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On the deep superstring spectrum

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Abstract: We propose a covariant method of constructing entire trajectories of physical states in superstring theory in the critical dimension. It is inspired by a recently developed covariant technology of excavating bosonic string trajectories, that is facilitated by the observation that the Virasoro constraints can be written as linear combinations of lowering operators of a bigger algebra, namely a symplectic algebra, which is Howe dual to the spacetime Lorentz algebra. For superstrings, it is the orthosymplectic algebra that appears instead, with its lowest weight states forming the simplest class of physical trajectories in the NS sector. To construct the simplest class in the R sector, the lowest weight states need to be supplemented with other states, which we determine. Deeper trajectories are then constructed by acting with suitable combinations of the raising operators of the orthosymplectic algebra, which we illustrate with several examples.

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Contents

1 Introduction

The spectrum of superstring theory is organized in supermultiplets of the $\mathcal{N} = 1$ super-Poincaré group in the critical dimension $d = 10$. For open superstrings, it contains a single massless supermultiplet, composed of a spin-1 and a spin-($1/2$) state, and *infinitely many* massive ones.[1] Each of these massive states is characterized by a highest weight of the little group SO(9), i.e. a collection $\mathbb{Y} = (s_1, s_2, s_3, s_4)$ of *ordered* non-negative numbers which are either *all integers* — for bosonic states — or *all half-integers* — for fermionic states, which we may think of as the analogue of spin in four dimensions. To construct these states explicitly in a systematic manner (for low level examples, see e.g. [3–9], and more recently [10, 11], for the bosonic string, and [12–21] for the superstring), one can for example work in the light cone, or use the Del Giudice-Di Vecchia-Fubini (DDF) construction [22–25], since both can generate the whole superstring spectrum. While both methods have led to great success in the development of (super)string theories (for instance, in the proof of the no-ghost theorem [26–28]), they also suffer from drawbacks: in the light cone, the output

[1] The spectrum can be extracted from the partition function given in terms of SO(9) characters and spelled out explicitly up to level 10 in [1] (see also [2]).

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of constructed states is not immediately Lorentz covariant, and in the DDF formalism it is a superposition of massive states with different spins.

For the bosonic string, a method for computing whole *trajectories* at once, and in a Lorentz covariant manner, was introduced in [29], based on a result from representation theory known as Howe duality [30, 31]. The idea of [29], which we extend to the superstring in this paper, goes as follows. First, one observes that a polarization tensor $\varepsilon_{\mathbb{Y}}$ which is transverse, traceless and has the symmetry of the Young diagram $\mathbb{Y}$,[2] when contracted in a specific way with d-dimensional oscillators of the string Fock space, defines a simple solution to the Virasoro constraints, i.e. gives rise to physical states that were called *principally embedded states*. They are characterized by the fact that they minimize the level for a fixed diagram $\mathbb{Y}$, i.e. they constitute the first occurrence of the diagram $\mathbb{Y}$ when going through the bosonic string spectrum in the direction of increasing mass level. For instance, the leading Regge trajectory, namely the set of highest-spins per level, consists of principally embedded states; in the bosonic string, its interactions were thoroughly explored in [33–35] and its transversality and tracelessness constraints were put in differential form in [35], while its interactions in the superstring were investigated in [36, 37].

The second observation of [29] is to recognize the Virasoro algebra as a subalgebra of $\mathfrak{sp}(\bullet)$, the limit of $\mathfrak{sp}(2K,\mathbb{R})$ when $K \longrightarrow \infty$. This algebra acts on the *modes* of the oscillators, and importantly, commutes with the Lorentz algebra. As a consequence, the Fock space of the bosonic string decomposes into a direct sum of tensor products of irreps of $\mathfrak{so}(d-1,1)$ and $\mathfrak{sp}(\bullet)$ wherein each irrep of one algebra appears together with a unique irrep of the other. Put differently, the bosonic string Fock space can be decomposed in subspaces, in which each state carries a given Lorentz representation $\mathbb{Y}$, and also forms an irreducible module of $\mathfrak{sp}(\bullet)$. One can therefore consider this $\mathfrak{sp}(\bullet)$-module, often called "Howe dual" to $\mathbb{Y}$, as its *multiplicity space*. Upon working in the transverse subspace,[3] all states with $\mathfrak{so}(d-1)$-irrep $\mathbb{Y}$ can be found in the $\mathfrak{sp}(\bullet)$-module, and hence one can focus on this module to look for physical states of spin $\mathbb{Y}$.

The last point is that the Howe dual representation to $\mathbb{Y}$ is a *lowest weight module* of $\mathfrak{sp}(\bullet)$, whose lowest weight vector is the principally embedded state discussed previously. Consequently, one can reach any other state with spin $\mathbb{Y}$ by acting on the principally embedded one with raising operators of $\mathfrak{sp}(\bullet)$. Building physical states of spin $\mathbb{Y}$ therefore amounts to finding polynomials in $\mathfrak{sp}(\bullet)$ raising operators which are compatible with the physicality conditions, meaning that they commute with the positive-modes Virasoro generators, up to lowering operators of $\mathfrak{sp}(\bullet)$, as the latter act trivially on principally embedded states. In doing so, the dependence of the algorithm on $\mathbb{Y}$ is fairly minimal: the lengths of its rows merely appear as parameters in the polynomials in raising operators, thereby allowing one to derive whole *trajectories of states* [29, section 4.3]. In fact, a more relevant parameter

[2]The Young symmetry and tracelessness condition ensure that the polarization tensor carries the irrep $\mathbb{Y}$ of the Lorentz algebra $\mathfrak{so}(d-1,1)$, while the transversality constraint reduces its $\mathfrak{so}(d-1)$ content to a single irrep, corresponding to the same Young diagram $\mathbb{Y}$ for the massive little algebra. In other words, it allows one to describe an $\mathfrak{so}(d-1)$-irrep in a Lorentz-covariant manner. For more details on this last point, see appendix B, or [32].

[3]This restriction is possible since the BRST cohomology of the bosonic string is included in the transverse subspace [38–40].

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for defining the notion of trajectory in the bosonic string seems to be the number of rows of a Young diagram. The other important quantity is the difference between the level of the principally embedded state with diagram $\mathbb{Y}$, and the level (necessarily higher) at which a clone of the same diagram appears, called *depth*.

In this work, we extend this method to the case of the superstring in the Ramond-Neveu-Schwarz (RNS) formulation. The reasoning behind the construction is almost identical to the bosonic string case, upon replacing the symplectic algebra $\mathfrak{sp}(\bullet)$ with its orthosymplectic counterpart $\mathfrak{osp}(\bullet|\bullet)$, and identifying the multiplicity space of a Lorentz irrep with a particular lowest weight module of it. There are however, a few noteworthy differences, owing to the specificities of the superstring compared to the bosonic one.

- To begin with, one should distinguish between the Neveu-Schwarz (NS) and the Ramond (R) sectors, the former consisting of spacetime bosons while the latter of spacetime fermions. While, in both sectors, the $\mathfrak{osp}(\bullet|\bullet)$-module dual to a given Lorentz irrep $\mathbb{Y}$ is of lowest weight type, its lowest weight vector is also the principally embedded state of diagram $\mathbb{Y}$ *only* in the NS sector. In the R sector, we find that one should consider a linear combination of the lowest weight vector together with other states, as elaborated on in subsection 4.2. This is related to the fact that, in the R sector, some $\mathfrak{osp}(\bullet|\bullet)$ raising operators do not change the level, and therefore the subspace of states at minimal level in the lowest weight $\mathfrak{osp}(\bullet|\bullet)$-module is not reduced to the lowest weight vector as in the NS sector, but is of finite dimension (greater than 1). This also leads to the appearance of non-trivial multiplicities for principally embedded fermions of mixed-symmetry, which are confirmed by the results of [1] for low levels. The situation is illustrated in figure 1 below.

- The spectrum of superstring theory in the RNS formalism becomes spacetime supersymmetric only after implementing the Gliozzi-Scherk-Olive (GSO) projection that is necessary for modular invariance [41]. In the NS sector, this means projecting out all half-integer levels. Our construction is formulated before the GSO projection and we find that it is both possible to uplift physical states that are projected out to ones that are not, and vice-versa.

- Finally, we observe that the number of boxes of the principal *diagonal* of Young diagrams seems to define a sensible notion of a superstring trajectory, instead of the number of rows in the bosonic string.

This paper is organized as follows. In section 2, we give a brief summary of superstring theory in the Ramond-Neveu-Schwarz formalism, focusing on the construction of physical vertex operators and the corresponding physical states. In section 3, we quickly review Howe duality for the bosonic string, and discuss its extension relevant for the superstring. In section 4, we start by deriving the physicality conditions in the NS sector and discuss their counterpart in the R sector, before identifying principally embedded states in both sectors. We then examine operators of small values of the depth, present various solutions for those and use them to derive examples of physical trajectories. We conclude in section 5. Appendices A and B contain a review of the construction of some of the lightest R states in the light cone gauge and our conventions for Young diagrams respectively.

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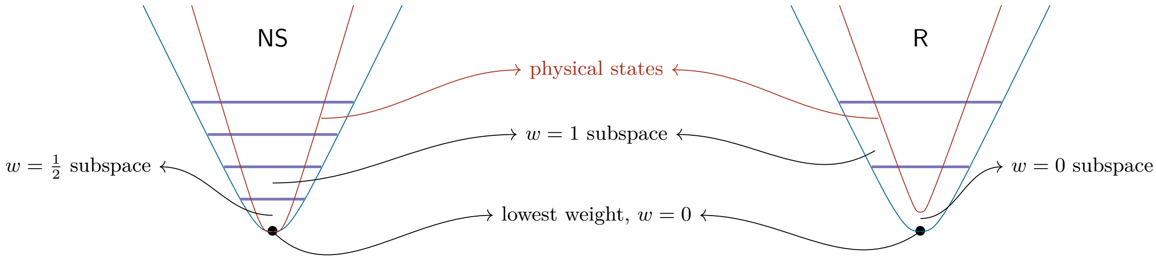

Figure 1. Schematic representation of an $\mathfrak{osp}(\bullet|\bullet)$ lowest weight module, Howe dual to an irrep $\mathbb{Y}$ of the Lorentz algebra $\mathfrak{so}(d-1,1)$, in the blue outline. The purple horizontal lines separate subspaces of different depths w, and the red outline the subspace of *physical states*.

Conventions. Spacetime indices are denoted by Greek letters, $\mu, \nu, \ldots = 0, \ldots, d-1$ and we use the mostly-plus metric signature. We work in the critical dimension of the superstring, namely $d = 10$ implicitly. The dot product "$\cdot$" denotes the Minkowski product in flat spacetime. For the open string, we set $\alpha' = \frac{1}{2}$ as is customary (and equivalently $\alpha' = 2$ for the closed string), which means that the string field X^μ has 0 mass dimension.

2 Superstring elements

2.1 Ingredients of the worldsheet SCFT

We will be employing the Ramond-Neveu-Schwarz (RNS) formulation of the superstring, which we briefly review in this subsection following [42, 43] (see also [44] for a pedagogical introduction). We focus on the open superstring, namely holomorphic fields on the string worldsheet, and recall that it is straightforward to obtain closed strings by introducing their antiholomorphic counterparts and imposing the level-matching condition. Fundamental ingredients are the "matter" spacetime vectors ∂X^μ and ψ^μ, that are superpartners and primary fields of weight 1 and $\frac{1}{2}$ respectively with respect to the worldsheet superconformal field theory (SCFT); they are functions of the (complex) worldsheet coordinate z. The corresponding worldsheet energy-momentum tensor and supercurrent read

$$T^{X,\psi}(z) = -\frac{1}{2}\,\partial X \cdot \partial X\,(z) - \frac{1}{2}\,\psi \cdot \partial\psi\,(z)\,, \qquad T_{\mathrm{F}}^{X,\psi}(z) = \frac{i}{2}\,\psi \cdot \partial X\,(z)\,, \tag{2.1}$$

and are naturally superpartners. In the BRST quantization, the conformal (b, c) and superconformal (β, γ) ghost systems, with weights equal to $(2, -1)$ and $(\frac{3}{2}, -\frac{1}{2})$ respectively, have to be introduced. The ghost energy-momentum tensor and supercurrent read respectively

$$T^{\mathrm{gh}}(z) = T^{b,c}(z) + T^{\beta,\gamma}(z) = -2\,b\partial c(z) - (\partial b)c(z) - \frac{3}{2}\,\beta\partial\gamma(z) - \frac{1}{2}\,(\partial\beta)\gamma(z)\,, \tag{2.2}$$

$$T_{\mathrm{F}}^{\mathrm{gh}}(z) = \frac{1}{2}\,b\gamma(z) - (\partial\beta)c(z) - \frac{3}{2}\,\beta\partial c(z)\,. \tag{2.3}$$

The (nilpotent) worldsheet BRST charge then takes the form

$$\mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2\,, \tag{2.4}$$

with

$$\mathcal{Q}_0 = \oint \frac{\mathrm{d}z}{2\pi i}\, c\left[T^{X,\psi} + T^{\beta,\gamma} + (\partial c)b\right], \tag{2.5a}$$

$$\mathcal{Q}_1 = -\oint \frac{\mathrm{d}z}{2\pi i}\, e^{-\chi}e^{\varphi}\, T_{\mathrm{F}}^{X,\psi}, \tag{2.5b}$$

$$\mathcal{Q}_2 = \frac{1}{4}\oint \frac{\mathrm{d}z}{2\pi i}\, b\, e^{-2\chi}e^{2\varphi}, \tag{2.5c}$$

where the superghost system is bosonized by the chiral bosons φ and χ as in

$$\beta = e^{-\varphi}e^{\chi}\partial\chi \qquad \text{and} \qquad \gamma = e^{-\chi}e^{\varphi}, \tag{2.6}$$

so that the index i of $\mathcal{Q}_i$ stands for its superghost charge.[4] Note that b and c are Grassmann-odd, while β and γ are Grassmann-even. The worldsheet $\mathcal{N}=1$ superconformal algebra reads

$$T(z)\,T(w) \sim \frac{1}{2}\frac{\mathsf{c}}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{2.7a}$$

$$T(z)\,T_F(w) \sim \frac{3}{2}\frac{T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w}, \tag{2.7b}$$

$$T_F(z)\,T_F(w) \sim \frac{1}{6}\frac{\mathsf{c}}{(z-w)^3} + \frac{1}{2}\frac{T(w)}{z-w}, \tag{2.7c}$$

where the symbol "$\sim$" stands for equality up to regular terms and the central charge c is equal to ± 15 for the matter fields and superghosts respectively. Note that the CFTs of X, ψ, (b,c) and (β,γ) are free and thought of as being decoupled.

Due to the two possibilities for the periodicity of ψ^μ, the open superstring comprises two sectors, the Neveu-Schwarz (NS) and the Ramond (R), in which ψ is periodic and antiperiodic on the complex plane respectively, which is mapped here to the two-dimensional disk. The boson OPEs read

$$X^\mu(z)X^\nu(w) \sim -\eta^{\mu\nu}\,\ln|z-w|, \tag{2.8}$$

and

$$\chi(z)\chi(w) \sim \ln|z-w|, \qquad \varphi(z)\varphi(w) \sim -\ln|z-w|, \tag{2.9}$$

whereas the fermion OPE reads

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}, \tag{2.10}$$

in both sectors. The R sector requires the introduction of a primary field S_A of weight $\frac{d}{16}$, which is a Dirac spinor of $\mathfrak{so}(d-1,1)$. Upon decomposing S_A into left– and right-handed Weyl spinors S_α and $S^{\dot\alpha}$, the relevant OPE read

$$S_\alpha(z)S_\beta(w) \sim \frac{(\gamma^\mu C)_{\alpha\beta}\,\psi_\mu(w)}{\sqrt{2}\,(z-w)^{3/4}}, \tag{2.11}$$

$$\begin{aligned}\psi^\mu(z)S_\alpha(w) = {}& \frac{(\gamma^\mu)_{\alpha\dot\beta}\,S^{\dot\beta}(w)}{\sqrt{2}\,(z-w)^{1/2}} \\ &+ (z-w)^{1/2}\left[\frac{\sqrt{2}}{d-2}\psi^\mu\psi - \frac{1}{\sqrt{2}(d-2)(d-1)}\gamma^\mu\psi\psi\right]_{\alpha\dot\beta} S^{\dot\beta}(w) + \dots,\end{aligned} \tag{2.12}$$

[4]The superghost charges of β and γ are -1 and $+1$ respectively.

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where we used the conventions of [16, appendix A & B] for the gamma and charge conjugation matrices, denoted by γ^μ and C respectively. The subleading term in (2.12) was first given in terms of contractions of $\psi^\mu\psi^\nu S^{\dot\beta}$ in [45] (see also [12]) and much later obtained in the covariant form in terms of S^μ_α and $\partial S^{\dot\beta}$ [16], with S^μ_α being a vector-spinor of weight ${}^{(d+16)}/_{16}$. Finally, we will also need the OPE

$$e^{q_1\varphi(z)}e^{q_2\varphi(w)} = (z-w)^{-q_1q_2} \\ \times\left[1+(z-w)\,q_1\partial\varphi+\frac{1}{2}(z-w)^2\,q_1\left[\partial^2\varphi+q_1(\partial\varphi)^2\right]+\dots\right]e^{(q_1+q_2)\varphi(w)}\,. \tag{2.13}$$

Physical states. A generic vertex operator creating asymptotic states of momentum p^μ takes the form

$$V_F^{(q)}(p,z) = g_{\rm o}\,T^a\,e^{q\,\varphi}\,F(\partial^k X,\partial^r\psi)\begin{cases} e^{i\,p\cdot X}\,, & \text{in the NS sector} \\ S_A\,e^{i\,p\cdot X}\,, & \text{in the R sector,}\end{cases} \tag{2.14}$$

where $g_{\rm o}$ is the open string coupling constant and T^a the brane gauge group generators, namely the Chan-Paton factors [46]; the factor $g_{\rm o}\,T^a$ will always be implicit from now on. Here, F is a polynomial in ∂X^μ, ψ^μ and their derivatives, and q is called the ghost picture of $V_F^{(q)}$. The factor $e^{q\,\varphi}$ and the momentum eigenstate $e^{i\,p\cdot X}$ contribute $-\frac{1}{2}q(q+2)$ and $\frac{1}{2}\,p^2$ units of conformal weight respectively, while the descendants $\partial^k X$ and $\partial^r\psi$ have weights k and $r+\frac{1}{2}$ respectively. Vertex operators are spacetime scalars, so the polynomial F further contains a priori arbitrary tensors (tensor-spinors) in the NS (R) sector, which are introduced to construct a Lorentz invariant quantity out of the previous ingredients. Superstring states belonging to the NS (R) sector are thus spacetime bosons (fermions). In the BRST quantization, the physical state condition takes the form

$$[\mathcal{Q},V_F^{(q)}] = \text{tot. deriv.}\,, \tag{2.15}$$

and implies separately

$$[\mathcal{Q}_0,V_F^{(q)}(z)] = \text{tot. deriv.}\,,\qquad [\mathcal{Q}_1,V_F^{(q)}(z)] = \text{tot. deriv.}\,,\qquad [\mathcal{Q}_2,V_F^{(q)}(z)] = +\text{tot. deriv.}\,, \tag{2.16}$$

since the three parts $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have different superconformal ghost charges.

Moreover, the vertex operator creating a physical state can be written in infinitely many different ways, which correspond to different values of the picture q. Different pictures are obtained via the operation

$$V_F^{(q+1)}(w) = 2\Big[\mathcal{Q},e^{\chi(w)}\,V_F^{(q)}(w)\Big]\,, \tag{2.17}$$

valid in both sectors, NS and R. F is simplest in what is called the canonical ghost picture, which corresponds to $q=-1$ and $q=-\frac{1}{2}$ (or $-\frac{3}{2}$) in the NS and R sector respectively.[5] Notice that, while (2.17) takes the form of a spurious state by construction, it *does* contribute to scattering amplitudes, namely all allowed pictures are physical. In addition, because of the

[5]In the NS (R) sector, it holds that $q\in\mathbb{Z}$ ($q\in\mathbb{Z}+\frac{1}{2}$) for the allowed pictures.

first of the constraints (2.16), every physical vertex operator has to have conformal weight equal to 1. Indeed, assuming that $V_F^{(q)}(w)$ has conformal weight h with respect to the total energy-momentum tensor, $T = T^{X,\psi} + T^{b,c} + T^{\beta,\gamma}$, namely

$$T(z)V_F^{(q)}(w) \sim \frac{h\, V_F^{(q)}(w)}{(z-w)^2} + \frac{\partial V_F^{(q)}(w)}{z-w}\,, \tag{2.18}$$

the first of (2.16) yields $h = 1$ for any vertex operator in any picture q. In the canonical ghost picture, this yields the conformal weight h_F of its corresponding polynomial as

$$h_F = \phi - \frac{1}{2}p^2\,, \tag{2.19}$$

where

$$\phi = \begin{cases} \frac{1}{2}\,, & \text{in the NS sector} \\ 0\,, & \text{in the R sector.} \end{cases} \tag{2.20}$$

2.2 Fock space and oscillators

The various fields of the worldsheet CFTs can be expanded in Fourier modes, for example

$$i\partial X^\mu(z) = \sum_{n\in\mathbb{Z}} \alpha_n^\mu\, z^{-n-1}\,, \qquad \psi^\mu(z) = \sum_{r\in\mathbb{Z}+\phi} b_r^\mu\, z^{-r-\frac{1}{2}}\,, \tag{2.21}$$

and

$$T^{X,\psi}(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}\,, \qquad T_{\mathrm{F}}^{X,\psi}(z) = \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} G_r\, z^{-r-\frac{3}{2}}\,, \tag{2.22}$$

where

$$\alpha_0^\mu = p^\mu\,. \tag{2.23}$$

The level number operator reads

$$N := \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m + \sum_{r\in\mathbb{Z}+\phi>0}^{\infty} r\, b_{-r}\cdot b_r\,, \tag{2.24}$$

while the modes of the energy-momentum tensor and supercurrent are given by the bosonic and fermionic oscillators via[6]

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_{-m}\cdot\alpha_{m+n}: + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\left(r+\frac{n}{2}\right) :b_{-r}\cdot b_{n+r}: -\delta_{n,0}\,\frac{d}{8}\left(\frac{1}{2}-\phi\right)\,, \tag{2.25}$$

$$G_r = \sum_{m\in\mathbb{Z}} :\alpha_{-m}\cdot b_{m+r}:\,, \tag{2.26}$$

where $:(\dots):$ denotes the normal ordering of oscillators. Explicitly separating the positive and negative modes, and implementing the normal ordering, the super-Virasoro generators read

$$\begin{aligned} L_{n>0} &= p\cdot\alpha_n + \sum_{m\geq 1}\alpha_{-m}\cdot\alpha_{m+n} + \frac{1}{2}\sum_{m=1}^{n-1}\alpha_m\cdot\alpha_{n-m} \\ &\quad + \sum_{r\geq 1}\left(r+\frac{n}{2}-\phi\right) b_{-r+\phi}\cdot b_{r-\phi+n} + \frac{1}{2}\sum_{r=2\phi}^{n}\left(-r+\frac{n}{2}+\phi\right) b_{r-\phi}\cdot b_{n-r+\phi}\,, \end{aligned} \tag{2.27}$$

[6] Note that, in the R sector, the L_0 has been shifted by $-\frac{d}{16}$ so as to have the standard realization of the super-Virasoro algebra (2.32a)–(2.32c). This shift is cancelled by the R vacuum's conformal weight upon action of L_0 in the physicality condition (2.34).

and

$$G_{r-\phi>0} = p \cdot b_{r-\phi} + \sum_{k\geq 1} \alpha_{-k} \cdot b_{k+r-\phi} + \sum_{k\geq 2\phi} b_{-k+\phi} \cdot \alpha_{k+r-2\phi} + \sum_{k=1}^{r-1} \alpha_k \cdot b_{r-\phi-k}\,, \tag{2.28}$$

for the positive modes, and the negative ones can be obtained by conjugation. Note that some sums in the above expressions start at 2ϕ, i.e. 1 in the NS sector and 0 in the R sector, due to the existence of *zero-modes* for the fermionic oscillators in the R sector. Finally, one finds

$$L_0 = \frac{1}{2}\,p^2 + \sum_{k\geq 1} \left(\alpha_{-k} \cdot \alpha_k + (k-\phi)\, b_{-k+\phi} \cdot b_{k-\phi}\right) - \frac{d}{8}\left(\frac{1}{2} - \phi\right) \equiv \frac{1}{2}\,p^2 + N - \frac{d}{8}\left(\frac{1}{2} - \phi\right), \tag{2.29}$$

where N is the level number operator introduced above. Requiring states to be eigenvectors of L_0 therefore fixes their mass $m^2 = -p^2$ in terms of the level N.

The analogue of the boson OPE (2.8) is the bosonic oscillator algebra

$$[\alpha^\mu_m, \alpha^\nu_n] = m\, \delta_{m+n,0}\, \eta^{\mu\nu}\,, \qquad (\alpha^\mu_m)^\dagger = \alpha^\mu_{-m}\,, \qquad m, n\, \in \mathbb{Z}\,, \tag{2.30}$$

while the analogue of the fermion OPE (2.10) is the fermionic oscillator algebra

$$\{b^\mu_m, b^\nu_n\} = \delta_{m+n,0}\, \eta^{\mu\nu}\,, \qquad (b^\mu_m)^\dagger = b^\mu_{-m}\,, \qquad m, n \in \mathbb{Z} + \phi\,, \tag{2.31}$$

and the analogue of (2.7a)–(2.7c) is the super-Virasoro algebra

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{d}{8}\, m(m^2-1)\, \delta_{m+n}\,, \tag{2.32a}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}\,, \tag{2.32b}$$

$$\{G_r, G_s\} = 2\, L_{r+s} + \frac{d}{2}\left(r^2 - \frac{1}{4}\right) \delta_{r+s}\,. \tag{2.32c}$$

The Fock space is generated by the action of the oscillators with negative modes on a vacuum state $|0\rangle$, whose defining property is to be annihilated by positive-mode oscillators

$$\alpha^\mu_m\;|0\rangle = 0 = b^\mu_r\;|0\rangle\;, \qquad \forall\, m, r > 0\,. \tag{2.33}$$

The physicality conditions (2.16) on a general state $|\text{phys}\rangle$, built out of non-positive-mode oscillators acting on the vacuum, now take the form

$$(L_0 - \phi)|\text{phys}\rangle = 0\,, \tag{2.34}$$

$$L_m|\text{phys}\rangle = 0\,, \qquad m > 0\,, \tag{2.35}$$

$$G_r|\text{phys}\rangle = 0\,, \qquad r \geq \phi\,. \tag{2.36}$$

Moreover, sufficient are the following constraints

$$\mathsf{NS}: \quad \left(L_0 - \frac{1}{2}\right)|\text{phys}\rangle = 0\,, \qquad G_{1/2}\,|\text{phys}\rangle = 0\,, \qquad G_{3/2}\,|\text{phys}\rangle = 0\,, \tag{2.37}$$

$$\mathsf{R}: \quad G_0\,|\text{phys}\rangle = 0\,, \qquad G_1\,|\text{phys}\rangle = 0\,, \tag{2.38}$$

as a consequence of the super-Virasoro algebra (2.32a)–(2.32c).

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2.3 Building the spectrum and the lightest states

Using the creation operators α_{-n} and $b_{-r+\frac{1}{2}}$ in the NS sector and α_{-n} and b_{-r} in the R sector, a general candidate for a physical state can be written as

$$\varepsilon^{\mathsf{NS}}_{\mu_1(m_1)|\ldots|\mu_I(m_I)|\nu_1[n_1]|\ldots|\nu_J[n_J]}\,\alpha^{\mu_1(m_1)}_{-k_1}\ldots\alpha^{\mu_I(m_I)}_{-k_I}\,b^{\nu_1[n_1]}_{-r_1+\frac{1}{2}}\ldots b^{\nu_J[n_J]}_{-r_J+\frac{1}{2}}\;|p;0\rangle_{\mathsf{NS}}\;, \tag{2.39}$$

$$\varepsilon^{\mathsf{R},A}_{\mu_1(m_1)|\ldots|\mu_I(m_I)|\nu_1[n_1]|\ldots|\nu_J[n_J]}\,\alpha^{\mu_1(m_1)}_{-k_1}\ldots\alpha^{\mu_I(m_I)}_{-k_I}\,b^{\nu_1[n_1]}_{-r_1}\ldots b^{\nu_J[n_J]}_{-r_J}\;|p;A;0\rangle_{\mathsf{R}}\;, \tag{2.40}$$

in the NS and in the R sector respectively, where we have used the notation

$$\alpha^{\mu(m)}_{-k} := \alpha^{(\mu_1}_{-k}\ldots\alpha^{\mu_m)}_{-k}\,,\qquad b^{\nu[n]}_{-r+\phi} := b^{[\nu_1}_{-r+\phi}\ldots b^{\nu_n]}_{-r+\phi}\,, \tag{2.41}$$

both as a way to shorten the expression and to highlight the commuting, respectively anticommuting, nature of α, respectively b. The polarization tensor $\varepsilon^{\mathsf{NS}}$ and the polarization tensor-spinor $\varepsilon^{\mathsf{R},A}$ have I groups of m_i symmetric indices, and J groups of n_j antisymmetric indices, which a priori do not obey any particular symmetry condition between one another — requiring that a state be physical will result in imposing certain symmetry conditions on its polarization tensor. Using (2.24), the respective levels at which the states (2.39) and (2.40) find themselves at are

$$N = \sum_{i=1}^{I} k_i\,m_i + \sum_{j=1}^{J}(r_j-\phi)\,n_j\,. \tag{2.42}$$

State-operator correspondence. By means of the state-operator correspondence, string states can be mapped to vertex operators, that is the language most suitable for the computation of string scattering amplitudes, as the latter can be written as vacuum expectation values of the product of the vertex operators creating all external states in a given diagram. To recall how the correspondence is realized, let us start with the vacuum. The momentum eigenstate vacuum in the NS sector is created by

$$|p;0\rangle_{\mathsf{NS}} = \lim_{z\to 0} e^{i\,p\cdot X(z)}\,|0\rangle\;, \tag{2.43}$$

while in the R sector, it is created by

$$|p;A;0\rangle_{\mathsf{R}} = \lim_{z\to 0} S_A\,e^{i\,p\cdot X}(z)\,|0\rangle\;. \tag{2.44}$$

In fact, the interpolation between the two sectors by means of the spin-field S_A can be expressed via

$$|p;A;0\rangle_{\mathsf{R}} = \lim_{z\to 0} S_A(z)\,|p;0\rangle_{\mathsf{NS}}\;. \tag{2.45}$$

Moreover, the expansions (2.21) can be inverted to yield, for example, the creation operators as integrals of the fields ∂X and ψ:

$$\alpha^{\mu}_{-n} = \oint\frac{\mathrm{d}z}{2\pi i}\frac{1}{z^n}\,i\partial X^{\mu}(z)\qquad\text{and}\qquad b^{\mu}_{-r+\phi} = \oint\frac{\mathrm{d}z}{2\pi i}\frac{1}{z^{r+1/2-\phi}}\,\psi^{\mu}(z)\,. \tag{2.46}$$

Consequently, the bosonic dictionary in both sectors is easily derived from (2.46) as

$$\alpha^{\mu}_{-n}\,|0\rangle \quad\longleftrightarrow\quad \frac{1}{(n-1)!}\,i\partial^n X^{\mu}(0)\,, \qquad \alpha^{\mu}_{n} \quad\longleftrightarrow\quad n!\,\eta^{\mu\nu}\frac{\delta}{i\delta\partial^n X^{\nu}}\,, \qquad n\geqslant 1\,, \tag{2.47}$$

as is the fermionic dictionary in the NS sector:

$$b^{\mu}_{-r+\frac{1}{2}}\,|0\rangle \quad\longleftrightarrow\quad \frac{1}{(r-1)!}\,\partial^{r-1}\psi^{\mu}(0)\,, \qquad b^{\mu}_{r-1/2} \quad\longleftrightarrow\quad (r-1)!\,\eta^{\mu\nu}\frac{\delta}{\delta\partial^{r-1}\psi^{\nu}}\,, \qquad r\geqslant 1\,. \tag{2.48}$$

Applying the dictionary (2.47) and (2.48) on the creation operators of (2.39), the vertex operator for a general NS state is given by

$$\varepsilon^{\mathsf{NS}}_{\mu_1(m_1)|\dots|\mu_I(m_I)|\nu_1[n_1]|\dots|\nu_J[n_J]}\,i\partial^{k_1}X^{\mu_1(m_1)}\dots i\partial^{k_I}X^{\mu_I(m_I)}\,\partial^{r_1-1}\psi^{\nu_1[n_1]}\dots\partial^{r_J-1}\psi^{\nu_J[n_J]}\,, \tag{2.49}$$

where we have absorbed the factors $(n_i-1)!$ and $(r_j-1)!$ in the polarization tensor ε.

To obtain the fermionic dictionary in the R sector, we have to calculate

$$b^{\mu}_{-r}\,|p;A;0\rangle_{\mathsf{R}} = \lim_{w\to 0}\oint\frac{\mathrm{d}z}{2\pi i}\frac{1}{z^{r+1/2}}\psi^{\mu}(z)S_A(w)\,|p;0\rangle_{\mathsf{NS}}\;. \tag{2.50}$$

To perform this calculation for any r, knowledge of the OPE of $\psi^{\mu}(z)S_A(w)$ to *arbitrary* order is required to obtain an integrand with non-vanishing integral, so, with the means presented so far, we cannot present a closed formula that relates the action of the creation operators b_{-r}, for any r, on the vacuum to ψ and its descendants. However, once a certain value of r is chosen, the computation is feasible at finite time. For example,

$$b^{\mu}_{-1}\,|p;\alpha;0\rangle_{\mathsf{R}} \quad\longleftrightarrow\quad \Big[\frac{\sqrt{2}}{d-2}\psi^{\mu}\not{\psi} - \frac{1}{\sqrt{2}(d-2)(d-1)}\gamma^{\mu}\not{\psi}\not{\psi}\Big]_{\alpha\dot{\beta}}\,S^{\dot{\beta}}(0)\,|p;0\rangle_{\mathsf{NS}}\;, \tag{2.51}$$

where we needed the first subleading term in the OPE $\psi^{\mu}(z)S_A(w)$, given in (2.12). Nevertheless, knowledge of (2.51) is not sufficient once more than one b are present, since for example

$$b^{\mu}_{-1}b^{\nu}_{-1}\,|p;A;0\rangle_{\mathsf{R}} = \lim_{w\to 0}\oint\frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}y}{2\pi i}\frac{1}{(zy)^{3/2}}\psi^{\mu}(z)\psi^{\mu}(y)S_A(w)\,|p;0\rangle_{\mathsf{NS}}\;, \tag{2.52}$$

which cannot be calculated with the standard version of Wick's theorem.[7] To avoid having to find the dictionary for the states/trajectory in question, we will be concentrating on the Fock space realization of R states.

Instead, in the NS sector, writing the most general Ansatz for a vertex operator at level N boils down to enumerating the half-integer partitions of $N-\frac{1}{2}$ (i.e. all possible ways of obtaining $N-\frac{1}{2}$ as a sum of positive integers *and half-integers*). Indeed, such partitions encode monomials of the form (2.49). Now let us recall how the string spectrum is built using the traditional methods. In the covariant language of vertex operators, to build states at a level N, one writes all possible terms with conformal weight N contributing to the respective polynomial F that appears in the generic expression (2.14) and imposes the BRST

[7]Here we use the term "standard" to distinguish Wick's theorem for non-interacting fields from the generalized Wick's theorem for interacting fields described for example in [47, Chap. 6.B].

constraints (2.16). For example, at $N = \frac{3}{2}$ namely $p^2 = -\frac{1}{2}$ in the NS sector, there are three possible monomials that can contribute,

$$F_{3/2} = \varepsilon_{\mu|\nu}\, i\partial X^\mu\, \psi^\nu + \varepsilon_\mu\, \partial\psi^\mu + \varepsilon_{\mu\nu\lambda}\, \psi^\mu\psi^\nu\psi^\lambda\,, \tag{2.53}$$

where $\varepsilon_{\mu|\nu}$, ε_μ and $\varepsilon_{\mu\nu\lambda}$ are a priori arbitrary (the latter is by construction totally antisymmetric). Imposing (2.16) yields [12, 14] the lightest massive spin-2 and the lightest spin-$(1,1,1)$ of the open superstring spectrum, with vertex operators given by

$$V^{(-1)}(p,z) = e^{-\varphi}\, \varepsilon_{\mu\nu}(p)\, i\partial X^\mu\, \psi^\nu\, e^{ip\cdot X}\,, \quad p^\mu \varepsilon_{\mu\nu} = 0\,, \quad \varepsilon_{[\mu\nu]} = 0\,, \tag{2.54}$$

and

$$V^{(-1)}(p,z) = e^{-\varphi}\, \varepsilon_{\mu\nu\lambda}(p)\, \psi^\mu\psi^\nu\psi^\lambda\, e^{ip\cdot X}\,, \quad p^\mu \varepsilon_{\mu\nu\lambda} = 0\,, \tag{2.55}$$

respectively. At level 2, one finds for instance the lightest hook,

$$V^{(-1)}(p,z) = e^{-\varphi}\, \varepsilon_{\mu|\nu\rho}(p)\, i\partial X^\mu\, \psi^\nu\psi^\rho\, e^{i\,p\cdot X}\,, \qquad p^\mu\, \varepsilon_{\mu|\nu\rho} = 0\,, \qquad \varepsilon_{[\mu|\nu\rho]} = 0\,, \tag{2.56}$$

where ε is also traceless. The same procedure can be employed to construct levels of the R sector. Alternatively, one can apply the light cone technology, namely use the transverse creation oscillators α^i_{-n} and b^i_{-r} , $i = 0, \ldots, d-2$ to construct all possible $\mathfrak{so}(d-2)$ irreps, which are then to be recombined into $\mathfrak{so}(d-1)$ irreps, on a level-by-level basis. For the example of the lightest levels of the R sector, this procedure is illustrated in appendix A. The lightest physical states in both sectors are displayed in table 1.

GSO projection and spacetime supersymmetry. The previously described methodology leads to a spectrum which is not supersymmetric and contains a tachyonic state. A way of treating these pathologies is via the GSO projection, which we can think of essentially as projecting out half the states, namely all half-integer NS levels, displayed in gray in table 1, and half of the "generalized" chirality of all R states, see appendix A for a very brief review. The number of propagating d.o.f. after the GSO projection is given in the last column of table 1. $\mathcal{N} = 1$ spacetime super-Poincaré symmetry is generated by

$$P^\mu = \oint \frac{\mathrm{d}z}{2\pi i}\, i\partial X^\mu\,, \qquad J^{\mu\nu} = \oint \frac{\mathrm{d}z}{2\pi i} \left[iX^{[\mu}\partial X^{\nu]} + \psi^{[\mu}\psi^{\nu]} \right], \tag{2.57}$$

and

$$\mathbf{Q}_\alpha^{(-1/2)} = 2^{1/4} \oint \frac{\mathrm{d}z}{2\pi i}\, S_\alpha\, e^{-\varphi/2}\,, \qquad \mathbf{Q}_\alpha^{(+1/2)} = \frac{1}{2^{1/4}} \oint \frac{\mathrm{d}z}{2\pi i}\, i\partial X \cdot (\gamma\, S)_\alpha\, e^{+\varphi/2}\,, \tag{2.58}$$

with algebra

$$\left\{ \mathbf{Q}_\alpha^{(-1/2)}, \mathbf{Q}_\beta^{(+1/2)} \right\} = (\gamma\, C)_{\alpha\beta} \cdot P\,, \quad \text{etc}\,. \tag{2.59}$$

After the GSO projection, the open superstring states at every mass level form multiplets of $\mathcal{N} = 1$ supersymmetry, with the supersymmetry generators creating R from NS states (at the same level) and vice versa according to

$$V_{\mathsf{R}}^{(-1/2)}(p,z) = \left[\eta^\alpha\, \mathbf{Q}_\alpha^{(+1/2)}, V_{\mathsf{NS}}^{(-1)}(p,z) \right], \tag{2.60a}$$

$$V_{\mathsf{NS}}^{(-1)}(p,z) = \left[\eta^\alpha\, \mathbf{Q}_\alpha^{(-1/2)}, V_{\mathsf{R}}^{(-1/2)}(p,z) \right]. \tag{2.60b}$$

The existence of these operations suggests that, once a physical state or trajectory is constructed in one sector, they can be used to find its superpartner in the other sector.

The lowest states form the massless vector supermultiplet, whose vector's vertex operator in the canonical ghost picture reads

$$V^{(-1)}(p,z) = e^{-\varphi}\,\varepsilon_\mu(p)\,\psi^\mu\,e^{ip\cdot X}\,. \tag{2.61}$$

Its superpartner is given by

$$V^{(-1/2)}(p,z) = \Big[\eta^\alpha\,\mathbf{Q}_\alpha^{(+1/2)}, V^{(-1)}(p,z)\Big] = e^{-\varphi/2}\,\upsilon^\alpha(p)\,S_\alpha\,e^{ip\cdot X}\,, \tag{2.62}$$

where we have used the OPEs (2.13) and (2.12) to show that the integrand has a single pole and the spinor wavefunction υ^α is identified with

$$\upsilon^\alpha(p) = \frac{1}{2^{3/4}}\,\varepsilon_\nu(p)\,p_\mu\,\eta^\beta\,\gamma^\mu_{\beta\dot\beta}\,\overline{\gamma}^{\nu\dot\beta\alpha}\,. \tag{2.63}$$

Notice that the vertex operator (2.62) of the spin-(1/2) is produced in the canonical ghost picture for the R sector with this method. More generally, the spectrum can be visualised in terms of Regge trajectories, as in the bosonic string. The leading Regge trajectory consists of the highest spins of every level. In the NS sector, it thus contains the states with symmetric rank-s polarisation tensors $\varepsilon_{\mu_1\ldots\mu_s}$ that find themselves at level $s-1$, namely $\frac{1}{2}p^2 = 1-s$, with vertex operator [37]

$$V_{\mathsf{NS}}^{(-1)}(p,z) = e^{-\varphi}\,\varepsilon_{\mu_1\ldots\mu_s}(p)\,i\partial X^{\mu_1}\ldots i\partial X^{\mu_{s-1}}\,\psi^{\mu_s}\,e^{ip\cdot X}\,, \quad p^\mu\varepsilon_{\mu\mu_2\ldots\mu_s} = 0\,, \quad \varepsilon^\mu{}_{\mu\mu_3\ldots\mu_s} = 0\,, \tag{2.64}$$

and is highlighted in red in table 1. Its superpartner trajectory, namely the leading Regge in the R sector that is highlighted in blue in table 1, can be found via

$$\begin{aligned} V_{\mathsf{R}}^{(-1/2)}(p,z) &= \Big[\eta^\alpha\,\mathbf{Q}_\alpha^{(+1/2)}, V_{\mathsf{NS}}^{(-1)}(p,z)\Big] \\ &= e^{-\varphi/2}\bigg[\upsilon^\alpha_{\mu_1\ldots\mu_{s-1}}(p)\,i\partial X^{\mu_1}\ldots i\partial X^{\mu_{s-1}} \\ &\qquad + \rho_{\dot\beta\,\mu_1\ldots\mu_{s-1}}(p)\,i\partial X^{\mu_1}\ldots i\partial X^{\mu_{s-2}}\,\psi^{\mu_{s-1}}\,\psi^{\dot\beta\alpha}\bigg]\,S_\alpha\,e^{ip\cdot X}\,, \end{aligned} \tag{2.65}$$

see also [12, 13, 37], where we have used the identity $\gamma^{(\mu}_{\alpha\dot\beta}\,\overline{\gamma}^{\nu)\,\dot\beta\gamma} = -\eta^{\mu\nu}\delta^\gamma_\alpha$ and that $\varepsilon_{\mu_1\ldots\mu_s}$ is symmetric and traceless and we have defined

$$\upsilon^\alpha_{\mu_1\ldots\mu_{s-1}}(p) = \frac{1}{2^{3/4}}\varepsilon_{\mu_1\ldots\mu_s}(p)p_\mu\eta^\beta\gamma^\mu_{\beta\dot\beta}\overline{\gamma}^{\mu_s\dot\beta\alpha}, \quad \rho_{\dot\beta\mu_1\ldots\mu_{s-1}}(p) = \frac{1}{2^{1/4}}(s-1)\varepsilon_{\mu_1\ldots\mu_s}(p)\eta^\alpha\gamma^{\mu_s}_{\alpha\dot\beta}. \tag{2.66}$$

Notice that, for $s=1$, the second term in (2.65) disappears and we recover (2.62).

At this point, it is clear that, as in the bosonic string, the construction of physical superstring states on a level-by-level basis can quickly become cumbersome as the level increases. In the following, we will extend to the superstring the methodology recently developed in [29] using Howe duality. To this end, we will first recall how Howe duality appears in the bosonic string spectrum, namely for the bosonic oscillators α^μ_n, and proceed to analyse how it can be realised once the fermionic oscillators b^μ_n are added.

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m^2	NS	R	propagating d.o.f.
$-\frac{1}{2}$	$\bullet$		
0	$\ydiagram{1}^{\,so(d-2)}$	$\bullet^{\,so(d-2)}_{1/2}$	$\mathbf{8}_{\mathrm{B}} + \mathbf{8}_{\mathrm{F}}$
$\frac{1}{2}$	$\ydiagram{1,1}$		
1	$\ydiagram{2} \oplus \ydiagram{1,1,1}$	$\ydiagram{1}_{\,1/2}$	$\mathbf{128}_{\mathrm{B}} + \mathbf{128}_{\mathrm{F}}$
$\frac{3}{2}$	$\ydiagram{2} \oplus \bullet \oplus \ydiagram{1,1,1,1} \oplus \ydiagram{2,1}$		
2	$\ydiagram{3} \oplus \ydiagram{1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1,1}$	$\ydiagram{2}_{\,1/2} \oplus \ydiagram{1}_{\,1/2} \oplus \bullet_{1/2} \oplus \ydiagram{1,1}_{\,1/2}$	$\mathbf{1152}_{\mathrm{B}} + \mathbf{1152}_{\mathrm{F}}$

Table 1. Open superstring, physical content of the first few levels. All Young diagrams appearing should be understood as irreducible representations of the massive little group $\mathrm{SO}(d-1)$ in d-dimensions, except at mass level $m^2 = 0$ where they denote irreps of the massless little group $\mathrm{SO}(d-2)$. The bosonic part of the leading Regge trajectory is indicated in red and its fermionic superpartner in blue.

3 Bosonic and fermionic versions of Howe duality

Let us consider a generic Fock space $\mathfrak{F}$ generated by pairs of creation-annihilation operators. $\mathfrak{F}$ can be thought of as the oscillator representation of a group G, which is either the symplectic group Sp if they are all bosonic, the orthogonal group O if they are all fermionic, or the orthosymplectic group OSp if both are present. A pair of reductive groups H and $\widetilde{H}$, both of which are subgroups of G and which are each other's centralisers,[8] is then called a "reductive dual pair". The reductive dual pair correspondence, also known as "Howe duality" [30, 31], see also [48, 49], refers then to a "1-1" correspondence between the irreducible representations of H and $\widetilde{H}$. In particular, the important property of the decomposition of $\mathfrak{F}$ under $H \times \widetilde{H}$ is that it is *strongly multiplicity-free*, meaning that not only pairs of representations of H and $\widetilde{H}$ appear only once in the decomposition, but also that any irrep, appears in one such pair at most. In other words, as a $H \times \widetilde{H}$ representation, the Fock space $\mathfrak{F}$ is a direct sum of the form[9]

$$\mathfrak{F}\big|_{H\times\widetilde{H}} \cong \bigoplus_{\lambda} \pi^{H}_{\lambda} \boxtimes \pi^{\widetilde{H}}_{\theta(\lambda)}\,, \tag{3.1}$$

where the sum runs over a subset of irreps π^{H}_{λ} of H (those appearing in $\mathfrak{F}$) labelled by λ, and $\theta(\lambda)$ is the label for the corresponding $\widetilde{H}$ representation $\pi^{\widetilde{H}}_{\theta(\lambda)}$. More importantly, the map θ between H-irreps and $\widetilde{H}$-irreps appearing in $\mathfrak{F}$ is a *bijection*, meaning that each representation π^{H}_{λ} of H appears only once in the above decomposition of $\mathfrak{F}$, accompanied by a unique irrep $\pi^{\widetilde{H}}_{\theta(\lambda)}$ of $\widetilde{H}$.

In the remainder of this section, we will briefly review [29], and recall how the bosonic version of Howe duality is tied to the description of arbitrary symmetry Lorentz tensors *in*

[8]This last condition means that, not only H and $\widetilde{H}$ commute, but also that H is the maximal subgroup commuting with $\widetilde{H}$ in G, and vice versa.

[9]The symbol $\boxtimes$ denotes the *external* tensor product of a H-irrep and a $\widetilde{H}$-irrep, resulting in a $(H \times \widetilde{H})$-irrep.

the symmetric basis. Next, we will discuss the fermionic version of Howe duality, i.e. the duality between irreps of dual pairs in fermionic Fock spaces and argue that it is related to the description of mixed-symmetry Lorentz tensors *in the antisymmetric basis*. Finally, as both bosonic and fermionic oscillators are needed for the construction of superstring states, we will put the two versions together and explain how Howe duality in a Fock space generated by both types of oscillators allows one to describe Lorentz tensors in a mixture of the symmetric and antisymmetric basis.

3.1 Bosonic version and the symmetric basis

To discuss physical states in the superstring spectrum, we will only need to use Howe duality between *Lie algebra representations*, so that we will work at the level of algebras from now on.

For the bosonic string, the relevant Howe dual pair is the one formed by the Lorentz algebra $\mathfrak{so}(d-1,1)$ and the symplectic algebra $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$, in the oscillator representation of $\mathfrak{sp}(2\mathsf{N}d,\mathbb{R})$, i.e. the bosonic Fock space defined by $\mathsf{N}\times d$ pairs of creation and annihilation operators. Howe duality was first observed in the bosonic string spectrum in [29] and allows one to characterise and construct tensors of $\mathfrak{so}(d-1,1)$ with the symmetry of a N-row Young diagram in terms of lowest weight irreps of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$, thereby constructing entire physical string trajectories efficiently and covariantly.

Let us recall the bosonic version of Howe duality, following [29, section 3.3].[10] We will work on the subspace of the Fock space generated by the negative modes of the bosonic oscillators (2.30), i.e. the space of linear combinations of states of the form

$$|\varepsilon\rangle := \varepsilon_{\mu_1(\ell_1)|\cdots|\mu_p(\ell_p)}\, \alpha_{-i_1}^{\mu_1(\ell_1)} \dots \alpha_{-i_p}^{\mu_k(\ell_p)}\, |0\rangle\ , \qquad 1\leq i_j\leq \mathsf{N}\,, \quad i_j\neq i_k\,, \tag{3.2}$$

where $\varepsilon_{\mu_1(\ell_1)|\cdots|\mu_l(\ell_p)}$ is a tensor with p groups of *symmetric* indices (with $0\leq p\leq \mathsf{N}$), without additional symmetries among indices of different groups. N is an integer that enumerates the different types of creation operators excited; its value depends on the level or trajectory in question, while it can become arbitrarily large in principle, reflecting the infinite size of the string spectrum — it will be thought of as implicitly finite throughout this work. The action of the Lorentz algebra $\mathfrak{so}(d-1,1)$ commutes with that of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$, with the latter being generated *on this subspace* by

$$T^{mn} := \frac{1}{m\,n}\,\alpha_{-m}\cdot\alpha_{-n}\,, \qquad T^m{}_n := \frac{1}{m}\,\alpha_{-m}\cdot\alpha_n + \frac{d}{2}\,\delta_{mn}\,, \qquad T_{mn} := \alpha_m\cdot\alpha_n\,, \tag{3.3}$$

where $m,n=1,2,\dots,\mathsf{N}$. The $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ indices correspond to the various groups of symmetric spacetime indices carried by a given type of creation operator.[11] Consequently, the $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ generators encode various operations on these groups of indices, namely:

- The generators $T^k{}_k$, without implicit summation, count how many indices are in the kth group,
$$T^k{}_k\,|\varepsilon\rangle = \left(\ell_k + \frac{d}{2}\right)\,|\varepsilon\rangle\ , \tag{3.4}$$
modulo a constant shift by $\frac{d}{2}$.

[10] Bosonic Howe duality has also been used in the context of higher spin gravity, see e.g. [50–56] and [57, section 3] for a concise review of its use to treat arbitrary tensors.

[11] Our convention for the prefactors in the definition of the generators (3.3) or (4.2a) of the $\mathfrak{sp}(\bullet)$ algebra is slightly different from the one used in [29].

- The generators ${T^m}_n$, with $m \neq n$, symmetrize an index of the nth group with *all* indices of the mth group. In particular, it allows one to reformulate the condition that ε is irreducible under $\mathfrak{gl}_d$. Let us assume that ε is of the form $|\varepsilon\rangle = \varepsilon_{\mu_1(s_1)|\cdots|\mu_l(s_l)}\, \alpha_{-1}^{\mu_1(\ell_1)} \dots \alpha_{-p}^{\mu_p(\ell_p)}\, |0\rangle$ with

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p > 0\,, \tag{3.5}$$

and $0 \leq p \leq \mathsf{N}$. Then, in the symmetric basis, irreducibility of ε reads

$${T^m}_n\,|\varepsilon\rangle = 0\,, \quad 1 \leq m < n \leq \mathsf{N} \qquad \Longleftrightarrow \qquad \varepsilon_{\cdots|\mu_m(\ell_m)|\cdots|\mu_m\mu_n(\ell_n-1)|\cdots} = 0\,, \tag{3.6}$$

or in plain words, the symmetrization of all indices in the mth group with one index of the *subsequent*, nth group, should vanish.

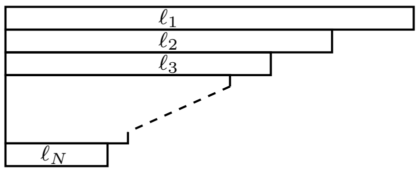
- The generators T_{mn} take a trace between one index in the mth group and another one in the nth group. Requiring that ε be traceless therefore translates into

$$T_{mn}\,|\varepsilon\rangle = 0 \qquad \Longleftrightarrow \qquad \eta^{\alpha\beta}\,\varepsilon_{\cdots,\mu_m(\ell_m-1)\alpha,\cdots,\mu_n(\ell_n-1)\beta,\cdots} = 0\,, \tag{3.7}$$

for all pairs $1 \leq m, n \leq \mathsf{N}$.

- Finally, the generators T^{mn} multiply the tensor by a metric η, and symmetrizes one of its indices with all those of the mth group, and the other index with all indices of the nth group.

The action of these generators gives us a way to define an irreducible tensor ε of the Lorentz algebra in a bosonic Fock space by a vector $|\varepsilon\rangle$, verifying

$\varepsilon \;\simeq$ 

$$\Longleftrightarrow \qquad \begin{cases} {T^k}_k\,|\varepsilon\rangle = (\ell_k + \frac{d}{2})\,|\varepsilon\rangle\ , & k = 1, \dots, \mathsf{N}\,, \\ {T^m}_n\,|\varepsilon\rangle = 0\,, & 1 \leq m < n \leq \mathsf{N}\,, \\ T_{mn}\,|\varepsilon\rangle = 0\,, & m, n = 1, \dots, \mathsf{N}\,, \end{cases} \tag{3.8}$$

i.e. $|\varepsilon\rangle$ is a *lowest weight vector* for $\mathfrak{sp}(2\mathsf{N}, \mathbb{R})$.[12] This is due to the fact that the Lorentz and symplectic algebras form a reductive dual pair, and hence Howe duality tells us that the subspace of states generated by the first N pairs of modes of ∂X can be decomposed into irreducible representations of their direct sum, $\mathfrak{so}(d-1,1) \oplus \mathfrak{sp}(2\mathsf{N}, \mathbb{R})$, such that each representation of a given algebra appearing in this decomposition does so together with a *unique* irrep of the other algebra. In other words, the infinitely many disparate and physical apparitions of a given $\mathfrak{so}(d-1,1)$ irrep in the bosonic string spectrum correspond to different irreps of $\mathfrak{sp}(2\mathsf{N}, \mathbb{R})$ and can be reached by acting with the raising operators, T^{mn} and ${T^{m>n}}_n$, of the latter on lowest weight states as defined in (3.8).

[12] Indeed, the first equation fixes the $\mathfrak{sp}(2\mathsf{N}, \mathbb{R})$ weight of $|\varepsilon\rangle$ — its eigenvalues under the Cartan subalgebra generators ${T^k}_k$ — while the other two equations mean that $|\varepsilon\rangle$ is annihilated by all lowering operators.

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3.2 Fermionic version and the antisymmetric basis

Now let us turn to the fermionic version of Howe duality (see e.g. [48, 58] for reviews). In this case, the Fock space in question can be thought of as the oscillator representation of the orthogonal group, in the sense that "fermionic dual pairs" are found as mutual centralisers in it. We will be interested in the dual pairs composed of the Lorentz algebra $\mathfrak{so}(d-1,1)$ and another orthogonal algebra, either $\mathfrak{o}(2\mathsf{M})$ or $\mathfrak{o}(2\mathsf{M}+1)$. The action of these algebras on a fermionic Fock space is generated by bilinears in the oscillators, as in the bosonic case. There are, however, a couple of specificities to the fermionic case: (i) the Fock space is finite-dimensional which restricts the type of irreps of the dual pairs that can appear in the decomposition, and (ii) whether the dual algebra is $\mathfrak{o}(2\mathsf{M})$ or $\mathfrak{o}(2\mathsf{M}+1)$ determines the bosonic or fermionic nature of the corresponding Lorentz representations.[13]

NS sector. Let us start with the subspace $\mathfrak{F}^b_{\mathsf{NS}}$ of the NS sector of the Fock space that is generated by the negative modes half-integer $-\frac{1}{2}, -\frac{3}{2}, \dots, -\mathsf{M}+\frac{1}{2}$ of the fermionic oscillators (2.31), i.e. the space of linear combinations of states of the form

$$|\varepsilon\rangle := \varepsilon_{\mu_1[h_1]|\dots|\mu_M[h_l]}\, b^{\mu_1[h_1]}_{-r_1+\frac{1}{2}} \dots b^{\mu_l[h_l]}_{-r_l+\frac{1}{2}}\, |0\rangle_{\mathsf{NS}}\,, \qquad 1 \leq r_j \leq \mathsf{M}\,, \quad r_j \neq r_k\,, \tag{3.9}$$

where $\varepsilon_{\mu_1[h_1]|\dots|\mu_l[h_l]}$ is a tensor of l groups of antisymmetric indices (with $0 \leq l \leq \mathsf{M}$), without additional symmetries among indices of different groups. As in the bosonic case, M is implicitly thought of as a finite integer. The Lorentz algebra $\mathfrak{so}(d-1,1)$, generated by the bilinears

$$J^{\mu\nu} = 2\sum_{r\geq 1} b^{[\mu}_{-r+\frac{1}{2}}\, b^{\nu]}{}_{r-\frac{1}{2}}\,, \tag{3.10}$$

on this subspace commutes with the action of the orthogonal algebra $\mathfrak{o}(2\mathsf{M})$, generated by

$$M^{rs} := b_{-r+\frac{1}{2}} \cdot b_{-s+\frac{1}{2}}\,, \qquad M^r{}_s := b_{-r+\frac{1}{2}} \cdot b_{s-\frac{1}{2}} - \frac{d}{2}\,\delta_{rs}\,, \qquad M_{rs} = b_{r-\frac{1}{2}} \cdot b_{s-\frac{1}{2}}\,, \tag{3.11}$$

where $r, s = 1, 2, \dots, \mathsf{M}$. For the sake of completeness, let us recall the commutation relation of $\mathfrak{o}(2\mathsf{M})$ in the above basis,

$$[M^r{}_s, M^{tu}] = 2\,\delta^{[u}_s\, M^{t]r}\,, \qquad [M^r{}_s, M^t{}_u] = \delta^t_s\, M^r{}_u - \delta^r_u\, M^t{}_s\,, \tag{3.12a}$$

$$[M^r{}_s, M_{tu}] = -2\,\delta^r_{[u}\, M_{t]s}\,, \qquad [M^{rs}, M_{tu}] = -4\,\delta^{[r}_{[t}\, M^{s]}{}_{u]}\,. \tag{3.12b}$$

Now in parallel with the bosonic case, the $\mathfrak{o}(2\mathsf{M})$ indices correspond to the various groups of *antisymmetric* Lorentz indices, and hence the $\mathfrak{o}(2\mathsf{M})$ generators encode operations on these groups. More precisely:

- The generators $M^r{}_r$, without summation implied, returns the number of indices in the rth group,

$$M^r{}_r\, |\varepsilon\rangle = \left(h_r - \frac{d}{2}\right) |\varepsilon\rangle\,, \tag{3.13}$$

  modulo a shift by $-\frac{d}{2}$.

[13] See also, e.g., [59, section VI] for a similar discussion in the context of constructing lowest weight representation of $\mathfrak{osp}(2m+1|2n,\mathbb{R})$.

- The generators ${M^r}_s$ with $r \neq s$ antisymmetrize an index of the sth group with *all* indices of the rth group. Here again, it allows one to reformulate the condition that ε is irreducible under $\mathfrak{gl}_d$. Assuming that the groups of indices of ε are ordered by decreasing length, i.e. $h_1 \geq h_2 \geq \dots \geq h_M$, irreducibility of ε reads

$$
{M^r}_s \,|\varepsilon\rangle = 0\,, \quad 1 \leq r < s \leq \mathsf{M} \qquad \Longleftrightarrow \qquad \varepsilon_{\dots|\mu_r[h_r]|\cdots|\mu_r\mu_s[h_s-1]|\dots} = 0\,, \tag{3.14}
$$

  in the *antisymmetric basis*.

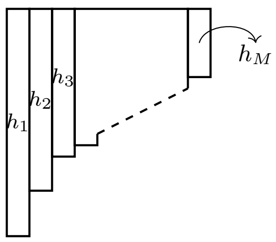
- The generators M_{rs} take a trace between an index of the rth group and another one of the sth group. In particular, tracelessness of ε reads

$$
M_{rs}\,|\varepsilon\rangle = 0 \qquad \Longleftrightarrow \qquad \eta^{\alpha\beta}\,\varepsilon_{\dots,\mu_r[h_r-1]\alpha,\cdots,\mu_s[h_s-1]\beta,\dots} = 0\,, \tag{3.15}
$$

  for any $1 \leq r, s \leq \mathsf{M}$.

- Finally, the generators M^{rs} act by multiplying the tensor ε with a metric η, and antisymmetrizing one of its index with all those of the rth group of indices in ε, and the other one with all indices of the sth group.

In complete parallel with the bosonic case, we can single out an irreducible Lorentz tensor in $\mathfrak{F}^b_{\mathsf{NS}}$ from a state $|\varepsilon\rangle$ that verifies

$\varepsilon \simeq$ 

$$
\Longleftrightarrow \qquad \begin{cases} {M^r}_r\,|\varepsilon\rangle = (h_r - \dfrac{d}{2})\,|\varepsilon\rangle\,, & r = 1, \dots, \mathsf{M}\,, \\ {M^r}_s\,|\varepsilon\rangle = 0\,, & 1 \leq r < s \leq \mathsf{M}\,, \\ M_{rs}\,|\varepsilon\rangle = 0\,, & r, s = 1, \dots, \mathsf{M}\,, \end{cases} \tag{3.16}
$$

which is to say that $|\varepsilon\rangle$ is a lowest weight state for $\mathfrak{o}(2\mathsf{M})$. This is again in accordance with the previous claim that the Lorentz and the orthogonal algebras form a dual pair, in $O(d \times 2\mathsf{M})$ here, and hence their irreps appear in mutually distinct couples. Here we have seen that the Lorentz irrep corresponding to a Young diagram with columns of height $[h_1, \dots, h_M]$ is dual to the lowest weight irrep of $\mathfrak{o}(2\mathsf{M})$ with lowest weight $[h_1 - \frac{d}{2}, \dots, h_M - \frac{d}{2}]$.

R sector. Now let us turn our attention to the subspace of the Ramond sector generated by fermionic oscillators (2.31) of modes $0, -1, \dots, -\mathsf{M}$. The zero modes form a Clifford algebra and it can be shown that they can be defined as the γ-matrices in d-dimensions

$$
b^\mu_0 := \frac{1}{\sqrt{2}}\,\gamma^\mu\,, \qquad \{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,, \tag{3.17}
$$

which implicitly carry Dirac indices $A, B, \dots$ and which we suppress. Let us also keep in mind that they anticommute with the other fermionic oscillators, $\{\gamma^\mu, b^\nu_{\pm r}\} = 0\,$, for $r = 1, 2, \dots, \mathsf{M}$. Correspondingly, the generators of the Lorentz algebra take the form

$$
J^{\mu\nu} = \frac{1}{2}\,[\gamma^\mu, \gamma^\nu] + 2\sum_{r\geq 1} b^{[\mu}_{-r}\,b^{\nu]}_{+r}\,, \tag{3.18}
$$

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on this subspace. These γ-matrices can be split into creation and annihilation operators as well, as explained in, e.g., [43, Chap. 8.1]. Assuming that the spacetime dimension $d=2n$ is even, we can define the creation and annihilation operators

$$\mathrm{b}^I_\pm := \frac{1}{2\sqrt{2}}\,(\sigma_I\gamma^I \pm i\,\gamma^{2n-1-I})\,, \qquad I=0,\dots,n-1\,, \qquad \sigma_I := \begin{cases} i & \text{if} \quad I=0\,, \\ 1 & \text{otherwise}\,, \end{cases} \tag{3.19}$$

where *no summation* is implied in the term $\sigma_I\,\gamma^I$, and which verify

$$\{\mathrm{b}^I_+,\mathrm{b}^J_-\} = \delta^{IJ}\,, \qquad \{\mathrm{b}^I_+,\mathrm{b}^J_+\} = 0 = \{\mathrm{b}^I_-,\mathrm{b}^J_-\}\,. \tag{3.20}$$

Requiring that the R vacuum be annihilated by half of the zero-modes, say $\mathrm{b}^I_-\,|0\rangle = 0$, we obtain a subspace generated by the action of the other half of the zero-modes, b^I_+ here, subspace of states which are all annihilated by the positive modes b^μ_r with $r=1,2,\dots,\mathsf{M}$, and that is left invariant by the action of the Lorentz algebra, thereby defining a finite-dimensional representation of it. Note that this representation is reducible, due to the fact that the Lorentz generators $\gamma_{[\mu}\,\gamma_{\nu]}$ are quadratic in $\mathrm{b}^I_\pm$ so that they will leave invariant the subspaces with an even or odd number of creation operators. These two invariant subspaces form the spin-($1/2$) positive and negative chirality representations of $\mathfrak{so}(1,2n-1)$.

The Fock space generated by the creation operators b^μ_{-r} and b^I_+ is then a linear combination of states of the form

$$|\varepsilon\rangle := \varepsilon^A_{\mu_1[h_1]|\dots|\mu_l[h_l]}\,b^{\mu_1[h_1]}_{-r_1}\dots b^{\mu_l[h_l]}_{-r_l}\,|A;0\rangle_{\mathsf{R}}\,, \qquad 1\leq r_i\leq \mathsf{M}\,, \qquad r_i\neq r_j\,, \tag{3.21}$$

where ε is a tensor-spinor with l groups of antisymmetric indices (again, $0\leq l\leq\mathsf{M}$), a priori without particular symmetries between indices of different groups and A is a spinor index. The Lorentz algebra commutes with bilinears of the form[14]

$$M^r := \frac{1}{\sqrt{2}}\,\gamma\cdot b_{-r}\,, \qquad M_r := \frac{1}{\sqrt{2}}\,\gamma\cdot b_r\,, \qquad r=1,2,\dots,\mathsf{M}\,, \tag{3.22}$$

as well as with the previously defined M^{rs}, $M^r{}_s$ and M_{rs}; note that the Lorentz and $\mathfrak{o}(2\mathsf{M})$ generators are still given by the expressions (3.10) and (3.11) respectively, where now the fermionic oscillators have integer modes. Altogether, they define a representation of $\mathfrak{o}(2\mathsf{M}+1)$, whose commutation relations read

$$[M^r{}_s, M^t] = \delta^t_s\,M^r\,, \qquad [M^r{}_s, M_t] = -\delta^r_t\,M_s\,, \qquad [M^r, M_s] = -\,M^r{}_s\,, \tag{3.23a}$$

$$[M^{rs}, M_t] = 2\,\delta^{[s}_t\,M^{r]}\,, \qquad [M_{rs}, M^t] = 2\,\delta^t_{[s}\,M_{r]}\,, \tag{3.23b}$$

on top of those recalled in (3.12). These new generators also encode operations on the groups of antisymmetric Lorentz indices, and on the spinor index now carried by the "vacuum space". To be explicit:

- The generators M_r act on ε by contracting the spacetime index of γ_μ with an index in the rth group. In particular, γ-tracelessness of ε amounts to

$$M_r\,|\varepsilon\rangle = 0 \qquad \Longleftrightarrow \qquad \gamma^{\nu\,A}{}_B\,\varepsilon^B{}_{\dots,\nu\mu_r[h_r-1],\dots} = 0\,, \tag{3.24}$$

  for any $1\leq r\leq\mathsf{M}$.

[14]The bilinears M^r of course also carry Dirac indices $A,B,\dots$ due to the gamma matrices they contain.

- The generators M^r act by multiplying the tensor-spinor ε by γ_μ and antisymmetrizing its index with those of rth group of ε.

So now we have a way to characterize irreducible Lorentz tensor-spinors, namely by imposing the conditions (3.16) together with the γ-tracelessness (3.24).

In summary, the lowest weight conditions for $\mathfrak{o}(2\mathsf{M})$ or $\mathfrak{o}(2\mathsf{M}+1)$ can be thought of as a way to glue columns generated by fermionic creation operators of a given species (meaning with a fixed value of the $\mathfrak{o}(2\mathsf{M})$ or $\mathfrak{o}(2\mathsf{M}+1)$ index), and well as imposing (γ–)tracelessness. A couple of differences with respect to the bosonic case are worth repeating. First, both bosonic and fermionic irreps of the Lorentz algebra appear, and the distinction between the two is tied to the nature of the dual algebra, $\mathfrak{o}(2\mathsf{M})$ or $\mathfrak{o}(2\mathsf{M}+1)$, and therefore to the dimension of the fermionic Fock space under consideration (upon restricting to certain levels or trajectories). Indeed, we have seen that to go from the former to the latter, one needs to add one family of gamma matrices, which reflects the fact that $\mathfrak{so}(d-1,1)$ form a dual pair with $\mathfrak{o}(2\mathsf{M})$ in $\mathfrak{o}(d\times 2\mathsf{M})$, and with $\mathfrak{o}(2\mathsf{M}+1)$ in $\mathfrak{o}(d\times(2\mathsf{M}+1))$. The relevant fermionic Fock space then requires $d\times\mathsf{M}$ pairs of oscillators in the first case, and $d\times\mathsf{M}+\frac{d}{2}$ in the second case. Second, the use of fermionic oscillators naturally leads us to describing tensors(–spinors) in the antisymmetric basis. As a consequence, the types of Young diagrams that can appear is restricted in both the horizontal and the vertical: each column cannot be higher than d as in the bosonic case, but the length of each row is also bounded by M — the maximal number of columns that we can create.

3.3 Super-Howe duality

Finally, let us consider the "super-version" of Howe duality,[15] by which we mean the duality between representations of dual pairs in an orthosymplectic algebra (for more details, see e.g. [60–63] and references therein).[16] More specifically, we will be interested in the pair formed by the Lorentz algebra $\mathfrak{so}(d-1,1)$ and the orthosymplectic algebras $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$ or $\mathfrak{osp}(2\mathsf{M}+1|2\mathsf{N},\mathbb{R})$.

NS sector. Let us start with the former pair, which acts on the Fock space generated by *both* the bosonic and fermionic oscillators in the NS sector. As before, we will focus on the subspace generated by the first N bosonic modes and M fermionic modes, so that a generic state in this subspace will be a linear combination of elements of the form

$$|\varepsilon\rangle=\varepsilon_{\mu_1(\ell_1)|\cdots|\mu_p(\ell_p)|\nu_1[h_1]|\cdots|\nu_l[h_l]}\,\alpha^{\mu_1(\ell_1)}_{-i_1}\dots\alpha^{\mu_p(\ell_p)}_{-i_p}\,b^{\nu_1[h_1]}_{-r_1+\frac{1}{2}}\dots b^{\nu_l[h_l]}_{-r_l+\frac{1}{2}}\,|0\rangle_{\mathsf{NS}}\,, \tag{3.25}$$

where ε is a tensor with p groups of symmetric indices *and* l groups of antisymmetric indices (with $0\leq p\leq\mathsf{N}$ and $0\leq l\leq\mathsf{M}$), without any symmetry properties between indices of difference groups a priori. The algebras $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ and $\mathfrak{o}(2\mathsf{M})$ act on such states, via the bilinears in bosonic and fermionic oscillators spelled out in (3.3) and (3.11). The important

[15] We will refrain from referring to this duality as "supersymmetric", since it does not relate bosonic and fermionic representations of some spacetime isometry algebra, but instead relates irreps of the Lorentz algebra to irreps of a dual superalgebra (at least in the cases of interest for us in this work).

[16] See also, e.g. [64–66] for early use of it to build lowest weight modules of the orthosymplectic algebra.

difference is that now the direct sum of these two algebras form the bosonic subalgebra of the superalgebra $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$, whose fermionic generators are represented by

$$Q^{nr} := \frac{1}{n}\,\alpha_{-n}\cdot b_{-r+\frac{1}{2}}\,, \qquad Q^{n}{}_{r} := \frac{1}{n}\,\alpha_{-n}\cdot b_{r-\frac{1}{2}}\,, \qquad Q_{n}{}^{r} := b_{-r+\frac{1}{2}}\cdot\alpha_{n}\,, \qquad Q_{nr} := \alpha_{n}\cdot b_{r-\frac{1}{2}}\,, \tag{3.26}$$

i.e. all possible bilinears made out of one bosonic and one fermionic oscillator, and whose Lorentz indices are contracted. The anticommutator of the fermionic generators gives back the bosonic ones,

$$\{Q^{m}{}_{r}, Q^{ns}\} = \delta^{s}_{r}\, T^{mn}\,, \qquad \{Q_{m}{}^{r}, Q^{ns}\} = \delta^{n}_{m}\, M^{rs}\,, \tag{3.27a}$$

$$\{Q^{m}{}_{r}, Q_{ns}\} = -\delta^{m}_{n}\, M_{rs}\,, \qquad \{Q_{m}{}^{r}, Q_{ns}\} = \delta^{r}_{s}\, T_{mn}\,, \tag{3.27b}$$

$$\{Q^{m}{}_{r}, Q_{n}{}^{s}\} = \delta^{s}_{r}\, T^{m}{}_{n} + \delta^{m}_{n}\, M^{s}{}_{r}\,, \qquad \{Q^{mr}, Q_{ns}\} = \delta^{r}_{s}\, T^{m}{}_{n} - \delta^{m}_{n}\, M^{r}{}_{s}\,, \tag{3.27c}$$

and fermionic generators transform under the action of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ ones as

$$[T^{k}{}_{l}, Q^{m}{}_{r}] = \delta^{m}_{l}\, Q^{k}{}_{r}\,, \qquad [T^{kl}, Q_{m}{}^{r}] = -2\,\delta^{(k}\, Q^{l)r}\,, \tag{3.28a}$$

$$[T^{k}{}_{l}, Q_{m}{}^{r}] = -\delta^{k}_{m}\, Q_{l}{}^{r}\,, \qquad [T^{kl}, Q_{mr}] = -2\,\delta^{(k}_{m}\, Q^{l)}{}_{r}\,, \tag{3.28b}$$

$$[T^{k}{}_{l}, Q^{mr}] = \delta^{m}_{l}\, Q^{kr}\,, \qquad [T_{kl}, Q^{m}{}_{r}] = 2\,\delta^{m}_{(k}\, Q_{l)r}\,, \tag{3.28c}$$

$$[T^{k}{}_{l}, Q_{mr}] = -\delta^{k}_{m}\, Q_{lr}\,, \qquad [T_{kl}, Q^{mr}] = 2\,\delta^{m}_{(k}\, Q_{l)}{}^{r}\,, \tag{3.28d}$$

and under $\mathfrak{o}(2\mathsf{M})$ as

$$[M^{r}{}_{s}, Q^{m}{}_{t}] = -\delta^{r}_{t}\, Q^{m}{}_{s}\,, \qquad [M^{rs}, Q^{m}{}_{t}] = -2\,\delta^{[r}_{t}\, Q^{ms]}\,, \tag{3.29a}$$

$$[M^{r}{}_{s}, Q_{m}{}^{t}] = \delta^{t}_{s}\, Q_{m}{}^{r}\,, \qquad [M^{rs}, Q_{mt}] = -2\,\delta^{[r}_{t}\, Q_{m}{}^{s]}\,, \tag{3.29b}$$

$$[M^{r}{}_{s}, Q^{mt}] = \delta^{t}_{s}\, Q^{mr}\,, \qquad [M_{rs}, Q_{m}{}^{t}] = -2\,\delta^{t}_{[r}\, Q_{ms]}\,, \tag{3.29c}$$

$$[M^{r}{}_{s}, Q_{mt}] = -2\,\delta^{r}_{t}\, Q_{ms}\,, \qquad [M_{rs}, Q^{mt}] = -2\,\delta^{t}_{[r}\, Q^{m}{}_{s]}\,. \tag{3.29d}$$

These commutation relations simply reflect the fact that the fermionic generators are in the tensor product of the fundamental irrep of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ and the fundamental irrep of $\mathfrak{o}(2\mathsf{M})$.

We have discussed previously how the generators of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ encode symmetrization and trace operations on rows of Young diagrams associated with different types of bosonic creation operators, and how the generators of $\mathfrak{o}(2\mathsf{M})$ encode antisymmetrization and trace operations on columns of Young diagrams associated with different types of fermionic operators. In particular, lowering operators of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ allow us to express how to “glue” rows together, while lowering operators of $\mathfrak{o}(2\mathsf{M})$ allow us to express how to “glue” columns together. Now that we have both bosonic and fermionic oscillators together, it is only natural that these new generators encode operations involving both rows and columns of Lorentz Young diagrams:

- The generators $Q^{m}{}_{r}$ symmetrize an index of the rth group of antisymmetric indices with all those of the mth group of symmetric indices.
- The generators $Q_{m}{}^{r}$ antisymmetrize an index of the mth group of symmetric indices with all those of the rth group of antisymmetric indices.
- The generators Q_{mr} takes a trace between an index of the mth group of symmetric indices and one of the rth group of antisymmetric indices.

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- Finally, the generators Q^{mr} multiply the tensor ε with a metric η, and symmetrize one of its indices with those of the mth group of symmetric indices, and anitsymmetrize the other index with those of the rth group of antisymmetric indices.

The condition that ε be traceless is therefore equivalent to

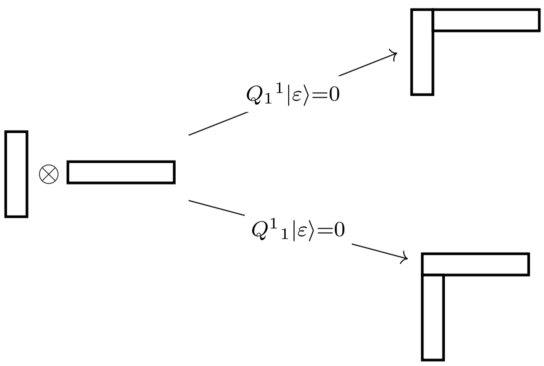
$$T_{mn}\,|\varepsilon\rangle = 0 = M_{rs}\,|\varepsilon\rangle \qquad \text{and} \qquad Q_{mr}\,|\varepsilon\rangle = 0\,, \tag{3.30}$$

since T_{mn} take traces between two groups of symmetric indices, M_{rs} between two groups of antisymmetric indices, and Q_{mr} between a group of symmetric indices and a group of antisymmetric ones. We also know from the two previous sections that imposing

$$T^m{}_n\,|\varepsilon\rangle = 0 \quad \text{for} \quad m < n \qquad \text{and} \qquad M^r{}_s\,|\varepsilon\rangle = 0 \quad \text{for} \quad r < s \tag{3.31}$$

will respectively glue together rows (corresponding to the groups of symmetric indices of ε), and columns (corresponding to the groups of antisymmetric indices of ε), in decreasing length order. It remains to understand how the block of rows and the block of columns are glued together.

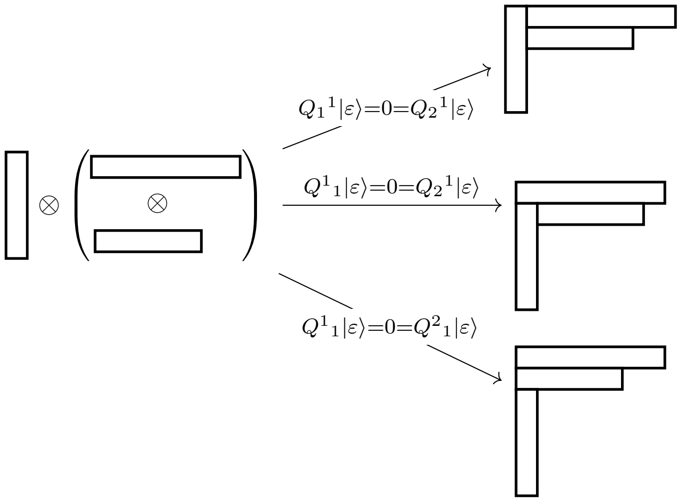
To do so, let us have a closer look at the action of $Q^m{}_r$ and $Q_m{}^r$, starting with the simplest case $\mathsf{M} = 1 = \mathsf{N}$. This means that the tensor ε has one group of antisymmetric indices, and one of symmetric indices. In terms of Young diagram, it corresponds to the tensor product of a column with a row. As a representation of $\mathfrak{gl}_d$, this tensor product will generally contain two irreducible pieces, one obtained by gluing the column on the left of the row, and another one by gluing it below. Requiring that the generators $Q^1{}_1$ or $Q_1{}^1$ annihilates the state $|\varepsilon\rangle$, selects one of these two $\mathfrak{gl}_d$ irreducible tensors,



(3.32)

so that, together with the tracelessness conditions $T_{11}\,|\varepsilon\rangle = M_{11}\,|\varepsilon\rangle = Q_{11}\,|\varepsilon\rangle = 0$, either one of the conditions $Q_1{}^1\,|\varepsilon\rangle = 0$ or $Q^1{}_1\,|\varepsilon\rangle = 0$ define an irreducible Lorentz tensor.

Now let us try to add, for example, one row, i.e. we consider the case $\mathsf{M} = 1$ and $\mathsf{N} = 2$. We therefore have four odd generators which implement anti/symmetrization between indices of a column and a row, namely $Q_1{}^1$, $Q_2{}^1$, $Q^1{}_1$ and $Q^2{}_1$ (and four odd generators adding/removing traces). Choosing a pair of generators among those four allows us to select one of three possible ways we can glue these rows and column,



(3.33)

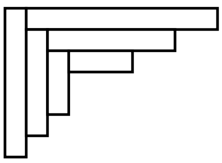
assuming that we also impose the rows to be glued one below the other[17] — i.e. that we also imposed $T^1{}_2\,|\varepsilon\rangle = 0$. Imposing that $|\varepsilon\rangle$ is also annihilated by the trace-type operators T_{11}, T_{12}, T_{22}, M_{11}, Q_{11} and Q_{21} makes ε into a Lorentz irreducible tensor as before.

These different options correspond to different choices of positive/negative roots for the orthosymplectic algebra $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$. Indeed, the odd generators Q are all ladder operators with respect to the Cartan subalgebra of the bosonic subalgebra, and hence should be divided into two groups, of raising and of lowering operators. Having in mind that Q_{mr} take traces, we would prefer to think of them as lowering operators, like T_{mn} and M_{rs}. It therefore remains to split the set of generators $Q_m{}^r$ and $Q^m{}_r$ into raising and lowering operators. The pairs singled out in the previous example correspond to different such splittings. For arbitrary ranks M and N of the orthogonal and symplectic subalgebras respectively, this freedom in the choice of positive and negative roots associated with $Q_m{}^r$ and $Q^m{}_r$ will lead to different ways of gluing columns and rows and, crucially, no choice can be favoured without additional criteria.

For the superstring, we know that additional criteria on the structure of string states do exist: they are the super-Virasoro constraints, enforcing that general states be physical. As we will see in the next section, their form in the NS sector is consistent with the choice of

$$Q_m{}^r \quad \text{with} \quad m \geq r \qquad \text{and} \qquad Q^m{}_r \quad \text{with} \quad m < r\,, \tag{3.34}$$

as being part of the lowering operators. For the moment, let us point out that the lowest weight conditions associated with this choice corresponds to constructing a Young diagram out of columns and rows as in the following example,



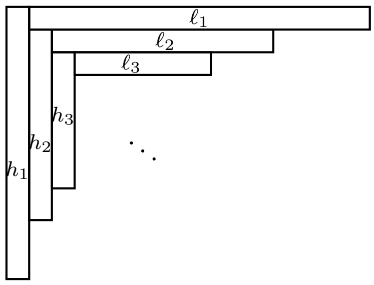
(3.35)

[17]Note that the pair of generators $Q_1{}^1$ and $Q^2{}_1$ does not appear here, as it would glue the second row on top of the first one, thereby violating the convention for the construction of a Young diagram, whose rows should be of decreasing lengths.

namely putting the ith row on the top right of the ith column so as to form a "hook", and then inserting this ith hook in the wedge of the $(i-1)$th hook, thereby creating a diagram with a "*nested-hook-like*" structure.

To summarize, we found that a tensor ε is irreducible for the Lorentz algebra if and only if the state $|\varepsilon\rangle$ built out of it as in (3.25) is a lowest weight for $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$ with respect to the previously identified root system, i.e.

$$\varepsilon \simeq$$



$$\Longleftrightarrow \quad \begin{cases} T^m{}_m\,|\varepsilon\rangle = \left(\ell_m + \frac{d}{2}\right)|\varepsilon\rangle\,, & \\ M^r{}_r\,|\varepsilon\rangle = \left(h_r - \frac{d}{2}\right)|\varepsilon\rangle\,, & \\ Q^m{}_r\,|\varepsilon\rangle = 0\,, & m<r\,, \\ Q_m{}^r\,|\varepsilon\rangle = 0\,, & m \geq r\,, \\ Q_{mr}\,|\varepsilon\rangle = 0\,, & \\ T_{mn}\,|\varepsilon\rangle = 0 = M_{rs}\,|\varepsilon\rangle\,, & \\ T^m{}_n\,|\varepsilon\rangle = 0\,, & m<n\,, \\ M^r{}_s\,|\varepsilon\rangle = 0\,, & r<s\,, \end{cases} \qquad (3.36)$$

and with $m,n = 1,\dots,\mathsf{N}$ and $r,s = 1,\dots,\mathsf{M}$. The pattern is similar to what we have seen previously for the bosonic and fermionic versions of Howe duality: conditions fixing the lowest weight of $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$ determine the total number of boxes making up the Young diagram labelling the dual $\mathfrak{so}(d-1,1)$ irrep, while the action of lowering operators impose that the tensor be traceless and have the symmetry of the Young diagram displayed above. The main difference is that now the boxes of the final Young diagram are arranged both into column and rows, so that on top of the symmetry and tracelessness conditions on rows or columns only, which are encoded by lowering operators of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$ or $\mathfrak{o}(2\mathsf{M})$ respectively, one should add similar conditions for both rows and columns using the *odd* lowering operators of $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$. In particular, some of these conditions tell us *how to glue rows with columns together.*

The Frobenius notation. Note that the relation between the Young diagram labelling the Lorentz irrep and the lowest weight of the dual $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$ irrep is a bit more difficult to express than before, since neither the symmetric nor the antisymmetric basis for Young diagram is adapted. To solve this problem, it is useful to turn to what is called the *Frobenius notation* for Young diagrams. The idea is the following: say that we are given a Young diagram denoted by $(s_1,\dots,s_n)$ in the symmetric basis (i.e. s_i denotes the length of the ith row), and $[h_1,\dots,h_m]$ in the antisymmetric basis (i.e. h_i is the height of the ith column), and whose diagonal contains K boxes. In the Frobenius notation, this diagram will be denoted

$$(\ell_1,\dots,\ell_{\mathsf{K}} \mid \bar{\ell}_1,\dots,\bar{\ell}_{\mathsf{K}})\,, \qquad (3.37)$$

where

$$\ell_i = s_i - i\,, \qquad \text{and} \qquad \bar{\ell}_i = h_i - i\,, \qquad i = 1,\dots,\mathsf{K}\,. \qquad (3.38)$$

In plain words, the first k entries correspond to the length of the ith row *on the right of the diagonal*, and the next K entries to the height of the ith column *below the diagonal.*

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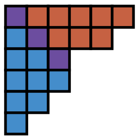
For instance, the diagram



(3.39)

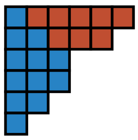
is denoted

$$(6,5,3,2,2,1) \qquad \text{and} \qquad [6,5,4,2,2,1]\,, \tag{3.40}$$

in the symmetric and antisymmetric basis respectively, and

$$(5,3,0|5,3,1)\,, \tag{3.41}$$

in the Frobenius notation. This notation is convenient to describe the $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$ lowest weight states identified above, since ℓ_i entries will correspond to the number of α^μ_{-i}, and $\bar{\ell}_i+1$ to the number of b^μ_{-i}. This last shift by one unit is a consequence of the fact that in the principal embedding, the diagonal consists of fermionic oscillators b, and hence are not taken into account in the "Frobenius labels". We will therefore use a slightly modified version of this notation, namely we will denote a diagram whose diagonal is composed of K boxes by $(\ell_1,\dots,\ell_{\mathsf{K}} \mid h_1,\dots,h_{\mathsf{K}})$, where ℓ_i is the length of the ith row on the right of the diagonal, and h_i is the height of the ith column *starting from the diagonal*, i.e. including the ith box of the diagonal. In this notation, the labels used to characterise a diagram are simply related to the components of the $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$ lowest weight by a shift of $\pm\frac{d}{2}$, as in the bosonic or fermionic version of Howe duality discussed previously. For instance, the previous diagram (3.39) would be denoted $(5,3,0|6,4,2)$, or to put it visually, this modified notation would correspond to the diagram,



(3.42)

where the first three entries are the length of the rows in red, and the next three entries the height of the columns in blue.

R sector. Let us now turn to the subspace of the Ramond sector generated by the first N modes of the bosonic oscillators, and the first M modes of the fermionic ones, acting on the R vacum $|A;0\rangle_{\mathsf{R}}$. On top of the generators of $\mathfrak{o}(2\mathsf{M}+1)\oplus\mathfrak{sp}(2\mathsf{N},\mathbb{R})$, as well as the odd generators (3.26) introduced previously (upon making the implicit replacement $b_{\pm(r-\frac{1}{2})} \to b_{\pm r}$), the additional odd generators[18]

$$Q^m := \frac{1}{\sqrt{2}\,m}\,\gamma\cdot\alpha_{-m}\,, \qquad Q_m := \frac{1}{\sqrt{2}}\,\gamma\cdot\alpha_m\,, \tag{3.43}$$

also act on this subspace, and commute with the Lorentz algebra generators, and altogether these generators make up a representation of $\mathfrak{osp}(2\mathsf{M}+1|2\mathsf{N},\mathbb{R})$. The last new addition

[18]The generators Q^m of course also carry Dirac indices $A,B,\dots$ due to the gamma matrices they contain.

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corresponds to creating or taking γ-traces *in rows* of a tensor-spinor. Naturally, the generators Q_m which take γ-traces should be added to the lowering operators previously identified.

For the sake of completeness, let us also recall the anticommutation relations obeyed by these new generators,

$$\{Q^m, Q^n\} = T^{mn}\,, \qquad \{Q^m, Q_n\} = T^m{}_n\,, \qquad \{Q_m, Q_n\} = T_{mn}\,, \tag{3.44a}$$

$$\{Q^m{}_r, Q_n\} = \delta^m_n\, M_r\,, \qquad \{Q_m{}^r, Q^n\} = -\,\delta^n_m\, M^r\,, \tag{3.44b}$$

$$\{Q^{mr}, Q_n\} = \delta^m_n\, M^r\,, \qquad \{Q_{mr}, Q^n\} = -\delta^n_m\, M_r\,, \tag{3.44c}$$

as well as the action of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$,

$$[T^k{}_l, Q^m] = \delta^m_l\, Q^k\,, \qquad [T^k{}_l, Q_m] = -\delta^k_m\, Q_l\,, \tag{3.45a}$$

$$[T_{kl}, Q^m] = 2\,\delta^m_{(k}\, Q_{l)}\,, \qquad [T^{kl}, Q_m] = -2\,\delta^{(k}_m\, Q^{l)}\,, \tag{3.45b}$$

and of $\mathfrak{o}(2\mathsf{M}+1)$

$$[M^r, Q^m] = -Q^{mr}\,, \qquad [M^r, Q_m] = -Q_m{}^r\,, \tag{3.46a}$$

$$[M_r, Q^m] = -Q^m{}_r\,, \qquad [M_r, Q_m] = -Q_{mr}\,, \tag{3.46b}$$

on them. On top of that, the generators M^r and M_r of $\mathfrak{o}(2\mathsf{M}+1)$ act on the odd generators $Q^m{}_r$, $Q_m{}^r$, Q^{mr} and Q_{mr} as

$$[M^r, Q^m{}_s] = \delta^r_s\, Q^m\,, \qquad [M_r, Q_m{}^s] = \delta^s_r\, Q_m\,, \tag{3.47a}$$

$$[M^r, Q_{ms}] = \delta^r_s\, Q_m\,, \qquad [M_r, Q^{ms}] = \delta^s_r\, Q^m\,. \tag{3.47b}$$

Again, the set of relations (3.45) merely expresses that the additional odd generators Q^m and Q_m are in the fundamental representation of $\mathfrak{sp}(2\mathsf{N},\mathbb{R})$.

Let us stress that, contrarily to the situation in the NS sector, the choice of lowering operators given by (3.36) together with Q_m and M_r is *not* singled out by the super-Virasoro constraints in the R sector, as we will discuss in details in section 4.2.

4 Superstring spectrum sections

4.1 Physicality conditions

NS sector. We begin with the NS sector. Let us introduce the shorthand notation

$$X^{(n)}_\mu(z) := \frac{i}{(n-1)!}\,\partial^n X_\mu(z)\,, \qquad \psi^{(r)}_\mu(z) := \frac{1}{(r-1)!}\,\partial^{r-1}\psi_\mu(z)\,, \qquad n, r \geq 1\,, \tag{4.1a}$$

$$\delta^{(n)}_X(z) := -i\,n!\,\frac{\delta}{\delta\partial^n X(z)}\,, \qquad \delta^{(r)}_\psi(z) := (r-1)!\,\frac{\delta}{\delta\partial^{r-1}\psi(z)}\,, \tag{4.1b}$$

with $X^{(n)}$, $\delta^{(n)}_X$, $\psi^{(r)}$ and $\delta^{(r)}_\psi$ having conformal weights equal to n, $-n$, $r-\frac{1}{2}$ and $-r+\frac{1}{2}$ respectively. Note that all $X^{(n)}$ and all $\psi^{(r)}$ are treated as independent variables. Using the dictionary (2.47) and (2.48), the generators (3.3), (3.11) and (3.26) of the $\mathfrak{osp}(\bullet|\bullet)$ algebra now take the form

$$T^{kl} = \frac{1}{k\,l}\,X^{(k)}\cdot X^{(l)}\,, \qquad T^k{}_l = \frac{1}{k}\,X^{(k)}\cdot\delta^{(l)}_X\,, \qquad T_{kl} = \delta^{(k)}_X\cdot\delta^{(l)}_X\,, \tag{4.2a}$$

with conformal weights equal to $k+l$, $k-l$ and $-k-l$ respectively,

$$M^{rs} = \psi^{(r)} \cdot \psi^{(s)}\,, \qquad M^r{}_s = \psi^{(r)} \cdot \delta_\psi^{(s)}\,, \qquad M_{rs} = \delta_\psi^{(r)} \cdot \delta_\psi^{(s)}\,, \tag{4.2b}$$

with conformal weights equal to $r+s-1$, $r-s$ and $-r-s+1$ respectively, and

$$Q^{nr} = \frac{1}{n} X^{(n)} \cdot \psi^{(r)}\,, \qquad Q^n{}_r = \frac{1}{n} X^{(n)} \cdot \delta_\psi^{(r)}\,, \qquad Q_n{}^r = \psi^{(r)} \cdot \delta_X^{(n)}\,, \qquad Q_{nr} = \delta_X^{(n)} \cdot \delta_\psi^{(r)}\,, \tag{4.2c}$$

with conformal weights equal to $n+r-\frac{1}{2}$, $n-r+\frac{1}{2}$, $-n+r-\frac{1}{2}$ and $-n-r+\frac{1}{2}$ respectively.

In the above, we have omitted the terms proportional to $\frac{d}{2}$ in the definition of the generators (3.3) and (3.11), as it is the structures without the terms in question that will naturally appear inside the Virasoro constraints, as we will see. This modification affects no (anti)commutators of the orthosymplectic algebra other than those involving one trace-creation and one trace-annihilation operator,

$$[T^{mn}, T_{kl}] = -4\,\delta^{(m}_{(k}\,T^{n)}{}_{l)} - 2d\,\delta^m_{(k}\delta^n_{l)}\,, \qquad [M^{rs}, M_{tu}] = -4\,\delta^{[r}_{[t}\,M^{s]}{}_{u]} + 2d\,\delta^r_{[t}\delta^s_{u]}\,, \tag{4.3a}$$

$$\{Q^{mr}, Q_{ns}\} = \delta^r_s\,T^m{}_n - \delta^m_n\,M^r{}_s + d\,\delta^m_n\delta^r_s\,, \tag{4.3b}$$

$$\{Q^m, Q_n\} = T^m{}_n + \frac{d}{2}\,\delta^m_n\,, \qquad [M^r, M_s] = -M^r{}_s + \frac{d}{2}\,\delta^r_s\,, \tag{4.3c}$$

which obtain a contribution containing the number of spacetime dimensions. These contributions to (4.3a) will not influence the derivation of physical solutions to the (necessary and sufficient) Virasoro constraints given by G_ϕ and $G_{\phi+1}$, as the latter will only involve the lowering operators (3.34) and not T_{mn} neither M_{rs}, as we will see. The d-dependent contribution to (4.3b) will also not influence the construction of the physical states considered in this work, unlike the first of (4.3c), as we will see in an example in the R sector later.

Let us now consider the most general (integrated) vertex operator in the canonical ghost picture of the NS sector

$$\mathbb{V}_F^{(-1)}(p,z) = e^{-\varphi}\,F_{\mathsf{NS}}\,(X^{(1)}, X^{(2)}, \dots, \psi^{(1)}, \psi^{(2)}, \dots)\,e^{i\,p\cdot X}\,, \tag{4.4}$$

which we may think of as an operator that can create *any* state or trajectory of the open superstring. We will derive the consequences of physicality conditions (2.16) on the general $\mathbb{V}_F^{(-1)}$. Let us first recall that

$$[\mathcal{Q}_0, \mathbb{V}_F^{(-1)}] = \partial(c\mathbb{V}_F^{(-1)})\,, \tag{4.5}$$

for $\mathbb{V}_F^{(-1)}$ of conformal weight $h=1$. Now let us plug in the general form (4.4) and employ the fact that all CFTs in question are decoupled. Since the energy-momentum tensor $T^{X,\psi}$ given in (2.1) is the sum of two terms,

$$T^X(z) = -\frac{1}{2}\,\partial X \cdot \partial X(z)\,, \qquad T^\psi(z) = -\frac{1}{2}\,\psi \cdot \partial\psi(z)\,, \tag{4.6}$$

we can then use the result of [29] for the contribution of the bosonic part $T^X(z)$ (see also [67] for an early, and partial, version in the language of oscillators) and focus only on the contribution of $T^\psi(z)$, given by

$$T^\psi(z)\,F_{\mathsf{NS}}\,(w) \sim \frac{1}{2}\sum_{r,s\geq 1}\left[\frac{(r+s-1)}{(z-w)^{r-s+2}}\,\psi^{(s)}\cdot\delta_\psi^{(r)} + \frac{s}{(z-w)^{r+s+1}}\,\delta_\psi^{(r)}\cdot\delta_\psi^{(s)}\right]F_{\mathsf{NS}}\,(w)\,. \tag{4.7}$$

Gathering this result with the contributions from T^X and $T^{\beta,\gamma}$, we can compare the coefficients of the poles of every order with (2.18) for $h = 1$. At order 1, we find the coefficient

$$\left(\sum_{n\geq 1}\left[X^{(n+1)}\cdot\delta_X^{(n)} + n\,\psi^{(n+1)}\cdot\delta_\psi^{(n)}\right] - \partial\varphi + ip\cdot\partial X\right)\mathbb{V}_F^{(-1)} = \partial\mathbb{V}_F^{(-1)}\,, \tag{4.8}$$

so agreement is achieved identically. At order 2, we obtain the on-shell condition

$$\left(\sum_{n\geq 1}\left[X^{(n)}\cdot\delta_X^{(n)} + \left(n-\frac{1}{2}\right)\psi^{(n)}\cdot\delta_\psi^{(n)}\right] + \frac{1}{2}\left(p^2-1\right)\right)F_{\mathsf{NS}}\,(w) = 0\,, \tag{4.9}$$

which we can rewrite as

$$\left(L_0 - \frac{1}{2}\right)F_{\mathsf{NS}} = \left(\sum_{n\geq 1}\left[n\,T^n{}_n + \left(n-\frac{1}{2}\right)M^n{}_n\right] + \frac{1}{2}\left(p^2-1\right)\right)F_{\mathsf{NS}} = 0\,, \tag{4.10}$$

i.e. in terms of the Cartan subalgebra generators of the algebra $\mathfrak{osp}(\bullet|\bullet)$. At order $n+2>2$, we find the differential constraints

$$\begin{aligned}&\left(p\cdot\delta_X^{(n)} + \sum_{m\geq 1}\left[X^{(m)}\cdot\delta_X^{(n+m)} + \left(m+\frac{n-1}{2}\right)\psi^{(m)}\cdot\delta_\psi^{(n+m)}\right]\right.\\ &\left.+\frac{1}{2}\sum_{m=1}^{n-1}\delta_X^{(m)}\cdot\delta_X^{(n-m)} + \frac{1}{2}\sum_{m=1}^{n}\left(\frac{n+1}{2}-m\right)\delta_\psi^{(m)}\cdot\delta_\psi^{(n+1-m)}\right]\right)F_{\mathsf{NS}} = 0\,,\end{aligned} \tag{4.11}$$

which we may rewrite as

$$\begin{aligned}L_n F_{\mathsf{NS}} \equiv &\left(T^0{}_n + \sum_{m\geq 1}\left[m\,T^m{}_{n+m} + \left(m+\frac{n-1}{2}\right)M^m{}_{n+m}\right]\right.\\ &\left.+\frac{1}{2}\sum_{m=1}\left[T_{m,n-m} + \left(\frac{n+1}{2}-m\right)M_{m,n-m+1}\right]\right)F_{\mathsf{NS}} = 0\,,\end{aligned}\qquad \forall\, n>0\,, \tag{4.12}$$

where we have introduce the notation $T^0{}_n = p\cdot\delta_X^{(n)}$, following [29, section 3.3]. Next, let us turn to the action of $\mathcal{Q}_1$. We first have that

$$\begin{aligned}T_{\mathrm{F}}^{X,\psi}(z)\,(F_{\mathsf{NS}}\,e^{i\,p\cdot X})(w) \sim \frac{1}{2}&\left(\sum_{n,m\geq 1}\frac{1}{(z-w)^{n-m+1}}\left[\psi^{(m)}\cdot\delta_X^{(n)} + X^{(m)}\cdot\delta_\psi^{(n)}\right]\right.\\ &\left.-\frac{1}{z-w}\,p\cdot\psi + \sum_{m,n\geq 1}\frac{1}{(z-w)^{n+m+1}}\,\delta_X^{(n)}\cdot\delta_\psi^{(m)}\right)(F_{\mathsf{NS}}\,e^{i\,p\cdot X})(w)\,.\end{aligned} \tag{4.13}$$

Now let us recall that requiring $F_{\mathsf{NS}}\,e^{i\,p\cdot X}$ to be a primary field of weight h with respect to the $\mathcal{N}=1$ super-Virasoro algebra imposes that it has a superpartner $\widetilde{F}_{\mathsf{NS}}$ of weight $h+\frac{1}{2}$ with respect to worldsheet supersymmetry, and that the two are related by the supercurrent, in the sense that

$$T_{\mathrm{F}}^{X,\psi}(z)\,(F_{\mathsf{NS}}\,e^{i\,p\cdot X})(w) \sim \frac{1/2}{z-w}\,(\widetilde{F}_{\mathsf{NS}}\,e^{i\,p\cdot X})(w)\,, \tag{4.14a}$$

$$T_{\mathrm{F}}^{X,\psi}(z)\,(\widetilde{F}_{\mathsf{NS}}\,e^{i\,p\cdot X})(w) \sim \left[\frac{h}{(z-w)^2} + \frac{1/2}{z-w}\partial\right](F_{\mathsf{NS}}\,e^{i\,p\cdot X})(w)\,, \tag{4.14b}$$

with $h = 1/2$ here. Comparing (4.13) and (4.14a), we find that the worldsheet superpartner of F_{NS} is given by

$$\widetilde{F}_{\mathsf{NS}} = \Big[Q_0{}^1 + \sum_{n\geq1}(Q_n{}^{n+1} + n\,Q^n{}_n)\Big]\,F_{\mathsf{NS}} = G_{-1/2}F_{\mathsf{NS}}\,, \tag{4.15}$$

where we introduced $Q_0{}^1 := p\cdot\psi^{(1)}$, and that the following constraints

$$G_{1/2}F_{\mathsf{NS}} = \frac{1}{2}\Big(Q_{01} + \sum_{k\geq1}\Big[k\,Q^k{}_{k+1} + Q_k{}^k\Big]\Big)F_{\mathsf{NS}} = 0\,, \tag{4.16a}$$

$$G_{3/2}F_{\mathsf{NS}} = \frac{1}{2}\left(Q_{02} + Q_{11} + \sum_{k=1}\Big[k\,Q^k{}_{k+2} + Q_{k+1}{}^k\Big]\right)F_{\mathsf{NS}} = 0\,, \tag{4.16b}$$

$$G_{r-\frac{1}{2}}F_{\mathsf{NS}} = \frac{1}{2}\left(Q_{0\,r} + \sum_{k=1}^{r-1}Q_{r-k,k} + \sum_{k\geq1}\Big[k\,Q^k{}_{k+r} + Q_{k+r-1}{}^k\Big]\right)F_{\mathsf{NS}} = 0\,, \qquad \forall\, r > 2\,, \tag{4.16c}$$

are necessary in order to ensure the absence of poles of order $r > 1$ in the OPE (4.13). Note that we introduced $Q_{0\,r} := p\cdot\delta_\psi^{(r)}$. At this point, let us observe that the form of (4.16a) and (4.16c) justifies the choice (3.34) of lowering operators. In addition, we find that

$$[\mathcal{Q}_1, \mathbb{V}_F^{(-1)}] = 0\,, \quad [\mathcal{Q}_2, \mathbb{V}_F^{(-1)}] = 0\,, \tag{4.17}$$

as there are no poles in the integrands (where we have used (2.13) for the former), so the second and third of the conditions (2.16) are also satisfied. Finally, since only L_0, $G_{1/2}$ and $G_{3/2}$ are sufficient, we will only need (4.10), (4.15), (4.16a) and (4.16b).

Picture changing. Applying the picture-changing operation (2.17) on (4.4) gives

$$\Big[\mathcal{Q}_0, e^\chi\,\mathbb{V}_F^{(-1)}\Big] = \partial\big(c\,e^\chi\,\mathbb{V}_F^{(-1)}\big)\,, \quad \Big[\mathcal{Q}_1, e^\chi\,\mathbb{V}_F^{(-1)}\Big] = -\frac{1}{2}\,\widetilde{F}_{\mathsf{NS}}\,e^{ip\cdot X}\,, \quad \Big[\mathcal{Q}_2, e^\chi\,\mathbb{V}_F^{(-1)}\Big] = 0\,, \tag{4.18}$$

so only $\mathcal{Q}_1$ contributes to

$$\mathbb{V}_F^{(0)} = 2\,[\mathcal{Q}, e^\chi\,\mathbb{V}_F^{(-1)}] = -\widetilde{F}_{\mathsf{NS}}\,e^{ip\cdot X}\,, \tag{4.19}$$

with $\widetilde{F}_{\mathsf{NS}}$ given by (4.15). We can now verify physicality on $\mathbb{V}_F^{(0)}$, and find

$$[\mathcal{Q}_0, \mathbb{V}_F^{(0)}] = \partial(c\mathbb{V}_F^{(0)})\,, \tag{4.20}$$

which is a total derivative as should be the case, with the on-shell condition now taking the form

$$\left(L_0 - \frac{1}{2}\right)\widetilde{F}_{\mathsf{NS}} = \left\{\sum_{n\geq1}\left[n\,T^n{}_n + \left(n - \frac{1}{2}\right)M^n{}_n\right] + \frac{1}{2}(p^2 - 2)\right\}\widetilde{F}_{\mathsf{NS}} = 0\,, \tag{4.21}$$

and the form of the differential constraints (4.12) is unaffected, but now act on $\widetilde{F}_{\mathsf{NS}}$. We also find that

$$[\mathcal{Q}_1, \mathbb{V}_F^{(0)}] = -\frac{1}{2}\partial\big(e^{-\chi}e^{\varphi}\,F_{\mathsf{NS}}\,e^{ip\cdot X}\big)\,, \qquad [\mathcal{Q}_2, \mathbb{V}_F^{(0)}] = 0\,, \tag{4.22}$$

with the form of (4.14a) and (4.15) along with the constraints (4.16a)–(4.16c) is unaffected. Notice that the two superghost pictures, (-1) and (0), are sufficient to calculate scattering amplitudes of NS states, as the sum of the pictures of all external states has to be equal to -2 in order to cancel the background superghost charge of 2.

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R sector. To derive the super-Virasoro conditions on a general vertex operator,

$$\mathbb{V}_F^{(-1)}(p,z) = e^{-\varphi/2}\, F_{\mathsf{R}}\left(X^{(1)}, X^{(2)}, \ldots, \psi^{(1)}, \psi^{(2)}, \ldots\right) S_\alpha\, e^{i\,p\cdot X}\,, \tag{4.23}$$

in the R sector, two (equivalent) possibilities appear a priori: impose the physicality condition (2.16) on (4.23) or *supersymmetrize*, via (2.60a), a general *physical* vertex operator in the NS sector, namely one of the form (4.4) which also *satisfies* the constraints (4.16a)–(4.16c). These procedures involve OPEs of the form

$$\psi(z)\cdot\partial\psi(z)\, F_{\mathsf{R}}\,(w) S_A(w)\, e^{i\,p\cdot X(w)}\,, \quad \psi(z)\cdot\partial X(z)\, F_{\mathsf{R}}\,(w) S_A(w)\, e^{i\,p\cdot X(w)}\,, \tag{4.24}$$

and

$$i\partial X(z)\cdot\left(\gamma\, S(z)\right)_A F_{\mathsf{NS}}\,(w)\, e^{i\,p\cdot X(w)}\,, \tag{4.25}$$

respectively. However, as indicated by the form of their 2-point functions (2.11), ψ and S are *interacting* fields: in the R sector the fermion CFT is *not* free, unlike in the NS sector. Consequently, the standard version of Wick's theorem cannot be applied to calculate OPEs that involve ψ and S, a problem that can in principle be bypassed by means of the technique of bosonization for ψ and S in particular [42, 45]. More specifically, ψ and S can be bosonized by means of a system of free chiral bosons $H_i\,, i = 1,\ldots,5\,,$ with 2-point function naturally given by

$$\langle H_k(z)\, H_l(w)\rangle = -\delta_{kl}\,\ln|z-w|\,, \tag{4.26}$$

via

$$\psi^\mu = e^{i\mu\cdot H}\,, \qquad S_A = e^{iA\cdot H}\,, \tag{4.27}$$

(up to cocycle factors à la Jordan-Wigner, which guarantee that all components of ψ and S anticommute), where now μ and A stand both for vector/spinor spacetime indices and for the vectors

$$\mu = (0,\ldots,0,\pm1,0,\ldots,0)\,, \qquad A = \left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right)\,, \tag{4.28}$$

where μ is a 5-component vector, and the previous notation is meant to highlight that only one of its components is non-vanishing (and equal to ±1). Let us then attempt to calculate the OPE of S_A with a general function of ψ and its derivatives, $F(\psi^{(1)},\psi^{(2)},\ldots)$. Since we need to bosonize all S and ψ, we must first calculate the OPE of S_A with the transform $F_{\mathsf{bos}}(\mu\cdot H^{(1)},\mu\cdot H^{(2)},\ldots)$ of F, where

$$\mu\cdot H^{(n)} := \mu\cdot\partial^{n-1}H\,. \tag{4.29}$$

We thus have

$$\begin{aligned}
&S_A(z)\,F_{\mathsf{bos}}(w)\\
&\sim \sum_{l\geq 0}\frac{(z-w)^l}{l!}\partial^l S_A(w)\Bigg\{\pm\frac{i}{2}\bigg[-\ln|z-w|\frac{\delta}{\delta(\mu\cdot H^{(1)})}+\sum_{n\geq 2}\frac{(n-2)!}{(z-w)^{n-1}}\frac{\delta}{\delta(\mu\cdot H^{(n)})}\bigg]\\
&\quad-\frac{1}{4}\bigg[\ln^2|z-w|\frac{\delta^2}{\delta(\mu\cdot H^{(1)})\delta(\mu\cdot H^{(1)})}\\
&\qquad-2\ln|z-w|\sum_{n=2}\frac{(n-2)!}{(z-w)^{n-1}}\frac{\delta^2}{\delta(\mu\cdot H^{(1)})\delta(\mu\cdot H^{(n)})}\\
&\qquad+\sum_{n,m\geq 2}\frac{(n-2)!(m-2)!}{(z-w)^{n+m-2}}\frac{\delta^2}{\delta(\mu\cdot H^{(n)})\delta(\mu\cdot H^{(m)})}\bigg]+\dots\Bigg\}F_{\mathsf{bos}}\,.
\end{aligned}\tag{4.30}$$

For the leading Regge trajectory, there appears only ψ in F according to (2.64), so $F_{\mathsf{bos}}=e^{i\mu\cdot H}$ and only (an infinite series in) the derivatives w.r.t. $\mu\cdot H^{(1)}$ survive, that can be resummed to the result (2.65). For general subleading trajectories, there appears a double infinity: one can act with infinitely many derivatives w.r.t. to each of the infinitely many $\mu\cdot H^{(n)}$.

Instead of attempting to transform such a general operator acting on F_{bos} into its form, in terms of ψ and its derivatives, acting on F, we can turn to the Fock space description of states: in this language, we can easily derive the super-Virasoro constraints from (2.28):

$$G_0\,F_{\mathsf{R}}=\left[\frac{1}{\sqrt{2}}\not{p}+\sum_{k=1}^{\infty}\left(k\,Q^k{}_k+Q_k{}^k\right)\right]F_{\mathsf{R}}=0\,,\tag{4.31a}$$

$$G_1\,F_{\mathsf{R}}=\left[Q_{01}+Q_1+\sum_{k=1}^{\infty}\left(k\,Q^k{}_{k+1}+Q_{k+1}{}^k\right)\right]F_{\mathsf{R}}=0\,,\tag{4.31b}$$

with the operators Q of the $\mathfrak{osp}(\bullet|\bullet)$ algebra given by (3.26) in terms of the oscillators $\alpha_{\pm n}$ and $b_{\pm r}$. While the former can be mapped to the field ∂X and its descendants by means of the dictionary (2.47) in the constraints (4.31a)–(4.31b), as explained around equation (2.50), we have no closed formula available that maps b_{-r} to ψ and its descendants. Consequently, at this moment there is no obvious way of writing the constraints (4.31a)–(4.31b) in the language most adapted to constructing physical vertex operators, which are necessary for the calculation of string scattering amplitudes; we wish to revisit this point in the future.

Transverse subspace simplification. As argued for example in [40] for the bosonic string (see also [38, 39] for arguments obtained in the BRST formalism), restricting to the transverse subspace, with metric

$$\eta^{\perp}_{\mu\nu}=\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\,,\tag{4.32}$$

is sufficient to construct physical states, which we may understand intuitively as eliminating spurious longitudinal modes. This argument can be extended to the case of the superstring since the BRST cohomology is known, and has been shown to consists of only transverse states, generated by DDF operators, see e.g. [26, 68, 69]. Since one has simply $p_\perp^2=0=\not{p}_\perp$ in the transverse subspace, so the operators p^2 and $\not{p}$, which form the Laplacian and Dirac operators respectively and which are necessary for the Klein-Gordon and Dirac equations (in momentum space) to be satisfied in the NS and R sectors respectively, remain unchanged. The Virasoro constraints become then

$$\left(L_0 - \frac{1}{2}\right) F_{\mathsf{NS}} = \left[N_{\mathsf{NS}} + \frac{1}{2}(p^2 - 1)\right] F_{\mathsf{NS}} = 0\,, \tag{4.33a}$$

$$G_{1/2} F_{\mathsf{NS}} = \sum_{n\geq 1} \left[n\, Q^{n}{}_{n+1} + Q_{n}{}^{n}\right] F_{\mathsf{NS}} = 0\,, \tag{4.33b}$$

$$G_{3/2} F_{\mathsf{NS}} = \left[Q^{\perp}_{11} + \sum_{k\geq 1} \left(k\, Q^{k}{}_{k+2} + Q_{k+1}{}^{k}\right)\right] F_{\mathsf{NS}} = 0\,, \tag{4.33c}$$

in the NS sector, and

$$G_0 F_{\mathsf{R}} = \left[\frac{1}{\sqrt{2}}\not{p} + \sum_{k\geq 1} \left(k\, Q^{k}{}_{k} + Q_{k}{}^{k}\right)\right] F_{\mathsf{R}} = 0\,, \tag{4.34a}$$

$$G_1 F_{\mathsf{R}} = \left[Q_1 + \sum_{k\geq 1} \left(k\, Q^{k}{}_{k+1} + Q_{k+1}{}^{k}\right)\right] F_{\mathsf{R}} = 0\,, \tag{4.34b}$$

in the R sector, with the level being given by

$$N_{\mathsf{NS}} F_{\mathsf{NS}} = \sum_{n\geq 1} \left[n\, T^{n}{}_{n} + \left(n - \frac{1}{2}\right) M^{n}{}_{n}\right] F_{\mathsf{NS}} \qquad \text{and} \qquad N_{\mathsf{R}} F_{\mathsf{R}} = \sum_{n\geq 1} n \left(T^{n}{}_{n} + M^{n}{}_{n}\right) F_{\mathsf{R}}\,. \tag{4.35}$$

4.2 Principal embedding

As in the bosonic string, a state characterized by a given polarization tensor, visualised as a certain Young diagram $\mathbb{Y}$, can and will appear *infinitely* many times in the spectrum, but at *different* mass levels, up to a possible non-trivial multiplicity. A given Young diagram $\mathbb{Y}$ can namely be embedded in infinitely many different ways in the string spectrum, starting from a lowest possible level $N_{\min}$, whose value depends on the former's symmetries. Using the terminology of [29], we will refer to the polynomial dressing $\mathbb{Y}$ at $N_{\min}$, such that the corresponding state be physical, as its *principal embedding.* We will now determine its form in the two sectors of the superstring. In both sectors, the anticommuting character of the fermionic modes b_{-r} implies that, if k copies of a certain mode are used in the construction of a state, the corresponding polarization will be labelled by a diagram having at least k rows.

Neveu-Schwarz sector. Let us start with the simplest example, namely one-row Young diagrams of length s, i.e. totally symmetric rank-s tensors. To minimise the level, we should restrict ourselves to the conformal ingredients with the lowest conformal weights, i.e. α^{μ}_{-1} and $b^{\mu}_{-1/2}$. Since the b are anticommuting, there cannot be more than one, since a second would create a second row and so on. There are, therefore, only two possibilities, namely to use s times α_{-1}, or to use $s-1$ copies of α^{μ}_{-1} and one $b_{-1/2}$. Considering that $b_{-1/2}$ has weight $\frac{1}{2}$ and α_{-1} has weight 1, the second option is the one minimising the level, with the latter being then equal to $N_{\min}(s) = s - \frac{1}{2}$. Schematically, we can represent the polynomial in $b_{-1/2}$ and α_{-1}, out of which the corresponding state will be constructed, as

$$\begin{array}{|c|c|c|c|}\hline b_{-1/2} & \alpha_{-1} & \qquad\cdots\qquad & \alpha_{-1} \\ \hline \end{array} \tag{4.36}$$

where the position of $b_{-r+\frac{1}{2}}$ and α_{-n} in the diagram is meant to denote the index of the polarization tensor with which their spacetime index is contracted. Such vertex operators build the leading Regge trajectory, discussed in e.g. [37].

Now let us proceed to two-row diagrams (s_1, s_2). The first row, of length s_1, will naturally be built according to (4.36), so a single $b_{-1/2}$ and several α_{-1} have been used. For the second row, again a single $b_{-1/2}$ can be used, but no α_{-1}, since powers of α_{-1} are automatically symmetric in their spacetime indices and, as such, cannot be used to construct yet another row in the Young diagram. Consequently, to fill the second row, we can use a single $b_{-1/2}$ as well as the next possible modes, namely $b_{-3/2}$ and α_{-2}, of weight $\frac{3}{2}$ and 2 respectively. Again, only a single $b_{-3/2}$ can appear, but several α_{-2}, so there are four options: s_2 copies of α_{-2}, one $b_{-1/2}$ or $b_{-3/2}$ and $s_2 - 1$ copies of α_{-2}, a pair of $b_{-1/2}$ and $b_{-3/2}$ and $s_2 - 2$ copies of α_{-2}. Obviously, the last option is the one minimising the level, so that a two-row diagram in the principal embedding can be schematically represented as

$$\begin{array}{|c|c|c|c|c|c|}
\hline
b_{-1/2} & \alpha_{-1} & \multicolumn{3}{c|}{\cdots} & \alpha_{-1} \\
\hline
b_{-1/2} & b_{-3/2} & \alpha_{-2} & \cdots & \alpha_{-2} & \multicolumn{1}{c}{} \\
\cline{1-5}
\end{array} \tag{4.37}$$

from which we can read off the level,

$$N_{\min}(s_1, s_2) = \begin{cases} s_1 & \text{if} \quad s_2 = 1\,, \\ s_1 + 2\,s_2 - \dfrac{5}{2} & \text{if} \quad s_2 \geq 2\,. \end{cases} \tag{4.38}$$

Note that we need to distinguish between two cases, either the second row has only one box, or it has two or more, in order to account for the presence or absence of $b_{-3/2}$.

One can convince oneself, following the same line of arguments, that for three row diagrams, the principal embedding is given by

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
b_{-1/2} & \alpha_{-1} & \multicolumn{5}{c|}{\cdots} & \alpha_{-1} \\
\hline
b_{-1/2} & b_{-3/2} & \alpha_{-2} & \multicolumn{3}{c|}{\cdots} & \alpha_{-2} & \multicolumn{1}{c}{} \\
\cline{1-7}
b_{-1/2} & b_{-3/2} & b_{-5/2} & \alpha_{-3} & \cdots & \alpha_{-3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-6}
\end{array} \tag{4.39}$$

and at this point one can infer the following pattern: the ith row of a Young diagram in the principal embedding is created by one copy of $b_{-r+\frac{1}{2}}$ for every $r = 1, \ldots, i$, and $s_i - i$ copies of α_{-i}. Taking into account that $b_{-r+\frac{1}{2}}$ has weight $r - \frac{1}{2}$, and $\partial^i X$ has weight i, we find that the lowest possible level at which a given Young diagram can be embedded in the spectrum reads

$$N_{\min}^{\mathbb{Y},\mathsf{NS}} = \sum_{i=1}^{\mathsf{K}} i\,\ell_i + \left(i - \frac{1}{2}\right) h_i\,, \tag{4.40}$$

where we use the (modified) Frobenius notation introduced in the previous section for the diagram, namely $\mathbb{Y} = (\ell_1, \ldots, \ell_{\mathsf{K}} \mid h_1, \ldots, h_{\mathsf{K}})$, where K is the length of its diagonal. Let us highlight that the derived principal embedding automatically satisfies the lowest weight conditions (3.36) of $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N}, \mathbb{R})$.

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Ramond sector. Things are slightly more subtle in the Ramond sector, mainly due to the fact that the bosonic and fermionic oscillators carry now the same amount of energy units. Consider for instance the leading Regge trajectory, consisting of totally symmetric tensor-spinors of rank $s \in \mathbb{N}$: there are two ways of creating such states at level s, either with s bosonic oscillators α^{μ}_{-1}, or with one fermionic oscillator b^{μ}_{-1} and $s-1$ bosonic α^{μ}_{-1}, which can be schematically represented as

$$\begin{array}{|c|c|c|c|}\hline b_{-1} & \alpha_{-1} & \quad\cdots\quad & \alpha_{-1} \\ \hline\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|}\hline \alpha_{-1} & \quad\cdots\quad & \alpha_{-1} \\ \hline\end{array}\ . \tag{4.41}$$

Unlike in the NS sector, where the analogue of the first option was favoured as the one minimizing the level as in (2.64), here the two possibilities cannot be distinguished on this ground and *both* contribute to the trajectory, as is clear from (2.65). For mixed-symmetric tensor-spinors, the situation becomes more ambiguous: for example, for two-row diagrams, there are 4 *distinct* possibilities

$$\begin{array}{l} \begin{array}{l}\begin{array}{|c|c|c|c|}\hline b_{-1} & \alpha_{-1} & \quad\cdots\quad & \alpha_{-1} \\ \hline\end{array}\\ \begin{array}{|c|c|c|c|c|}\hline b_{-1} & b_{-2} & \alpha_{-2} & \cdots & \alpha_{-2} \\ \hline\end{array}\end{array}\ , \qquad \begin{array}{l}\begin{array}{|c|c|c|c|}\hline b_{-1} & \alpha_{-1} & \quad\cdots\quad & \alpha_{-1} \\ \hline\end{array}\\ \begin{array}{|c|c|c|c|}\hline b_{-1} & \alpha_{-2} & \cdots & \alpha_{-2} \\ \hline\end{array}\end{array}\ , \\[1em] \begin{array}{l}\begin{array}{|c|c|c|}\hline \alpha_{-1} & \quad\cdots\quad & \alpha_{-1} \\ \hline\end{array}\\ \begin{array}{|c|c|c|c|c|}\hline b_{-1} & b_{-2} & \alpha_{-2} & \cdots & \alpha_{-2} \\ \hline\end{array}\end{array}\ , \qquad \begin{array}{l}\begin{array}{|c|c|c|}\hline \alpha_{-1} & \quad\cdots\quad & \alpha_{-1} \\ \hline\end{array}\\ \begin{array}{|c|c|c|c|}\hline b_{-1} & \alpha_{-2} & \cdots & \alpha_{-2} \\ \hline\end{array}\end{array}\ , \end{array} \tag{4.42}$$

and for three-row diagrams, we can depict the 2^3 possibilities as

$$\begin{array}{l}\begin{array}{|c|c|c|c|}\hline \phantom{b_{-1}} & \alpha_{-1} & \qquad\quad\cdots\qquad\quad & \alpha_{-1} \\ \hline\end{array}\\ \begin{array}{|c|c|c|c|c|}\hline b_{-1} & \phantom{b_{-1}} & \alpha_{-2} & \quad\cdots\quad & \alpha_{-2} \\ \hline\end{array}\\ \begin{array}{|c|c|c|c|c|c|}\hline b_{-1} & b_{-2} & \phantom{b_{-1}} & \alpha_{-3} & \cdots & \alpha_{-3} \\ \hline\end{array}\end{array}\ , \tag{4.43}$$

where the empty box i in the diagonal is to be understood as accommodating either an α_{-i} or a b_{-i}, with $i = 1, 2, 3$. It is thus straightforward to see that, more generally, for the principal embedding in the R sector and a diagram with a diagonal of length K, there are 2^{K} possible *disparate* configurations of the bosonic and fermionic oscillators, which differ only along the diagonal and which all lead to the *same* lowest possible level, that is given by

$$\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}} = \sum_{i=1}^{\mathsf{K}} i\,(\ell_i + \bar{\ell}_i + 1) = \sum_{i=1}^{\mathsf{K}} i\,(\ell_i + h_i)\,, \tag{4.44}$$

where in the first and in the second equality we use the Frobenius (3.37) and the modified Frobenius notation respectively.

Consequently, the candidate physical states belonging to the principal embedding in the R sector are a priori linear combinations of the 2^{K} configurations. Moreover, since the operators ${Q_n}^n$ and ${Q^n}_n$ interchange bosonic with fermionic oscillators of the same conformal weight along the diagonal of a general Young diagram, the 2^{K} configurations can be represented by starting from any among them and then acting with ${Q_n}^n$ and ${Q^n}_n$ to obtain the rest. A

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simple way of constructing such a representation is to start from the configuration whose diagonal is filled by fermionic oscillators only. This is a state with spin $\mathbb{Y}_{1/2}$, namely a representation of the Lorentz group carried by a (γ–)traceless tensor-spinor, whose tensor part has the symmetry of the Young diagram $\mathbb{Y}$ in precisely the same way as in the NS sector. That is to say, we consider a polarisation tensor-spinor with symmetry $\mathbb{Y}_{1/2}$ and contract its indices corresponding to the kth row (column) above (below) the diagonal with the bosonic (fermionic) oscillators α^{μ}_{-k} (b^{μ}_{-k}, and its spinor index with that of the Ramond vacuum). We will denote such a state by the symbol $|\mathbb{Y}\rangle_{1/2}$; its level $N_{\mathbb{Y}}$ is given by (4.44). As discussed previously, the irreducibility of this tensor-spinor will be reflected in the fact that the state constructed in this manner is annihilated by the generators listed in (3.36), as well as Q_n (encoding its γ-tracelessness), namely

$$Q^{m}{}_{r}\,|\mathbb{Y}\rangle_{1/2} = 0\,,\; m < r\,,\quad Q_{m}{}^{r}\,|\mathbb{Y}\rangle_{1/2} = 0\,,\; m \geq r\,,\quad Q_n\,|\mathbb{Y}\rangle_{1/2} = 0\,,\; n > 0\,. \tag{4.45}$$

The candidate physical states of the principal embedding can then be represented as

$$|\mathrm{cand}\rangle_{1/2} := P_{\mathbb{Y}}\,|\mathbb{Y}\rangle_{1/2} := \left(\mathtt{b}_0 + \sum_{n=1}^{\mathsf{K}} \frac{1}{n!}\,\mathtt{b}_{i_1\dots i_n}\,Q^{i_1}{}_{i_1}\dots Q^{i_n}{}_{i_n}\right)|\mathbb{Y}\rangle_{1/2}\,, \tag{4.46}$$

where, because of the anticommuting nature of the $Q^{i}{}_{i}$, the $\mathtt{b}$'s are a priori arbitrary totally antisymmetric coefficients, that account for 2^{K} parameters. By construction, $P_{\mathbb{Y}}$ has conformal weight equal to 0.

With a candidate (4.46) available for the principal embedding, let us look at the super-Virasoro constraints (4.34a)–(4.34b), which are respectively equivalent to imposing that physical states be annihilated by G_0 and G_1. First let us notice that the product of G_1 and $P_{\mathbb{Y}}$ has conformal weight equal to -1, so it has to be proportional to the lowering operators of $\mathfrak{osp}(\bullet|\bullet)$; the latter annihilate $|\mathbb{Y}\rangle_{1/2}$ by construction, so the constraint (4.34b) can only produce identities. We thus turn to (4.34a): G_0 contains the Dirac operator $\not{p}$, so it is this constraint that will yield the Dirac equation, namely

$$G_0\,|\mathrm{phys}\rangle_{1/2} = 0 \qquad \Longleftrightarrow \qquad (\not{p} - m)\;|\mathrm{phys}\rangle_{1/2} = 0\,, \tag{4.47}$$

as is expected for physical R states. This observation is suggestive to recasting (4.34a) as a mass-shell condition on the polarisation tensor-spinor and imply that principally embedded states in the R sector are those eigenstates of the "mass operator",

$$\boldsymbol{m} := G_0 - \frac{1}{\sqrt{2}}\,\not{p} \equiv \sum_{n\geq 1} n\,Q^{n}{}_{n} + Q_{n}{}^{n}\,, \tag{4.48}$$

which appear at the *lowest possible level*, for a given spin. Since G_0 "squares to" L_0 according to the super-Virasoro algebra (2.32c), we have that

$$\boldsymbol{m}^2 + \frac{1}{\sqrt{2}}\{\boldsymbol{m}, \not{p}\} = N_{\mathsf{R}} \quad \Rightarrow \quad \boldsymbol{m}\,|\mathrm{phys}\rangle_{1/2} = \pm\sqrt{N_{\mathsf{R}}}\,|\mathrm{phys}\rangle_{1/2}\,, \tag{4.49}$$

so the eigenvalues of $\boldsymbol{m}$ are square roots of the mass level, given by (4.44) for principally embedded eigenstates. Consequently, we have to consider the equation (4.49) on the candidate (4.46) and solve for the parameters $\mathtt{b}$.

To this end, we first consider the action of $\boldsymbol{m}$ on the state $|\mathbb{Y}\rangle_{1/2}$. By definition, the generators $Q_n{}^n$ annihilate such a state, while the $Q^n{}_n$ do not, nor do they act diagonally on it, so that

$$\boldsymbol{m}\,|\mathbb{Y}\rangle_{1/2} = \sum_{n\geq 1} n\, Q^n{}_n\,|\mathbb{Y}\rangle_{1/2}\,. \tag{4.50}$$

Considering now the action of $\boldsymbol{m}$ on the candidate (4.46), and that $\{Q^m{}_m, Q^n{}_n\} = 0$, we can simply replace $Q^m{}_m$ with anticommuting variables, say θ^m. We can also use these variables to collect the $\mathsf{b}_{i_1\dots i_n}$ coefficients as

$$\mathsf{b}_{[n]} := \frac{1}{n!}\,\mathsf{b}_{i_1\dots i_n}\,\theta^{i_n}\dots\theta^{i_n}\,, \tag{4.51}$$

where summation is implied. Moreover,

$$\{Q_m{}^m, Q^n{}_n\} = \delta_{mn}\,(T^n{}_n + M^n{}_n)\,, \qquad [T^k{}_k + M^k{}_k, Q^m{}_m] = 0 = [T^k{}_k + M^k{}_k, Q_m{}^m]\,, \tag{4.52}$$

so that $Q_m{}^m$ effectively acts as a derivation on $P^{\pm}_{\mathbb{Y}}\,|\mathbb{Y}\rangle_{1/2}$, and $T^k{}_k + M^k{}_k$ by multiplication with $\ell_k + h_k$. This means that we can replace these depth zero $\mathfrak{osp}(\bullet|\bullet)$ generators by

$$Q^m{}_m \quad\longleftrightarrow\quad \theta_m\,, \qquad\qquad Q_m{}^m \quad\longleftrightarrow\quad (\ell_m + h_m)\,\frac{\partial}{\partial\theta_m}\,, \tag{4.53a}$$

$$T^k{}_k \quad\longleftrightarrow\quad \theta_k\,\frac{\partial}{\partial\theta_k} + \ell_k\,, \qquad\qquad M^k{}_k \quad\longleftrightarrow\quad -\theta_k\,\frac{\partial}{\partial\theta_k} + h_k\,, \tag{4.53b}$$

when acting on the space of states of the form (4.46). Introducing the operators

$$\delta := \sum_{m\geq 1} m\,\theta^m\,, \qquad \delta^* := \sum_{m\geq 1} (\ell_m + h_m)\,\frac{\partial}{\partial\theta^m}\,, \tag{4.54}$$

which obey

$$\delta^2 = 0 = (\delta^*)^2\,, \qquad \{\delta, \delta^*\} = \mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}\,, \tag{4.55}$$

we can now rewrite the constraint (4.49) on $|\mathrm{cand}\rangle_{1/2}$ simply as

$$\pm\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}\,\mathsf{b}_{[n]} = \delta\mathsf{b}_{[n-1]} + \delta^*\mathsf{b}_{[n+1]}\,, \quad \forall\, 0 \leq n \leq \mathsf{K}\,. \tag{4.56}$$

As the notation suggests, we may think of δ and δ^* as a differential and a codifferential on the spaces of forms, with basis given by monomials in $\theta^k \leftrightarrow Q^k{}_k$.

The system (4.56) contains as many equations as there are of unknowns, which correspond to the sum of the number of components of antisymmetric tensors in K dimensions of all ranks, i.e. $\sum_{n=0}^{\mathsf{K}}\binom{\mathsf{K}}{n} = 2^{\mathsf{K}}$. Note, however, that the system (4.56) contains some redundancy, i.e. not all equations are independent. Indeed, suppose we are given the equations

$$\pm\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}\,\mathsf{b}_{[n]} = \delta\mathsf{b}_{[n-1]} + \delta^*\mathsf{b}_{[n+1]} \qquad \text{and} \qquad \pm\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}\,\mathsf{b}_{[n+2]} = \delta\mathsf{b}_{[n+1]} + \delta^*\mathsf{b}_{[n+3]}\,, \tag{4.57}$$

for some value of n, and let us act with δ on the first one. This yields

$$\pm\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}\;\mathsf{b}_{[n+1]} = \delta\mathsf{b}_{[n]} \pm \frac{1}{\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}}\,\delta^*\delta\mathsf{b}_{[n+1]}\,, \tag{4.58}$$

which, upon using the second equation of (4.57) to replace $\delta \mathsf{b}_{[n+1]}$, leads to

$$\pm\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}\ \mathsf{b}_{[n+1]} = \delta \mathsf{b}_{[n]} + \delta^* \mathsf{b}_{[n+2]}\,, \tag{4.59}$$

i.e. we obtain the equation for $\mathsf{b}_{[n+1]}$. One can check that, upon acting on the second equation of (4.57) with δ^* and using the first one also produces the equation for $\mathsf{b}_{[n+1]}$. Since this mechanism works for all values of n, this tells us that roughly "half" of the equations are independent. More precisely, the subsets of equations (4.56) with n taking only *even* or *odd* values are independent. For the sake of definiteness, let us say that we keep only the latter, i.e. our system of equations is

$$\pm\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}\ \mathsf{b}_{[2k+1]} = \delta \mathsf{b}_{[2k]} + \delta^* \mathsf{b}_{[2k+2]}\,, \quad \forall\, 0 \leq k \leq [\mathsf{K}/2]\,, \tag{4.60}$$

which can be read as the *definitions* for the unknowns $\mathsf{b}_{[2k+1]}$ in terms of *free* parameters, the $\mathsf{b}_{[2k]}$. In other words, the above equation is the most general solution of the defining equation for $P_{\mathbb{Y}}$. The space of solutions is therefore parameterized by the antisymmetric tensors $\mathsf{b}_{[2k]}$ of even ranks, and hence of dimension

$$\sum_{n=0}^{[\mathsf{K}/2]} \binom{\mathsf{K}}{2n} = \frac{1}{2}\sum_{n=0}^{\mathsf{K}} \binom{\mathsf{K}}{n} \left(1+(-1)^n\right) = 2^{\mathsf{K}-1}\,. \tag{4.61}$$

For example, for $K = 2$, up to an overall coefficient, the principal embedding takes the form

$$\begin{aligned} &\Big\{\mathsf{b}_0\, b^{\nu_1}_{-1} b^{\nu_2}_{-2} \pm \frac{1}{\sqrt{\mathsf{N}}}\Big[h_1\big(\mathsf{b}_0-(\ell_2+h_2)\mathsf{b}_{12}\big)\,\alpha^{\mu_1}_{-1} b^{\nu_2}_{-2} \\ &\qquad + h_2\left(\mathsf{b}_0+\frac{1}{2}(\ell_1+h_1)\mathsf{b}_{12}\right) b^{\nu_1}_{-1}\alpha^{\mu_2}_{-2}\Big] + \frac{1}{2}h_1 h_2 \mathsf{b}_{12}\,\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-2}\Big\} \\ &\qquad \times \alpha^{\mu_1(\ell_1)}_{-1}\alpha^{\mu_2(\ell_2)}_{-2} b^{\nu_1[h_1-1]}_{-1} b^{\nu_2[h_2-1]}_{-2}\ \varepsilon^{A}_{\mu_1(\ell_1),\mu_2(\ell_2)|\nu_1[h_1],\nu_2[h_2]}\, |p;A;0\rangle_{\mathsf{R}}\ , \end{aligned} \tag{4.62}$$

with $\mathsf{N} = \ell_1 + h_1 + 2(\ell_2 + h_2)$ and multiplicity equal to 2, namely there are two distinct such trajectories.

The multiplicity (4.61) may seem surprising at first glance, considering that it concerns *principally embedded* states, as it contrasts drastically with the situation in the NS sector, or in the bosonic string [29], where principally embedded states all appear with multiplicity one. This unexpected feature of the R sector in fact seems to be in accordance with the result of [1], where the superstring spectrum has been computed up to level 10. More specifically, using the *factorized* partition function (3.18)–(3.19) of [1], one can compute the content of each supermultiplet appearing at a given mass level. In the R sector, the first diagram with $\mathsf{K} = 2$, namely the "window" $\yng(2,2)_{1/2}$, appears at level $N = 5$ and in two *distinct* supermultiplets,[19]

$$\left(\yng(2) \oplus \yng(1,1,1) \oplus \yng(1)_{1/2}\right) \otimes \yng(2)_{1/2} \quad \supset \quad \yng(2,2)_{1/2} \quad \subset \quad \left(\yng(2) \oplus \yng(1,1,1) \oplus \yng(1)_{1/2}\right) \otimes \yng(1,1), \tag{4.63}$$

[19]Recall that any supermultiplet is obtained by tensoring the little group irreps of the fundamental supermultiplet with any other little group irrep, so that the latter can serve as a label for the supermultiplet in question.

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i.e. the first time this state appears, it is with multiplicity two. This is precisely in agreement with our result: the window in question is the lightest member of the trajectory (4.62) and is given by

$$\varepsilon^{A}_{\mu|\nu_1\nu_2,\nu}\,\alpha^{\mu}_{-1}\Big\{\mathsf{b}_0\,b^{[\nu_1}_{-1}b^{\nu_2]}_{-1}b^{\nu}_{-2}\pm\frac{1}{\sqrt{5}}\Big[2(\mathsf{b}_0-\mathsf{b}_{12})\alpha^{[\nu_1}_{-1}b^{\nu_2]}_{-1}b^{\nu}_{-2} + \Big(\mathsf{b}_0+\frac{3}{2}\mathsf{b}_{12}\Big)\,b^{[\nu_1}_{-1}b^{\nu_2]}_{-1}\alpha^{\nu}_{-2}\Big]+\mathsf{b}_{12}\,\alpha^{[\nu_1}_{-1}b^{\nu_2]}_{-1}\alpha^{\nu}_{-2}\Big\}\,|p;A;0\rangle_{\mathsf{R}}\ , \tag{4.64}$$

so that it indeed appears with multiplicity two, as encoded in the two free parameters b_0 and b_{12}. Note, however, that, to associate the two states with their respective supermultiplets, we would have to determine those two linear combinations of (4.64) that close the spacetime supersymmetry algebra on the two supermultiplets that appear in (4.63). Similarly, we find two other principally embedded diagrams with $\mathsf{K}=2$ at level $N=6$,

$$\Big(\ydiagram{2}\oplus\ydiagram{1,1,1}\oplus\ydiagram{1}_{1/2}\Big)\otimes\ydiagram{3}_{1/2}\quad\supset\quad\ydiagram{3,2}_{1/2}\quad\subset\quad\Big(\ydiagram{2}\oplus\ydiagram{1,1,1}\oplus\ydiagram{1}_{1/2}\Big)\otimes\ydiagram{3,1}\,, \tag{4.65}$$

and

$$\Big(\ydiagram{2}\oplus\ydiagram{1,1,1}\oplus\ydiagram{1}_{1/2}\Big)\otimes\ydiagram{1,1}_{1/2}\quad\supset\quad\ydiagram{2,2,1}_{1/2}\quad\subset\quad\Big(\ydiagram{2}\oplus\ydiagram{1,1,1}\oplus\ydiagram{1}_{1/2}\Big)\otimes\ydiagram{2,1,1}\,, \tag{4.66}$$

both of which also appear with multiplicity 2, in accordance with the previous discussion. At level $N=7$, one finds principally embedded states with spin

$$\ydiagram{4,2}_{1/2}\,,\qquad\ydiagram{3,3}_{1/2}\,,\qquad\ydiagram{3,2,1}_{1/2}\,,\qquad\ydiagram{2,2,2}_{1/2}\qquad\text{and}\qquad\ydiagram{2,2,1,1}_{1/2}\,, \tag{4.67}$$

and which all appear with multiplicity 2 again. This trend at relatively low levels seem to confirm the degeneracy (4.61), at least for $\mathsf{K}=2$. In order to check this for $\mathsf{K}=3$, one would need to push the computation of the superstring spectrum as outlined in [1] up to level $N=14$ at the minimum.[20]

Let us conclude by remarking that there exists a fairly simple solution for $P^{\pm}_{\mathbb{Y}}$ which is only linear in $Q^{n}{}_{n}$, namely

$$P^{\pm}_{\mathbb{Y}}\propto\mathbf{1}\pm\frac{1}{\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}}}\sum_{n=1}^{\mathsf{K}}n\,Q^{n}{}_{n}\,, \tag{4.68}$$

obtained by setting the free parameters $\mathsf{b}_{[2k]}=0$ for all $k>0$. Notice also that this solution exists for any diagram $\mathbb{Y}$. For example, we find the principally embedded totally symmetric tensor-spinor as the general solution (4.68), restricted to a one-row diagram of length s, to be

$$|\varepsilon\rangle_{\pm}:=\Big(1\pm\frac{1}{\sqrt{s}}\,Q^{1}{}_{1}\Big)\,\varepsilon^{A\pm}_{\mu(s)}\,b^{\mu}_{-1}\,\alpha^{\mu(s-1)}_{-1}|A\rangle=\varepsilon^{A\pm}_{\mu(s)}\,\Big(b^{\mu}_{-1}\pm\frac{1}{\sqrt{s}}\,\alpha^{\mu}_{-1}\Big)\,\alpha^{\mu(s-1)}_{-1}|A\rangle\ , \tag{4.69}$$

[20] The diagram with $\mathsf{K}=3$ and the lowest principal embedding level is $\ydiagram{3,3,3}_{1/2}$, which has $\mathsf{N}=14$.

where the polarization tensor-spinor ε is traceless and γ-traceless. Now the generator G_1 does annihilate such a state, while the generator G_0 produces the equation

$$(\not{p} \pm \sqrt{2s})\, \varepsilon^{A\pm}_{\mu(s)} = 0\,, \tag{4.70}$$

so after the GSO projection we obtain

$$\not{p}_{\alpha\dot{\beta}}\, \varepsilon^{\dot{\beta}\pm}_{\mu(s)} \pm \sqrt{2s}\, \varepsilon^{\pm}_{\mu(s)\,\alpha} = 0\,, \tag{4.71}$$

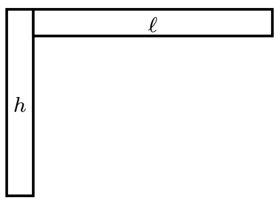
upon using the chiral representation of γ-matrices available in even dimensions, which matches known results for leading Regge states [12, 37].

4.3 Non-principal embedding

We can now extend the notion of "*depth*" w in both sectors, defined [29] in the bosonic string as the difference between the level at which a certain diagram appears from the lowest possible level at which the same diagram can be physically embedded, to the case of the superstring, via:

$$w = N - \mathsf{N}_{\mathbb{Y}} = N - \sum_{i=1}^{\mathsf{K}} i\,\ell_i + (i - \phi)\, h_i\,. \tag{4.72}$$

In addition, while a bosonic string "trajectory" can be defined as the set of Young diagrams with a fixed number of rows that appear at a fixed value of w, for the case of the superstring the former integer should be replaced by the length K of their diagonal. We thus define a superstring "*trajectory*" as the set of Young diagrams with a fixed pair (K, w). The analogue of the simplest, namely 1-row trajectory, in the bosonic string is hence the hook



(4.73)

in both sectors of the superstring, of the 2-row it is the "nested" hook of two rows and two columns and so on. Consequently, we can view 1-row and 1-column diagrams as a subclass of the hook trajectory. The principal embedding of this first trajectory, consisting of all hooks at $w = 0$, is highlighted in red in table 2 and its states take the form

$$\varepsilon^{\mathsf{NS}}_{\mu(\ell),\nu[h]}\, \alpha^{\mu(\ell)}_{-1}\, b^{\nu[h]}_{-1/2}\, |p;0\rangle_{\mathsf{NS}}\ \ , \tag{4.74}$$

$$\varepsilon^{\mathsf{R},A\pm}_{\mu(\ell),\nu[h]} \left(\alpha^{\mu(\ell)}_{-1}\, b^{\nu[h]}_{-1} \pm \frac{h}{\sqrt{\ell+h}} \alpha^{\mu(\ell)\nu}_{-1} b^{\nu[h-1]}_{-1}\right) |p;A;0\rangle_{\mathsf{R}}\ \ . \tag{4.75}$$

By applying the dictionary (2.47), (2.48) on (4.74), the vertex operators of the principally embedded NS hooks are very easily found to be

$$\mathbb{V}^{(-1)}_{w=0} = \varepsilon^{\mathsf{NS}}_{\mu(\ell),\nu[h]}\, e^{-\varphi}\, i\partial X^{\mu(\ell)}\, \psi^{\nu[h]}\, e^{ip\cdot X}\,, \tag{4.76}$$

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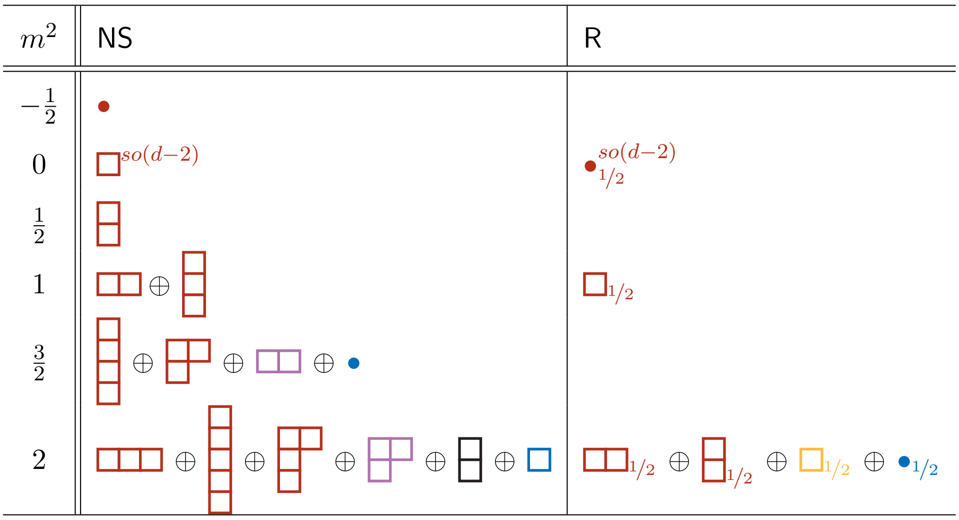

Table 2. Open superstring, physical content of the first few levels. The principal embedding is indicated in red, while the embeddings of depth $w = \frac{1}{2}, 1, \frac{3}{2}, 2$ in orchid, dandelion, black and blue respectively.

of which the leading Regge trajectory (2.64) is a subcase. Notice also that all principally embedded trajectories are built *before* the GSO projection, as is for example the hook (4.74), on which the GSO has to be applied; the leading Regge (2.64) is already given after the GSO, as every time a box is added to the row, the level changes by an integer. On the other hand, constructing the vertex operators of the principally embedded R hooks would necessitate the dictionary for n copies of b_{-1}, which, as explained around (2.51), requires knowledge of the OPE $\psi^\mu(z) S_A(w)$ to *arbitrary* order, even though only one type of b is present. For the subcase of the leading Regge trajectory, there appears only one copy of b_{-1}, so we can apply the dictionary (2.48), (2.51) which yields (2.65).

In the rest of this work, we will restrict ourselves to the Fock space description of R states. By construction, all trajectories of the principal embedding can be uplifted to both half-integer and integer depths in the NS and only to integer depths in the R sector. We will be considering specific cases of small values of nontrivial depth that are highlighted in table 2 and construct the Ansätze for the respective dressing functions, on which we will then apply the super-Virasoro constraints to find the physical trajectories, and once obtained, check whether they are spurious or not. Proceeding to deeper trajectories is straightforward by following the same algorithm.

NS sector at depth $w = \frac{1}{2}$. An operator of depth $w = \frac{1}{2}$ is generically of the form

$$f = \sum_{k\geq 1} \left(\frac{1}{k}\, \mathtt{u}_k \, {Q_k}^{k+1} + \mathtt{v}_k \, {Q^k}_k \right) , \tag{4.77}$$

where the factor $\frac{1}{k}$ before the first term has been added solely for later convenience and $\mathtt{u}_k$, $\mathtt{v}_k$ are a priori arbitrary coefficients. The anticommutator of f with $G_{1/2}$ is given by

$$\{G_{1/2}, f\} = \sum_{k\geq 1} (\mathtt{u}_{k-1} + \mathtt{v}_k)\, {M^k}_k + (\mathtt{u}_k + \mathtt{v}_k)\, {T^k}_k \,, \tag{4.78}$$

whereas the anticommutator $\{G_{3/2}, f\}$ is of depth -1, and hence necessarily proportional to the lowering operators of the orthosymplectic algebra. As a consequence, when evaluating these anticommutators on a principally embedded state with Young diagram $(\ell_1, \ldots \ell_{\mathsf{K}}|h_1, \ldots, h_{\mathsf{K}})$, one finds that the result vanishes provided that

$$\sum_{k=1}^{\mathsf{K}} \mathtt{u}_k\,(h_{k+1}+\ell_k) + \mathtt{v}_k\,(h_k+\ell_k) = 0\,. \tag{4.79}$$

Note that $h_k + \ell_k$ is the "hook length" at the kth box on the principal diagonal of the diagram.

Let us consider specific examples that correspond to explicit solutions of (4.79) for the first trajectory.

- For the leading Regge subcase, the solution is (up to an overall non-zero prefactor),

$$f = Q^1{}_1 - \frac{\ell+1}{\ell} Q_1{}^2\,, \tag{4.80}$$

  whose lightest member is the spin-2

$$F = \varepsilon_{\mu\nu}(p)\left(\frac{1}{2}\,i\partial X^\mu\, i\partial X^\nu + \psi^\mu \partial\psi^\nu\right). \tag{4.81}$$

  This is the lightest clone of (2.54). Notice that this trajectory does *not* survive the GSO projection.

- For hook-type diagrams, the solution is (up to an overall non-zero prefactor),

$$f = -\frac{\ell}{\ell+h}\, Q^1{}_1 + Q_1{}^2\,. \tag{4.82}$$

  For the smallest diagram of this type, , this yields the polynomial

$$F = \varepsilon_{\mu|\nu\rho}(p)\left(-\frac{2}{3}\,i\partial X^\mu\, i\partial X^\nu \psi^\rho + \partial\psi^\mu \psi^\nu \psi^\rho\right) \tag{4.83}$$

  of the vertex operator in the (-1) superghost picture of the lightest clone of the hook (2.56) in the principal embedding, highlighted in orchid in table 2. Using (4.15), the polynomial of the vertex operator in the (0) picture is found to be given by

$$\widetilde{F} = \varepsilon_{\mu|\nu\rho}(p)\left(\frac{4}{3}\,\partial\psi^\mu\, i\partial X^\nu \psi^\rho - \frac{2}{3}\,i\partial X^\mu \partial\psi^\nu \psi^\rho + i\partial^2 X^\mu \psi^\nu \psi^\rho\right). \tag{4.84}$$

- For p-forms, namely when $\ell = 0$ and h is arbitrary, there is no non-trivial solution for the $w = \frac{1}{2}$ dressing function, meaning that one cannot uplift principally embedded p-forms at level $\frac{p}{2}$ to physical p-forms at level $\frac{p+1}{2}$, which is in accordance with table 2.

More generally, since the most general operator of depth $w = \frac{1}{2}$ is characterized by $2\mathsf{K}$ parameters, subject to a single constraint (4.79), the space of depth-$\frac{1}{2}$ dressing functions is of dimension $2\mathsf{K} - 1$. The general solution can be simply obtained by solving one of the $2\mathsf{K}$ parameters $\mathtt{u}_k$ and $\mathtt{v}_k$ in the general Ansatz (4.77) in terms of the others in (4.79). Another way to proceed is to notice that *only two* non-zero terms in the Ansatz (4.77) are necessary

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in order to produce a solution. We can therefore find a basis of this space of solutions made out of simple linear combinations of two operators, such as for instance,

$$f_k = \begin{cases} \frac{1}{k}\, Q_k{}^{k+1} - \frac{\ell_k + h_{k+1}}{\ell_k + h_k}\, Q^k{}_k\,, & 1 \le k \le \mathsf{K}\,, \\ Q^{k+1}{}_{k+1} - \frac{1}{k}\, \frac{\ell_{k+1} + h_{k+1}}{\ell_k + h_{k+1}}\, Q_k{}^{k+1}\,, & \mathsf{K}+1 \le k \le 2\mathsf{K}-1\,. \end{cases} \tag{4.85}$$

In particular, the dressing functions presented above for the leading Regge trajectory and the hook-type diagrams are multiples of f_1.

NS sector at depth $w = 1$. To derive dressing functions of depth $w = 1$, we can repeat the above steps, starting with spelling out the most general form of such a dressing function,

$$\begin{aligned} f &= \sum_{k\ge1} \frac{1}{k}\, (\mathtt{u}_k\, T^{k+1}{}_k + \mathtt{v}_k\, M^{k+1}{}_k) \\ &\quad + \frac{1}{2} \sum_{i,j\ge1} \left(\mathtt{u}_{ij}\, Q^i{}_i\, Q^j{}_j + \frac{1}{ij}\, \mathtt{v}_{ij}\, Q_i{}^{i+1}\, Q_j{}^{j+1} + \frac{1}{j}\, \mathtt{t}_{ij}\, [Q^i{}_i, Q_j{}^{j+1}] \right), \end{aligned} \tag{4.86}$$

where $\mathtt{u}_{ij}$ and $\mathtt{v}_{ij}$ are antisymmetric, and computing its anti/commutator with $G_{1/2}$ and $G_{3/2}$. The former reads

$$\begin{aligned} [G_{1/2}, f] = \sum_{k\ge1} \Bigg(&\left[\mathtt{v}_k - \mathtt{u}_{k-1} + \frac{1}{2}\, (\mathtt{t}_{k\,k-1} - \mathtt{t}_{kk}) \right] Q^k{}_k + \frac{1}{k} \left[\mathtt{u}_k - \mathtt{v}_k + \frac{1}{2} (\mathtt{t}_{k+1\,k} - \mathtt{t}_{kk}) \right] Q_k{}^{k+1} \Bigg) \\ - \sum_{k,l\ge1} \Bigg(&Q^k{}_k \left[\mathtt{u}_{kl}\, (T^l{}_l + M^l{}_l) + \mathtt{t}_{kl}\, (T^l{}_l + M^{l+1}{}_{l+1}) \right] \\ &+ Q_k{}^{k+1}\, \frac{1}{k} \left[\mathtt{v}_{kl}\, (T^l{}_l + M^{l+1}{}_{l+1}) - \mathtt{t}_{lk}\, (T^l{}_l + M^l{}_l) \right] \Bigg) \end{aligned} \tag{4.87}$$

and hence leads to the conditions

$$0 = \mathtt{v}_k - \mathtt{u}_{k-1} + \frac{1}{2}\, (\mathtt{t}_{k\,k-1} - \mathtt{t}_{kk}) - \sum_{l=1}^{\mathsf{K}} \Big(\mathtt{u}_{kl} (\ell_l + h_l) + \mathtt{t}_{kl} (\ell_l + h_{l+1}) \Big)\,, \tag{4.88a}$$

$$0 = \mathtt{u}_k - \mathtt{v}_k + \frac{1}{2}\, (\mathtt{t}_{k+1\,k} - \mathtt{t}_{kk}) - \sum_{l=1}^{\mathsf{K}} \Big(\mathtt{v}_{kl} (\ell_l + h_{l+1}) - \mathtt{t}_{lk}\, (\ell_l + h_l) \Big)\,, \tag{4.88b}$$

when evaluated on a principally embedded state with diagram $(\ell_1, \ell_2, \ldots | h_1, h_2, \ldots)$. As for the depth $w = \frac{1}{2}$ case, the anti/commutator of f with $G_{3/2}$ vanishes identically on a principally embedded state and hence does not yield any new condition on the dressing function. Although we will not discuss the general solutions of these conditions here, let us consider a few examples.

- For diagrams of one row, the solution is unique and given by

$$f = \left(\ell - \frac{1}{2} \right) M^2{}_1 + \frac{1}{2}\, [Q^1{}_1, Q_1{}^2]\,, \tag{4.89}$$

  up to a normalization coefficient. However, it is easy to see that f vanishes identically upon acting on the one-row subclass of (4.74). The leading Regge cannot namely be uplifted to $w = 1$ and this is in accordance with the fact that we see no states of spin $s = 0, 1, 2$ at $w = 1$ in table 2.

- For hook-shaped diagrams $(\ell|h)$ in the (modified) Frobenius notation, one also finds a unique solution,

$$f = -(\ell-1)\left((h-1)\,T^2{}_1 - \left(\ell+\frac{1}{2}\right)\,M^2{}_1 - \frac{1}{2}[Q^1{}_1, Q_1{}^2]\right)\,, \tag{4.90}$$

up to normalization. In the simple case of the smallest hook diagram $\yng(2,1)$, this yields a vertex operator whose polynomial is given by

$$F = \varepsilon_{\mu|\nu\rho}(p)\left(i\partial X^\mu\,\partial\psi^{[\nu}\,\psi^{\rho]} - \frac{1}{2}\,i\partial^2 X^\mu\,\psi^\nu\psi^\rho - \partial\psi^\mu\,i\partial X^{[\nu}\,\psi^{\rho]}\right)\,, \tag{4.91}$$

which account for the hook-type physical state that appears (with multiplicity 1) at level $N=3$. The latter is however absent after the GSO projection.

- For p-forms of height $h=p$, we find that $f=0$: like the leading Regge, p-forms cannot be uplifted to $w=1$ and this is in accordance with table 2.

R sector at depth $w=1$. In the Ramond sector, a dressing function of depth $w=1$ is of the form

$$f = \sum_{k\geq 1}\left(T^{k+1}{}_k\,P^1_k + M^{k+1}{}_k\,P^2_k + Q^{k+1}{}_k\,P^3_k + Q_k{}^{k+1}P^4_k\right) + M^1P^5 + Q^1P^6\,, \tag{4.92}$$

where the polynomials P are a priori arbitrary polynomials of $Q^n{}_n$. For the first trajectory, namely the hook of one row and one column, the dressing function simplifies to[21]

$$\begin{aligned} f = {}& T^2{}_1(\mathtt{u}_1+\mathtt{u}_2\,Q^1{}_1) + M^2{}_1(\mathtt{v}_1+\mathtt{v}_2\,Q^1{}_1) + Q^2{}_1(\mathtt{s}_1+\mathtt{s}_2\,Q^1{}_1) + Q_1{}^2\,(\mathtt{t}_1+\mathtt{t}_2\,Q^1{}_1) \\ & + M^1(\mathtt{a}_1+\mathtt{a}_2\,Q^1{}_1) + Q^1(\mathtt{c}_1+\mathtt{c}_2\,Q^1{}_1)\,. \end{aligned} \tag{4.93}$$

The constraint (4.34a), $G_0 f\,|\mathbb{Y}\rangle_{1/2}=0$, enforces then the conditions[22]

$$\begin{aligned} &\pm\sqrt{\ell+h+1}\,\mathtt{u}_2 = \mathtt{u}_1+\mathtt{s}_2+2\mathtt{t}_2\,, && \pm\sqrt{\ell+h+1}\,\mathtt{v}_2 = \mathtt{t}_2+\mathtt{v}_1+\mathtt{s}_2\,, \\ &\pm\sqrt{\ell+h+1}\,\mathtt{s}_2 = -\mathtt{s}_1-\mathtt{u}_2+2\mathtt{v}_2\,, && \pm\sqrt{\ell+h+1}\,\mathtt{t}_2 = \mathtt{u}_2-\mathtt{t}_1-\mathtt{v}_2\,, \\ &\pm\sqrt{\ell+h+1}\,\mathtt{a}_2 = \mathtt{a}_1-\mathtt{c}_2\,, && \pm\sqrt{\ell+h+1}\,\mathtt{c}_2 = -\mathtt{a}_2-\mathtt{c}_1\,, \end{aligned} \tag{4.94}$$

while the constraint (4.34b) $G_1 f\,|\mathbb{Y}\rangle_{1/2}=0$ imposes that

$$\begin{aligned} \left(\ell+\frac{d}{2}\right)\mathtt{c}_1 + \ell\mathtt{t}_1 + h\mathtt{s}_1 + (\ell+h)(\mathtt{u}_2+\mathtt{a}_2) &= 0\,, \\ \mathtt{v}_1 + \left(\ell+\frac{d}{2}+1\right)\mathtt{c}_2 + (\ell+1)\mathtt{t}_2 + (h-1)\mathtt{s}_2 &= 0\,. \end{aligned} \tag{4.95}$$

[21] Notice that the dressing function f should be applied on the lowest weight state of the R sector, namely on $|\mathbb{Y}\rangle_{1/2}$. However, this is equivalent to having f act on *physical* principally embedded states, as the latter take the form $P\,|\mathbb{Y}\rangle_{1/2} = \left(1\pm\frac{1}{\sqrt{\ell+h}}Q^1{}_1\right)|\mathbb{Y}\rangle_{1/2}$ for example for the first trajectory, and so this option amounts to mere linear redefinitions of the coefficients of f acting on $|\mathbb{Y}\rangle_{1/2}$. On the other hand, as previously highlighted, the lowest weight states of the NS sector *are* physical, so there is no such ambiguity in the NS sector.

[22] More precisely, each of the terms in the Ansatz (4.93) induces one condition, hence we have twelve conditions in total, half of which are consequences of the other half.

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The solutions can be written as

$$
\begin{aligned}
&\mathtt{s}_1 = (h+\ell)(\mathtt{u}_2-2\mathtt{v}_2)\mp\sqrt{h+\ell+1}(\mathtt{u}_1-2\mathtt{v}_1)\,, \quad \mathtt{s}_2=\mp\sqrt{h+\ell+1}(\mathtt{u}_2-2\mathtt{v}_2)+\mathtt{u}_1-2\mathtt{v}_1\,,\\
&\mathtt{t}_1=\pm\sqrt{h+\ell+1}(\mathtt{u}_1-\mathtt{v}_1)-(h+\ell)(\mathtt{u}_2-\mathtt{v}_2)\,, \quad \mathtt{t}_2=\pm\sqrt{h+\ell+1}(\mathtt{u}_2-\mathtt{v}_2)-\mathtt{u}_1+\mathtt{v}_1\,,\\
&\mathtt{a}_1=\frac{1}{d-2h}\Big\{-2(3h+\ell)\,\mathtt{u}_1+6(2h+\ell)\,\mathtt{v}_1\pm6(h+\ell)\sqrt{h+\ell+1}(\mathtt{u}_2-\mathtt{v}_2)\Big\}\,,\\
&\mathtt{a}_2=\frac{2}{(d-2h)(2+d+2\ell)}\Big\{\pm2\sqrt{h+\ell+1}\Big[-(d+h+\ell)\mathtt{u}_1+(2d+2h+3\ell)\mathtt{v}_1\Big]+2(dh+2d\ell\\
&\qquad+d+h^2+2h\ell+h+3\ell^2+3\ell)\,\mathtt{u}_2-(dh+4d\ell+3d+4h^2+4h\ell+6\ell^2+6\ell)\,\mathtt{v}_2\Big\}\,,\\
&\mathtt{c}_1=\frac{2}{(d-2h)(2+d+2\ell)}\Big\{\pm\sqrt{h+\ell+1}\Big[-(-dh+d\ell+2h^2-2h\ell-6h-2\ell)\,\mathtt{u}_1\\
&\qquad+(-2dh+d\ell+4h^2-2h\ell-12h-6\ell)\,\mathtt{v}_1\Big]+(h+\ell)\Big[(-dh+d\ell-d+2h^2\\
&\qquad-2h\ell-4h-6\ell-6)\,\mathtt{u}_2-(-2dh+d\ell+4h^2-2h\ell-6h-6\ell-6)\,\mathtt{v}_2\Big]\Big\}\,,\\
&\mathtt{c}_2=\frac{2}{2+d+2\ell}\Big\{\Big[(2-h+\ell)\,\mathtt{u}_1+(-4+2h-\ell)\,\mathtt{v}_1\Big]\\
&\qquad\pm\sqrt{h+\ell+1}\Big[(h-\ell-2)\mathtt{u}_2-(2h-\ell-3)\,\mathtt{v}_2\Big]\Big\}\,,
\end{aligned}
\tag{4.96}
$$

where the d-dependence is due to the d-contribution to the anticommutator $\{Q^m, Q_n\}$ given in (4.3c).

To consider an example, let as look at the lightest member of this trajectory, namely the vector-spinor highlighted in dandelion in table 2, for which $\ell = 0\,, h = 1$. Its Ansatz simplifies further to

$$
f = \mathtt{u}_2\, T^2{}_1 Q^1{}_1 + \mathtt{v}_1\, M^2{}_1 + \mathtt{s}_1\, Q^2{}_1 + \mathtt{t}_2\, Q_1{}^2 Q^1{}_1 + M^1(\mathtt{a}_1 + \mathtt{a}_2\, Q^1{}_1) + Q^1(\mathtt{c}_1 + \mathtt{c}_2\, Q^1{}_1)\,. \tag{4.97}
$$

The first four conditions of (4.94) now have to be replaced by the system

$$
\pm\sqrt{2}\,\mathtt{u}_2 = 2\mathtt{t}_2\,, \qquad \pm\sqrt{2}\,\mathtt{v}_1 = \mathtt{s}_1\,, \tag{4.98}
$$

as the choice of independent constraints in (4.94) involves constraints that are not present for the vector-spinor, since four terms of (4.93) now vanish trivially.[23] In the remaining two conditions of (4.94), as well as the conditions (4.95), it suffices to set $\mathtt{u}_1 = \mathtt{v}_2 = \mathtt{s}_2 = \mathtt{t}_1 = 0$. Solving the complete system yields the physical vector-spinor

$$
\begin{aligned}
\varepsilon^{\mathsf{R},A\pm}_{\mu}\Big[&\pm\frac{1}{\sqrt{2}}\,b^{\mu}_{-2}+\frac{1}{2}\alpha^{\mu}_{-2}\pm\frac{3}{d-2}\gamma\cdot b_{-1}b^{\mu}_{-1}\mp\frac{1}{d+2}\gamma\cdot\alpha_{-1}\alpha^{\mu}_{-1}\\
&+\frac{\sqrt{2}}{(d^2-4)}\left[2(d+1)\,\gamma\cdot b_{-1}\,\alpha^{\mu}_{-1}-(d+4)\,\gamma\cdot\alpha_{-1}\,b^{\mu}_{-1}\right]\Big]\,|p;A;0\rangle_{\mathsf{R}}\ ,
\end{aligned}
\tag{4.99}
$$

up to an overall coefficient. Let us sketch how to find the polynomial of the respective vertex operator in the $(-1/2)$ picture after the GSO projection. First, for the terms of (4.99) which

[23] More precisely, the first four terms in (4.97) induce four conditions, half of which are consequences of the other half, which can be written as (4.98).

contain one or no oscillator b_{-1}, the dictionary (2.47) and (2.51) yields[24]

$$\begin{aligned}
\alpha^{\mu}_{-2}\,|\alpha;0\rangle_{\mathsf{R}} \quad &\rightarrow \quad \partial^2 X^{\mu} S_{\alpha}\,, \\
\gamma\cdot\alpha_{-1}\alpha^{\mu}_{-1}\,|\dot\alpha;0\rangle_{\mathsf{R}} \quad &\rightarrow \quad \partial X^{\mu}(\partial\not{X})_{\beta\dot\alpha}S^{\dot\alpha}\,, \\
\overline{\gamma}\cdot\alpha_{-1}\,b^{\mu}_{-1}\,|\alpha;0\rangle_{\mathsf{R}} \quad &\rightarrow \quad \partial X^{\nu}\,\overline{\gamma}_{\nu}^{\dot\gamma\alpha}(\psi^{\mu}\not{\psi})_{\alpha\dot\beta}S^{\dot\beta}\,, \quad \partial X^{\nu}\,\overline{\gamma}_{\nu}^{\dot\gamma\alpha}(\gamma^{\mu}\not{\psi}\not{\psi})_{\alpha\dot\beta}S^{\dot\beta}\,, \\
\overline{\gamma}\cdot b_{-1}\,\alpha^{\mu}_{-1}\,|\alpha;0\rangle_{\mathsf{R}} \quad &\rightarrow \quad \partial X^{\mu}\,(\not{\psi}\not{\psi})^{\dot\gamma}{}_{\dot\beta}\,S^{\dot\beta}\,,
\end{aligned} \tag{4.100}$$

so, up to the spin-field, terms of the schematic form $\partial^2 X^{\mu}$, $\partial X^{\mu}\partial X^{\nu}$ and $\partial X^{\mu}\psi^{\nu}\psi^{\lambda}$. In addition, using the dictionary (2.50) for $r=1$, as well as the covariant form, in terms of S^{μ}_{α} and $\partial S^{\dot\beta}$, of the OPE of $\psi(z)S_A(w)$ [16], finding the terms arising from $b^{\mu}_{-2}\,|p;\dot\alpha;0\rangle_{\mathsf{R}}$ and $\gamma\cdot b_{-1}b^{\mu}_{-1}\,|p;\alpha;0\rangle_{\mathsf{R}}$ amounts to calculating

$$\oint\frac{\mathrm{d}z}{2\pi i}\frac{1}{z^{5/2}}\psi^{\mu}(z)S^{\dot\alpha}(0)\,,\quad \oint\frac{\mathrm{d}z}{2\pi i}\frac{1}{z^{3/2}}\psi^{\mu}(z)S_{\alpha}(0)\qquad\text{and}\qquad \oint\frac{\mathrm{d}z}{2\pi i}\frac{1}{z^{3/2}}\psi^{\mu}(z)\partial S_{\alpha}(0)\,. \tag{4.101}$$

Up to subleading order, the OPEs $\psi^{\mu}(z)S^{\nu}_{\alpha}(0)$ and $\psi^{\mu}(z)\partial S_{\alpha}(0)$ can be found in [16]; subsubleading orders, however, can also obviously contribute and they would have to be determined using the bosonization (4.27) of the fermions ψ and spin-fields S. Schematically, it is easy to see the evaluation of the integrals (4.101) will produce terms involving S^{μ}_{α}, $\partial S^{\dot\beta}$ and their derivatives, which can be rewritten in the form $\psi^{\mu}\psi^{\nu}\psi^{\lambda}\psi^{\kappa}$ and $\psi^{\mu}\partial\psi^{\nu}$, suitably contracted to produce a vector-spinor. Picture-changing is then straightforward by means of (2.17).

For a state $|\mathbb{Y}\rangle_{1/2}$ with an arbitrary number of boxes $\mathsf{K}\geq 1$ on the diagonal of the Young diagram $\mathbb{Y}$, the condition

$$G_0\,f\,|\mathbb{Y}\rangle_{1/2} = 0\,, \tag{4.102}$$

with f the general $w=1$ Ansatz (4.92), yields

$$\begin{aligned}
\mu_{\pm}P^1_k &= -P^3_k-(k+1)\,P^4_k\,, & \mu_{\mp}P^3_k &= -k\,P^1_k+(k+1)\,P^2_k\,, \\
\mu_{\pm}P^2_k &= -P^3_k-k\,P^4_k\,, & \mu_{\mp}P^4_k &= P^1_k-P^2_k\,, \\
\mu_{\pm}P^5 &= P^6\,, & \mu_{\mp}P^6 &= -P^5\,,
\end{aligned} \tag{4.103}$$

with $\mu_{\pm}:=\boldsymbol{m}\mp\sqrt{\mathsf{N}^{\mathsf{R}}_{\mathbb{Y}}+1}$. In fact, only half of these equations are independent: a simple way to see it is to notice that three equations on the right column are obtained as consequences of those on the right, upon using $\mu_+\,\mu_-P\,|\mathbb{Y}\rangle_{1/2}=-P\,|\mathbb{Y}\rangle_{1/2}$ for P any of the polynomials in the Ansatz (4.92). The second condition,

$$G_1\,f\,|\mathbb{Y}\rangle_{1/2} = 0\,, \tag{4.104}$$

amounts to

$$\begin{aligned}
\sum_{k\geq 1}\Big(&Q^{k}{}_{k}\,[-(k-1)\,P^1_{k-1}+k\,P^2_k]+Q_{k}{}^{k}\,[P^1_k-P^2_{k-1}+\delta_{k,1}\,P^5] \\
&+T^{k}{}_{k}\,[P^3_{k-1}+k\,P^4_k+\delta_{k,1}\,P^6]+\frac{d}{2}\,P^6+M^{k}{}_{k}\,[P^3_k+(k-1)\,P^4_{k-1}]\Big)\,|\mathbb{Y}\rangle_{1/2}=0\,,
\end{aligned} \tag{4.105}$$

[24] We use the symbol γ for all gamma matrices, irrespectively of whether they carry Dirac or Weyl indices.

where the dimension-dependent term $\frac{d}{2}\,P^6$ comes from the anticommutator between Q_1 and Q^1, which receives a correction as mentioned previously. Since this equation is a polynomial in $Q^k{}_k$, we can use the dictionary (4.53a) to rewrite it in terms of the anticommuting variables θ_k introduced earlier, so as to simplify computations. Since we mainly aim at outlining how to produce physical subleading trajectories in our formalism, we will not derive the general solution of this equation. Instead, let us simply stress that using the representation of the above equation as a polynomial in θ_k, one obtains a linear system consisting of the conditions that the coefficient of each monomial vanishes, and this system can be solved as shown in the case of the vector-spinor discussed previously.

5 Conclusions

String spectra comprise infinitely many *physical* states, the vast majority of which are massive tensors of mixed symmetry. Traditional methodologies of constructing physical states, such as the light cone quantization or the DDF formulation, come with limitations such as lack of manifest covariance or the output not immediately given in terms of irreducible representations of the massive little group. On the other hand, the partition function [1, 2], offers access to the characters of the representations that appear, in principle, at any level along with their multiplicities, while the exact form of the respective states remains obscure. In [29], it was observed that the Virasoro constraints in bosonic string theory, which select physical states from functions of bosonic oscillators, are given in terms of linear combinations of lowering operators of a bigger (infinite-dimensional) symplectic algebra. The fact that the latter commutes with the spacetime Lorentz algebra, with which it forms a Howe dual pair, was further employed to develop a covariant and efficient technology of excavating entire physical trajectories deeper in the spectrum by dressing simpler trajectories with suitable combinations of the raising operators of the symplectic algebra in question.

In this work, we address the same question for superstring theory, namely, how do excited superstring states look like and how can we construct physical states deep inside the spectrum in a systematic manner? More specifically, we explore the realization of Howe duality in the case of the superstring and develop a covariant technology of excavating physical open superstring trajectories. With respect to the bosonic string, a first difference is that now the Fock space of states is constructed by means of pairs of bosonic *and* fermionic oscillators. While the former are integer modes in both sectors of the superstring, the latter are half-integer and integer in the NS and R sectors respectively. Now, the superstring Virasoro constraints are expressed in terms of lowering operators of the orthosymplectic algebra, whose generators are formed by pairs of bosonic and fermionic oscillators and which commutes with the spacetime Lorentz algebra. In particular, the algebra $\mathfrak{osp}(\bullet|\bullet)$ is realized in the NS and R sectors as a limit of the algebras $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$ and $\mathfrak{osp}(2\mathsf{M}+1|2\mathsf{N},\mathbb{R})$ respectively, when $\mathsf{M},\mathsf{N}\longrightarrow\infty$; the bullets stand for the infinity of the pairs of oscillators that are necessary to describe the full spectrum, while they correspond to specific integers, enumerating the subsets of oscillators that are excited, once a certain trajectory is chosen. After working out the form of the physical trajectories whose member-states appear at the lowest possible mass level that the respective Young diagrams can find themselves at, we can dress the former with suitable combinations of the raising operators of the orthosymplectic algebra. With this dressing, we can construct entire physical trajectories deeper in the spectrum in a completely covariant way in the critical dimension of the superstring.[25] We illustrate the algorithm for small values of the "depth", which parametrizes how "deep" inside the spectrum a certain diagram or set of diagrams find themselves, as in [29] for the bosonic string.

Interestingly, while in both the NS sector and the bosonic string the lowest weight states of the orthosymplectic algebra are *by construction* physical, this is no longer true for the R sector: there, a linear combination of the lowest weight states together with other states at the same level is necessary to form physical states in the principal embedding. Crucially, this implies a nontrivial multiplicity for states of depth zero in the R sector, unlike NS states, which matches the prediction of the partition function [1], at least at low levels. Intuitively, we can motivate the difference in multiplicities through spacetime supersymmetry: to close the supersymmetry algebra on massive supermultiplets, fermions of a given spin appear in several disparate copies in the same multiplet, as can be illustrated in the toy example of the massive spin-2 multiplet of $\mathcal{N}=1$, $d=4$, that takes the form $(2, 3/2, 3/2, 1)$. In addition, the technology is developed in the Fock space of superstring states, which are then suitable functions of the creation modes of bosonic and fermionic oscillators, with no gauge-fixing, light cone or otherwise, necessary. In the NS sector, the expressions for physical trajectories can easily be translated into their respective vertex operators by means of the simple dictionary between oscillators and the primary fields ∂X and ψ and their descendants, which we illustrate by specific examples, including the two superghost pictures, (-1) and (0), that are necessary and sufficient for the calculation of scattering amplitudes of NS states.[26] In the R sector, to the best of our knowledge, the dictionary is not currently available in a closed form for any fermionic creation mode, because of the fact that the CFT of the field ψ and the spin-field S is interacting. Consequently, in order to find the vertex operator creating a certain physical trajectory, the dictionary for every mode that is excited in order to construct the trajectory has to be computed first. This implies a necessity for the knowledge of the corresponding subleading terms in the OPE of, for example, $\psi(z)S_A(w)$, which have to be computed first; we sketch the procedure for the second-lightest vector-spinor in the main text. Let us also note that the technology can be applied to construct closed superstring trajectories in a straightforward manner: one has to employ two copies of the technology, one to the left– and one to the right-moving sector of the closed string and then apply the level-matching condition.

In view of the method for the construction of entire physical trajectories deep in the superstring spectrum presented in this work, several future directions can be discussed, as for example its formulation for compactifications. In addition, the decay of *coherent* or of *superpositions of highly excited* bosonic string states has been under thorough investigation, with evidence for chaotic traits in its dependence on the angle between the emitted tachyons and initial states having been presented in bosonic strings [70–78].[27] The states built with our technology correspond by construction to irreducible Lorentz representations and the question of whether their decay and, more generally, string scattering amplitudes can exhibit

[25] We do not explicitly set $d = 10$, but it is implicit that no compactification has been effectuated throughout this work.

[26] We also produce a formula for computing the (0) picture of any NS vertex operator, once the latter is given in the (-1) picture.

[27] See [8, 9, 79] and [80] for the construction of bosonic and of superstring *coherent* states respectively.

chaotic characteristics remains open. Moreover, it would be interesting to explore whether trajectories or sets of states deeper in the superstring spectrum can reproduce, for example, amplitudes of Kerr black holes, given that it was recently shown that leading Regge NS states, in the classical limit, do not [81], while there has been an aggregation of results indicating that higher-spin *fields* can model aspects of black hole scattering [82–88]. Lastly, it would be fascinating to use massive higher-spin superstring states deep inside the spectrum to probe black-hole microstates, with past investigations having been focused on the massless level [89, 90]. After all, the (super)massive states constructed with our technology are excitations of (here spacetime-filling) D-branes and, as such, their scattering amplitudes with closed string states can, in certain configurations, be thought of as describing the absorption or emission of Hawking radiation from black holes, while for suitable compactifications to four spacetime dimensions they can be neglected [91, 92]. Ultimately, a deeper understanding of the string spectrum may perhaps shed light on string theory per se.

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A Light cone construction of the lightest fermions

In this appendix, to illustrate how light cone oscillators combine into physical Wigner irreps on a level-by-level fashion in the superstring, we give the construction of the part of the first three levels with $(-)^F = +1$ "generalized" chirality in the R sector in table 3, with the "generalized" chirality operator given by

$$(-1)^F = 16 b_0^2 \dots b_0^9 \, (-1)^{\sum_{n>0} b^i_{-n} b^i_n} \,, \tag{A.1}$$

see for example [43]. Let us further recall that, after the GSO projection, the physical Wigner irreps can also be found by taking the tensor product of the lightest massive supermultiplet, namely the spin-2 multiplet at level 1, with another $\mathfrak{so}(9)$ representation, see for example [1]. The simplest case is to take the product

$$\left(\square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \oplus \square_{1/2} \right) \otimes \square \,, \tag{A.2}$$

which produces precisely the physical state content at level 2.

N	$\mathfrak{gl}(d-2)$ tensors	$\mathfrak{so}(d-2)$ irreps	Wigner
0	$\lvert\alpha\rangle$ $\mathbf{8}_s$	$\mathbf{8}_s$	$\mathbf{8}_s$
1	$\alpha^i_{-1}\lvert\alpha\rangle$ $\quad b^i_{-1}\lvert\dot{\alpha}\rangle$ $\mathbf{8}_v\otimes\mathbf{8}_s$ $\quad \mathbf{8}_v\otimes\mathbf{8}_c$	$\mathbf{8}_c\oplus\mathbf{8}_s\oplus\mathbf{56}_c\oplus\mathbf{56}_s$	$\mathbf{128}$
2	$\alpha^i_{-1}\alpha^j_{-1}\lvert\alpha\rangle$ $\quad \alpha^i_{-2}\lvert\alpha\rangle$ $\mathbf{36}_v\otimes\mathbf{8}_s$ $\quad \mathbf{8}_v\otimes\mathbf{8}_s$	$(\mathbf{8}_s\oplus\mathbf{56}_s\oplus\mathbf{224}_{vs})\oplus(\mathbf{8}_c\oplus\mathbf{56}_c)$	$\mathbf{576}\oplus\mathbf{432}$
	$b^i_{-2}\lvert\dot{\alpha}\rangle$ $\quad b^i_{-1}\alpha^j_{-1}\lvert\dot{\alpha}\rangle$ $\quad b^i_{-1}b^j_{-1}\lvert\alpha\rangle$ $\mathbf{8}_v\otimes\mathbf{8}_c$ $\quad \mathbf{8}_v\otimes\mathbf{8}_v\otimes\mathbf{8}_s$ $\quad \mathbf{28}_v\otimes\mathbf{8}_s$	$\oplus(\mathbf{8}_s\oplus\mathbf{56}_s)\oplus(\mathbf{8}_c\oplus\mathbf{8}_c\oplus\mathbf{56}_c\oplus\mathbf{56}_c$ $\oplus\mathbf{160}_c\oplus\mathbf{224}_{vc})\oplus(\mathbf{8}_s\oplus\mathbf{56}_s\oplus\mathbf{160}_s)$	$\oplus\mathbf{128}\oplus\mathbf{16}$

Table 3. Decomposition and recombination up to $N=2$, $(-)^F=1$ section of the R sector.

B Tensors and Young diagrams

In this paper, we have to deal with tensors of arbitrary ranks and symmetries. To do so, it is convenient to denote indices that are all symmetrized, respectively antisymmetrized, by the same letter, with the number of indices in round, respectively square, brackets. More explicitly,

$$S_{\mu(m)}=\frac{1}{m!}\sum_{\sigma\in\mathcal{S}_m}S_{\mu_{\sigma_1}\dots\mu_{\sigma_m}}\,,\qquad A_{\nu[n]}=\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(-1)^\sigma A_{\nu_{\sigma_1}\dots\nu_{\sigma_n}}\,,\tag{B.1}$$

where $(-1)^\sigma$ is the signature of the permutation σ. We use a vertical bar to separate different groups of symmetrized or antisymmetrized indices, groups which do not have any symmetry properties between one another: for instance, the tensor $T_{\mu_1(m_1)|\mu_2(m_2)|\dots|\nu_1[n_1]|\nu_2[n_2]|\dots}$ has several groups of totally symmetric indices, denoted by μ_i and containing m_i indices, and several groups of totally antisymmetric indices, denoted by ν_i and containing n_i indices. There is, however, no symmetry assumed between the indices with different names, i.e. in different groups.

Such tensors are *not* irreducible under the action of the general linear algebra $\mathfrak{gl}_d$, as can be easily seen in the simplest case of a rank-2 tensor $T_{\mu|\nu}$: both its symmetric and antisymmetric parts, $T_{(\mu|\nu)}$ and $T_{[\mu|\nu]}$, are preserved independently by the action of $\mathfrak{gl}_d$, and one can check that there exists no other invariant subspace. For tensors of rank $r\geq 3$, the situation becomes slightly more subtle, in that their decomposition under $\mathfrak{gl}_d$ irreps contains more general projections than the totally symmetric or totally antisymmetric ones. For instance, for a rank-3 tensor of the form $T_{\mu(2)|\nu}$, i.e. symmetric in its first two indices, one finds on top of its symmetric part,

$$S_{\mu\nu\rho}:=T_{(\mu\nu|\rho)}\equiv\frac{1}{3}\big(T_{\mu\nu|\rho}+2\,T_{\rho(\mu|\nu)}\big)\,,\tag{B.2}$$

the projection

$$H_{\mu\nu,\rho}:=\frac{2}{3}\,\big(T_{\mu\nu|\rho}-T_{\rho(\mu|\nu)}\big)\,,\tag{B.3}$$

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which is again symmetric in the first two indices, but now obeys the over-symmetrization condition

$$H_{(\mu\nu,\rho)} = 0\,. \tag{B.4}$$

This makes it clear that this projection is orthogonal to the symmetric one, and one readily sees that the original tensor is given by $T_{\mu\nu|\rho} = S_{\mu\nu\rho} + H_{\mu\nu,\rho}$. One can also check that the projection H does not contain any invariant subspace so that the previous decomposition does correspond to that of the tensor T into irreps of $\mathfrak{gl}_d$. Similarly, one could start from a rank-3 tensor of the form $\widetilde{T}_{\mu[2],\nu}$, i.e. one whose first two indices are antisymmetric, and find in addition to its antisymmetric part

$$A_{\mu\nu\rho} := \widetilde{T}_{[\mu\nu|\rho]} \equiv \frac{1}{3}\,(\widetilde{T}_{\mu\nu|\rho} + 2\,\widetilde{T}_{\rho[\mu|\nu]})\,, \tag{B.5}$$

the projection

$$\widetilde{H}_{\mu\nu,\rho} := \frac{1}{3}\,(\widetilde{T}_{\mu\nu,\rho} - \widetilde{T}_{\rho[\mu|\nu]})\,, \tag{B.6}$$

which now obeys an over-antisymmetrization condition,

$$\widetilde{H}_{[\mu\nu,\rho]} = 0\,. \tag{B.7}$$

The situation mirrors that of the previous example: we see that $\widetilde{H}$ is orthogonal to the antisymmetric projection, and such that $\widetilde{T}_{\mu\nu|\rho} = A_{\mu\nu\rho} + \widetilde{H}_{\mu\nu,\rho}$ with $\widetilde{H}$ corresponding to another irrep of $\mathfrak{gl}_d$. Summing the two tensors,

$$\mathcal{T}_{\mu|\nu|\rho} = T_{\mu\nu|\rho} + \widetilde{T}_{\mu\nu|\rho} = S_{\mu\nu\rho} + H_{\mu\nu,\rho} + \widetilde{H}_{\mu\nu,\rho} + A_{\mu\nu\rho}\,, \tag{B.8}$$

we obtain a generic tensor of rank-3, i.e. one whose indices have no symmetry, and its full decomposition into irreducible representations of $\mathfrak{gl}_d$. The two projections H and $\widetilde{H}$ are examples of *mixed-symmetry* tensors, which are tensors with several groups of symmetric or antisymmetric indices, which obey conditions of over-anti/symmetrization similar to, but generalizing, conditions (B.4) and (B.7).

The symmetry pattern of such tensors is encoded by *Young diagrams*, which are graphical representations of an ordered sequence of integers, $n_1 \geq n_2 \geq \ldots n_p \geq 0$, has p rows (respectively columns) of n_k boxes stacked in decreasing order from top to bottom (respectively from left to right). Young diagrams with a total of n boxes classify the irreps of the symmetric group $\mathcal{S}_n$. For instance, if $n = 3$ there exists three Young diagrams,

$$\yng(3) \qquad \yng(2,1) \qquad \text{and} \qquad \yng(1,1,1)\,, \tag{B.9}$$

corresponding to the three inequivalent irreps of $\mathcal{S}_3$. Young diagrams also classify finite-dimensional irreps of $\mathfrak{gl}_d$, a result known as Schur-Weyl duality (which can be considered a precursor of Howe duality, see e.g. the classical textbook of Weyl [93], or the more recent [94]). The correspondence is obtained by Young projectors which, given a rank-n tensor and a Young diagram with n boxes, produce an irreducible $\mathfrak{gl}_d$ tensor by

(i) First filling in the diagram with indices of the tensor such that the position of the index both increases when read from left to right and from top to bottom;

(ii) Second, symmetrizing over the indices in each row;

(iii) And third, antisymmetrizing the result over indices in each column.

Alternatively, one can swap the last two steps, i.e. first antisymmetrize indices in each column, and then symmetrize over the result in each row. Tensors obtained with the first method (symmetrization, then antisymmetrization) are said to be in the *symmetric basis*, whereas those obtained by the second method in the *antisymmetric basis*, but both belong to the *same* $\mathfrak{gl}_d$-irrep — as the name suggest, the two method merely correspond to different basis for constructing the same representation.

In the previous $n = 3$ example, the totally symmetric tensor corresponds to the one-row diagram,

$$S_{\mu\nu\rho} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|}\hline \mu & \nu & \rho \\ \hline \end{array}\,, \tag{B.10}$$

the totally antisymmetric one to the one-column diagram,

$$A_{\mu\nu\rho} \qquad \longleftrightarrow \qquad \begin{array}{|c|}\hline \mu \\ \hline \nu \\ \hline \rho \\ \hline \end{array}\,, \tag{B.11}$$

and the last two projections to the two possibilities of filling in the "hook" diagram,

$$H_{\mu\nu,\rho} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|}\hline \mu & \nu \\ \hline \rho & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \qquad \text{and} \qquad \widetilde{H}_{\mu\nu,\rho} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|}\hline \mu & \rho \\ \hline \nu & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}\,. \tag{B.12}$$

Note that $H_{\mu\nu,\rho}$ is written in the symmetric basis, whereas $\widetilde{H}_{\mu\nu,\rho}$ is in the antisymmetric basis. Let us also stress once more that both tensors correspond to the same $\mathfrak{gl}_d$-irrep, the one characterized by the Young diagram $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$. For more details on mixed-symmetry tensors and Young diagrams, see the pedagogical review [32].

Concretely, in the symmetric basis we will write

$$\phi_{\mu(s_1),\nu(s_2),\dots,\rho(s_p)}\,, \tag{B.13}$$

for a tensor which possesses a group of s_1 indices collectively denoted by μ, a group of s_2 indices collectively denoted by ν, etc, all of which symmetric with respect to permutations with other indices in the same group. On top of that, these groups of symmetric indices obey the conditions

$$\phi_{\dots,\mu(s_i),\dots,\mu\nu(s_j-1),\dots} = 0\,, \qquad 1 \leq i < j \leq p\,, \tag{B.14}$$

i.e. the symmetrization of all indices in the ith group with any one index in the jth group, where $i < j$, identically vanishes. We usually say such a tensor has the symmetries of a Young diagram whose kth row has length s_k. In the antisymmetric basis, we will write

$$F_{\mu[h_1],\nu[h_2],\dots,\rho[h_p]}\,, \tag{B.15}$$

for a tensor which possesses a group of h_1 indices collectively denoted by μ, a group of h_2 indices collectively denoted by ν, etc, all of which antisymmetric with respect to permutations

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with other indices in the same group. On top of that, these groups of antisymmetric indices obey the conditions

$$F_{\ldots,\mu[s_i],\ldots,\mu\nu[s_j-1],\ldots} = 0\,, \qquad 1 \le i < j \le p\,, \tag{B.16}$$

i.e. the antisymmetrization of all indices in the ith group with any one index in the jth group, where $i < j$, identically vanishes. We usually say such a tensor has the symmetries of a Young diagram whose kth column has height h_k.

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