chiral higher spin gravity, its existence almost implies the $3d$ bosonization duality

Plan

- Chern-Simons vector models and bosonization duality
- Slightly-broken higher spin symmetry
- Chiral higher spin gravity and $3d$ bosonization duality
- Unbroken higher spin symmetry: from canonical QFT/CFT to algebraic viewpoint on free CFT's
- L_∞-algebra as a physical symmetry and $3d$ bosonization duality

Chern-Simons Matter Theories and bosonization duality



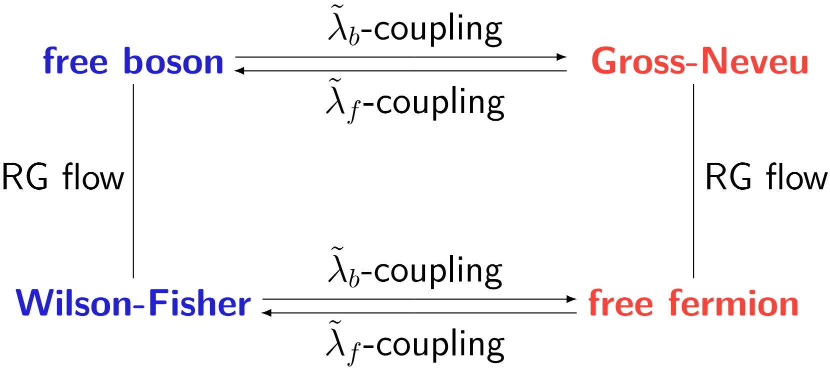


CFT$_3$: Chern-Simons Matter theories, which span CFTs from vector models to ABJ(M). Let's consider the simplest 4 vector models

$$\frac{k}{4\pi} S_{CS}(A) + \text{Matter} \begin{cases} (D\phi^i)^2 & \textbf{free boson} \\ (D\phi^i)^2 + g(\phi^i\phi^i)^2 & \textbf{Wilson-Fisher (Ising)} \\ \bar{\psi}^i \not{D} \psi_i & \textbf{free fermion} \\ \bar{\psi}^i \not{D} \psi_i + g(\bar{\psi}^i\psi_i)^2 & \textbf{Gross-Neveu} \end{cases}$$

- describe physics (Ising, quantum Hall, ...)
- break parity in general (due to Chern-Simons)
- two parameters $\lambda = N/k$, $1/N$ (λ continuous for N large)
- exhibit remarkable dualities, e.g. ***3d* bosonization duality** (Aharony, Alday, Bissi, Giombi, Jain, Karch, Maldacena, Minwalla, Prakash, Seiberg, Tong, Witten, Yacobi, Yin, Zhiboedov, ...)

Chern-Simons Matter theories and dualities



$3d$ **bosonization: these** 4 **families/theories are just** 2 **theories** (Giombi et al; Maldacena, Zhiboedov; many checks by many people, but no proof)

The simplest gauge-invariant operators are **higher spin currents**:

$$J_s = \phi D ... D\phi \qquad \text{and} \qquad J_s = \bar{\psi}\gamma D ... D\psi$$

which are conserved to the leading order in $1/N$ $\rightarrow$ higher symmetry

There are many other operators, e.g. $[JJ]$, $[JJJ]$, etc., correlators thereof and anomalous dimensions, all should be the same in the duals

To see bosonization one needs all orders in λ even at large N, so it is a weak/strong duality in a sense

Since everything appears in the OPEs of Js with themselves, it is sufficient to concentrate on **higher spin currents**, i.e. to prove that

$$\langle J_{s_1} J_{s_2} ... J_{s_n} \rangle$$

are the "same" in the dual theories, which is a job for some symmetry ...

Slightly-broken higher spin symmetry

What is going on in CS-matter theories?

HS-currents are responsible for their own non-conservation:

$$\partial \cdot J_s = \sum_{s_1,s_2} C_{s,s_1,s_2}(\lambda) \frac{1}{N} [J_{s_1} J_{s_2}] + F(\lambda) \frac{1}{N^2} [JJJ]$$

which is an exact non-perturbative quantum equation. In the large-N we can use classical (representation theory) formulas for $[JJ]$.

The worst case $\partial \cdot J =$ some other operator. The symmetry is gone, the charges are not conserved, do not form Lie algebra.

In our case the non-conservation operator $[JJ]$ is made out of J themselves, but charges are still not conserved.

Slightly-broken higher spin symmetry

MZ applied the non-conservation equation to study the 3-point functions. The idea is to combine $\partial \cdot J = \frac{1}{\tilde{N}}[JJ]$ with the very constrained form of 3-pt correlators and $[Q, J] = J + [JJ]$ and use large-N. The result is

$$\langle \boldsymbol{J_{s_1} J_{s_2} J_{s_3}} \rangle \sim \cos^2 \boldsymbol{\theta} \langle \boldsymbol{JJJ} \rangle_b + \sin^2 \boldsymbol{\theta} \langle \boldsymbol{JJJ} \rangle_f + \cos \boldsymbol{\theta} \sin \boldsymbol{\theta} \langle \boldsymbol{JJJ} \rangle_o$$

θ is related to N, k in a complicated way.

The correlators of J_s's get fixed irrespective of what the constituents are! Sign of an ∞-dimensional symmetry ... **What is the right math?**

Slightly-broken higher spin symmetry seems to work (Alday, Zhiboedov, Turiaci, Jain et al, Li, Racobi, Silva and many others!); $\gamma(J_s)$ at order $1/N$ (Giombi, Gurucharan, Kirillin, Prakash, E.S.) confirm the duality. 4, 5-loop $1/N^2$ results in Gross-Neveu and Wilson-Fisher (Manashov, E.S., Strohmaier) seem hard to extend in λ.

Higher spin gravity dual?

Remarks on Higher Spin Gravity

AdS/CFT duals of (Chern-Simons) vector models are HiSGRA since conserved tensor J_s is dual to (massless) gauge field in AdS_4 (Sundborg; Klebanov, Polyakov; Sezgin Sundell; Leight, Petkou; Giombi, Yin, ...)

$$\partial^m J_{ma_2\ldots a_s} = 0 \qquad \Longleftrightarrow \qquad \delta\Phi_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}$$

Instead of tedious quantum calculations in CS-matter one could do the standard holographic computation in the HiSGRA dual, (Giombi, Yin)

However, Vasiliev's equations are incomplete (∞-many free params, non-locality) (Boulanger et al). Independently, this HiSGRA was shown to be too non-local to be constructed by field theory tools (Bekaert, Erdmenger, Ponomarev, Sleight, Taronna). It can be reconstructed (Jevicki et al; Aharony et al) from the very CFT, but no λ. Nevertheless, it has been quite useful to think of HiSGRA dual (Giombi et al)

Chiral HiSGRA

Given J_s, $s=0,...\infty$ we are looking for a HiSGRA in AdS_4 ...

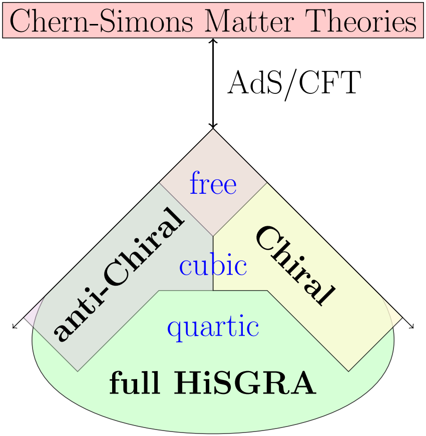
There is a unique local HiSGRA for any value of cosmological constant with such a spectrum — Chiral HiSGRA, which was first constructed in the light-cone gauge in flat space (Metsaev; Ponomarev, E.S.). It is a HS-extension of both SDYM and SDGR. It is at least one-loop UV-finite (E.S., Tran, Tsulaia); it is integrable (Ponomarev); covariant equations (Sharapov, E.S., Sukhanov, Van Dongen); **Chiral $\in$ any $4d$ HiSGRA**

$$\boldsymbol{\mathcal{L}} = \sum_{\boldsymbol{\lambda}} \boldsymbol{\Phi}^{-\boldsymbol{\lambda}} \Box \boldsymbol{\Phi}^{+\boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda}_i} \frac{\boldsymbol{g}\, \boldsymbol{l}_{\mathrm{Pl}}^{\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_3-\mathbf{1}}}{\boldsymbol{\Gamma}(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_3)} \boldsymbol{V}^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3} + \mathcal{O}(\Lambda)$$

where the three-point

$$\boldsymbol{V}^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3} \sim [\mathbf{12}]^{\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_3}[\mathbf{23}]^{\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_3-\boldsymbol{\lambda}_1}[\mathbf{13}]^{\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_3-\boldsymbol{\lambda}_2}$$

Chiral HiSGRA and Secrets of Chern-Simons Matter



Chiral HiSGRA is a HS extension of SDYM/SDGR, which is local;

yep, there is anti-Chiral as well;

has the right spectrum to be dual to CS-matter, but it is short of some interactions to achieve that ...

The very existence of Chiral HiSGRA implies: (a) two more (non-unitary) solutions of the slightly-broken HS; (b) there are two closed subsectors of CS-matter theories, maybe to all orders in $1/N$, hence, Ising?

The very existence of Chiral HiSGRA implies $3d$ bosonization duality at least up to the 4-point correlators of J_s

Input: (i) chiral and anti-chiral interactions are complete at 3-pt; (ii) (anti)-chiral HiSGRA do not have free params save for coupling g.

$$\boldsymbol{V_3 = g\, V_{chiral} \oplus \bar{g}\, \bar{V}_{chiral} \quad \leftrightarrow \quad \langle JJJ\rangle}$$

How to glue (anti)-chiral bricks while imposing unitarity? Simple EM-duality phase rotation $\Phi_{\pm s} \to e^{\pm i\theta}\Phi_{\pm s}$ does the job and we get

$$\boldsymbol{\langle J_{s_1} J_{s_2} J_{s_3}\rangle \sim \cos^2\theta \langle JJJ\rangle_b + \sin^2\theta\langle JJJ\rangle_f + \cos\theta\sin\theta\langle JJJ\rangle_o}$$

which can be seen to consists of pieces of the limiting theories in the helicity basis. **Bosonization is manifest!** Can be pushed to 4-pt to show the existence of a one-param family of CFTs save for uniqueness

Very-unbroken higher-spin symmetry

Let's take any free CFT, e.g. free boson $\Box\phi = 0$ or free fermion $\not{\partial}\psi = 0$. In each of them we find (global symmetry current), the stress-tensor J_{ab} and infinitely many *higher spin conserved tensors* $J_{a_1...a_s}$ (aka **higher spin currents**, old name — Zilch):

$$J_s = \phi\partial...\partial\phi + ... \qquad\qquad J_s = \bar{\psi}\gamma\partial...\partial\psi + ...$$

They are quasi-primary at the unitarity bound and have $\Delta = d + s - 2$.

Stress-tensor is responsible for conformal symmetry $so(d, 2)$

$$Q_v = \int d^{d-1}x\, J_{0m}(x)v^m(x) \qquad\qquad \partial^n v^m + \partial^m v^n \sim \eta^{mn}$$

What are higher spin currents responsible for?

Unbroken higher spin symmetry

Any symmetry is certainly useful unless too much ...

Imagine a CFT $d \geq 3$ with $J_2 \equiv T_{ab}$ and $J_s \equiv J_{a_1 \ldots a_s}$, all being traceless and conserved. Is it interesting?

One can show (Maldacena, Zhiboedov; Boulanger, Ponomarev, E.S, Taronna; Alba, Diab) that there are J_s with arbitrarily high spin (at least all even spins), and the correlators are

$$\langle \boldsymbol{J \ldots J} \rangle = \textbf{some free CFT}$$

In $3d$ there are two choices: free boson $\Box \phi = 0$ and free fermion $\not{\partial} \psi = 0$. Note: they have different correlators of J!

When something is completely fixed, usually it is thanks to some symmetry. What is the symmetry behind?

Conserved tensor → current → symmetry charge → invariants=correlators

$$j_m(v) = J_{ma_2\ldots a_s} v^{a_2\ldots a_s} \qquad \partial^{(a_1} v^{a_2\ldots a_s)} = \eta^{(a_1 a_2} u^{a_3\ldots a_s)}$$

where $v^{a_1\ldots a_{s-1}}$ is a conformal Killing tensor (CKT). **Higher spin charges** form some ∞-dimensional extension of $so(d,2)$

$$Q = \int d^{d-1}p\; a_p^\dagger\, f(p,\partial_p)\, a_p \qquad [Q,Q] = Q$$

Miracle 1: Lie algebra of $Q_s = \int J_s$ originates from an associative one

Free CFT = Associative algebra

It can be understood as $U(so(d,2))/I$ (Gunaydin; Eastwood; ...). In $3d$ it is just the algebra of even operators $f(a^\alpha, a^\dagger_\beta)$ of $2d$ Harmonic oscillator. Note that $sp(4) \sim so(3,2)$ and $a^\alpha a^\beta$, $\{a^\alpha, a^\dagger_\beta\}$, $a^\dagger_\alpha a^\dagger_\beta$ form $sp(4)$.

Indeed, $\Box\Phi = 0$ is $so(d,2)$-invariant:

$$\delta_v\Phi = v^m\partial_m\Phi + \frac{d-2}{2d}(\partial_m v^m)\Phi$$

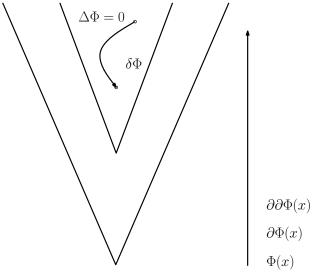
The latter means that $\Box\delta_v\Phi = L_v\Box\Phi = 0$ for some L_v, i.e. solutions are mapped to solutions. We can multiply such symmetries

$$\delta\Phi = \delta_{v_1}...\delta_{v_n}\Phi$$

For example, we find hyper-translations

$$\delta\Phi = \epsilon^{a_1...a_k}\partial_{a_1}...\partial_{a_k}\Phi$$

As a result the Lie bracket $[Q,Q]$ originates from some associative algebra, higher spin algebra, $\mathfrak{hs}$ via $a\star b - b\star a$.



Define V as the space of one-particle states $P_a ... P_c|\phi\rangle$, where $|\phi\rangle \equiv \phi(0)|0\rangle$

Higher spin algebra $\mathfrak{hs}$ is $\mathrm{End}(V)$, i.e. linear maps $V \to V$, which is $\mathfrak{hs} \sim V \otimes V^*$

Higher spin currents are bilinear in ϕ or ψ, i.e. $J \sim V \otimes V$

Miracle 2: $J \leftrightarrow \mathfrak{hs}$ upon identifying $|\phi\rangle|\phi\rangle$ with $|\phi\rangle\langle\phi|$ by inversion $\boldsymbol{R}$

There is a simple generating function (non primary)

$$\bar{\phi}(x-y)\phi(x+y) = \bar{\phi}\phi + \sum_s j_{a_1...a_s} y^{a_1} ... y^{a_s}$$

Unbroken higher spin symmetry: correlators

Conserved tensor $\to$ current $\to$ symmetry $\to$ invariants=correlators

Is higher spin symmetry powerful enough to fix correlators?

All correlators are invariants (Sundell, Colombo; Didenko, E.S.; ...)

$$\langle J...J\rangle = \mathrm{Tr}(C\star...\star C) \qquad\qquad C\leftrightarrow J$$

where cyclic symmetry is due to possibility to have ${J^i}_j \sim \bar{\phi}^i\partial...\partial\phi_j$, add permutations/projections if needed. The correlators are invariant under conformal and full HS symmetry, $\delta C = [C,\xi]_\star$:

Easy to say, but can we compute them?

Unbroken higher spin symmetry: correlators

Coherent states $J \leftrightarrow C$ in the Moyal-Weyl star-product algebra are Gaussians, hence, $C_1 \star C_2 ...$ is about Gaussian integrals. As a result one finds (Giombi, Yin; Sundell, Colombo) e.g. for free boson

$$\langle JJJ\rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} \cos(Q^2_{13} + Q^3_{21} + Q^1_{32}) \cos(P_{12}) \cos(P_{23}) \cos(P_{31})$$

and there is a simple formula for all n-point (Didenko, E.S.; Mei, Didenko, E.S.; Boulanger et al), e.g. free boson 4-point

$$\begin{aligned}\langle JJJJ\rangle_{F.B.} = &\frac{1}{|x_{12}||x_{23}||x_{34}||x_{41}|}\times \\ &\times \cos(Q^2_{13} + Q^3_{24} + Q^4_{31} + Q^1_{43}) \cos(P_{12}) \cos(P_{23}) \cos(P_{34}) \cos(P_{41}) \\ &+ \text{permutations}\end{aligned}$$

It would be hard to get these results 'just by Wick contractions'.

Unbroken higher spin symmetry: Summary

In every free CFT one finds ∞-many higher spin currents $J_s \equiv J_{a_1 \ldots a_s}$, which generate HS-charges $Q_s = \int J$. By construction, Q generate an ∞-dim Lie algebra, an extension of $so(3,2)$

$$[Q, Q] = Q \qquad\qquad [Q, J] = J \qquad\qquad [Q, \phi] = \phi$$

Miracle 1: the algebra originates from an associative HS-algebra $\mathfrak{hs}$ via $[a, b] = a \star b - b \star a$. **Free CFT = associative algebra**. **Miracle 2**: HS-currents J are isomorphic to $\mathfrak{hs}$ twisted by inversion $\boldsymbol{R}$.

Correlators are invariants of this HS-algebra $\mathfrak{hs}$

$$\langle J \ldots J \rangle = \mathrm{Tr}(C \star \ldots \star C) \qquad\qquad C \leftrightarrow J$$

Important: in $3d$ $\mathfrak{hs}_{F.B.} \sim \mathfrak{hs}_{F.F.} \sim$ Weyl algebra of $f(a_i^\dagger, a^j)$ and the invariants are the unique invariants of HS-algebra (Sharapov, E.S.)!

Slightly-broken higher-spin symmetry

Slightly-broken higher spin symmetry: what is it?

Initially: charges = higher spin algebra $\mathfrak{hs}$ and J = its module

$$\partial \cdot J_s = 0 \quad \Longrightarrow \quad Q_s = \int J_s \quad \Longrightarrow \quad [Q,Q] = Q \quad \& \quad [Q,J] = J$$

$$\boldsymbol{l(\xi_1,\xi_2)} \quad \& \quad \boldsymbol{l(\xi,J)}$$

The higher spin symmetry does not disappear completely:

$$\partial \cdot J = \frac{1}{N}[JJ] \qquad\qquad [Q,J] = J + \frac{1}{N}[JJ]$$

What is the right math?

Slightly-broken higher spin symmetry: what is it?

Initially we have well-defined charges and higher spin algebra $\mathfrak{hs}$

$$\partial \cdot J_s = 0 \quad \Longrightarrow \quad Q_s = \int J_s \quad \Longrightarrow \quad [Q, Q] = Q \quad \& \quad [Q, J] = J$$

The higher spin symmetry does not disappear completely:

$$\partial \cdot J = \tfrac{1}{N}[JJ] \qquad\qquad [Q, J] = J + \tfrac{1}{N}[JJ]$$

What is the right math?

We should deform the algebra together with its action on the module, so that the module (currents) can 'backreact':

$$\delta_\xi J = \boldsymbol{l(\xi, J)} + \boldsymbol{l(\xi, J, J)} + \dots\,, \qquad\qquad [\delta_{\xi_1}, \delta_{\xi_2}] = \delta_\xi\,,$$

where $\xi = \boldsymbol{l(\xi_1, \xi_2)} + \boldsymbol{l(\xi_1, \xi_2, J)} + \dots$.

The consistency of such a structure leads to $\boldsymbol{L_\infty}$-algebras

Slightly broken higher spin symmetry: summary

- necessary to bosonize: $\mathfrak{hs}$ (boson) $\sim$ $\mathfrak{hs}$ (fermion) (Dirac, 1963)
- there exist exactly one invariant, $\mathrm{Tr}(\Psi\star\ldots\star\Psi)$, to serve as n-point correlator $\langle J \ldots J\rangle$ for free/large-N limit
- L_∞ depends on two pheno parameters, to be related to k, N
- invariants are unobstructed and have a quasi-free form

$$\mathrm{Tr}_\circ \log_\circ[1-\Psi] \mod \text{irr}\,, \qquad a\circ b = a\star b + \phi_1(a,b)\boldsymbol{R} + \ldots$$

- a simple consequence is that correlators are very special

$$\langle J \ldots J\rangle = \sum \langle \text{fixed}\rangle_i \times \text{params}$$

This implies $3d$ bosonization since $\langle J \ldots J\rangle$ know everything and it does not matter what matter J are made of, ϕ or ψ

Summary/Remarks/Comments/Speculations

- **Higher spin symmetry is new ($d = 3, \ldots$) Virasoro**
- Slightly-broken symmetry should be understood as $\boldsymbol{L_\infty}$
- Uniqueness of $\boldsymbol{L_\infty}$-invariants implies the $3d$ bosonization duality and makes specific predictions for their structure
- **New type of a physical symmetry** where transformations (algebra) and the object (module) deform together
- Can we push slightly-broken symmetry beyond large N? (at least anomalous dimensions of HS-currents can be extracted from the non-conservation)
- Anomalous dimensions of HS-currents are small even for Ising model, $N = 1$, e.g. $\Delta(J_4) = 5.02$ instead of 5

Summary/Remarks/Comments/Speculations

- Chiral Higher Spin Gravity is dual to a closed subsector of (Chern-Simons) vector models. **There exists two such subsectors!** How to find them? It should extend to small N due to integrability, implications for Ising (low N)?
- The very existence of Chiral HiSGRA implies $3d$ bosonization at 3-pt and gives a one-parameter family of correlators at 4-pt. **Speculation: $3d$-bosonization is thanks to Chiral HiSGRA**
- Strings on $AdS_4 \times \mathbb{CP}^3$ are dual to ABJ theory = Chern-Simons (k) matter theories with bi-fundamental matter, $N \times M$, (Chang, Minwalla, Sharma, Yin). In the vector-like limit $N \gg M$ it is dual to $\mathcal{N} = 6$ $U(M)$-gauged HiSGRA (non-local). Inside there is $\mathcal{N} = 6$ $U(M)$-gauged Chiral HiSGRA. **Is it possible to directly identify the Chiral subsector of tensionless strings on $AdS_4 \times \mathbb{CP}^3$?**

That's all!

Thank you for your attention!

Unbroken higher spin symmetry: $3d$ specifics

In $3d$ the module of one-particle states of free boson/fermion CFT's is just the $2d$ harmonic oscillator (Dirac, 1963):

$$\begin{aligned} P_a ... P_a|\phi\rangle &\sim a^\dagger_\alpha a^\dagger_\beta ... a^\dagger_\alpha a^\dagger_\beta|0\rangle \\ P_a ... P_a|\psi\rangle &\sim a^\dagger_\alpha a^\dagger_\beta ... a^\dagger_\alpha a^\dagger_\beta \boldsymbol{a}^\dagger_\gamma|0\rangle \end{aligned}$$

This is thanks to $so(3,2) \sim sp(4,\mathbb{R})$ and thanks to the oscillator realization of $sp(2n)$, e.g. $P_m P^m \sim 0$, $P_m = \sigma^{\alpha\beta}_m a^\dagger_\alpha a^\dagger_\beta$, $\alpha, \beta, ... = 1, 2$

Now it is obvious that $\mathfrak{hs}$ is formed by even functions $f(a^\dagger, a)$. Formally, it is the even subalgebra of Weyl algebra A_2. Passing to p_i, q^j the product on $\mathfrak{hs}$ is the familiar Moyal-Weyl star-product:

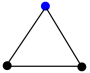
$$(f \star g)(q,p) f(q,p) \exp \frac{i\hbar}{2}(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q) g(q,p)$$

General structure of spinning correlators

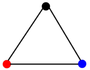
Following (Giombi, Prakash, Yin) in $3d$, $x^{\alpha\beta} = x^{\beta\alpha} = x^m \sigma_m^{\alpha\beta}$.

$$O_\Delta^{a_1 \dots a_s}(x) \qquad \longrightarrow \qquad O_\Delta(\mathrm{x}, \eta) = O_\Delta^{\alpha_1 \dots \alpha_{2s}}(\mathrm{x}) \eta_{\alpha_1} \dots \eta_{\alpha_{2s}} \; .$$

In $3d$ correlators of tensor operators can always be expressed in terms of conformally-invariant P, Q, S on top of functions of cross-ratios:

Q^i_{jk} :  $\langle O O J_s \rangle \sim Q^s$ $\quad Q^i_{jk} = \eta_i [\check{x}_{ij} - \check{x}_{ik}] \eta_i$

P_{ij} :  $\langle J_s J_s \rangle \sim P^s$ $\quad P_{ij} = \eta_i \check{x}_{ij} \eta_j$

S^i_{jk} :  $\langle J_s J_s O \rangle$ $\quad S^i_{jk} = \dfrac{\eta_j x_{ki} x_{ij} \eta_k}{|x_{ij}| |x_{ik}| |x_{jk}|}$

P and Q are parity-even, S is parity-odd. $\check{x} \equiv x^{\alpha\beta} / |x|^2$

Strong homotopy algebra is a graded space, e.g. $V = V_{-1} \oplus V_0$ equipped with multilinear maps $l_k(x_1, ..., x_k)$ of degree-one. In our case

$$l_k(\xi, \xi, J, ..., J) \qquad\qquad l_k(\xi, J, ..., J)$$

that allow us to encode the deformed action

$$\delta_\xi J = l_2(\xi, J) + l_3(\xi, J, J) + ... , \qquad\qquad [\delta_{\xi_1}, \delta_{\xi_2}] = \delta_\xi ,$$

where $\xi = l_2(\xi_1, \xi_2) + l_3(\xi_1, \xi_2, J) + ...$. The maps obey 'Jacobi' relations

$$\sum_{i+j=n} (\pm) l_i(l_j(x_{\sigma_1}, ..., x_{\sigma_j}), x_{\sigma_{i+1}}, ..., x_{\sigma_n}) = 0$$

L_∞ originates from A_∞ constructed from a certain deformation of $\mathfrak{hs}$, which is related to para-statistics/fuzzy sphere (Sharapov, E.S.)

Slightly-broken higher spin symmetry: L_∞

We need to construct L_∞ that 'deforms' our initial data = algebra + module, both originating from an associative algebra $A = \mathfrak{hs} \rtimes \mathbb{Z}_2$.

One can show (Sharapov, E.S.) that such L_∞ can be constructed as long as A is soft, i.e. can be deformed as an associative algebra:

$$a \circ (b \circ c) = (a \circ b) \circ c \qquad\qquad a \circ b = a \star b + \sum_{k=1} \phi_k(a,b)\, \hbar^k$$

The maps can be obtained from an auxiliary A_∞

$$\begin{aligned} m_3(a,b,u) &= \phi_1(a,b) \star u \quad &\rightarrow \quad l_3 \\ m_4(a,b,u,v) &= \phi_2(a,b) \star u \star v + \phi_1(\phi_1(a,b),u) \star v \quad &\rightarrow \quad l_4 \end{aligned}$$

Our algebra can be deformed thanks to para-statistics/anyons ...

Everyone knows that the Weyl algebra A_1 is rigid

$$[q,p]=i\hbar \qquad \text{no deformation of} \qquad f(q,p)\star g(q,p)$$

Suppose that $\boldsymbol{R}f(q,p)=f(-q,-p)$, i.e. we can realize it as

$$\boldsymbol{R}^2=1 \qquad \boldsymbol{R}q\boldsymbol{R}=-q \qquad \boldsymbol{R}p\boldsymbol{R}=-p$$

The crossed-product algebra $A_1\ltimes\mathbb{Z}_2$ is soft (Wigner; Yang; Mukunda; ...):

$$[q,p]=i\hbar+i\nu\boldsymbol{R}$$

Also known as para-bose oscillators. Even $\boldsymbol{R}(f)=f$ lead to $gl_\lambda=U(sp_2)/(C_2-\lambda(\lambda-1))$ (Feigin), also (Madore; Bieliavsky et al) as fuzzy-sphere, NC hyperboloid, also (Plyushchay et al) as anyons.

Orbifold $\mathbb{R}^2/\mathbb{Z}_2$ admits 'second' quantization on top of the Moyal-Weyl $\star$-product, (Pope et al; Joung, Mrtchyan; Korybut; Basile et al; Sharapov, E.S., Sukhanov)

Symmetry

99.99%: **Lie group** G/**Lie algebra** $\mathfrak{g}$ acting on some physical states. Group/Algebra $=$ transformations without any info on what they act

Yangian: deformation of $U(\mathfrak{g}[z])$ as a Hopf algebra. Spin-chains, planar $\mathcal{N}=4$ SYM and scattering amplitudes therein

Strong homotopy algebras: multi-linear products on graded spaces (Lie and associative algebras are examples). Nice organizing tool for $\boldsymbol{QQ=0}$: BV-BRST, string field theory, higher spin gravities, ...

New: (Chern-Simons) vector models (e.g. $3d$ Ising, ...) have ∞-many almost conserved tensors $\partial^m J_{ma_2...a_s}\approx 0$ — **slightly-broken higher spin symmetry** (Maldacena, Zhiboedov). The right structure are certain L_∞-algebras. **Symmetry gets entangled with its representation.** Explains $3d$-bosonizationd duality