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RECHERCHE EN DIDACTIQUE DES SCIENCES

What teaching practices should be used to introduce the limits of functions in the first year of university ? A case study

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Context of the research

- An ongoing research about university teachers' practices.
- Limit of functions : the formal definition of a limit, the first examples and the first results.
- A focus on teachers' discourse (lectures are not necessarily a source of inactivity for students, Bridoux et al., in press).

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Plan

1. Theoretical tools and main issue
2. Analysis of a lecture
3. Discussion

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Theoretical tools

Activity Theory (Vygotsky, Leontiev)

- AT adapted to the didactics of mathematics (Vandebrouck, 2008).
- Students' learning are studied through their mathematical activities.
- They are difficult to observe during the lectures.
- A focus on the teachers' discourse.
- Hypothesis : in order to advance the students' knowledge, the teacher's discourse has to be close to students' work to introduce new knowledge.

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$\Rightarrow$ **discursive proximities** (Robert and Vandebrouck, 2014).

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Theoretical tools

Three types of proximities (Bridoux et al., 2016) :

- **Bottom-up proximities** lie between what students have already done and the introduction of a new object or property.
- **Top-down proximities** are situated between what has been explained and examples or exercises.
- **Horizontal proximities** do not lead to any change between contextualised and decontextualised. They consist of reformulations, explanations of the links between concepts, comments on the structure of the course,...

Theoretical tools

- The researcher needs a reference to study the teachers' discourse.
- The **relief on the concepts to be taught** (Bridoux et al., 2016) combines epistemological, curricular and cognitive analyses.
- The relief helps the researcher to identify, a priori, opportunities of proximities.

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Main issue

What are the relief elements that give rise to proximities in the teachers' discourse during the course ? What are the links between the content organisation choices made by teachers and the proximities attempted in their discourse ?

Elements of relief

High school :

- Limits of functions are intuitively introduced in Première (grade 11, students aged 16-17) on the basis of examples and without formalization.
- A definition is given in Terminale (grade 12) in terms of intervals.
- The objective is for students to apply the operative aspects of limits.
- High school students often develop a dynamic conception (Robert, 1983) : "getting closer to".

Elements of relief

At university :

- $\forall \epsilon > 0 \; \exists \eta > 0 \; \forall x \in D_f (|x - a| \leq \eta \implies |f(x) - l| \leq \epsilon)$.
- It is difficult to make the students feel the need for introducing this definition.
- Dynamic conceptions must evolve towards static conceptions (Robert, 1982, Mamona-Downs, 2001).
- Complexity of the logic structure, knowledge probably not avalaible to a large number of students (absolute values, real numbers and inequalities).
- It is therefore difficult for the teacher to find an initial problem where the notion of limit would be the optimal tool to solve it and where the students could construct the new notion independently.
- Important role of the teacher's discourse.

Opportunities of proximities

- It is unlikely to find any bottom-up proximities in the teacher's discourse during the introduction of this definition.
- Horizontal proximities : formalizing the intuitive idea "$f(x)$ approaches l if x approaches a" to build the definition, using graphics and interpretring the inequalities in the definition in terms of intervals or distances.
- Top-down proximities in the teacher's discourse with first examples.

Teacher's discourse to introduce the formal definition

The teacher introduces intuitively the notion with a question : *How would you define an intuitive notion of limit ?*
A student : $f(x)$ *gets as close as you want to* l, then adds *when* x *gets close to* x_0.
Teacher shows a continuous function on the board and comments : *We're trying to look at a diagram to explore this concept. So* x *is approaching* x_0 *the point* M *is approaching the point* M_0, $f(x)$ *is the ordinate of* M *is approaching* l *there, OK ?*.
He writes and says : $f(x)$ *is as close as we want to* l *if* x *is close enough to* x_0.

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He writes and says : $f(x)$ *is as close as we want to* l *if* x *is close enough to* x_0.
$\Rightarrow$ **opportunities of horizontal proximities** (the teacher combines the graphic and natural language registers to give a static conception, see the relief).

Teacher's discourse to introduce the formal definition

Then he gives an example to desmontrate the need for a definition :
$f(x) = 0 \quad \text{if} \quad x \in \mathbb{R} \backslash \mathbb{Q}, f(x) = 1/q \quad \text{if} \quad x = p/q \in \mathbb{Q}$ where p/q is irreductible, and says :
Here, the intuitive notion becomes complicated, because you're going to have trouble tracing this curve. So we're not going to be able to use a geometric notion of the limit. So, to solve a certain number of problems, we need a more mathematical, more rigorous definition.

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$\Rightarrow$ **bottom-up proximities are missed** (the students did not have to work with this kind of function at high school).

Teacher's discourse to introduce the formal definition

On the board	What teacher says
$\lvert f(x) - l \rvert < \epsilon$	*What is the distance from $f(x)$ to l ? Yes, it's the absolute value of $f(x) - l$ OK. So we want $f(x) - l$ to be as small as we want, $f(x)$ is going to be as close as we want to l. What does that mean ? It means that the absolute value of $f(x) - l$ is less than epsilon, for epsilon to be as small as we want, we agree.*

Teacher's discourse to introduce the formal definition

On the board	What teacher says
$\forall\epsilon > 0 \; \exists\alpha \quad \lvert x - x_0\rvert < \alpha \implies \lvert f(x) - l\rvert < \epsilon$	*So alpha, how do we introduce it, because here we're introducing a notation, which means that alpha has to be in which set ? It has to increase a distance so... [student answer] positive, that's it. Whatever epsilon is, as soon as x is close enough to x_0 i.e. if the distance from x to x_0 is less than alpha, so behind this is the notion that there exists alpha such that.*

Plan

Theoretical tools and main issue

Analysis of a lecture

Discussion

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Teacher's discourse to introduce the formal definition

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$\Rightarrow$ **opportunities of horizontal proximities** (reformulations of the closeness idea in terms of an inequality and in terms of distance, see the relief), but there is a discrepancy between what is said orally and what is written.

Discussion

- The relief helped us to identify horizontal proximities : reformulations to link different semiotic registers (words, graphs, symbols).
- There are (probably) no possible bottom-up proximities as we had anticipated in the relief.
- This research shows how the tools lead us to study the teacher's discourse.
- The work needs to be extended to more lectures to compare the proximities contained in the teachers' discourse.