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Of asymptotic charges and renormalizations

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À mon grand-père

Abstract

Since the early 20th century, physicists have pursued a quantum theory of gravitation, with notable breakthroughs such as 1970 Bekenstein's proposal linking black hole entropy to the event horizon surface area. By leveraging upon this observation, 't Hooft and Susskind then introduced the holographic principle, proposing that quantum gravity degrees of freedom might be encoded in lower-dimensional surfaces. Maldacena's work extended this idea with the AdS/CFT correspondence, connecting gravitational theories on Anti de Sitter (AdS) spacetime to conformal field theories (CFT) defined on the boundary of AdS. However, this duality involves a negative cosmological constant, conflicting with its observed positive value. This PhD thesis aims to explore selected aspects of the AdS/CFT correspondence and their generalization in the limit of vanishing cosmological constant.

In particular, this manuscript focuses on asymptotic symmetries and corner or, equivalently, surface charges through the Lagrangian approach to general relativity and covariant phase space. This framework offers insights into observables in gravity and dual gauge theories. Identifying physical asymptotic symmetries allows indeed one to identify the global symmetries of the dual conformal field theory and thus sets up crucial constraints allowing to identify the latter. In their turn, the relevant symmetries are selected by non-trivial surface charges. However, determining the surface charges faces challenges due to divergences as one approaches the asymptotic boundary. To tackle this, we confront variational and symplectic structure "renormalization schemes", opting for the latter for a systematic study.

To illustrate these techniques, we analyze asymptotic symmetries of Maxwell theory in both Anti de Sitter and flat backgrounds, aiming to recover the flat space results from AdS. This leads to studying the relaxation of the standard Fefferman-Graham gauge within Einstein gravity, resulting in the Weyl-

Fefferman-Graham gauge, which restores the broken boundary Weyl covariance and introduces new charges associated with the underlying Weyl geometry. This raises questions about new charges related to different available choices for the underlying symplectic structure. These issues are also linked with the current efforts in the literature to transition towards gauge-free analyses. As a general guideline, one could argue that the more physical charges the better, as this would lead to larger symmetry algebras that are more powerful to organize the observables of the theory. While the Fefferman-Graham gauge is suited to AdS/CFT, it falls short for asymptotically flat spaces. In contrast, the Bondi gauge, designed for flat spacetimes and gravitational waves, is universally applicable. Introducing a relaxation, the covariant Bondi gauge combines advantages of all aforementioned gauges, providing insights into boundary anomalies through a fluid/gravity representation and deepening the understanding of the holographic duality through new finite corner physical charges.

Contents

CHAPTER 1

Background and motivations

"L'horizon souligne l'infini."

Victor Hugo

Since the beginning of the 20th century, physicists have sought a quantum theory of gravitation. In the early 1970s, Bekenstein noted that black holes possess an entropy, which scales with the area of their event horizon, unlike the volume they enclose (Bekenstein, 1972). This unexpected observation was subsequently reinforced by the research of Hawking, demonstrating that gravity operates in a distinct manner compared to other field theories. For instance, the entropy of a gas, even one composed of photons, scales proportionally to the volume of the container enclosing it (Hawking, 1975). Around two decades later, 't Hooft and Susskind proposed that this entropy behavior could be a fundamental property of gravitational theories, suggesting that all quantum gravity degrees of freedom might reside on a surface with one fewer dimension than spacetime ('t Hooft, 1993; Susskind, 1995). This led to the concept of the Universe as a "hologram" defined in two plus one dimensions (two in space and one in time). In the late 1990s, Maldacena provided a concrete realization of this holographic principle by linking type-IIB string theory in a five-dimensional Anti de Sitter (AdS) space with a four-dimensional conformal quantum field theory (CFT) (Maldacena, 1998). This became known as the AdS/CFT correspondence, which posits a duality between a gravitational theory in $D+1$ dimensions and a conformal quantum field theory in D dimensions (Gubser et al., 1998; Witten, 1998; Aharony et al., 2000; D'Hoker & Freedman, 2002).

Over the past two decades, the AdS/CFT correspondence has significantly influenced theoretical high-energy physics research. However, it relies on the AdS space and consequently on a negative cosmological constant, posing a fundamental question: "why does a practical application of the holographic principle appear to demand such a value, especially when our Universe cosmological constant is positive?" This PhD thesis tackles these inquiries and aligns with ongoing endeavors to explore alternative manifestations of the holographic principle, encompassing gravitational theories featuring a vanishing cosmological constant. Especially, in order to pave some way in this direction, let us recall that while AdS/CFT originates from a specific microscopic model (Maldacena, 1998; Witten, 1998), this is not the case for other attempts at applying the holographic principle to different signs of the cosmological constant. From then on, current attempts to holography in Minkowski and de Sitter (dS) backgrounds, corresponding to null and positive values respectively of the cosmological constant, primarily rely on the symmetries expected to underlie the duality. Symmetries play indeed a critical role also in AdS/CFT: the asymptotic symmetries of the gravitational theory, which represent specific diffeomorphisms affected by the presence of a boundary (even at infinity), correspond to global symmetries in the dual theory. This principle guides the search for CFT duals, about which little is currently understood regarding Minkowski and dS backgrounds.

Specifically, the manuscript focuses on holography in Minkowski space, often via a study of a flat limit of selected AdS/CFT features. Although it is uncertain whether this could provide the necessary tools to enhance our understanding of the conjectured dS/CFT (Strominger, 2001; Anninos et al., 2017) in future research projects, it does appear to be a promising compromise for the transition from a negative to a positive value of the cosmological constant. Besides, beyond the theoretical significance of enhancing our comprehension of the holographic principle, accomplishing this objective will enable the direct use of holography in phenomenologically compelling gravitational scenarii, such as gravitational scattering in asymptotically flat space or cosmology. This avenue holds particular interest because AdS/CFT is a strong/weak duality, wherein a weak coupling constant limit in gravity corresponds to a strong coupling constant limit in the dual theory, and vice versa. Moreover, this duality is presumed to apply at the quantum level. Consequently, any holographic depiction of gravity in asymptotically flat or cosmological environments is anticipated to furnish valuable tools for addressing issues involving strong fields or quantum corrections—extreme conditions that are nevertheless pertinent in black hole physics and early cosmology.

In this pursuit, the main idea explored in this thesis is to leverage on the well developed study of asymptotic symmetries in AdS and then investigating their counterparts in Minkowski space. This is in keeping with the modern idea of obtaining information about this flat background from AdS. We determine these symmetries through the Lagrangian approach to gauge theories and the covariant phase space formalism, originally developed in (Gawędzki, 1972; Kijowski, 1973; Kijowski & Szczyrba, 1976). Although less commonly employed in the AdS/CFT correspondence, these frameworks have yielded significant insights into the understanding of observables in gravity and gauge theories over recent years. Notably, when considering spacetimes with relevant boundaries, the second Noether theorem offers a means to distinguish the physical asymptotic symmetries from the gauge transformations of the system (Arnowitt et al., 1962; Regge & Teitelboim, 1974; Benguria et al., 1977). For example, in gravity, these symmetries constitute a subgroup of bulk diffeomorphisms compatible with specific falloffs and boundary conditions, yielding a non-zero Noether's charge localized on codimension-2 surfaces referred to as corners. This distinction is crucial compared to the "standard" first Noether theorem, which pertains to global symmetries and provides a conserved quantity obtained upon integrating over a codimension-1 Cauchy surface.

Along these lines, specific boundary conditions have to be imposed on the metric to regulate permissible metric fluctuations at infinity, as elucidated in the seminal paper by Brown and Henneaux (Brown & Henneaux, 1986). In the latter, it was demonstrated that the asymptotic symmetries of asymptotically AdS spaces in two plus one dimensions are an enhancement in comparison to vacuum isometries. This insight has subsequently been interpreted as the algebra of modes of the stress tensor in the boundary bidimensional conformal field theory (Strominger, 1998), showcasing a particularly significant instance of the AdS/CFT correspondence. The aforementioned choice of boundary conditions does not require fixing any particular gauge but it is often convenient to select such a specific gauge to discuss the behavior of a reduced set of metric components at infinity. With the same spirit, in this manuscript, our main focus will be on these three-dimensional aspects of general relativity in the gauge-fixing approach. Note that the phenomenon of enhanced asymptotic symmetry group has been observed in higher dimensions as well (see, e.g., (Compère et al., 2020)). The objective is logically to progress towards analyses in such cosmologically more realistic dimensional spaces.

Therefore, despite the absence of propagating degrees of freedom, Einstein gravity in 3D provides an ideal arena for investigating techniques and intuitions applicable in dimensions four and beyond due to the possibility to explore other gravitational phenomena in a simplified context, as emphasized in (Staruszkiewicz, 1963; Deser et al., 1984; Deser & Jackiw, 1984). In addition to the asymptotic symmetry enhancement (Barnich & Brandt, 2002; Carlip, 2005; Barnich & Compère, 2008), one crucial aspect of this model is the existence of black holes in presence of a negative cosmological constant. (Banados et al., 1992, 1993; Carlip, 1995). Moreover, the topological nature of this three-dimensional theory allows it to be rewritten as a Chern-Simons theory (Achucarro & Townsend, 1986; Witten, 1988; Banados, 1996), which proves useful at the gravitational level for calculating asymptotic charges and analysing asymptotic symmetries, but also for establishing higher-spin theories coupled to gravity (Henneaux & Rey, 2010; Campoleoni et al., 2010)[1].

In this sense, the Chern-Simons formulation of gravity in two plus one dimensions brings about a notable simplicity, particularly in the ability to gauge away the radial dependence of asymptotic charges[2]. Actually, as already mentioned, an asymptotic behavior is assigned to the fields at the boundary. The latter is approached by following the evolution of a radial holographic coordinate to infinity. Since the asymptotic charges depend on this behavior of the fields, we understand that radial divergences may affect their correct definition. Similar phenomena can also occur at the level of the associated variational principle. The topological nature of three-dimensional gravity allows us to address such issues, but it becomes essential to understand how to do so in a general manner for gauge theories. Specifically, in our particular case of interest, in the metric formulation of Einstein's gravitational theory, such a workaround is not applicable. To address this, chapter 2 reviews two procedures for renormalizing such quantities, with a particular emphasis on asymptotic corner charges, as outlined in sections 2.3 and 2.4. The former procedure focuses on the renormalization of the variational principle (Henningson & Skenderis, 1998; de Haro et al., 2001; Bianchi et al., 2002; Compere & Marolf, 2008), while the latter concentrates on the renormalization of the underlying symplectic structure (Freidel et al., 2019; McNees & Zwikel, 2023).

These procedures rely on techniques reviewed in sections 2.1 and 2.2 of the second chapter, which are dedicated to a contemporary study of the covariant phase space formalism following the approaches of Iyer-Wald (Lee & Wald, 1990; Wald, 1993; Wald & Zoupas, 2000) and Barnich-Brandt (Barnich & Brandt, 2002). To exemplify these techniques, chapter 3 is devoted

[1] We refer to (Campoleoni & Fredenhagen, 2024) for a recent review.

[2] This is true most of the time, but may require further investigation in some cases, see for example (Banados, 1996; Grumiller & Riegler, 2016).

to exploring asymptotic symmetries within the simplest non-trivial gauge theory, Maxwell electromagnetism. In this way, our aim is to glean insights for Einstein-Hilbert gravitation, the subject of the chapter 4, by investigating the propagation of a non-massive spin 1 field in an AdS background.

Regarding general relativity, in addition to the potential to explicitly demonstrate the application of symplectic renormalization in a gravitational context, the three-dimensionality aspect also enables exploration of a recent avenue in the literature of asymptotic symmetries, specifically regarding standard gauge and boundary conditions relaxations that lead to novel finite corner charges (Compère et al., 2013; Pérez et al., 2016; Ojeda & Pérez, 2019; Grumiller et al., 2020a). It is also of interest to provide a physical interpretation to these new finite surface charges from the boundary perspective. We first investigate this avenue in the context of Maxwell theory, successfully addressed by employing a manifestly gauge-invariant prescription. Notably in (Grumiller & Riegler, 2016; Grumiller et al., 2017), for Einstein-Hilbert theory, the most general gravitational solution space aligned with a well-defined variational principle was derived in three dimensions, encompassing the maximum number of asymptotic charges. The chapter 4 then delves into a field content falling within the scope of these works, with the distinction that the boundary geometry is not constrained. This approach opens up intriguing interpretations of the novel charges from an on-boundary perspective, as well as insights into certain dual anomalies arising from the non-conservation of the variational principle.

In particular, in the theory of asymptotic symmetries, a key result is that a charged diffeomorphism constitutes a physical symmetry, mapping inequivalent physical configurations. Fixing a specific gauge is therefore a delicate procedure, as it can constrain the physical content of the theory. In this context, the corner proposal provides an interesting shift of paradigm by identifying universal structures associated with corners. This has been proposed in (Donnelly & Freidel, 2016; Speranza, 2018; Geiller, 2017, 2018)[3]. It raises the crucial question of classifying new charges associated with choices of symplectic spaces. Having more physical charges is indeed advantageous since it leads to larger algebras that are more effective in organizing the observables of the theory. Furthermore, to progress toward a quantum gravity theory, it is essential to move away from the gauge-fixing approach and construct a gauge-free analysis. Taking several steps forward, this opens the door (or rather the corner of a door, to make a pun) to classifying charges stemming from partial gauge fixings, which are yet to be unveiled.

[3]For further exploration, see also (Freidel et al., 2020; Donnelly et al., 2021; Ciambelli & Leigh, 2021, 2023).

Following this way of thinking, we proceed to examine the relaxation of certain conventional gauges found in the literature to investigate these diverse aspects. The first gauge we examine is rooted in the framework proposed by the Fefferman-Graham ambient construction (Starobinsky, 1983; Fefferman & Graham, 1985, 2011). This is derived from a mathematical theorem put forth by Fefferman and Graham, asserting that any asymptotically AdS spacetime can be reconstructed by fixing a boundary metric and an energy-momentum tensor. It features a radial direction (referred to as the holographic direction) that parametrizes a family of time-like hypersurfaces, with radial evolution interpreted as the renormalization flow of the boundary theory. The bulk metric induces a conformal class of metrics and an energy-momentum tensor on the boundary. Notably, the Fefferman-Graham gauge utilizes all available diffeomorphism freedom to fully fix the radial structure of the bulk metric. As a result, this gauge has been widely employed in holography (Balasubramanian & Kraus, 1999; Skenderis, 2001). Furthermore, in hindsight, it becomes possible to comprehend and rediscover the findings of the seminal work by Brown and Henneaux (Brown & Henneaux, 1986) by imposing this gauge. Specifically, we will delve further into this aspect, uncovering the double copy of the Virasoro algebra within the algebra of asymptotic symmetries of AdS in three dimensions.

In more concrete terms, in the section 4.2, our focus is on revisiting the examination of asymptotic symmetries within the 3D Fefferman-Graham gauge. Historically, following the reformulation of the Brown-Henneaux boundary conditions within this gauge – equivalent to Dirichlet conditions imposed on the boundary metric – these conditions were relaxed in (Troessaert, 2013) to allow for fluctuations of its conformal factor. In the latter, a flatness condition on the boundary curvature was imposed in order to ensure the well-posedness of the variational problem. This relaxation, in turn, results in an enhancement of the asymptotic symmetry algebra by the inclusion of two additional affine $\mathfrak{u}(1)$ algebras. Subsequently, in (Alessio et al., 2021), a proposition was made to permit all conceivable configurations of the boundary conformal factor.

However, as we have already mentioned, it is well understood from (Grumiller & Riegler, 2016) that not all conceivable charges within this framework have been accounted for. Consequently, investigating relaxations of the Fefferman-Graham gauge becomes intriguing to explore these new charges. Specifically, as noted in (Ciambelli & Leigh, 2020), reintroducing certain degrees of diffeomorphism freedom within the Fefferman-Graham gauge allows for the realization of a connection associated with Weyl rescalings as an integral component of the induced boundary structure. This, in turn, facilitates the restoration of boundary Weyl covariance, which is a natural holographic expectation since the asymptotic boundary sits at conformal infinity but which is explicitly broken in the conventional Fefferman-Graham setup (Henningson & Skenderis, 1998). Consequently, this gauge relaxation has been dubbed the Weyl-Fefferman-Graham gauge. At the level of the line element, it consists of the relaxation of the bulk metric mixed component which is set to zero in Fefferman-Graham. We group the dual Weyl geometric aspects associated with this relaxation in the appendix B.1. Our asymptotic symmetry analysis highlights that the diffeomorphism mapping from Weyl-Fefferman-Graham to Fefferman-Graham can carry charge, thereby becoming non-trivial (Ciambelli et al., 2023). This generalization extends previous enhancements of boundary conditions in Fefferman-Graham to the Weyl-Fefferman-Graham scenario, affirming that while the standard gauge is always attainable, it may impose constraints on the bulk physical solution space. Therefore, a fascinating possibility arises: a more comprehensive holographic dictionary formulated geometrically in the relaxed gauge, which incorporates extra charges and observables playing a role in the dual field theory.

As we have just seen, the Fefferman-Graham gauge garnered significant attention in the context of the AdS/CFT correspondence towards the end of the previous millennium and the beginning of this one. However, this gauge has no smooth flat boundary to describe flat space through, which is one of our goals in this thesis. The Bondi gauge, originally proposed in the 1960s by Bondi, van der Burg, Metzner and Sachs (Bondi et al., 1962; Sachs, 1962a,b) for metrics of asymptotically flat spaces, serves as the foundation for this discussion. Notably, it was designed to investigate gravitational waves in dimensions four and higher, and is implemented along a null direction. Furthermore, this gauge has also been instrumental in studying asymptotically AdS spacetimes and their flat limit, facilitated by a set of defined boundary conditions (Barnich et al., 2012; Barnich & Lambert, 2013).

Consequently, in the last decade, there has been a renewed interest in the Bondi gauge. Explicitly, it has facilitated the understanding that gravity without a cosmological constant is quite distinctive, as its asymptotic symmetries do not align with the isometries of Minkowski space. In four dimensions they are given by the infinite-dimensional Bondi-Metzner-Sachs (BMS) group (Barnich & Compere, 2007), and in (Barnich & Troessaert, 2010) it has been proposed to further extend it to include local conformal transformations on the two-dimensional celestial sphere. The appearance of the symmetries of 2D CFTs, which are among the best understood quantum field theories, triggered an intense effort to try to formulate a BMS/CFT correspondence, describing gravity near null infinity via a 2D CFT (Barnich & Troessaert, 2010; Ball et al., 2019). These investigations brought a remarkable and unexpected byproduct, uncovering a triangular equivalency between BMS asymptotic symmetries, soft-theorems involving spin-2 particles and memory effects in gravity (He et al., 2015; Campiglia & Laddha, 2014; Ashtekar et al., 2018; Adamo et al., 2019): all these phenomena have been interpreted as different facets of the same infrared effect (Strominger, 2018).

In this context, similar to the mathematical justification of the Fefferman-Graham gauge through construction in ambient space, the geometric rationale behind the Bondi gauge can be elucidated via conformal completion à la Penrose (Penrose, 1963, 1964)[4]. Continuing along the same trajectory as previously emphasized, our focus remains on the gauge-fixing approach. This method proves advantageous in our analysis due to its flexibility in modifying boundary and gauge conditions imposed on the dynamical field. Besides, our subsequent objective lies in relaxing the constraints of the Bondi gauge to explore new avenues for uncovering finite corner charges. In the section 4.3, we begin by reviewing this standard gauge and note that, while it is valid regardless of the value of the cosmological constant, it is not covariant with respect to the pseudo-Riemannian boundary as the Fefferman-Graham gauge can be.

To reconcile these advantages, we introduce the covariant Bondi gauge, which integrates the strengths of the previously mentioned gauges (Ciambelli et al., 2018b; Campoleoni et al., 2019a, 2023b). This relaxed gauge implementation involves a null bulk congruence, akin to the Bondi gauge, albeit with a boundary-to-bulk approach. This method relies on an expansion in inverse powers of the radial light-like coordinate, driven by Weyl covariance as in the Weyl-Fefferman-Graham gauge. More precisely, the covariant modification in the Bondi line element originates from the fluid/gravity correspondence (Bhattacharyya et al., 2008a; Haack & Yarom, 2008; Bhattacharyya et al., 2008b; Hubeny et al., 2012). In the appendix B.2, we provide a concise overview of this duality, elucidating the geometric interpretation achievable through this gauge relaxation from the boundary. This aspect underscores the significance of the covariant Bondi framework. Firstly, the fluid/gravity duality extends the holographic AdS/CFT correspondence via a long-wavelength approximation, simplifying the field theory to an effective representation using fluid mechanics. Consequently, we can articulate the boundary in terms of relativistic hydrogeometric concepts, which proves simpler compared to a conventional CFT

[4]For detailed insights into this geometric perspective, we refer to works such as (Hansen et al., 1978; Ashtekar & Streubel, 1981; Dray & Streubel, 1984; Ashtekar, 2014; Ashtekar et al., 2015; Herfray, 2020).

representation.

Secondly, we discover the potential to derive new corner charges (Campoleoni et al., 2022) and, consequently, novel asymptotic symmetries beyond the scope of Bondi related literature. This second novel feature is intertwined with the choice to describe the boundary metric in terms of bulk congruence, which is interpreted as the fluid velocity within the fluid/gravitational analogy. We will refer to this maneuver as the selection of the boundary Cartan frame. Furthermore, we explicitly observe that such frame dependence, which restores Lorentz symmetry – that is broken in the standard Bondi case – is pure gauge within the Fefferman-Graham setup. Thus, another perspective on introducing the relaxed covariant gauge can be seen by establishing the connection between Fefferman-Graham and Bondi through gauge transformation, aligning the two independent data of the Fefferman-Graham gauge (the metric and the energy-momentum tensor) to Bondi data. Indeed, establishing such a dictionary necessitates selecting a boundary frame. Therefore, this endeavor aims to reinstate the broken frame covariance within the Bondi framework, achieved by allowing the metric component that combines the null radial and spatial directions to be non-zero, giving rise to the so-called covariant Bondi gauge.

Thirdly and finally, we detail in the section 4.4 that the covariant Bondi gauge is instrumental in revealing the emerging boundary Carrollian structure (Jankiewicz, 1954; Vogel, 1965; Lévy-Leblond, 1965; Sen Gupta, 1966; Isham, 1976; Henneaux, 1979; Dautcourt, 1998), when the cosmological constant approaches zero. In fact, the so-called ultrarelativistic or Carrollian fluids represent a facet of broader holographic investigations known as Carrollian holography (Hartong, 2015; Bagchi et al., 2016; Bergshoeff et al., 2017), inherently suited for describing null hypersurfaces, including null infinity and black hole horizons (Donnay & Marteau, 2019). Notably, the conformal Carroll group is isomorphic to the BMS group (Duval et al., 2014b). In the conjecture of Carrollian flat space holography, the dual theory manifests as a conformal Carrollian field theory at null infinity. This holographic correspondence intersects with celestial holography (Strominger, 2018; Pasterski, 2019; Raclariu, 2021; Pasterski et al., 2021; Donnay et al., 2022, 2023)[5], which has recently garnered substantial attention following the discovery of the soft graviton theorem (Cachazo et al., 2006). This theorem underscores a profound connection between soft gravitons and 2D CFT stress tensor Virasoro-Ward identities (Kapec et al., 2017). Given that these two dual approaches are widely recognized and suc-

[5]See, e.g., (Pasterski, 2021; McLoughlin et al., 2022; Donnay, 2024) for pedagogical reviews.

cessful in the literature for investigating holographic correspondence involving asymptotically flat spaces, the covariant Bondi gauge adaptation into a Carrollian rewriting represents another significant advantage. Last but not least, the new charges derived in this gauge give rise to new anomalies in the relativistic dual theory, which in turn result in fresh Carrollian anomalies in the flat limit.

Structure of the manuscript

All of these considerations and motivations lead us to establish the following outline of this thesis, serving as a summary of our introductory discussion and facilitating navigation through the manuscript.

Chapter 2 reviews techniques to compute the asymptotic corner charges in gauge theories through the covariant phase space formalism, respectively in 2.1 and 2.2. Subsequent sections, 2.3 and 2.4, introduce two approaches of the literature to holographic renormalization. The first focuses on boundary counterterms to be incorporated into the variational principle, while the second concentrates on corner contributions in order to renormalize the underlying symplectic structure. These formal methods are exemplified in chapter 3, which is dedicated to Maxwell theory, serving as the simplest example of a non-trivial gauge theory. Specifically, in 3.1, we introduce two coordinate systems suitable for holography à la AdS/CFT and for subsequent investigations of flat space via a smooth limit. The detailed definition of the theory under consideration is provided in section 3.2, followed by an exploration of photon propagation in an AdS background in 3.3. The chapter concludes with the exploration of the flat limit in section 3.4.

This leads to similar analyses for three-dimensional gravitational theory in chapter 4. The formal definition is provided in section 4.1, and further explored in the Fefferman-Graham and Bondi gauges (and their variations) of asymptotically AdS spaces in 4.2 and 4.3, respectively. The transition to the flat limit is addressed in the final section 4.4. The main findings are summarized in the concluding chapter 5, which also outlines potential future perspectives and continuations of the research conducted in this thesis. Lastly, in appendix A, we provide a concise summary of the different notations and conventions utilized throughout the manuscript. Additionally, in appendix B, we present reviews of the dual geometric aspects associated with the various gauge relaxations used. Specifically, in section B.1, we consolidate Weyl aspects, whereas in section B.2, we cover the ones related to relativistic and Carrollian hydrogeometry.

CHAPTER 2

Covariant phase space formalism

> "La meccanica è il paradiso delle scienze matematiche, perché con quella si viene al frutto matematico."
>
> Leonardo da Vinci

In the introduction, one of the objectives of this thesis is outlined: to ascertain whether a residual gauge symmetry in a spacetime, with boundary, constitutes a physical symmetry or a mere gauge transformation. This determination is made by examining whether the associated Noether charge, as derived from the Noether theorem, vanishes or not as one approaches the boundary. In the context of gauge theory, the standard textbook Noether theorem is adapted into its second version, the so-called second Noether theorem, asserting that each gauge symmetry of the theory is linked to a conserved codimension-2 quantity termed surface or corner charge. These charges may diverge since they are computed as integrals over a slice of the boundary of spacetime and depend on the behaviour of the fields while approaching the boundary. This is built into the inherent divergence of the associated variational principle in such limits.

The first part of the chapter provides a succinct review of two frameworks for a covariant analysis of boundary charges. While one could adopt a non-covariant approach, utilizing methods from the Hamiltonian perspective as seen in references such as (Regge & Teitelboim, 1974; Crnkovic & Witten, 1987; Gawedzki, 1991; Banados, 1999; Henneaux et al., 2000)[1], which splits time and space and explores trajectory evolution in phase space, the preference in this thesis is to maintain covariance in spacetime and thus rely on the Lagrangian approach. In the latter, the set of solutions to the equations of motion defines the phase space, simplifying the approach but introducing potential ambiguities and challenges in understanding certain physical quantities.

The main idea of the Lagrangian analysis of asymptotic symmetries is to unify spacetime and phase space, giving rise to the covariant phase space formalism, introduced in (Gawędzki, 1972; Kijowski, 1973; Kijowski & Szczyrba, 1976) and further developed in (Crnkovic, 1988; Lee & Wald, 1990; Wald, 1993; Wald & Zoupas, 2000; Barnich & Brandt, 2002)[2]. This formulation, also known as the Iyer-Wald prescription, is discussed in section 2.1. It comes with ambiguities, known as Iyer-Wald ambiguities. Another covariant phase space formulation, the Barnich-Brandt prescription, is introduced in section 2.2. While it fixes the Iyer-Wald ambiguities, its symplectic structure depends solely on the constitution of the equations of motion. The connections between these two prescriptions are explored in section 2.2. Note that the Iyer-Wald formulation derives only a general form of diffeomorphism charges, while the Barnich-Brandt formulation does so for an arbitrary gauge theory.

The second part of this chapter delves into renormalization of charges, addressing the need for boundary actions to complete the bulk theory. It leads to the introduction of the holographic renormalization procedure by Skenderis and collaborators (Henningson & Skenderis, 1998; de Haro et al., 2001; Bianchi et al., 2002; Skenderis, 2002; Papadimitriou & Skenderis, 2005a; Hollands et al., 2005; Mann & Marolf, 2006) in section 2.3. This traditional method involves renormalizing both the variational principle and the Iyer-Wald symplectic structure via the so-called Compère-Marolf prescription (Compere & Marolf, 2008). In section 2.4, a new systematic approach, symplectic renormalization, is reviewed. This method, initiated in (Freidel et al., 2019; McNees & Zwikel, 2023) for Maxwell and diffeomorphism invariant gauge theories, is generalized in this thesis to encompass any gauge theory in the presence of an asymptotic boundary. We implement these procedures within the framework of the Iyer-Wald covariant phase space, as the Barnich-Brandt formalism can only undergo renormalization through these guidelines when connected to the Iyer-Wald formulation.

[1] We also refer to, for example, (Bunster et al., 2014; Perez et al., 2015; Riegler & Zwikel, 2018).

[2] See also (Compère & Fiorucci, 2018; Ruzziconi, 2020; Ciambelli, 2023) for pedagogical reviews.

2.1. Iyer-Wald prescription

In this section, we present a concise review of the covariant phase space formalism, leading to the Iyer-Wald prescription for determining conserved surface (or corner) charges via Noether's second theorem. Instead of providing proofs for the various results, our focus is on facilitating the understanding of the outcomes through useful tools for subsequent sections with a special focus on renormalization. We encourage readers to explore the original literature or specialized educational reviews, as exemplified by (Barnich & Brandt, 2002; Barnich & Del Monte, 2018; Fiorucci, 2021), for a more in-depth examination. It is crucial to highlight that, while the general covariant phase space method is applicable to any gauge theory, the Iyer-Wald charge derivation remains valid exclusively for diffeomorphism-invariant theories, such as Einstein's theory of gravitation. We will delve into this aspect further in the relevant discussions.

2.1.1 Variational bicomplex

Spacetime calculus

We commence by recalling the bases of differential calculus in spacetime to set the notation. Consider a differentiable Lorentzian $D-$dimensional manifold $\mathcal{M}$ with coordinates denoted by $(x^\mu) = (r, x^a)$, where r is a radial coordinate, and the boundary is located at $r \to \infty$. Assuming that $\mathcal{M}$ possesses a regulating boundary $\partial\mathcal{M}$ with a radial isosurface component, we label this surface as $\mathcal{B}$ and its coordinates as x^a. A summary of the various conventions and notations we use throughout this manuscript can be found in appendix A. At any point of the spacetime manifold, a tangent space $T\mathcal{M}$ can be constructed with a natural basis $\{\partial_\mu\}$. Its dual space, known as the cotangent space $T^\star\mathcal{M}$, contains 1-forms spanned by the corresponding natural basis $\{\mathrm{d}x^\mu\}$.

The space of all differential forms constitutes the de Rham complex (De Rham, 1955):

$$\Omega(\mathcal{M}, \mathbb{R}) = \bigoplus_{n=0}^{\dim \mathcal{M}} \wedge^n T^\star\mathcal{M} \,. \tag{2.1}$$

Here, $\wedge$ represents the skew-symmetric product. We denote by $\mathrm{d} = \mathrm{d}x^\mu\partial_\mu$ (ensuring $\mathrm{d}^2 = 0$) and $\iota_\xi = \xi^\mu \frac{\partial}{\partial \mathrm{d}x^\mu}$, where $\xi = \xi^\mu\partial_\mu \in T\mathcal{M}$, the exterior derivative and the interior product on this complex, respectively. These operations, when applied in sequence, increment and decrement the degree of the spacetime forms to which they are applied. When enforced in different orders, the Lie derivative with respect to a vector $\xi \in T\mathcal{M}$ compares these two operations:

$$\mathscr{L}_\xi = \mathrm{d}\iota_\xi + \iota_\xi \mathrm{d}\,. \tag{2.2}$$

This relation is known as Cartan's (magic) formula. It is important to note that the interior product of a 0-form (a scalar) is zero, and the exterior derivative of a top-form vanishes.

Field space calculus

Secondly, we aim to replicate these outcomes in the field space Γ, defined as the space comprising all potential field configurations, assumed to be a differentiable manifold. Consequently, we can establish a calculus on the manifold space of forms. At this juncture, we treat these fields as abstract entities, devoid of any reference to spacetime coordinates. If we designate the set of fields as $\varphi = (\varphi^i)$ and their symmetrized derivatives as φ^i_μ, $\varphi^i_{\mu\nu}$, and so forth, a point in Γ is expressed as $(\varphi_{(\mu)})$, and the cotangent space $T^\star\Gamma$ at this point consists of the abstract variations collection $(\delta\varphi_{(\mu)}) = (\delta\varphi^i, \delta\varphi^i_\mu, \delta\varphi^i_{\mu\nu}, \dots)$.

Emulating the characteristics of spacetime calculus, this leads to the following definition of the exterior derivative on the field space:

$$\delta = \sum_{(\mu)} \delta\varphi^i_{(\mu)} \frac{\partial}{\partial\varphi^i_{(\mu)}} \tag{2.3}$$

where we use the convention that

$$\frac{\partial\varphi^i_{(\mu)}}{\partial\varphi^j_{(\nu)}} = \delta^{(\mu)}_{(\nu)}\delta^i_j\,, \tag{2.4}$$

and the $\delta\varphi^i_{(\mu)}$ are considered Grassmann odd. This last odd-property implies that $\delta^2 = 0$. The operation (2.3) can be conceptualized as a field variation, incrementing the degree of forms on Γ. We will be more specific about this comment in the discussion around the equations (2.16) and (2.17), written in terms of Grassmann even objects.

Actually, the introduction of Grassmann odd quantities, as opposed to Grassmann even, may seem unnatural. However, it proves convenient for defining the analog of an exterior derivative on the field space, particularly for maintaining a nilpotent aspect. Additionally, it aids in obtaining more concise and manageable expressions in subsequent discussions than if they were formulated in terms of Grassmann even quantities. Further comments on this will be provided in the following sections, offering a comparison with more standard relations.

Despite its formal nature, the introduction of a covariant phase space enables the treatment of gauge symmetries and their associated conserved quantities in a general manner, as opposed to their case-by-case derivation through integration by parts à la Abbott-Deser (Abbott & Deser, 1982b). Ultimately, the charges from both approaches coincide, prompting a consideration of the utility of such formality. Moreover, as we will explore later, the covariant phase space formalism à la Iyer-Wald (Lee & Wald, 1990; Wald, 1993; Iyer & Wald, 1994; Wald & Zoupas, 2000) facilitates a transparent renormalization procedure for divergent asymptotic charges. This justification underpins the subsequent definitions introduced in this section. More details will be provided as we delve into various examples in the next chapters, and this should improve clarity and understanding of these statements.

The space of all forms on the field space constitutes the variational complex:

$$\Omega(\Gamma, F) = \bigoplus_{n=0}^{\dim \Gamma} \wedge^n T^\star \Gamma \,, \tag{2.5}$$

where $F = C^\infty(\Gamma)$ represents the space of functionals. The 0-forms in this complex are what we usually call local functionals of the fields.

Variational bicomplex

Thirdly and finally, the primary concept of the covariant phase space formalism is to combine the calculi of spacetime and field space, aiming to obtain the jet bundle or variational bicomplex denoted as $(\mathcal{M}, \Gamma)$. The associated space is a manifold with local coordinates $(x^\mu, \varphi^i_{(\mu)})$, where the fields act as fibers above the target manifold. Extracting a section of the fiber reveals the coordinate-dependent fields along with their derivatives.

The spacetime exterior derivative d remains defined on the jet bundle, where the notation ∂_μ should now be interpreted as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} + \sum_{(\nu)} \varphi^i_{\mu(\nu)} \frac{\partial}{\partial \varphi^i_{(\nu)}} = \frac{\partial}{\partial x^\mu} + \varphi^i_\mu \frac{\partial}{\partial \varphi^i} + \varphi^i_{\mu\nu} \frac{\partial}{\partial \varphi^i_\nu} + \dots \,, \tag{2.6}$$

signifying that d is also a Grassmann-odd differential operator and anticommutes with the other, $\{\mathrm{d}, \delta\} = 0$. Drawing inspiration from spacetime calculus, the interior product in the field space with respect to a vector V tangent to the latter[3] is defined as

$$I_V = \sum_{(\mu)} \partial_{(\mu)} V^i \frac{\partial}{\partial \delta \varphi^i_{(\mu)}} \,. \tag{2.7}$$

[3]More precisely, V is the characteristic of the transformation $I_V \delta\varphi^i = \delta_V \varphi^i = V^i$.

This operation (2.7) can be viewed as a field contraction, and its application to a functional yields zero.

The Lie derivative along V, representing the transformation under V of any field space function to which it is applied, possesses the following property on Γ, akin to Cartan's spacetime formula

$$\mathfrak{L}_V = \delta I_V + I_V \delta \,. \tag{2.8}$$

In the following, the notation (p,q)-form refers to a p-form on the spacetime $\mathcal{M}$ and a q-form on the field space Γ.

2.1.2 Symplectic structure

Lagrangian and presymplectic potential

A field theory is characterized by an action

$$S = \int_{\mathcal{M}} L = \int_{\mathcal{M}} \mathrm{d}^D x \, \mathscr{L} \,, \tag{2.9}$$

where L represents the Lagrangian form, a natural object within the following variational bicomplex structure: a top form on the spacetime and a function of the fields and their derivatives. In the above convention, it corresponds to a $(D,0)$-form. Denoting $\mathscr{L}$ as the Lagrangian density, it equals $\sqrt{-g}$ times the corresponding scalar (which is then a $(0,0)$-form), where $g_{\mu\nu}$ is the metric (with Lorentzian signature) on the manifold $\mathcal{M}$ and g its determinant.

Under an arbitrary field variation, $\varphi \to \varphi + \delta\varphi$, the Lagrangian form undergoes the following transformation:

$$\delta L = \delta\varphi^i \frac{\delta L}{\delta \varphi^i} - \mathrm{d}\Theta[\varphi; \delta\varphi] \,, \tag{2.10}$$

where the minus sign, unconventional with respect to Grassmann even analogous expression (see (2.16)), arises from the Grassmann odd parity of the exterior derivatives. The first part of the right hand side encompasses the Euler-Lagrange derivatives responsible for deriving the equations of motion and is explicitly defined as

$$\frac{\delta L}{\delta \varphi^i} = \sum_{(\mu)} (-1)^{|\mu|} \partial_{(\mu)} \left(\frac{\partial L}{\partial \partial_{(\mu)} \varphi^i} \right) , \tag{2.11}$$

where $|\mu|$ denotes the cardinal of (μ). The boundary term in (2.10) is termed the local presymplectic potential, $\Theta = \Theta^\mu (\mathrm{d}^{D-1}x)_\mu$. It is a $(D-1,1)$-form depending on the fields and their derivatives.

Presymplectic form and ambiguities

We define the local Lee-Wald presymplectic $(D-1,2)$-form (Lee & Wald, 1990),

$$\omega[\varphi;\delta\varphi;\delta\varphi] = \delta\Theta[\varphi;\delta\varphi]\,. \tag{2.12}$$

Then, we have the tools to be more precise about what we mean by a field theory. In fact, defining a Lagrangian on the manifold $\mathcal{M}$ does not suffice to completely specify the theory. As we shall demonstrate, distinct additions of boundary and corner Lagrangians can result in different associated charges. Throughout the following discussion, we persist denoting these choices as ambiguities for historical reasons, while acknowledging that varying them implies distinct physical theories.

Actually, the presymplectic potential exhibits two types of ambiguities in its definition (2.10), which do not impact the equations of motion:

$$\Theta[\varphi;\delta\varphi] \to \Theta[\varphi;\delta\varphi] + \delta B[\varphi] - \mathrm{d}C[\varphi;\delta\varphi]\,. \tag{2.13}$$

Introducing a boundary term to the Lagrangian $L \to L + \mathrm{d}B$ gives rise to the first type of ambiguity, namely δB. Its contribution to the (Lee-Wald) presymplectic form ω vanishes due to $\delta^2 = 0$. The nilpotent nature of δ and the definition of Θ as a boundary term in δL result in the second type of ambiguity, i.e. $\mathrm{d}C$. It is noteworthy that the latter modifies the Lee-Wald presymplectic form (2.12):

$$\omega \to \omega - \delta\mathrm{d}C = \omega + \mathrm{d}\delta C =: \omega + \mathrm{d}\omega_C\,, \tag{2.14}$$

highlighting our uncertainty in selecting the boundary terms ω_C, referred to as corner terms, in the presymplectic form. This is associated with the corner proposal (Donnelly & Freidel, 2016; Speranza, 2018; Geiller, 2017, 2018; Ciambelli & Leigh, 2021).

For the sake of this thesis, we emphasize that the ambiguities can be used to renormalize the symplectic potential whenever the asymptotic charge diverges near the boundary (Papadimitriou & Skenderis, 2005b; Compere & Marolf, 2008; Papadimitriou, 2010) and can be further employed to restore integrability (Adami et al., 2021b; Geiller et al., 2021). In sections 2.3 and 2.4, we will examine two renormalization procedures that fix these ambiguities in different ways. This rationale also justifies the formal aspect of our formulation of the covariant phase space à la Iyer-Wald.

The (local) expression (2.12) can be integrated on a Cauchy slice Σ to yield the (global) presymplectic $(0,2)$-form:

$$\Omega[\varphi;\delta\varphi;\delta\varphi] = \int_\Sigma \omega[\varphi;\delta\varphi;\delta\varphi]\,. \tag{2.15}$$

We shall see in subsection 2.1.6 that this quantity is a crucial element of the theory as it carries the Poisson bracket.

Grassmann even convention

Let us now delve into a brief commentary, as we have previously emphasized the caution regarding Grassmann even quantities related to Grassmann odd ones. When contracting the presymplectic potential defining relation (2.10) with an arbitrary Grassmann even variation δ_e that is tangent to the solution space, we recover the conventional expression:

$$I_{\delta_e}\delta L \equiv \delta_e L = \delta_e\varphi^i \frac{\delta L}{\delta\varphi^i} + \mathrm{d}\Theta[\varphi;\delta_e\varphi]\,. \tag{2.16}$$

Here, we reestablish the familiar interpretation of $\delta_e L$ as a field variation of the Lagrangian form and I_{δ_e} as a field contraction. This relation illustrates how to connect this formal framework to concrete examples using standard expressions. Similarly, when contracting (2.12) with specific Grassmann even variations $\delta_1\varphi$ and $\delta_2\varphi$, both tangent to the solution space, one arrives at the more recognizable expression:

$$I_{\delta_2}I_{\delta_1}\omega := \omega[\varphi;\delta_1\varphi;\delta_2\varphi] = \delta_1\Theta[\varphi;\delta_2\varphi] - \delta_2\Theta[\varphi;\delta_1\varphi]\,. \tag{2.17}$$

2.1.3 Second Noether theorem

Gauge symmetries

In subsection 2.1.1, we introduced a general transformation of the fields $\varphi = (\varphi^i)$ as

$$\delta_V\varphi^i = I_V\delta\varphi^i = V^i\,, \tag{2.18}$$

where the characteristic V is generally a collection of local functions, i.e. a function of the coordinates, the fields and their derivatives. Such a transformation is a symmetry of the theory if it preserves the Lagrangian form up to a boundary term, expressed as

$$I_V\delta L = \mathfrak{L}_V L = \mathrm{d}B_V \tag{2.19}$$

for some codimension-1 form B_V. Within this set of transformations, some might be generated by arbitrary functions $\lambda = (\lambda^\alpha)$ of the coordinates and are known as gauge transformations. The latter act at the infinitesimal level on fields like

$$\delta_\lambda\varphi^i = I_\lambda\delta\varphi^i = R^i[\lambda] = \sum_{(\mu)} R^{i(\mu)}_\alpha \partial_{(\mu)}\lambda^\alpha\,, \tag{2.20}$$

where the characteristics $R^{i(\mu)}_\alpha$ are also local functions.

The global symmetries, satisfying both (2.18) and (2.19), are governed by the first Noether theorem. According to the latter, every continuous global symmetry is associated with a codimension-1 conserved current, referred to as the Noether current

$$J_V = B_V - I_V \Theta \,, \qquad \mathrm{d}J_V = V^i \frac{\delta L}{\delta \varphi^i} \approx 0 \,. \tag{2.21}$$

The notation $\approx$ signifies that the equality is evaluated on-shell, meaning that it is valid when the equations of motion are satisfied. Actually, the relationship (2.21) also involves gauge transformations, but the associated currents are trivial in this case since they can be written in terms of a total exterior spacetime derivative. This will become explicit just below in the subsection 2.1.4 where we attempt to write this current (see in particular (2.26)).

Second Noether theorem

With this disclaimer in mind, we proceed to adapt the above standard Noether's theorem (2.21) in the presence of gauge symmetries, i.e., when both (2.19) and (2.20) are fulfilled. This adaptation, known as the second Noether theorem, asserts that each gauge symmetry leads to an identity among the equations of motion of the Lagrangian:

$$R^i[\lambda] \frac{\delta L}{\delta \varphi^i} = \mathrm{d}S_\lambda \left[\frac{\delta L}{\delta \varphi} \right] , \tag{2.22}$$

where the weakly vanishing Noether current S_λ is defined as

$$S_\lambda \left[\frac{\delta L}{\delta \varphi^i} \right] = \lambda^\alpha \left[R^{i\mu}_\alpha \frac{\delta L}{\delta \varphi^i} - \partial_\nu \left(R^{i(\mu\nu)}_\alpha \frac{\delta L}{\delta \varphi^i} \right) + \dots \right] (\mathrm{d}^{D-1}x)_\mu \,. \tag{2.23}$$

This current is conserved on-shell but also vanishes on-shell. The refining of (2.21) also stipulates that a set of off-shell Noether identities for each gauge parameter can be derived:

$$N_\alpha \left[\frac{\delta L}{\delta \varphi^i} \right] = 0 \,, \tag{2.24}$$

where

$$N_\alpha \left[\frac{\delta L}{\delta \varphi^i} \right] = R^i_\alpha \frac{\delta L}{\delta \varphi^i} - \partial_\mu \left(R^{i\mu}_\alpha \frac{\delta L}{\delta \varphi^i} \right) + \partial_\mu \partial_\nu \left(R^{i(\mu\nu)}_\alpha \frac{\delta L}{\delta \varphi^i} \right) + \dots \,. \tag{2.25}$$

In the next subsection, we will look explicitly at how to construct a conserved quantity associated with gauge symmetries using the above statements, (2.22) and (2.24).

2.1.4 Iyer-Wald surface charge

Rewriting the conserved current of the first Noether theorem (2.21) for a gauge transformation λ, this consists of contracting the local presymplectic form ω via the interior product on the field space along λ. Then, one can demonstrate that it satisfies the following on-shell identity (Lee & Wald, 1990), called fundamental theorem of covariant phase space:

$$I_\lambda \omega[\varphi; \delta\varphi; \delta\varphi] = 2\omega[\varphi; \delta_\lambda\varphi; \delta\varphi] =: \omega_\lambda \approx \mathrm{d}k_\lambda[\varphi; \delta\varphi] \,, \tag{2.26}$$

where the infinitesimal surface charge k_λ is a $(D-2, 1)$-form. We therefore have the on-shell conservation of a codimension-1 current $\mathrm{d}\omega_\lambda \approx 0$ involving a codimension-2 quantity k_λ through an exterior total derivative d. Specifically, we will investigate this relationship in the next few pages for a gauge invariance under diffeomorphisms.

The form k_λ is unique up to the inclusion of a total derivative that does not impact the aforementioned equality (2.26). In other words, one can add to this $(D-2)$-form the divergence of a $(D-3)$-form without altering the charge, thanks to Stokes' theorem. Conversely, using (2.14), we observe that the corner ambiguity alters k_λ as follows:

$$k_\lambda[\varphi; \delta\varphi] \to k_\lambda[\varphi; \delta\varphi] - \delta_\lambda C[\varphi; \delta\varphi] \,. \tag{2.27}$$

Thus, it means that this ambiguity, in turn, impacts the corresponding surface charge that we will define starting from k_λ and will serve as the foundation for the discussed renormalization in sections 2.3 and 2.4, where it will be properly adjusted to cancel the divergences.

Encouragingly, these Iyer-Wald ambiguities do not influence the exact conserved quantities, specifically the charges associated with the rigid (or global) symmetries of the theory. These are generated by the following condition:

$$\delta_\lambda \varphi = 0 \,. \tag{2.28}$$

In our examples of interest, electromagnetism and gravitation, they correspond respectively to the constant global gauge transformations and the isometries of the metric (the Killing vectors). In such a scenario, the condition (2.28) implies, due to linearity, that

$$I_\lambda \omega_C[\varphi; \delta\varphi; \delta\varphi] = \delta_\lambda C[\varphi; \delta\varphi] = \omega_C[\varphi; \delta_\lambda\varphi; \delta\varphi] = 0 \,. \tag{2.29}$$

Consequently, adapting (2.27),

$$k_\lambda[\varphi; \delta\varphi] \to k_\lambda[\varphi; \delta\varphi] \,, \tag{2.30}$$

which implies that if such exact quantities diverge in the asymptotic limit, as we will address later in the context of asymptotic surface charges, we lack a procedure for renormalizing these quantities. Therefore, it becomes imperative to prevent such quantities from diverging, and this consideration becomes a criterion for the selection of gauge choices and the fixation of boundary conditions in the theories under consideration.

Diffeomorphism-invariant gauge theory

Up to the last paragraph, the entire discussion is applicable to any gauge theory. We now focus on a diffeomorphism-invariant theory, such as general relativity. In this case, it is possible to find an explicit expression for the conserved current (2.26) in the formalism used in this section. In the next section, we present an alternative method for obtaining such quantities for any gauge symmetry.

For diffeomorphisms, the gauge parameters are vectors ξ on spacetime, implying that (2.19) can be reformulated as

$$\mathfrak{L}_\xi L = \mathscr{L}_\xi L = \mathrm{d}(\iota_\xi L)\,, \tag{2.31}$$

where Cartan's spacetime formula (2.2) and the spacetime top form aspect of the Lagrangian form have been used. In such a case, the field space vector V referring to the spacetime vector ξ is defined as follows:

$$V_\xi = \int \mathrm{d}^D x\, \mathscr{L}_\xi \varphi^i \frac{\delta}{\delta \varphi^i}\,, \tag{2.32}$$

where

$$\delta_\xi \varphi^i = \mathscr{L}_\xi \varphi^i\,, \qquad \xi = \xi^\mu \partial_\mu\,. \tag{2.33}$$

Thus, the Noether current (2.21) reads

$$J_\xi = \iota_\xi L - I_{V_\xi}\Theta[\varphi;\delta\varphi] = \iota_\xi L - \Theta[\varphi;\delta_\xi\varphi] \tag{2.34}$$

and obeys Noether's second theorem (2.22)

$$\mathrm{d}(\iota_\xi L) - \mathrm{d}\Theta[\varphi;\delta_\xi\varphi] = \mathrm{d}S_\xi\left[\frac{\delta L}{\delta\varphi}\right], \tag{2.35}$$

which, using the generalized (or algebraic) Poincaré's lemma, yields

$$J_\xi = S_\xi + \mathrm{d}Q_\xi\,. \tag{2.36}$$

The last equation can be integrated to obtain an expression for the $(D-2,0)$-form Q_ξ, known as the Noether-Wald surface charge (Wald, 1993; Iyer & Wald, 1994):

$$Q_\xi = h_\xi^{D-1}(J_\xi - S_\xi)\,, \tag{2.37}$$

plus possible total derivative of a $(D-3)$-form, where the homotopy operator h_ξ^p (Barnich & Compère, 2008) is introduced. Its action on a spacetime p-form w is defined as

$$h_\xi^p w = \sum_{(\mu)} \frac{|\mu|+1}{D-p+|\mu|+1} \partial_{(\mu)} \left(\xi^\alpha \frac{\partial}{\partial \partial_\nu \partial_{(\mu)} \xi^\alpha} \frac{\partial}{\partial \mathrm{d}x^\nu} \right) w\,, \tag{2.38}$$

and it satisfies the property:

$$\mathrm{d}h_\xi^p + h_\xi^{p+1}\mathrm{d} = \mathbb{I}\,. \tag{2.39}$$

It therefore transforms a (p,q)-form into a $(p-1,q)$-form. In this context, $\mathbb{I}$ represents the identity operator. The definition (2.38) implies that, when the homotopy operator is applied to a form that is independent of the derivatives of the diffeomorphism parameter, the result is identically zero. It holds for both S_ξ and $\iota_\xi L$. Consequently, the standard expression for the Noether-Wald surface charge (2.37) is given by

$$Q_\xi[\varphi] = -h_\xi^{D-1}\Theta[\varphi; \delta_\xi \varphi]\,. \tag{2.40}$$

In the case of a diffeomorphism-invariant theory, it can be demonstrated that the codimension-2 form k_ξ, as introduced in (2.26), can be expressed on-shell in terms of the presymplectic potential and the Noether-Wald charge (2.40), as outlined in (Iyer & Wald, 1994):

$$k_\xi[\varphi; \delta\varphi] \approx \delta Q_\xi[\varphi] - Q_{\delta\xi}[\varphi] - \iota_\xi \Theta[\varphi; \delta\varphi]\,. \tag{2.41}$$

The expression $Q_{\delta\xi}[\varphi]$, where $\delta\xi[\varphi^i] = \xi[\delta\varphi^i]$, emerges when the diffeomorphism parameters exhibit field dependence. The equation (2.41) in the subsequent context is commonly referred to as the Iyer-Wald codimension-2 form.

According to (Wald, 1993), the variation of the charge linked to a vector symmetry ξ is defined in the following manner:

$$(\not{\delta} H_\xi)[\varphi; \delta\varphi] = I_\xi \Omega[\varphi; \delta\varphi; \delta\varphi] = 2\Omega[\varphi; \delta_\xi\varphi; \delta\varphi] := \Omega_\xi\,. \tag{2.42}$$

We would like to stress that it does not correspond to the charge but rather its variation, since it still depends on $\delta\varphi$. In the next subsection, we will explain the $\not{\delta}$-notation and how to integrate this variation where possible. The inspiration for the last equation comes from the classical mechanics definition of Hamiltonian charges using Hamilton's equation on phase space. The corresponding Noether current (2.36) remains well-conserved on-shell, facilitated by the spacetime exterior derivative nilpotent nature. By leveraging the fundamental theorem of covariant phase space formalism (2.26), applying Stokes' theorem, and choosing a Cauchy slice with a boundary $\partial\Sigma = \mathcal{C}$, the last equation can be reformulated as a codimension-2 charge:

$$\delta\!\!\!/ H_\xi[\varphi; \delta\varphi] = \int_{\mathcal{C}} k_\xi[\varphi; \delta\varphi] \,. \tag{2.43}$$

In conventional literature, this quantity is referred to as the variation of the surface charge, while more recent works may use the term corner charge due to the fact that it lives on the corner $\mathcal{C}$.

Asymptotic symmetry program

We can now articulate the notion of asymptotic symmetries in gauge theories. Specifically, to discern whether a symmetry qualifies as a asymptotic symmetry or merely a pure gauge transformation, one must ascertain the non-vanishing of the variation of the surface charge (2.43) as we approach the asymptotic boundary. These (physical) asymptotic symmetries are conceived as a proper generalization of global symmetries in the context of gauge theory and, as such they map, e.g., certain solutions of the equations of motion into physically inequivalent ones.

To embark on our study, the initial step involves defining the theory being examined, establishing the dynamics on the manifold $\mathcal{M}$. Following this, the second and third steps are to impose boundary conditions on the fields, i.e. the behavior, or falloffs, of the bulk fields near the boundary $\Gamma|_{\mathcal{B}}$, alongside enforcing gauge fixing conditions on these fields. The last aspect warrants careful consideration as it is often mishandled in existing literature. Ideally, one should compute the conserved charges (2.43) of the theory devoid of gauge fixing, demonstrating that the gauge symmetry required for fixing the gauge is merely a trivial symmetry transformation. However, physical and technical constraints sometimes hinder the computation of corner charges without such restrictions on the field space Γ. This thesis aims to address these challenges by relaxing boundary and gauge conditions.

The fourth step entails determining the residual symmetries, which are gauge parameters λ (2.20) that preserve the preceding steps. These symmetries are also known as allowed gauge transformations. These contribute to the (Lee-Wald) presymplectic form (2.12) through possible zero modes. Due to the latter, this form is not invertible, justifying the prefix "pre" in "presymplectic". Such modes can be removed by determining the surface charges (2.43) associated with each residual symmetry in the case the boundary $\mathscr{B}$ is asymptotic (i.e. in the limit $r \to \infty$). A vanishing charge signifies a trivial transformation, indicating redundancy within the system. These peculiar residual symmetries are alternatively termed proper or small gauge transformations and represent the zero modes of (2.15). Conversely, a non-vanishing value of the charge identifies the symmetry as an asymptotic symmetry. These are physical modifications to the field content of the theory, leading to distinct configurations. Such symmetries are referred to improper or large gauge transformations.

Since the trivial symmetry group acts as an ideal within the residual symmetry group, we subsequently define the asymptotic symmetry group via the following quotient:

$$\text{Asymptotic symmetries} = \frac{\text{Improper gauge transformations}}{\text{Proper gauge transformations}}\,. \tag{2.44}$$

As a consequence, by limiting the (Lee-Wald) presymplectic form (2.12) exclusively to asymptotic symmetries, it becomes invertible. This action effectively eradicates the zero modes, thereby leading to the emergence of a symplectic form (without the prefix) for the theory.

The final step, the fifth and ultimate, entails computing the Poisson bracket of charges, which yields the algebra organizing the physical observables in the theory. However, as we will explore shortly, executing this last step can be challenging due to complexities stemming from the charge calculation. Let us be more specific and list the three difficulties that can arise. In particular, we shall see that a priori there is no guarantee that the variation of the charge (2.43) is well-defined in the case of asymptotic boundaries due to radial divergences.

2.1.5 Properties of the surface charge

Integrability

For this particular discussion, let us return to the case of diffeomorphism-invariant theories. This can be done for any gauge theory, but the expressions in output are simpler and more widely used in this context with respect to the machinery à la Iyer-Wald.

In the preceding equations (2.42) and (2.43), we opted for the notation $\delta\!\!\!/$ over just δ. This decision stems from the fact that contracting the presymplectic 2-form with a diffeomorphism does not always result in a δ-exact term;

specifically, k_ξ may not be an exact $(D-2,1)$-form. For diffeomorphism-invariant theories, we can demonstrate the following on-shell relation using (2.26) and (2.34):

$$I_{V_\xi}\omega \approx \mathrm{d}\iota_\xi\Theta - \delta J_\xi\,. \tag{2.45}$$

This expression does not produce a δ-exact form unless $\mathrm{d}\iota_\xi\Theta$ is zero. In such cases, we label the charge as non-integrable, which significantly impacts the charge algebra. Actually, as discussed in the last paragraph of this subsection, if the corner charge is non-integrable, the associated algebra does not close on itself, and Poisson's bracket cannot be employed. It is important to note that this issue will not arise in the remainder of this manuscript except, apparently, in three-dimensional Bondi gravity in the chapter 4 but which can be solved by an adequate field redefinition via the associated Pfaff problem (Darboux, 1882; Barnich & Compère, 2008; Grumiller et al., 2020a). However, if it is feasible to write such a contraction as a δ-exact form, we write (2.43) with δH_ξ, which can be integrated subsequently into the corner charge H_ξ. In such instances, the on-shell expression for the Noether current (2.34) becomes:

$$\delta J_\xi \approx -\mathrm{d}k_\xi\,. \tag{2.46}$$

In higher-dimensional cases ($D>3$), which are not covered in this thesis, one possible physical interpretation of the non-integrability is that the system is open, suggesting the presence of physical degrees of freedom that originate from the bulk and extend to the boundary. Consequently, predicting the evolution on the boundary seems unpredictable from a single surface due to dissipation. Further details on this aspect are unnecessary for the thesis. To address this situation, one can interpret the non-integrable part as a symplectic flux,

$$F_\xi = \int_\Sigma \mathrm{d}\iota_\xi\Theta\,, \tag{2.47}$$

and partition the charges into integrable and flux components. For more information, the reader can refer to (Barnich & Troessaert, 2011; Troessaert, 2016; Wieland, 2022).

Another approach involves extending the field space, resulting in:

$$I_{V_\xi}\omega^{\text{ext}} \approx -\delta J_\xi\,. \tag{2.48}$$

This study is linked to edge modes and the so-called corner proposal, and readers interested in exploring this topic further are encouraged to refer to (Ciambelli & Leigh, 2021; Freidel, 2021; François et al., 2021; Ciambelli et al., 2022).

Conservation

Another recurring challenge in dealing with charges, which in turn is encountered frequently in the various examples explored in this thesis, is the charge non-conservation. When applied to an arbitrary gauge symmetry, assuming the integrability of surface charges, the conservation equation reads

$$\delta H_\lambda\Big|_{\mathcal{C}_2} - \delta H_\lambda\Big|_{\mathcal{C}_1} \approx \int_{\mathcal{S}} \mathrm{d}k_\lambda = \int_{\mathcal{S}} I_\lambda \omega \,. \tag{2.49}$$

Please note that we have now reverted to a discussion that employs notations applicable to all gauge theories.

Thanks to the last equation (2.49), we see that the charge is conserved if $I_\lambda \omega$ vanishes on $\mathcal{S}$, a codimension-1 surface delineating the two codimension-2 sections $\mathcal{C}_1$ and $\mathcal{C}_2$ on the Cauchy slice. This non-conservation is attributed to the leakage of physical information between the two codimension-2 sections. In the context of gravity, this is associated with the presence of gravitational fluxes through the surface. For more in-depth information and comprehensive literature on this subject, we refer to (Fiorucci, 2021). Another possible explanation is the presence of anomalies, arising from a non-stationary variational problem for the action principle. In this context, by anomaly we mean that the latter observation indicates the presence of a non-conserved current in dual theory. This is a problem discussed, for example, in (Alessio et al., 2021; Fiorucci & Ruzziconi, 2021; Campoleoni et al., 2022; Ciambelli et al., 2023). We will delve into this aspect more thoroughly in the holographic renormalization procedure and explicitly explore it in the various examples covered in the subsequent chapters.

Finiteness

The third and final challenge encountered when studying surface charges is radial divergence, specifically arising when considering an asymptotic boundary. This aspect constitutes one of the focal points in our approach to crafting this manuscript. The finite nature of charges can only be assured if the boundary is located at a finite distance in spacetime. This constraint is crucial as the action principle itself might diverge, impacting the derivation of corner charges.

To facilitate the following discussion, we reconsider the radial isosurface component $\mathcal{B}$ of $\partial\mathcal{M}$ and we break down the coordinates of this boundary into timelike and spacelike coordinates, denoted as $(x^a) = (t, x^i)$, where i pertains to spacelike coordinates on $\mathcal{B}$ – distinct from the collection of fields in the field space. Despite employing the same symbol, the context will consistently distinguish between the two. Note that these coordinates x^i are the ones along the corner $\mathcal{C}$. Moreover, in our asymptotic case of interest, one conceives this codimension-2 surface as the intersection of $\mathcal{B}$ with an isosurface of the time coordinate t on a neighborhood of $\mathcal{B}$. Indeed, in this context, assuming integrability (a condition verified throughout this thesis), the variation of the surface charge (2.43), associated with an arbitrary gauge symmetry for an asymptotic boundary located at $r \to \infty$, can be rewritten as:

$$\delta H_\lambda \approx \lim_{r\to\infty} \int_{\mathcal{C}} \mathrm{d}^{D-2}x \, k_\lambda^{tr} \,. \tag{2.50}$$

Due to the Iyer-Wald definition (2.26) of the presymplectic structure, the codimension-2 form k_λ satisfies the following relation:

$$\partial_r k_\lambda^{tr} + \partial_i k_\lambda^{ti} \approx \omega_\lambda^t \,. \tag{2.51}$$

This implies that if the timelike component of the presymplectic form ω_λ^t does not decay fast enough for large r, the above integration (2.50) of k_λ^{tr} over the corner $\mathcal{C}$ and thus the charge H_λ will diverge in the limit $r \to \infty$, rendering it ill-defined.

Recent literature has addressed these three above properties. Historically, a "good asymptotic charge" needed to be integrable, conserved, and finite. However, recent years have seen a relaxation of these conditions, and this thesis particularly focuses on relaxing the finiteness aspect of charges by proposing to renormalize charges that would not have been accepted in earlier literature.

2.1.6 Charge algebra

Charge algebra constitutes a compelling and crucial structure within the realm of asymptotic symmetries. Particularly, in the pursuit of developing a quantum theory of gravitation, the Poisson brackets that define the charge algebra serve as the commutators for quantum observables linked to charges through the conventional quantization. This algebra then plays a pivotal role in organizing quantum observables, a significance further emphasized in a holographic context.

The charge algebra is established based on the definition of a Hamiltonian vector field $\delta_\lambda \varphi$, which, as derived from (2.26) and (2.43), reads

$$I_\lambda \Omega[\varphi; \delta\varphi; \delta\varphi] \approx \delta H_\lambda[\varphi] \,. \tag{2.52}$$

At this juncture, we assume that the surface charges can be integrated. By contracting this expression once more, it can be demonstrated that it gives rise to a Poisson bracket structure:

$$\{H_{\lambda_1}[\varphi], H_{\lambda_2}[\varphi]\} := I_{\lambda_2} I_{\lambda_1} \Omega[\varphi; \delta\varphi; \delta\varphi] \approx \delta_{\lambda_2} H_{\lambda_1}[\varphi] \,. \tag{2.53}$$

Subsequently, one can show this Poisson bracket of charges can be isomorphically written as the modified Lie algebra of gauge parameters, with the addition of a field-independent central extension:

$$\{H_{\lambda_1}[\varphi], H_{\lambda_2}[\varphi]\} = H_{[\lambda_1,\lambda_2]_\star}[\varphi] + \kappa_{\lambda_1,\lambda_2} \,, \qquad \delta\kappa_{\lambda_1,\lambda_2} = 0 \,. \tag{2.54}$$

Thus, the charge algebra represents the symmetry algebra projectively. The modified Lie bracket accommodates the potential field dependence of gauge parameters by altering the standard Lie bracket $[\bullet, \bullet]$ as follows (Schwimmer & Theisen, 2008; Barnich & Troessaert, 2010),

$$[\lambda_1, \lambda_2]_\star = [\lambda_1, \lambda_2] - \delta_{\lambda_2}\lambda_1 + \delta_{\lambda_1}\lambda_2 \,. \tag{2.55}$$

Furthermore, the central extension satisfies a 2-cocycle condition on the modified Lie algebra:

$$\kappa_{[\lambda_1,\lambda_2]_\star,\lambda_3} + \text{cyclic}(1,2,3) = 0 \,. \tag{2.56}$$

However, it is worth noting that if the corner charges are non-integrable, the bracket needs modification into the Barnich-Troessaert bracket (Barnich & Troessaert, 2011), for example, which incorporates a bracket between integrable parts to resolve the ambiguity in the above-mentioned split between integrable charges and fluxes. This results in a field-dependent central extension.

2.2. Barnich-Brandt prescription

To comprehensively understand the procedures for determining asymptotic surface charges in the Lagrangian approach, although we will not be utilizing it further in this thesis due to the inherent lack of renormalization scheme, we briefly introduce the Barnich-Brandt prescription (Barnich & Brandt, 2002) in this section. We compare it with the Iyer-Wald approach of the last section for conserved quantities in general relativity.

The core concept behind Abbott-Deser's approach is to leverage the inherent structure of the equations of motion (Abbott & Deser, 1982b)[4], stemming from the Euler-Lagrange derivatives (2.11). This involves employing an

[4]This method was also employed in (Abbott & Deser, 1982a) to derive the conserved

integration-by-parts procedure to deduce a conserved charge, aligning with the foundational idea encapsulated in the Euler-Lagrange resolution (Tulczyjew, 2006). Instead of delving into this method, we focus on its formalization. Actually, the Barnich-Brandt formalism systematizes this approach for any gauge theory within the covariant phase space, akin to the Iyer-Wald method. It achieves this by utilizing a symplectic structure to construct a conserved surface charge. While refraining from delving into intricate details, it is worth noting that the former relies on a more formal homotopy operator compared to the latter when performing integrations by parts to encompass all gauge theories.

The Anderson's homotopy operator, defined in the Grassmann odd convention for the field space exterior derivative (Anderson, 1989), is articulated as follows:

$$\mathcal{H}^p_{\delta\varphi} w = \sum_{(\mu)} \frac{|\mu|+1}{D-p+|\mu|+1} \partial_{(\mu)} \left(\delta\varphi^i \frac{\partial}{\partial\partial_\nu \partial_{(\mu)}\varphi^i} \frac{\partial}{\partial \mathrm{d}x^\nu} \right) w \,. \tag{2.57}$$

It transforms the (p, q)-form w into a $(p-1, q+1)$-form. The operator (2.57) abides by the following relations:

$$\delta = \delta\varphi \frac{\delta}{\delta\varphi} - \mathrm{d}\mathcal{H}^D_{\delta\varphi} \quad \text{(when acting on a spacetime top-form)}, \tag{2.58}$$

$$\delta = \mathcal{H}^{p+1}_{\delta\varphi}\mathrm{d} - \mathrm{d}\mathcal{H}^p_{\delta\varphi} \quad \text{(when acting on a spacetime } p\text{-form } (p < D)). \tag{2.59}$$

It also commutes with δ, i.e. $[\mathcal{H}^p_{\delta\varphi}, \delta] = 0$. The definition (2.57) is more formal than (2.38) but shares some similarities with it. This is because it acts on expressions dependent on the fields, though not necessarily on the gauge parameters. One can view the operator (2.38) as a specific instance of (2.57) applied to gauge parameters, specifically the ones related to diffeomorphisms. To illustrate the use of this formal operator, consider the rewriting of the local presymplectic potential (2.10) as

$$\Theta[\varphi; \delta\varphi] = \mathcal{H}^D_{\delta\varphi} L[\varphi] \,, \tag{2.60}$$

plus a possible total spacetime derivative, which represents the aforementioned corner ambiguity in this formalism.

Whereas Lee-Wald's symplectic structure (2.12) relies on the action principle, Barnich-Brandt's is solely defined in terms of the equations of motion:

$$W[\varphi; \delta\varphi; \delta\varphi] = \frac{1}{2}\mathcal{H}^D_{\delta\varphi} \left(\delta\varphi^i \frac{\delta L}{\delta\varphi^i} \right) . \tag{2.61}$$

Due to this characteristic, we designate the latter as an "invariant" presymplectic form. This definition is unambiguous, in contrast to the earlier prescription (Barnich, 2003). Indeed, Barnich-Brandt's approach resolves the ambiguities present in the Iyer-Wald method:

$$\omega[\varphi;\delta\varphi;\delta\varphi] = W[\varphi;\delta\varphi;\delta\varphi] + \mathrm{d}E[\varphi;\delta\varphi;\delta\varphi]\,, \tag{2.62}$$

where the corner term reads

$$E[\varphi;\delta\varphi;\delta\varphi] = \frac{1}{2}\mathcal{H}^{D-1}_{\delta\varphi}\Theta = \frac{1}{2}\mathcal{H}^{D-1}_{\delta\varphi}\mathcal{H}^{D}_{\delta\varphi}L\,. \tag{2.63}$$

This establishes the connection between the two symplectic forms. By applying Noether's second theorem (2.22) and contracting (2.61) with an arbitrary gauge parameter, one can demonstrate the following on-shell relation akin to the fundamental theorem of the covariant phase space (2.26), namely

$$W[\varphi;\delta_\lambda\varphi;\delta\varphi] \approx \mathrm{d}k^{\mathrm{BB}}_\lambda[\varphi;\delta\varphi]\,. \tag{2.64}$$

In this context, an expression for the codimension-2 form can be derived for an arbitrary gauge symmetry (Barnich & Brandt, 2002):

$$k^{\mathrm{BB}}_\lambda[\varphi;\delta\varphi] \approx -\mathcal{H}^{D-1}_{\delta\varphi}S_\lambda\left[\frac{\delta L}{\delta\varphi}\right]. \tag{2.65}$$

Similarly to (2.62), it is possible to connect the codimension-2 forms of Iyer-Wald (2.26) and Barnich-Brandt (2.65) through the use of the corner term (2.63):

$$k_\lambda[\varphi;\delta\varphi] \approx k^{\mathrm{BB}}_\lambda[\varphi;\delta\varphi] + E[\varphi;\delta_\lambda\varphi;\delta\varphi]\,. \tag{2.66}$$

Comparison of various prescriptions

To conclude this first half of the chapter, we take the liberty of providing a personal comparison of the strengths and weaknesses of the various methodologies we have reviewed for determining asymptotic surface charges. The historically oldest and most intuitive approach is the case-by-case integration by parts based on the Euler-Lagrange equations à la Abbott-Deser (Abbott & Deser, 1982b). This method, while natural, was later generalized by Barnich and Brandt to accommodate arbitrary gauge theories (Barnich & Brandt, 2002) through the introduction of a covariant phase space (Gawedzki, 1991;

charge in a non-Abelian gauge theory and was further applied in (Deser & Tekin, 2002, 2003) to higher curvature theories.

Kijowski, 1973; Kijowski & Szczyrba, 1976) and formal operators within the variational bicomplex framework (Anderson, 1989).

While offering a formal and generic expression for conserved charges applicable to any gauge symmetry, its derivation relies on a symplectic structure dependent solely on the equations of motion. In the presence of radial surface charge divergences, as discussed in (2.50), finding a renormalization procedure for the charge becomes challenging. As previously mentioned and further detailed in the upcoming sections, these divergences arise due to the diverging nature of the underlying variational principle. Renormalizing the charges is then possible using action renormalization methods inspired by analogies in field theories. Note that it might be possible to manually find an appropriate corner counterterm at the level of the associated presymplectic form. This process can be quite tedious to ensure that it will effectively cancel the radial divergences of the charge.

To facilitate such an examination, it is preferable for the symplectic structure to depend explicitly on the action. This condition is met in the charge derivation prescription proposed by Lee-Iyer-Wald-Zoupas (Lee & Wald, 1990; Wald, 1993; Iyer & Wald, 1994; Wald & Zoupas, 2000). The formal nature of this approach finds significance in its dependence on the action and the presence of inherent ambiguities in its symplectic structure, which will be strategically utilized in charge renormalization. However, the Iyer-Wald procedure lacks a generic and formal expression for the codimension-2 form in the broader context of arbitrary gauge invariant Lagrangian. Original papers and reviews focus primarily on theories invariant under diffeomorphism, making it a case-by-case study for such expressions in other gauge theories. In order to extend this setup, due to its clarity in providing a renormalization procedure for the symplectic structure and associated surface charges, we favor the Iyer-Wald prescription throughout the remainder of the manuscript. However, we will try as far as possible to make the link with Barnich-Brandt's formulation.

The next section 2.3 will review the renormalization of the variational principle, known as holographic renormalization. This process fixes the Iyer-Wald ambiguities via the boundary counterterms to be added to the action, allowing for the subsequent renormalization of the symplectic structure and charges. Finally, in the last section 2.4, we will propose a systematic approach for eliminating divergent terms by profiting from ambiguities, termed symplectic renormalization. This approach enables the consistent renormalization of the presymplectic potential without delving into the more involved renormalization aspects of the Lagrangian form.

2.3. Holographic renormalization

For the remainder of this thesis, as justified in the previous paragraphs, we will consistently adopt the choice of charges aligned with renormalization schemes within the covariant phase space formalism, following the approach introduced by Iyer-Wald. As emphasized in subsection 2.1.4, where we explored challenges associated with surface charges linked to asymptotic symmetries, the action principle (2.9), the Lee-Wald symplectic structure (2.12), and the corresponding corner charges (2.43) may exhibit radial divergences (2.50).

Traditionally, one might have asserted that diverging charges are not viable as physical charges. However, a procedure at the level of the variational principle has proven to be both useful and necessary to make sense of these theories. This approach, known as holographic renormalization, was introduced and developped by Skenderis and collaborators (Henningson & Skenderis, 1998; Balasubramanian & Kraus, 1999; Skenderis, 2001; de Haro et al., 2001; Bianchi et al., 2002; Papadimitriou & Skenderis, 2005a). Remarkably developed over the past two decades, this method has yielded significant implications in the realm of holographic dualities such as the AdS/CFT correspondence and has contributed to the analysis of asymptotic symmetries (Mann & Marolf, 2006; Compere & Marolf, 2008; Papadimitriou, 2010; Compère et al., 2018; Anastasiou et al., 2020; Chandrasekaran et al., 2022).

The main idea of this prescription originates from the gauge/gravity correspondence (Maldacena, 1998; Gubser et al., 1998; Witten, 1998; Aharony et al., 2000; D'Hoker & Freedman, 2002). In the dual quantum field theories, it is well-established that correlation functions can display ultraviolet divergences. To address this issue, one must undertake the process of renormalization to make sense of these divergences. When analyzing asymptotic symmetries, we may encounter radial divergences, which are long-range and can be considered in the infrared (IR) regime. These radial divergences are connected to the ultraviolet (UV) divergences of dual quantum field theories through the UV/IR connection (Susskind & Witten, 1998). To handle these bulk infrared divergences and simultaneously address the dual ultraviolet ones, a proposal was made to renormalize the gravitational side based on the analogous process carried out on the dual quantum side (Henningson & Skenderis, 1998). In the subsequent discussion, we will not delve into the holographic dual details. We shall present the general outlines of this prescription so that it is applicable to any gauge theory.

The underlying concept of this renormalization approach involves adding to the variational principle (2.9), defined on the spacetime manifold $\mathcal{M}$ (referred to as the bulk action), appropriate counterterms at each boundary – specifically, at the asymptotic spacetime boundary $\mathcal{B}$ and at the corner $\mathcal{C}$:

$$S_{\text{ren}} = \int_{\mathcal{M}} L + \int_{\mathcal{B}} L_B + \int_{\mathcal{C}} L_C \,. \tag{2.67}$$

These counterterms serve the purpose of cancelling the radial divergences within the bulk action while simultaneously renormalizing the associated symplectic structure and charges, as discussed by (Compere & Marolf, 2008). We shall come back to this explicitly in the next few pages.

The success of this renormalization procedure is not guaranteed universally; rather, its efficacy has been demonstrated in specific cases in the literature. Consequently, this prescription lacks a systematic guarantee of success and operates on a case-by-case basis. While this approach is well-established and refined when the bulk manifold $\mathcal{M}$ is the Anti de Sitter space, particularly owing to its connections with the AdS/CFT correspondence and the advantageous feature that the asymptotic boundary is a timelike hypersurface, challenges arise when the boundary is null infinity. These challenges stem from computational complexities, as encountered with non-local terms when studying flat space (Mann & Marolf, 2006). We shall be more specific about these aspects as we progress through the manuscript.

Nevertheless, for the sake of technical comprehensiveness and historical context, we choose to initially present holographic renormalization in this section. It is important to note that this renormalization scheme holds paramount significance if one is pursuing a finite variational principle. This becomes particularly crucial in scenarii where one wants to obtain a holographic interpretation via, for example, the computation of correlation functions in dual conformal theory. Although this is not the central focus of this thesis, we will encounter this challenge multiple times in various examples, necessitating our engagement with this potentially intricate scheme. In chapter 3, we illustrate the simplest example by examining the variational principle associated with the propagation of a free massless spin 1 particle. This example serves to demonstrate, in a tangible and straightforward manner, the procedural steps involved in holographic renormalization.

First step: Asymptotic solution

We consider a bulk field theory (2.9) defined on a background manifold $\mathcal{M}$ with the metric $g_{\mu\nu}$, where the coordinates are represented as $(x^\mu) = (r, x^a)$. We recall that, according to the appendix A, r is a radial coordinate chosen such that the asymptotic boundary is located at $r \to \infty$, and x^a are the coordinates along this boundary. The set of bulk fields in this theory is denoted by $\varphi = (\varphi^i)$. The initial step involves assuming that near the boundary, all fields exhibit a polyhomogeneous asymptotic expansion in terms of boundary fields given by:

$$\varphi(r, x^a) = \sum_n \frac{1}{r^n} \left(\varphi^{(n)}(x^a) + \log r\, \widetilde{\varphi}^{(n)}(x^a)\right). \tag{2.68}$$

The field equations (2.11) are then solved iteratively by treating the $\frac{1}{r}$-variable as a small parameter. The summation range in (2.68) is a priori arbitrary. But it is set a fortiori by imposing boundary conditions on the fields, i.e., choosing how they behave near the asymptotic boundary $\mathcal{B}$. The fixing of boundary conditions on the fields can be established either manually, guided by the equations of motion, through associated charges with exact symmetries, or through a posteriori analysis of asymptotic charges. Moreover, we will observe that relaxing these conditions can be intriguing from the perspective of asymptotic symmetries. This concept will be elucidated through various examples in the thesis, with the Maxwell field providing initial insights.

In many cases, certain orders of the asymptotic expansion (2.68), or derivatives thereof, can remain undetermined by the equations of motion. These undetermined terms constitute the degrees of freedom of the theory. If one opts for a polynomial expansion by excluding the possibility of logarithmic terms, some arbitrary functions that are expected by other considerations may become fixed by the equations of motion. This justifies the inclusion of logarithmic terms, which, from a holographic standpoint, can be associated with anomalies in the dual theory (Skenderis, 2002).

Second step: Regularization of the theory

The next step consists in the computation of the on-shell value of the action, which provides a boundary term. If this value diverges as the asymptotic boundary $\mathcal{B}$ is approached ($r \to \infty$), it must be regularized. To achieve this, a regularization cut-off, denoted as $R > r$, is introduced, such that R is a large parameter. The boundary term is then evaluated at $r = R$. This is the regulated boundary, denoted as $\partial\mathcal{M}_R$ and introduced at the beginning of subsection 2.1.1. This process allows the isolation of a finite number of divergent terms, resulting in the following expression for the on-shell value of the bulk action (2.9):

$$S_{\text{reg}} \approx \int_{\partial\mathcal{M}_R} \mathrm{d}^{D-1}x \sqrt{-g} \left[R^k \sum_{n=0}^{k-1} \frac{1}{R^n} \left(a_{(n)} + \log R\, \widetilde{a}_{(n)}\right) + \log R\, \widetilde{a}_{(k)} + \mathcal{O}(1)\right]. \tag{2.69}$$

Here, k is a positive number determined by the specific fixed boundary conditions and the asymptotic resolution of the equations of motion, in the same way that $a_{(n)}$, $\widetilde{a}_{(n)}$ and $\widetilde{a}_{(k)}$ depend on the boundary fields and their derivatives.

In particular, these divergent terms must be local functions of the source. We shall clarify what we mean by the latter with a holographic interpretation. This requirement is so as not to break the basic cornerstone underlying the variational principle, namely locality. Indeed, otherwise the Lagrangian counterterm could be nonlocal in time.

Third step: Counterterms to the bulk action

The third step is to define the counterterms as minus the divergent terms present in the regularized action (2.69), i.e.

$$S_{\text{ct}} = -\int_{\partial\mathcal{M}_R} \mathrm{d}^{D-1}x\,\sqrt{-g}\left[R^k \sum_{n=0}^{k-1} \frac{1}{R^n}\left(a_{(n)} + \log R\,\widetilde{a}_{(n)}\right) + \log R\,\widetilde{a}_{(k)}\right]. \tag{2.70}$$

Subsequently, this expression is covariantized by writing it in terms of the fields living at the regulating surface $r = R$, where the induced metric is denoted as $\gamma_{\mu\nu} = g_{\mu\nu}/R$. To achieve this, the asymptotic expansions are inverted, such that $\varphi^{(n)} = \varphi^{(n)}(\varphi(R, x^a), R)$. The obtained result, along with the induced metric, is then substituted into (2.70). This procedure yields the covariant counterterms that are to be added to the bulk action.

Fourth step: Renormalized action

The subtracted action at the regularization cut-off is defined as

$$S_{\text{sub}} = S + S_{\text{ct}}\,. \tag{2.71}$$

In the limit $R \to \infty$, this on-shell action converges to a finite expression by construction. This finite term corresponds to the on-shell value of the renormalized action:

$$S_{\text{ren}} = \lim_{R\to\infty} S_{\text{sub}}\,. \tag{2.72}$$

Fifth step: Finite surface charges

The fifth and final step involves determining the renormalized surface charges based on the associated on-shell renormalized action (2.72). The objective is to extend this action and adhere to the entire Iyer-Wald prescription outlined in section 2.1. This process aims to derive a symplectic structure, inherently renormalized, by utilizing equations (2.9) and (2.10),

$$\delta S_{\text{ren}} \approx -\int_{\mathcal{B}} \Theta_{\text{ren}} \,, \tag{2.73}$$

which also encompasses the surface charges linked to the gauge symmetries of the theory. These surface charges are determined through the equations (2.12), (2.26), and (2.43):

$$\omega_{\text{ren}} = \delta\Theta_{\text{ren}} \,, \qquad I_\lambda \omega_{\text{ren}} \approx \mathrm{d}k_\lambda^{\text{ren}} \,, \qquad \delta\!\!\!/ H_\lambda^{\text{ren}} = \int_{\mathcal{C}} k_\lambda^{\text{ren}} \,. \tag{2.74}$$

It is essential to note that this procedure is not applicable to the Barnich-Brandt prescription since its symplectic structure is independent of bulk action, as previously elucidated.

At the level of the presymplectic potential, the holographic renormalization aligns with the Compère-Marolf prescription (Compere & Marolf, 2008), which resolves the Iyer-Wald ambiguities through covariant counterterms for the purpose of renormalization. To illustrate this alignment, let us express (2.71) in terms of Lagrangian forms as

$$L_{\text{sub}} = L + \mathrm{d}L_{\text{ct}} \,, \tag{2.75}$$

where

$$S_{\text{sub}} = \int_{\mathcal{M}_R} L_{\text{sub}} \,, \qquad S_{\text{ct}} = \int_{\partial\mathcal{M}_R} L_{\text{ct}} \,. \tag{2.76}$$

Under an arbitrary variation of fields (2.10), the subtracted Lagrangian form (2.75) transforms as follows,

$$\delta L_{\text{sub}} = \frac{\delta L}{\delta\varphi}\delta\varphi - \mathrm{d}\left(\Theta + \frac{\delta L_{\text{ct}}}{\delta\chi}\delta\chi\right) , \tag{2.77}$$

where the Grassmann odd convention is used and we leverage the nilpotent aspect of the spacetime exterior derivative, $\mathrm{d}^2 = 0$. The collection of boundary fields and background structures entering the formulation of L_{ct} is denoted as $\chi = (\chi^i)$. This involves:

$$\delta L_{\text{ct}} = \frac{\delta L_{\text{ct}}}{\delta\chi}\delta\chi - \mathrm{d}\Theta_{\text{ct}} \,. \tag{2.78}$$

Considering that the addition of a boundary Lagrangian to the bulk Lagrangian does not alter the equations of motion of the bulk theory, it follows that:

$$\delta L_{\text{sub}} = \frac{\delta L_{\text{sub}}}{\delta\varphi}\delta\varphi - \mathrm{d}\Theta_{\text{sub}} \,, \qquad \frac{\delta L_{\text{sub}}}{\delta\varphi} = \frac{\delta L}{\delta\varphi} \,. \tag{2.79}$$

This implies that the subtracted presymplectic potential can be expressed as:

$$\Theta_{\text{sub}} = \Theta + \frac{\delta L_{\text{ct}}}{\delta \chi}\delta\chi = \Theta + \delta L_{\text{ct}} + \mathrm{d}\Theta_{\text{ct}}\,. \tag{2.80}$$

Furthermore, in relation to the definition of the Iyer-Wald ambiguities (2.13), it is evident that these ambiguities are resolved as follows: the boundary ambiguity equals the counterterm Lagrangian, and the corner ambiguity corresponds to minus the presymplectic potential associated with the latter. This is the Compère-Marolf prescription to fix the above mentioned ambiguities (Compere & Marolf, 2008). In the limit $R \to \infty$, the renormalized presymplectic potential can be deduced, leading to (2.74).

2.4. Symplectic renormalization

Holographic renormalization can become cumbersome, especially when the goal resides solely on the computation of the asymptotic charges. To tackle this challenge, we present another prescription for resolving divergent Iyer-Wald ambiguities (2.13). This method, initially introduced for Maxwell theory in (Freidel et al., 2019), was later extended to diffeomorphism-invariant theories in (McNees & Zwikel, 2023, 2024)[5]. This prescription, known as symplectic renormalization, specifically concentrates on renormalizing the symplectic structure itself and does not address the associated variational principle. We will refine and elaborate on this final statement towards the end of the section. This is because valuable insights of the boundary counterterm action can still be gleaned within this procedure. What is more, we shall see that in some cases this can lead to a prescription of finite terms to be added to corner charges. In this section, we have opted to present this prescription in a manner that makes it applicable to any gauge theory.

The core concept is to leverage on Iyer-Wald ambiguities for the purpose of renormalizing the Lee-Wald symplectic structure (2.12) just as the Compère-Marolf prescription does, but without incorporating a boundary Lagrangian. In the subsequent part of this section, it is more advantageous to utilize the Lagrangian density $\mathscr{L}$. This choice is driven by the fact that certain equations we introduce have neater interpretations when expressed in standard terms, employing the spatiotemporal components of objects introduced in the covariant

[5]Notably, it has been recently employed in various studies, including (Geiller & Zwikel, 2022; Campoleoni et al., 2023a; Ciambelli et al., 2023; Geiller & Zwikel, 2024; Riello & Freidel, 2024).

phase space formalism. Then, the effective utilization of Iyer-Wald ambiguities is rooted in the equivalent on-shell equality (2.10), which can be written as

$$\delta\mathscr{L}[\varphi] \approx \partial_\mu \Theta^\mu[\varphi; \delta\varphi] \,. \tag{2.81}$$

In accordance with the conventions of the appendix A, we recall that $L = \mathscr{L} \mathrm{d}^D x$, $\Theta = \Theta^\mu (\mathrm{d}^{D-1}x)_\mu$, and the analysis is conducted on a background manifold $\mathcal{M}$ with coordinates $(x^\mu) = (r, x^a)$ where r serves as a radial coordinate, and the asymptotic boundary $\mathcal{B}$ is located at $r \to \infty$. The coordinates along this boundary are denoted as x^a. In this context, the operator δ signifies a field variation, previously denoted as δ_e in section 2.1, but for simplicity, we use the notation δ. The context will inherently clarify the distinction between the two notations.

Presymplectic potential

Substituting the solution of the equations of motion in the form of a polyhomogeneous expansion (2.68) of the fields, a total r-derivative form can be ascribed to the Lagrangian density and the component of the presymplectic potential along the radial isosurface $\mathcal{B}$:

$$\mathscr{L} \approx \partial_r \int \mathrm{d}r\, \mathscr{L} \,, \qquad \Theta^a \approx \partial_r \int \mathrm{d}r\, \Theta^a \,. \tag{2.82}$$

This holds true when the r-dependence of the fields is known: the on-shell integration over r yields a polyhomogeneous expansion, up to a r-independent constant. This corresponds to the indeterminacy of the finite ambiguities. Note that it will even be possible in some cases treated in this thesis, notably the gravitational Bondi gauge in the subsection 4.3, to write these components (2.82) directly as a total off-shell radial derivative just by applying the gauge conditions and integrating radially by parts. In such cases, we obtain an off-shell integral over r which, after the equations of motion have been enforced, fixes on-shell the r-independent constant in the expansions of $\mathscr{L}$ and Θ^a. Thus, it corresponds to a prescription for the finite ambiguities, potentially unveiling new finite charges. Throughout the remainder of the thesis, we term this resolution the McNees-Zwikel prescription.

As a consequence of (2.81) and (2.82), the Iyer-Wald ambiguities (2.13), that are written in this context as

$$\Theta^\mu[\varphi; \delta\varphi] \to \Theta^\mu[\varphi; \delta\varphi] + \delta B^\mu[\varphi] + \partial_\nu C^{\mu\nu}[\varphi; \delta\varphi] \,, \tag{2.83}$$

where $C^{\mu\nu}$ is an antisymmetric corner term linear in the field variations, are entirely determined for the radial divergent orders:

$$\partial_r \Theta^r_{\text{ren}} \approx 0\,, \qquad \Theta^r_{\text{ren}} \approx \Theta^r - \delta \int \mathrm{d}r\, \mathscr{L} + \partial_a \int \mathrm{d}r\, \Theta^a\,. \tag{2.84}$$

Specifically, this reveals that the divergent orders of the radial presymplectic potential are on-shell fixed to be total derivatives plus total variations, which can be systematically eliminated order by order using the Iyer-Wald ambiguities. Since Θ^r and ω^r are the quantities that appear naturally when integrating on the radial isosurface $\mathcal{B}$, this justifies our interest in renormalizing this component to follow the lines of the Iyer-Wald formalism for an asymptotic boundary ($r \to \infty$). However, the finite order of Θ^r remains undetermined by this procedure (2.84). Thus, finite boundary and corner terms can still be added. It means that there is then a choice to prescribe for this finished part. This can be motivated in several ways, as we shall illustrate in subsections 4.2.1 and 4.3, being the heart of the gravitational chapter. One possible proposal we have already come across is the McNees-Zwikel prescription mentioned just above.

Corner charge

Particularly, the above systematic approach (2.84) implies that the renormalized r-component of the Lee-Wald presymplectic form $\omega^\mu[\varphi; \delta_1\varphi; \delta_2\varphi]$ (2.17) is given by:

$$\omega^r_{\text{ren}} \approx \omega^r + \partial_a\Big(\delta_2 C^{ar}[\varphi; \delta_1\varphi] - \delta_1 C^{ar}[\varphi; \delta_2\varphi]\Big)\,, \quad \partial_r\, \omega^r_{\text{ren}} \approx 0\,. \tag{2.85}$$

Here, we define $C^{ra} = \int \mathrm{d}r\, \Theta^a$. This aligns with a corresponding adjustment to the renormalized mixed component of the codimension-2 form $k^{\mu\nu}_\lambda$ (see (2.27)):

$$k^{ra}_{\lambda,\text{ren}} \approx k^{ra}_\lambda + \delta C^{ar}[\varphi; \delta_\lambda\varphi] - \delta_\lambda C^{ar}[\varphi; \delta\varphi]\,. \tag{2.86}$$

This adjustment is subject to the earlier mentioned radial independent ambiguity.

The outlined procedure empowers us to renormalize the radial divergences of the surface charge. To grasp this intricacy, let us consider breaking down the asymptotic boundary coordinates as $(x^a) = (t, x^i)$, where t is a timelike coordinate and x^i are the coordinates along the corner $\mathcal{C}$. The fundamental theorem of the covariant phase space (2.26) then informs us that, on-shell,

$$\omega^r_\lambda \approx \partial_t k^{rt}_\lambda + \partial_i k^{ri}_\lambda\,. \tag{2.87}$$

On the left hand side, the Lee-Wald symplectic structure (2.17) is evaluated along the gauge parameter λ:

$$\omega^\mu_\lambda = \omega^\mu[\varphi; \delta_\lambda \varphi; \delta\varphi] = \delta_\lambda \Theta[\varphi; \delta\varphi] - \delta\Theta[\varphi; \delta_\lambda \varphi] \,. \tag{2.88}$$

At the renormalized level, thanks to (2.84) and (2.85), we have demonstrated, in particular, that one can determine finite charges from (2.87) by an integration by parts up to total derivatives on $\mathcal{C}$, since the right-hand side is radial independent. Given that $\mathcal{C}$ is closed, the total derivative term vanishes when we evaluate this result over the corner. In conclusion, this demonstrates that the limit $r \to \infty$ exists in the definition of the surface charge (2.50):

$$\oint\!\!\!\!/ H^{\text{ren}}_\lambda \approx \lim_{r\to\infty} \int_{\mathcal{C}} \mathrm{d}^{D-2}x \, k^{tr}_{\lambda,\text{ren}} = \int_{\mathcal{C}} \mathrm{d}^{D-2}x \, k^{tr}_{\lambda,\text{ren}} \,. \tag{2.89}$$

As a final comment on this symplectic procedure, we may mention that in (McNees & Zwikel, 2023) it was proposed to demonstrate the radial independence of $k^{tr}_{\lambda,\text{ren}}$ in an alternative way by conceiving the above prescription as a deft fixing of Iyer-Wald ambiguities (2.13), $B^a = 0$ and $C^{ab} = 0$, such that the components of the presymplectic potential along $\mathcal{B}$ are on-shell renormalized to zero:

$$\Theta^a_{\text{ren}} \approx \Theta^a + \partial_r C^{ar} = 0 \qquad \Rightarrow \qquad \omega^a_{\text{ren}} \approx 0 \,. \tag{2.90}$$

Expressing the boundary coordinates again as $(x^a) = (t, x^i)$, this implies in particular

$$\omega^t_{\lambda,\text{ren}} \approx 0 \,, \tag{2.91}$$

which, when injected in (2.26), yields:

$$\partial_r k^{tr}_{\lambda,\text{ren}} + \partial_i k^{ti}_{\lambda,\text{ren}} \approx 0 \,. \tag{2.92}$$

Integrating the latter expression over the corner $\mathcal{C}$ proves that $k^{tr}_{\lambda,\text{ren}}$ is well r-finite:

$$\partial_r \int_{\mathcal{C}} \mathrm{d}^{D-2}x \, k^{tr}_{\lambda,\text{ren}} + \int_{\mathcal{C}} \mathrm{d}^{D-2}x \, \partial_i k^{ti}_{\lambda,\text{ren}} \approx 0 \,. \tag{2.93}$$

Discussion

While the previous argument (2.89) to this one (2.90) is still valid, here it is based on the fact that $\Theta^t_{\lambda,\text{ren}} \approx 0$. While this condition holds true in the low-dimensional scenarii explored in (McNees & Zwikel, 2023), its applicability to higher-dimensional examples or more intricate theories with dynamic degrees of freedom that might extend to the asymptotic boundary requires qualification.

Even in our next examples, it is not clear that one can enforce it without impacting the physics of the system. In fact, it would remain valid if the fields decay rapidly enough near this boundary. However, the acceptability of such fall-off conditions raises questions. In such instances, Θ^a may consist of a leading part determined by boundary conditions and kinematics, alongside a subleading part involving dynamic degrees of freedom. While fixing Iyer-Wald ambiguities in the way outlined in this section might absorb the leading part as a boundary contribution of Θ^r[6], addressing the subleading part is more complex as it risks removing physical significance that can be used in other conserved quantities.

This type of possible issue has been studied in the literature, for example in (Capone et al., 2023; Riello & Freidel, 2024) via a covariant prescription with respect to boundary diffeomorphisms (called phase space renormalization), but requires further exploration in relevant cases. This presents a future avenue for refining the argument presented earlier. In particular, the symplectic counterterms must depend only locally on the free data allowed by the boundary conditions[7]. Similarly to holographic renormalization, the rationale behind this limitation stems from the foundational equation (2.81) of the symplectic approach, which is applicable solely to local theories. Consequently, incorporating non-local counterterms into the Lagrangian is deemed impermissible. One potential approach could involve relaxing (2.90) by not restricting the ambiguities of Θ^a to those outlined in (2.84), although we emphasize that this necessitates thorough investigation on a case-by-case basis. Given the complexities discussed above, these statements will not be further addressed formally in the remainder of this thesis. However, the examples treated in this thesis serve as an example, where we strive to have no non-covariance and no non-locality in the procedure.

Comparison of various prescriptions

To encapsulate and contrast the two approaches outlined in this chapter, we aim to analyze the strengths and weaknesses of each. First, let us provide a summary in the following manner. In the covariant phase space formalism as per Iyer-Wald's approach outlined in section 2.1, the determination of conserved charges via the second Noether theorem (2.22) inevitably introduces ambiguities in the associated presymplectic potential (2.13). These ambiguities

[6] This step itself could pose challenges for the same reason as encountered with subleading terms, similar to the case of logarithmic overleading terms (Fuentealba et al., 2023).

[7] For example, in the gravitational context, these terms cannot depend on mass or angular momentum.

are addressed in distinct ways in the preceding sections 2.3 and 2.4, effectively serving to renormalize the underlying symplectic structure.

The Compère-Marolf prescription, complemented by holographic renormalization, identifies the boundary ambiguity B in (2.13) through the counterterm added to the bulk action. Simultaneously, the corner ambiguity C is associated with the presymplectic potential derived from B. This determination not only renormalizes the charges (2.43) but also contributes to the renormalization of the variational principle. It achieves this by introducing a suitable boundary Lagrangian, thereby defining the entire bulk theory in a manner that yields finite asymptotic charges. This aspect gains significance for those interested not only in asymptotic symmetries but also in holographic Ward identities (de Boer et al., 2000; Corley, 2000; Kalkkinen et al., 2001), or the correlation functions of the dual theory (Skenderis, 2002), for example.

That being said, this process can become quite labor-intensive, especially for certain theories. This is particularly true when the analysis focuses solely on determining corner charges. The complexity of the hypersurface $\mathcal{B}$ on which the counterterm is defined can sometimes add to the challenge. Additionally, this task is not made easier by the fact that while this prescription serves as a useful tool, it needs to be reproduced from scratch for each theory under consideration.

Then comes the introduction of the second prescription in this section, precisely for the reasons mentioned in the last paragraph. Symplectic renormalization also identifies the same B as Compère-Marolf in the Iyer-Wald ambiguities (2.13), but instead designates C as the corner contribution of the bulk presymplectic potential. This approach focuses solely on the symplectic structure data in renormalizing asymptotic charges. Consequently, it does not concern itself with ensuring the finiteness of the variational principle by disregarding the boundary term necessary for completing the full Lagrangian.

Nevertheless, symplectic renormalization proves significantly more efficient in analyzing charges and asymptotic symmetries compared to the holographic one. It achieves this by offering a readily reproducible and systematized procedure for each gauge theory under consideration. Furthermore, constructed through the McNees-Zwikel prescription, it may provide a determination of finite ambiguities in specific cases where it is available. The philosophy of this feature is effectively utilized, for instance, in the gravitational context discussed in chapter 4, revealing the presence of certain new physical observables.

Considering these various factors, we propose to consider a blended approach, combining the strengths of both prescriptions to mitigate their respective weaknesses. This would facilitate a thorough and robust examination of asymptotic symmetries and their dual aspects. A recommended strategy involves initially examining the symplectic structure to directly compute surface charges, thereby discerning the presence or absence of residual symmetries arising from large gauge transformations. Subsequently, determining the corner contribution of the bulk presymplectic potential allows for the identification of crucial boundary counterterms to be incorporated into the bulk action through covariantization with respect to the boundary. Utilizing these insights, subsequent holographic renormalization can be conducted more efficiently than if this preliminary step were omitted. Finally, resolving finite ambiguities in the symplectic structure can yield new finite charges concerning the holographic approach. These can then be comprehensively understood in conjunction with the latter through an examination of the associated variational principle. That is the strategy we will be sticking to for the rest of the manuscript.

CHAPTER 3

Electromagnetism

"Tu es comme une bougie qu'on a
oublié d'éteindre dans une
chambre vide,
tu brilles entouré de gens sombres
voulant te souffler."

Abdoulaye Diarra

In this chapter, our goal is to elucidate the formal techniques introduced earlier for determining asymptotic symmetries and calculating their associated surface charges. To achieve this, we have selected the simplest non-trivial example of a gauge theory: the one-dimensional unitary Abelian Lie group, specifically Maxwell theory of electromagnetism -— that is comprehensive enough to address various challenges encountered in more complex scenarii, including the ones discussed in subsequent chapters.

We examine the propagation of a Maxwell free field, a massless spin-1 field, in a specific type of spacetime relevant for the remainder of the thesis. Our analysis of asymptotic symmetries is directed towards gaining a deeper understanding of dual theories in holographic correspondences, such as the AdS/CFT (Maldacena, 1998) or BMS/CFT correspondence (Barnich & Troessaert, 2010) for gravity in asymptotically Anti de Sitter (Fefferman & Graham, 1985; Henneaux & Teitelboim, 1985) and asymptotically flat spacetimes (Bondi et al., 1962; Sachs, 1962a,b), respectively. By focusing on Maxwell theory in both an AdS background and a Minkowski background, we aim to employ computational and systematic techniques to grasp the inherent difficulties in Einstein theory of gravitation. The reader can explore insightful literature on this subject in the AdS case through the following papers: (Taylor, 2000; Esmaeili et al., 2019; Esmaeili & Hosseinzadeh, 2021). Additionally, relevant literature for the flat case can be found in (Tamburino & Winicour, 1966; Mädler & Winicour, 2016; Campoleoni et al., 2018a; Strominger, 2018; Campoleoni et al., 2018b, 2019b; Freidel et al., 2019; Campoleoni et al., 2020). Our discussion will primarily draw upon the findings presented in (Campoleoni et al., 2023a). To closely mimic the gravitational setup, we consider coordinate patches akin to Fefferman-Graham or Bondi gauges.

Leveraging on the simplicity of Maxwell theory, we provide a detailed analysis of applying the formalisms outlined in the previous chapter, emphasizing the renormalization of surface charges from both the variational principle and the symplectic structure. This demonstration highlights the advantages and disadvantages discussed in relation to the two renormalization prescriptions. This practice will enable us to be more succinct in the upcoming chapters. We comment on the connection with standard Abbott-Deser expressions through the Barnich-Brandt formalism, revealing that charges coincide only in specific cases. Despite the theory versatility in accommodating any dimension of background spacetime, our analysis concentrates on two distinct dimensional examples due to their similarities with the gravitational context, showcasing features instrumental in understanding the procedures for determining Einstein-Hilbert asymptotic symmetries.

In section 3.1, we introduce the coordinate systems employed for AdS. The first, called Poincaré patch, serves as a well-established framework for holography. However, it does not encompass the entire AdS space in Lorentzian signature and is unsuitable for approaching the limit where the AdS radius tends to infinity. Subsequently, we transition to the second patch, referred to as Bondi coordinates, which are global and facilitate a straightforward smooth flat limit. We adopt this limit as the coordinate system for the flat background. Moving on to section 3.2, we delve into the theory under examination by defining the bulk action. This allows us to derive the equations of motion, symplectic structure, and the expression for the associated codimension-2 form defining the surface charge. Building on the justification from the previous chapter, we center our attention on the Iyer-Wald prescription for determining charges. When referring to the symplectic structure, we specifically mean the Iyer-Wald structure. If we ever invoke the Barnich-Brandt structure, we will explicitly state so. In the dimensional examples considered, i.e. for $D > 4$, the corner charge associated with Maxwell gauge symmetry diverges as one approaches the asymptotic boundary.

In section 3.3, we apply renormalization procedures successively within the AdS background, employing both variational and symplectic prescriptions. In Bondi coordinates, we achieve a smooth flat limit of our results and compare this with the analysis conducted in a flat background in section 3.4. In the latter, our focus is solely on symplectic renormalization, given that holographic renormalization becomes more intricate when dealing with a null manifold as the asymptotic boundary. We shall comment on these aspects in this final section.

3.1. Coordinate patches

In this first section, we provide an overview of the geometry of Anti de Sitter space and introduce relevant coordinate patches for the study of asymptotic symmetries in holography. This is the first step in studying asymptotic charges: we specify a spacetime manifold $\mathcal{M}$ and its metric $g_{\mu\nu}$ in the same conventions as in appendix A. We focus on a D-dimensional AdS space, which is commonly represented by starting with $(D+1)$-dimensional embedding flat space and with the line element:

$$\mathrm{d}s^2 = \mathrm{d}X^M \eta_{MN}\, \mathrm{d}X^N\,, \qquad \eta_{MN} = \mathrm{diag}(-1, 1, \dots, 1, -1)\,. \tag{3.1}$$

Here, $N, M = 0, 1, \dots, D-1, D$, and the AdS_D manifold is defined as

$$X^M \eta_{MN} X^N = -\ell^2\,, \tag{3.2}$$

where ℓ denotes the AdS radius.

To establish appropriate coordinates for AdS, we introduce a time coordinate T, a radial coordinate R, and $D-2$ angular coordinates x^i:

$$X^0 = \ell \cosh\frac{R}{\ell}\, \sin\frac{T}{\ell}\,, \qquad X^I = \ell\, \hat{X}^I(x^i)\, \sinh\frac{R}{\ell}\,, \qquad X^D = \ell\cosh\frac{R}{\ell}\, \cos\frac{T}{\ell}\,, \tag{3.3}$$

where $\hat{X}^I$ is a Euclidean unit vector with $\hat{X}^I\hat{X}^I = 1$. The resulting metric takes the form:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(\cosh\tfrac{R}{\ell})^2\mathrm{d}T^2 + \mathrm{d}R^2 + \ell^2(\sinh\tfrac{R}{\ell})^2\mathrm{d}\Omega^2\,, \tag{3.4}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}x^i\gamma_{ij}\,\mathrm{d}x^j\,, \quad \gamma_{ij} = \frac{\partial\hat{X}^I}{\partial x^i}\frac{\partial\hat{X}^J}{\partial x^j}\delta_{IJ}\,. \tag{3.5}$$

The coordinates (T, R, x^i) have the advantage of covering the entire AdS_D spacetime. To visualize this, we refer to figure 3.1 for a schematic representation in the case of $D = 2$. In order to compactify the radial variable, we introduce

$$\rho = \ell \arctan\left(\sinh\frac{R}{\ell}\right), \tag{3.6}$$

resulting in the metric:

$$\mathrm{d}s^2 = \frac{1}{(\cos\frac{\rho}{\ell})^2}\left[-\mathrm{d}T^2 + \mathrm{d}\rho^2 + \ell(\sin\tfrac{\rho}{\ell})^2\mathrm{d}\Omega^2\right]. \tag{3.7}$$

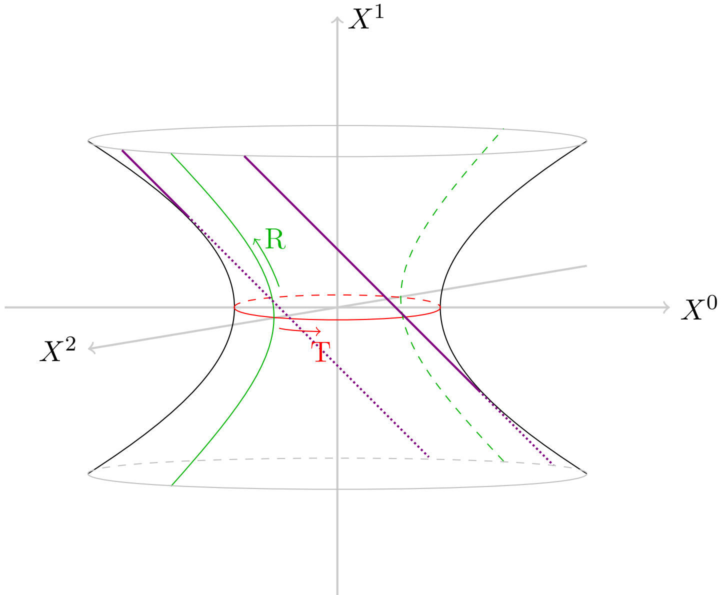
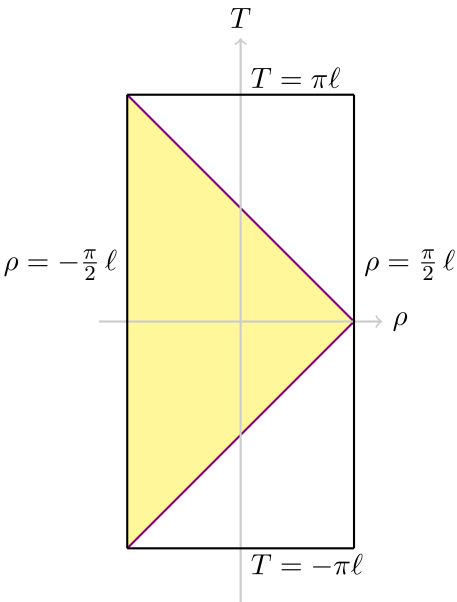
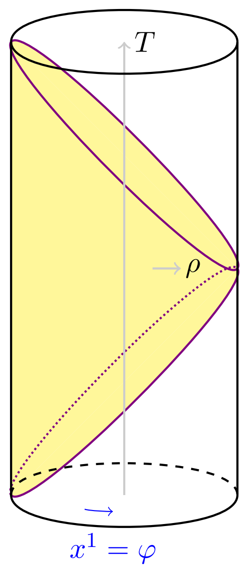


Figure 3.1: A standard illustration of the AdS_2 space. The red line represents the $R = 0$ surface, while the solid green line corresponds to $T = 0$, and the dashed green line corresponds to $T = \pi\ell$. The purple lines indicate the intersection with the plane $X^1 = X^2$. Figure taken from (Campoleoni et al., 2023a).



(a) The AdS_2 Penrose diagram.



(b) The AdS_3 Penrose diagram.

Figure 3.2: The AdS_2 and AdS_3 spaces drawn through their conformal completion. The coloured area is that covered by the Poincaré patch. The AdS boundary corresponds to the limit $z \to 0$ of this region. Figure taken from (Campoleoni et al., 2023a).

This form makes it clear that the conformal boundary is the surface of a cylinder located at $\rho = \pi\ell/2$ (see figure 3.2). The periodicity of the original time coordinate T is $T \sim T + 2\pi\ell$. To avoid closed timelike loops, we extend this cylinder to infinity and adopt a decompactified coordinate $-\infty < T < +\infty$. The AdS Riemann tensor indicates a constant curvature, with the Ricci tensor and scalar curvature given by:

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}\Big(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\Big)\,, \quad R_{\mu\nu} = -\frac{D-1}{\ell^2}g_{\mu\nu}\,, \quad R = -\frac{D(D-1)}{\ell^2}\,. \tag{3.8}$$

3.1.1 Poincaré

The Poincaré patch proves to be particularly suitable for the holographic exploration of AdS space, as evidenced by its widespread application within AdS/CFT (Maldacena, 1998). This utility will be explicitly demonstrated in the forthcoming chapter on general relativity. This coordinate system, denoted as (z, x^a) with $a = 0, 1, \ldots, D-1$, is defined by solving the AdS defining constraint (3.2) as follows,

$$\begin{aligned} X^a &= \frac{x^a}{z}\,, \\ X^{D-1} &= z\,\frac{\ell}{2}\left(1 + \frac{x^2}{\ell^2 z^2} - \frac{1}{z^2}\right), \\ X^D &= z\,\frac{\ell}{2}\left(1 + \frac{x^2}{\ell^2 z^2} + \frac{1}{z^2}\right), \end{aligned} \tag{3.9}$$

where $z > 0$, $x^2 = x^a \eta_{ab} x^b$ and $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$. The resulting AdS metric is then expressed as:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{1}{z^2}\left(\ell^2 \mathrm{d}z^2 + \eta_{ab}\mathrm{d}x^a \mathrm{d}x^b\right). \tag{3.10}$$

The Poincaré coordinates exclusively cover the AdS space in the half-space $X^D > X^{D-1}$ due to the condition $X^D - X^{D-1} = \frac{\ell}{z}$ for $z > 0$ (see the figure 3.2). One can see in figure 3.1 that the Poincaré coordinates cover only the section of spacetime beneath the plane $X^1 = X^2$. The metric determinant and the inverse metric are given by:

$$\sqrt{-g} = \frac{\ell}{z^D}\,, \qquad g^{\mu\nu}\partial_\mu\partial_\nu = z^2\left(\ell^{-2}\partial_z^2 + \eta^{ab}\partial_a\partial_b\right). \tag{3.11}$$

Furthermore, the non-zero Christoffel symbols take the following forms:

$$\Gamma^z_{zz} = -\frac{1}{z}\,, \qquad \Gamma^z_{ab} = \frac{1}{z\ell^2}\,\eta_{ab}\,, \qquad \Gamma^a_{bz} = -\frac{1}{z}\,\delta^a_b\,. \tag{3.12}$$

We notice that the flat limit $\ell \to \infty$ is ill defined in this context.

3.1.2 Bondi

Another valuable coordinate patch in holography, particularly in the context of the AdS/CFT to BMS/CFT transition (Barnich et al., 2012; Barnich & Lambert, 2013), is the Bondi coordinate system (Bondi et al., 1962; Sachs, 1962a,b) – a topic that will be explored more explicitly in the upcoming chapter. The significance of this secondary coordinate patch lies in its ability to establish a smooth flat limit as $\ell \to \infty$, despite its inherent complexity compared to Poincaré coordinates.

The approach involves reconsidering the embedding space and introducing polar coordinates for the spatial directions $I = 1, 2 \ldots, D-1$ based on $X^I = r\hat{X}^I(x^i)$, where $\hat{X}^I\hat{X}^I = 1$, and $i = 1, 2, \ldots, D-2$ represents the angular variables. The AdS defining constraint (3.2) is solved by expressing:

$$\begin{aligned} X^0 &= \ell\sqrt{1+\left(\frac{r}{\ell}\right)^2}\sin\left(\frac{u}{\ell}+\arctan\frac{r}{\ell}\right)\,,\\ X^D &= \ell\sqrt{1+\left(\frac{r}{\ell}\right)^2}\cos\left(\frac{u}{\ell}+\arctan\frac{r}{\ell}\right)\,. \end{aligned} \tag{3.13}$$

It defines Bondi coordinates (u, r, x^i) on AdS_D, and the resulting metric is given by:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\left(1+\frac{r^2}{\ell^2}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2\,, \tag{3.14}$$

where the metric on the unit sphere reads

$$\mathrm{d}\Omega^2 = \mathrm{d}x^i\gamma_{ij}\,\mathrm{d}x^j\,, \qquad \gamma_{ij} = \frac{\partial\hat{X}^I}{\partial x^i}\frac{\partial\hat{X}^I}{\partial x^j}\,. \tag{3.15}$$

Here, the coordinate u labels null hypersurfaces, and r serves as the affine parameter along the generating null geodesics. In these coordinates, the boundary of AdS is located at $r \to \infty$. The determinant is given by $\sqrt{-g} = r^{D-2}\sqrt{-\gamma}$, and the inverse metric is expressed as:

$$g^{\mu\nu}\partial_\mu\partial_\nu = -2\partial_u\partial_r + \left(1+\frac{r^2}{\ell^2}\right)\partial_r^2 + r^{-2}\gamma^{ij}\partial_i\partial_j\,. \tag{3.16}$$

The non-zero Christoffel symbols are detailed as follows:

$$\begin{aligned} \Gamma^i_{rj} &= \frac{1}{r}\,\delta^i_j\,, \quad \Gamma^u_{uu} = -\frac{r}{\ell^2}\,, & \Gamma^u_{ij} &= r\,\gamma_{ij}\,, \\ \Gamma^r_{ru} &= \frac{r}{\ell^2}\,, \quad \Gamma^r_{uu} = \frac{r}{\ell^2}\left(1+\frac{r^2}{\ell^2}\right), & \Gamma^r_{ij} &= -r\left(1+\frac{r^2}{\ell^2}\right)\gamma_{ij}\,, \end{aligned} \tag{3.17}$$

and

$$\Gamma^i_{jk} = \frac{1}{2}\,\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right). \tag{3.18}$$

3.2. Lagrangian

To complete the initial step in analyzing asymptotic symmetries, specifically, defining the bulk field theory, we specify the following Maxwell Lagrangian, which will be employed throughout the remainder of this chapter:

$$S = \int_{\mathcal{M}} \mathrm{d}^D x\,\mathscr{L}\,, \qquad \mathscr{L} = -\frac{1}{4}\sqrt{-g}\,F^{\mu\nu}F_{\mu\nu}\,, \tag{3.19}$$

where the Faraday tensor $F_{\mu\nu}$ is defined in terms of the Maxwell field $A_\mu(x^\nu)$ as

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3.20}$$

Moreover, we shall denote by ∇ the covariant derivative with respect to the background metric $g_{\mu\nu}$, and by g the determinant of the latter. Throughout this discussion, the use of round and square brackets around indices implies symmetrization and antisymmetrization, respectively, on these indices, without any overall factor.

The Lagrangian density given by (3.19) exhibits invariance under the following gauge symmetry:

$$\delta_\lambda A_\mu = \partial_\mu\lambda\,, \tag{3.21}$$

where $\lambda(x^\mu)$ is a scalar parameter. Since this symmetry is governed by (2.19) and (2.20), the conserved asymptotic charges of this theory adhere to the second Noether theorem (2.22). We can then apply the Iyer-Wald prescription outlined in section 2.1 to compute these charges. Under an arbitrary variation of fields $A_\mu \to A_\mu + \delta A_\mu$, the response of the Lagrangian (3.19) is given by:

$$\delta\mathscr{L} = E_\mu \delta A^\mu + \partial_\mu \Theta^\mu\,. \tag{3.22}$$

The equations of motion are

$$E_\mu[A] = -\partial^\nu\left(\sqrt{-g}\,F_{\mu\nu}\right) \approx 0\,, \tag{3.23}$$

and the presymplectic potential (2.10) is expressed as

$$\Theta^\mu[A;\delta A] = -\sqrt{-g}\, F^{\mu\nu}\delta A_\nu\,. \tag{3.24}$$

Here, and in subsequent discussions, the δ-operation should now be interpreted as a field variation.

The Noether identity (2.24) corresponds to the Bianchi identity for the Faraday field-strength and reads off-shell

$$N = \partial_\mu\partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right) = 0\,. \tag{3.25}$$

The (Lee-Wald) presymplectic form (2.12) is defined by:

$$\omega^\mu[A;\delta_1 A;\delta_2 A] = \delta_2\Theta^\mu[A;\delta_1 A] - \delta_1\Theta^\mu[A;\delta_2 A]\,. \tag{3.26}$$

Upon contracting it with a gauge parameter (3.21),

$$\omega^\mu_\lambda := \omega^\mu[A;\delta_\lambda A;\delta A] = \delta_\lambda\Theta^\mu[A;\delta A] - \delta\Theta^\mu[A;\delta_\lambda A]\,, \tag{3.27}$$

one can derive the following skew-symmetric codimension-2 quantity through the fundamental theorem of the covariant phase space (2.26):

$$\omega^\mu_\lambda \approx \partial_\nu k^{\mu\nu}_\lambda\,, \qquad k^{\mu\nu}_\lambda = -\sqrt{-g}\,\lambda\,\delta F^{\mu\nu}\,. \tag{3.28}$$

To provide a comprehensive understanding, the counterpart of the above expression in the Barnich-Brandt formulation (2.65) is articulated as follows:

$$k^{\mu\nu}_{\text{BB},\lambda} = -\sqrt{-g}\Big(\nabla^\mu\delta A^\nu\lambda + \delta A^\mu\nabla^\nu\lambda - (\mu\leftrightarrow\nu)\Big)\,, \tag{3.29}$$

which differs from Iyer-Wald's expression (3.28) due to the presence of the corner term (2.63):

$$E^{\mu\nu}[A;\delta_\lambda A;\delta A] = \sqrt{-g}\Big(\delta A^\mu\nabla^\nu - \delta A^\nu\nabla^\mu\Big)\lambda\,. \tag{3.30}$$

In certain instances, we will observe that this ambiguity vanishes, resulting in the alignment of the two expressions, (3.28) and (3.29).

In the upcoming sections, we will utilize the aforementioned relations to establish the asymptotic surface charge (2.43) within an AdS and flat background. This will be achieved by employing the coordinate patches introduced in the preceding section 3.1. We introduce two examples of different dimensions of interest for subsequent chapters, chosen due to their shared characteristics with more intricate examples such as gravitational theory. Notably, we will grapple with the occurrence of radial divergence (2.50) at the variational principle, symplectic structure, and charge levels, necessitating appropriate renormalization. To address this, we present in details the two prescriptions outlined in sections 2.3 and 2.4.

3.3. Anti de Sitter background

3.3.1 Solution space

Poincaré coordinates

The next phase in exploring asymptotic symmetries involves establishing a gauge and defining boundary conditions, specifying the behavior of fields in the vicinity of the asymptotic boundary situated at $z \to 0$ within the Poincaré patch (3.10) of AdS. In this quest, we utilize the equations of motion to guide our choices. This step is subsequently integrated with the third stage of asymptotic corner charge analysis.

Field equations. In our exploration, we initiate by utilizing the fact that the Maxwell Lagrangian (3.19) is articulated in terms of the gauge-invariant Faraday tensor. This enables us to investigate the equations of motion (3.23) without imposing any gauge. In Poincaré coordinates (3.10), these equations are expressed as:

$$E_z = \partial^a F_{az} \,, \tag{3.31}$$

$$E_a = \frac{1}{z\ell^2} \left(z\partial_z - D + 4 \right) F_{za} + \partial^b F_{ba} \,. \tag{3.32}$$

Throughout this chapter, we lower and raise boundary indices of the Poincaré framework using the flat metric η_{ab} and its inverse η^{ab}, respectively. Combining these equations with the Bianchi identity (3.25), $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$, yields a useful relation:

$$E_{ab} = \partial_z \left(z^{4-D} \partial_z F_{ab} \right) + \ell^2 z^{4-D} \Box F_{ab} \approx 0 \,. \tag{3.33}$$

Here, the symbol $\Box = \partial^a \partial_a = \eta^{ab} \partial_a \partial_b$ represents the boundary Laplacian.

Assuming a polyhomogeneous radial expansion (2.68) of the field strength in the form:

$$F_{\mu\nu}(z, x^a) = \sum_n z^n \left(F^{(n)}_{\mu\nu}(x^a) + \log z \, \widetilde{F}^{(n)}_{\mu\nu}(x^a) \right) \tag{3.34}$$

where the summation range is not fixed yet, we derive the following recurrence relations:

$$E_z^{(n)} = \partial^a F_{az}^{(n)}\,, \tag{3.35}$$

$$E_a^{(n)} = \frac{1}{\ell^2}(n-D+4)F_{za}^{(n)} + \frac{1}{\ell^2}\widetilde{F}_{za}^{(n)} + \partial^b F_{ba}^{(n-1)}\,, \tag{3.36}$$

$$E_{ab}^{(n)} = \frac{1}{\ell^2}(n-D+4)(n+1)F_{ab}^{(n+1)} + \frac{1}{\ell^2}(2n-D+5)\widetilde{F}_{ab}^{(n+1)} + \Box F_{ab}^{(n-1)}\,, \tag{3.37}$$

and similarly for the logarithmic terms:

$$\widetilde{E}_z^{(n)} = \partial^a \widetilde{F}_{az}^{(n)}\,, \tag{3.38}$$

$$\widetilde{E}_a^{(n)} = \frac{1}{\ell^2}(n-D+4)\widetilde{F}_{za}^{(n)} + \partial^b \widetilde{F}_{ba}^{(n-1)}\,, \tag{3.39}$$

$$\widetilde{E}_{ab}^{(n)} = \frac{1}{\ell^2}(n-D+4)(n+1)\widetilde{F}_{ab}^{(n+1)} + \Box \widetilde{F}_{ab}^{(n-1)}\,. \tag{3.40}$$

Boundary conditions. By scrutinizing the structure of these relationships, we can establish the boundary conditions to be imposed on the fields as follows:

$$F_{ab} \sim \mathcal{O}(1)\,, \qquad F_{za} \sim \mathcal{O}(z)\,. \tag{3.41}$$

This choice is substantiated by the observation from (3.36) and (3.37) that these orders are the first not contingent on overleading orders above (3.41), which we opt to set to zero. Indeed, each of these orders depends on orders even more overleading than itself, forming an infinite tower. This cascading effect causes all quantities derived from the field strength to diverge as one approaches the AdS boundary, i.e., in the limit $z \to 0$. To circumvent this issue, we have to prescribe boundary conditions as stated in (3.41).

With this foundation, we proceed to solve the equations of motion. While it is feasible to solve them for any dimension, our focus is on two examples, one with an even number of dimension and another with odd dimension, encompassing all the characteristics essential for understanding the subsequent chapters. This approach allows us to present more concrete results. For those interested in solutions for arbitrary dimensions, particularly for technical work concentrating on the calculation of holographic quantities from the perspective of dual quantum theory, such as correlation functions and Ward identities, we direct attention to the paper (Campoleoni et al., 2023a).

Asymptotic radial expansions. We now show two dimensional examples, each with a single divergent term, one linear and one logarithmic. Let us start with the second-to-last one. In the case of $D = 6$, the asymptotic solution space is described by

$$F_{ab} \approx F^{(0)}_{ab} + \ell^2 \frac{z^2}{2} \Box F^{(0)}_{ab} - \frac{z^3}{3} \partial_{[a} F^{(2)}_{b]z} + \mathcal{O}(z^4)\,, \tag{3.42}$$

$$F_{za} \approx \ell^2 z \partial^b F^{(0)}_{ab} + z^2 F^{(2)}_{za} + \mathcal{O}(z^3)\,, \tag{3.43}$$

where the free data are encapsulated in $F^{(0)}_{ab}$, an arbitrary antisymmetric tensor, and in the divergence-free tensor $F^{(2)}_{za}$, such that

$$\partial^a F^{(2)}_{za} \approx 0\,. \tag{3.44}$$

The appearance of new boundary data at these characteristic orders corresponds to the "source" and "VEV" according to standard terminology. This distinction becomes clearer when studying the symplectic structure and asymptotic charges, with the finite part providing information about conjugate pairs on the boundary of anti-de Sitter space.

In the even case, the logarithmic terms are constrained to zero by the equations of motion. However, in odd dimensions, the logarithmic branch is determined by the field equations in terms of the free data specified by the source. If we do not consider the presence of these log terms, it would constrain the source $F^{(0)}_{ab}$. For instance, in $D = 5$, the fall-offs are given by

$$F_{ab} \approx F^{(0)}_{ab} - \frac{z^2}{2} \left(\partial_{[a} F^{(1)}_{b]z} + \ell^2 \log z \, \Box F^{(0)}_{ab} \right) + \mathcal{O}(z^4)\,, \tag{3.45}$$

$$F_{za} \approx z F^{(1)}_{za} - \ell^2 z \log z \, \partial^b F^{(0)}_{ab} + \mathcal{O}(z^3)\,, \tag{3.46}$$

with the condition:

$$\partial^a F^{(1)}_{za} \approx 0\,. \tag{3.47}$$

These fall-offs in the odd case also capture the two expected branches of solutions associated with radiation (or source) and static (or VEV) contributions.

Discussion on the gauge field. These results can be replicated in terms of the Maxwell field. Building upon the argumentation surrounding (3.41) for the boundary conditions, which can be extended to the gauge potential, we already discern the ones that necessitate specification:

$$A_a \sim \mathcal{O}(1)\,, \qquad A_z \sim \mathcal{O}(z)\,. \tag{3.48}$$

However, to solve the equations of motion in this case, it is imperative to impose a gauge. This constraint should not be excessively restrictive, allowing for the possibility of non-trivial solutions. One viable option is the Lorenz gauge:

$$\nabla^\mu A_\mu = 0\,. \tag{3.49}$$

In this gauge, there is a residual symmetry that constrains the scalar parameter as

$$\nabla^\mu \delta_\lambda A_\mu = \nabla^\mu \nabla_\mu \lambda = 0\,. \tag{3.50}$$

Assuming, once again, a radial polyhomogeneous expansion (2.68) in the form

$$A_\mu(z, x^a) = \sum_{n\geq 0} z^n A_\mu^{(n)}(x^a) + \sum_{n\geq 0} z^n \log z\, \widetilde{A}_\mu^{(n)}(x^a)\,, \tag{3.51}$$

the equations of motion take the following form:

$$E_z^{(n)} = (n-1)(n-D+2)A_z^{(n)} + \ell^2 \Box A_z^{(n-2)} + (2n-D+1)\widetilde{A}_z^{(n)}\,, \tag{3.52}$$

$$E_a^{(n)} = n(n-D+3)A_a^{(n)} + \ell^2 \Box A_a^{(n-2)} - 2\ell^2 \partial_a A_z^{(n-1)} + (2n-D+3)\widetilde{A}_a^{(n)}\,, \tag{3.53}$$

and the gauge fixation condition (3.49) reads

$$(n-D+2)A_z^{(n)} + \ell^2 \partial \cdot A^{(n-1)} + \widetilde{A}_z^{(n)} \approx 0\,. \tag{3.54}$$

Similar expressions can be derived for logarithmic terms.

Solving these recursive relations for $D = 6$, we obtain:

$$A_a \approx A_a^{(0)} + \ell^2 \frac{z^2}{6}\left(3\Box A_a^{(0)} - 2\partial_a \partial \cdot A^{(0)}\right) + z^3 A_a^{(3)} + \mathcal{O}(z^4)\,, \tag{3.55}$$

$$A_z \approx \ell^2 \frac{z}{9} \partial \cdot A^{(0)} + \ell^4 \frac{z^3}{18} \Box \partial \cdot A^{(0)} + z^4 A_z^{(4)} + \mathcal{O}(z^5)\,, \tag{3.56}$$

where the spatial divergence reads $\partial \cdot A = \partial^a A_a = \eta^{ab} \partial_a A_b$. The arbitrary functions of the boundary coordinates are $A_a^{(0)}$, $A_z^{(4)}$, and the transverse part of $A_a^{(3)}$, while:

$$\partial \cdot A^{(3)} = 0\,. \tag{3.57}$$

When it comes to calculating charges, we will see that the order $A_z^{(4)}$ will not appear in the asymptotic charges and is therefore pure gauge, so that it can be removed without altering the physics content of the theory. In $D = 5$, the asymptotic solution space reads

$$A_a \approx A_a^{(0)} + z^2 A_a^{(2)} - \ell^2 \frac{z^2}{4} \log z \left(2\Box A_a^{(0)} - \ell^2 \partial_a \partial \cdot A^{(0)}\right) + \mathcal{O}(z^4)\,, \tag{3.58}$$

$$A_z \approx \ell^2 \frac{z}{4} \partial \cdot A^{(0)} + z^3 A_z^{(3)} - \ell^4 \frac{z^3}{8} \log z\, \Box \partial \cdot A^{(0)} + \mathcal{O}(z^5)\,, \tag{3.59}$$

such that

$$\partial \cdot A^{(2)} = \frac{\ell^2}{8}\Box\partial \cdot A^{(0)} \,. \tag{3.60}$$

In the Lorenz gauge (3.49), the scalar gauge parameter undergoes the following constraint (3.50) in Poincaré coordinates:

$$z\,\ell^2\,\Box\lambda + (z\partial_z - D + 2)\,\partial_z\lambda = 0\,. \tag{3.61}$$

According to (2.68), if we assume that

$$\lambda(z, x^a) = \sum_{n\geq 0} z^n \left(\lambda^{(n)}(x^a) + \log z\,\widetilde{\lambda}^{(n)}(x^a)\right), \tag{3.62}$$

we obtain the following relations:

$$\Box\lambda^{(n-2)} + \frac{1}{\ell^2}n(n-D+1)\lambda^{(n)} + \frac{1}{\ell^2}(2n-D+1)\widetilde{\lambda}^{(n)} = 0\,, \tag{3.63}$$

$$\Box\widetilde{\lambda}^{(n-2)} + \frac{1}{\ell^2}n(n-D+1)\widetilde{\lambda}^{(n)} = 0\,. \tag{3.64}$$

In $D = 6$ and $D = 5$, it leads to the following asymptotic radial expansions:

$$D = 6 : \lambda = \lambda^{(0)} + \ell^2\frac{z^2}{6}\Box\lambda^{(0)} + \ell^4\frac{z^4}{24}\Box^2\lambda^{(0)} + z^5\lambda^{(5)} + \mathcal{O}(z^6)\,, \tag{3.65}$$

$$D = 5 : \lambda = \lambda^{(0)} + \ell^2\frac{z^2}{4}\Box\lambda^{(0)} + z^4\lambda^{(4)} - \ell^4\frac{z^4}{16}\log z\,\Box^2\lambda^{(0)} + \mathcal{O}(z^6)\,. \tag{3.66}$$

An alternative choice for gauge fixing is the radial gauge:

$$A_z = 0\,. \tag{3.67}$$

This condition implies that the residual scalar parameter is radially independent:

$$\delta_\lambda A_z = \partial_z\lambda = 0\,. \tag{3.68}$$

In this scenario, the field equations can be expressed as

$$E_z^{(n)} = n\,\partial\cdot A^{(n)} + \partial\cdot\widetilde{A}^{(n)}\,, \tag{3.69}$$

$$E_a^{(n)} = n(D-3-n)A_a^{(n)} + (D-2n-3)\widetilde{A}_a^{(n)} - \ell^2\Box A_a^{(n-2)} + \ell^2\partial_a\partial\cdot A^{(n-2)}\,, \tag{3.70}$$

and the same applies to logarithmic terms.

In this gauge, in $D = 6$, the radial expansion of the solution space is as follows,

$$A_a \approx A_a^{(0)} + \ell^2 \frac{z^2}{2} \left(\Box A_a^{(0)} - \partial_a \partial \cdot A^{(0)} \right) + z^3 A_a^{(3)} + \mathcal{O}(z^4) \,, \tag{3.71}$$

while in $D = 5$, it takes the form:

$$A_a \approx A_a^{(0)} + z^2 A_a^{(2)} - \frac{z^2}{2} \log z \left(\Box A_a^{(0)} - \partial_a \partial \cdot A^{(0)} \right) + \mathcal{O}(z^4) \,. \tag{3.72}$$

In both dimensional examples, we encounter an additional constraint:

$$\partial \cdot A^{(D-3)} \approx 0 \,, \tag{3.73}$$

ensuring that the radial orders $A_a^{(0)}$ and the divergence-free part of $A_a^{(D-3)}$ remain unconstrained by the aforementioned equations of motion.

Bondi coordinates

We aim to replicate the above solution space results in the Bondi patch (3.14) of AdS. One motivation for this choice is that, within this framework, computing the flat limit of the charge becomes essentially straightforward. However, it is noteworthy that the analysis in Bondi coordinates proves to be more intricate than in the Poincaré patch. We adhere to the same philosophy as applied in these coordinates.

Field equations. In terms of the gauge-invariant Faraday tensor, the field equations (3.23) can be expressed as follows:

$$E_u = \left(\partial_r + \frac{D-2}{r} \right) F_{ur} - \frac{1}{r^2} \partial^i F_{ir} \,, \tag{3.74}$$

$$E_r = \partial_u F_{ru} - \frac{1}{r^2} \partial^i F_{iu} + \left(\frac{1}{r^2} + \frac{1}{\ell^2} \right) \partial^i F_{ir} \,, \tag{3.75}$$

$$\begin{aligned} E_i = & \frac{1}{r^2} \left(\partial_r + \frac{D-4}{r} \right) (F_{ri} - F_{ui}) + \frac{1}{\ell^2} \left(\partial_r + \frac{D-2}{r} \right) F_{ri} \\ & - \frac{1}{r^2} \partial_u F_{ri} - \frac{1}{r^4} \partial^j F_{ij} \,. \end{aligned} \tag{3.76}$$

Throughout the manuscript, when Bondi coordinates are used, we recall that the spherical indices are lowered and raised using the metric γ_{ij} and its inverse γ^{ij} according to appendix A. Expanding the field strength in the radial coordinate as

$$F_{\mu\nu}(r, u, x^i) = \sum_n r^{-n} \left(F_{\mu\nu}^{(n)}(u, x^i) + \log r \, \widetilde{F}_{\mu\nu}^{(n)}(u, x^i) \right) , \tag{3.77}$$

we derive recursive relations from the equations of motion:

$$E_u^{(n)} = (D-n-2)\,F_{ur}^{(n)} + \widetilde{F}_{ur}^{(n)} - \partial^i F_{ir}^{(n-1)}\,, \tag{3.78}$$

$$E_r^{(n)} = \partial_u F_{ru}^{(n)} + \partial^i \left(F_{ir}^{(n-2)} - F_{iu}^{(n-2)}\right) + \frac{1}{\ell^2}\partial^i F_{ir}^{(n)}\,, \tag{3.79}$$

$$\begin{aligned} E_i^{(n)} &= (D-n-4)\left(F_{ri}^{(n)} - F_{ui}^{(n)}\right) + \left(\widetilde{F}_{ri}^{(n)} - \widetilde{F}_{ui}^{(n)}\right) - \partial_u\, F_{ri}^{(n+1)} \\ &\quad + \frac{1}{\ell^2}\widetilde{F}_{ri}^{(n+2)} + \frac{1}{\ell^2}\,(D-n-4)\,F_{ri}^{(n+2)} - \partial^j F_{ij}^{(n-1)}\,. \end{aligned} \tag{3.80}$$

The Bianchi identity (3.25) imposes additional constraints:

$$\partial_u F_{ir}^{(n)} - \partial_i F_{ur}^{(n)} = -(n-1)F_{iu}^{(n-1)} + \widetilde{F}_{iu}^{(n-1)}\,, \tag{3.81}$$

$$\partial_i F_{uj}^{(n)} - \partial_j F_{ui}^{(n)} = \partial_u F_{ij}^{(n)}\,, \tag{3.82}$$

$$\partial_i F_{rj}^{(n)} - \partial_j F_{ri}^{(n)} = -(n-1)F_{ij}^{(n-1)} + \widetilde{F}_{ij}^{(n-1)}\,, \tag{3.83}$$

$$\partial_i F_{kj}^{(n)} - \partial_j F_{ki}^{(n)} = \partial_k F_{ij}^{(n)}\,. \tag{3.84}$$

Similar relations for logarithmic terms can be obtained. By injecting (3.82), (3.83), and (3.84) into the antisymmetric spherical derivative of (3.80), the recursive relation for the radial orders $F_{ij}^{(n)}$ is streamlined:

$$\begin{aligned} 0 &\approx \frac{1}{\ell^2}(D-n-4)(n+1)F_{ij}^{(n+1)} - \frac{1}{\ell^2}(D-2n-5)\widetilde{F}_{ij}^{(n+1)} + 2\partial_u \widetilde{F}_{ij}^{(n)} \\ &\quad + (D-2n-4)\partial_u F_{ij}^{(n)} - \left(\Delta - (D-n-4)(n-1)\right) F_{ij}^{(n-1)} \\ &\quad - (D-2n-3)\widetilde{F}_{ij}^{(n-1)}\,, \end{aligned} \tag{3.85}$$

where $\Delta = D^i D_i = \gamma^{ij} D_i D_j$ is the Laplacian operator with respect to the unit spherical metric and D_i the associated covariant derivative. Combining (3.78) and (3.81) yields, for $n \neq 0, D-3$:

$$F_{iu}^{(n)} \approx -\frac{1}{n}\left(\partial_u F_{ir}^{(n+1)} - \frac{\partial_i \partial^j F_{jr}^{(n)}}{D-n-3}\right) + \frac{1}{n}\widetilde{F}_{iu}^{(n)}\,, \tag{3.86}$$

providing a recursive relation for $F_{ir}^{(n)}$ in terms of $F_{ij}^{(n)}$ when injected into (3.80):

$$\begin{aligned} 0 &\approx \frac{1}{\ell^2}(D-n-4)F_{ir}^{(n+2)} + \frac{1}{\ell^2}\widetilde{F}_{ir}^{(n+2)} - \partial_u F_{ir}^{(n+1)} + (D-n-4)F_{ir}^{(n)} \\ &\quad + \left(\widetilde{F}_{ir}^{(n)} - \widetilde{F}_{iu}^{(n)}\right) + \frac{D-n-4}{n}\left(\partial_u F_{ir}^{(n+1)} - \frac{\partial_i \partial^j F_{jr}^{(n)}}{D-n-3} - \widetilde{F}_{iu}^{(n)}\right) \\ &\quad + \partial^j F_{ij}^{(n-1)}\,. \end{aligned} \tag{3.87}$$

Asymptotic radial expansions. Applying a similar rationale as in the Poincaré coordinates, we enforce the subsequent boundary conditions:

$$F_{ij} \sim \mathcal{O}(1)\,, \quad F_{ir} \sim \mathcal{O}(r^{-2})\,, \quad F_{ur} \sim \mathcal{O}(r^{-3})\,, \quad F_{iu} \sim \mathcal{O}(1)\,. \tag{3.88}$$

For $D = 6$, the equations of motion admit the following solutions:

$$F_{ij} \approx F^{(0)}_{ij} - \frac{\ell^2}{r}\partial_u F^{(0)}_{ij} + \frac{\ell^2}{2r^2}\Delta F^{(0)}_{ij} + \frac{1}{r^3}F^{(3)}_{ij} + \mathcal{O}(r^{-4})\,, \tag{3.89}$$

$$F_{ir} \approx \frac{\ell^2}{r^2}F^{(0)}_{iu} - \frac{\ell^2}{r^3}\partial^j F^{(0)}_{ij} + \frac{1}{r^4}F^{(4)}_{ir} + \mathcal{O}(r^{-5})\,, \tag{3.90}$$

$$F_{ur} \approx \frac{\ell^2}{r^3}\partial^i F^{(0)}_{iu} + \frac{1}{r^4}F^{(4)}_{ur} + \mathcal{O}(r^{-5})\,, \tag{3.91}$$

$$F_{iu} \approx F^{(0)}_{iu} - \frac{\ell^2}{r}\partial_u F^{(0)}_{iu} + \mathcal{O}(r^{-2})\,, \tag{3.92}$$

where $F^{(0)}_{ij}$, $F^{(3)}_{ij}$, $F^{(4)}_{ir}$, and $F^{(0)}_{iu}$ are arbitrary functions of (u, x^i), while

$$\partial_u F^{(4)}_{ur} \approx \partial^i \left(\frac{1}{\ell^2}F^{(4)}_{ir} + \ell^2 F^{(0)}_{iu} - \frac{\ell^2}{2}\partial^j \partial_i F^{(0)}_{ju} \right). \tag{3.93}$$

In the case of $D = 5$, the solution space is characterized by the following asymptotic expansion:

$$F_{ij} \approx F^{(0)}_{ij} - \frac{\ell^2}{r}\partial_u F^{(0)}_{ij} + \frac{1}{r^2}\left[F^{(2)}_{ij} - \frac{\ell^2}{2}\log r \left(\ell^2 \partial_u^2 - \Delta\right) F^{(0)}_{ij} \right] + \mathcal{O}(r^{-3})\,, \tag{3.94}$$

$$F_{ir} \approx \frac{\ell^2}{r^2}F^{(0)}_{iu} + \frac{1}{r^3}\left[F^{(3)}_{ir} + \ell^2 \log r \left(\ell^2 \partial_u F^{(0)}_{iu} - \partial^j F^{(0)}_{ij} \right) \right] + \mathcal{O}(r^{-4})\,, \tag{3.95}$$

$$F_{ur} \approx \frac{1}{r^3}\left(F^{(3)}_{ur} + \ell^2 \log r\, \partial^i F^{(0)}_{iu} \right) + \mathcal{O}(r^{-4})\,, \tag{3.96}$$

$$F_{iu} \approx F^{(0)}_{iu} + \mathcal{O}(r^{-1})\,, \tag{3.97}$$

where the free data are $F^{(0)}_{ij}$, $F^{(2)}_{ij}$, $F^{(3)}_{ir}$, and $F^{(0)}_{iu}$, and

$$\partial_u F^{(3)}_{ur} \approx \partial^i \left(\frac{1}{\ell^2}F^{(3)}_{ir} + \ell^2 \partial_u F^{(0)}_{iu} \right). \tag{3.98}$$

Discussion on the gauge field. In the context of the Maxwell field and within the radial gauge $A_r = 0$, the equations of motion manifest as follows:

$$
\begin{aligned}
E_u^{(n)} &= \frac{1}{\ell^2}(n+1)(n-D+4)A_u^{(n+1)} + n\partial_u A_u^{(n)} - \partial_u D\cdot A^{(n-1)} \\
&+ \left[\Delta + (n-1)(n-D+2)\right] A_u^{(n-1)} + (D-2n-1)\widetilde{A}_u^{(n-1)} \\
&+ \frac{1}{\ell^2}(D-2n-5)\widetilde{A}_u^{(n+1)} - \partial_u \widetilde{A}_u^{(n)} \,,
\end{aligned}
\tag{3.99}
$$

$$
\begin{aligned}
E_r^{(n)} &= (n-1)(n-D+2)A_u^{(n-1)} + (n-2)D\cdot A^{(n-2)} \\
&+ (D-2n-1)\widetilde{A}_u^{(n-1)} - D\cdot \widetilde{A}^{(n-2)} \,,
\end{aligned}
\tag{3.100}
$$

$$
\begin{aligned}
E_i^{(n)} &= (2n-D+4)\partial_u A_i^{(n)} + (D-n-4)\partial_i A_u^{(n)} - D_i D\cdot A^{(n-1)} \\
&+ (D-2n-1)\widetilde{A}_i^{(n-1)} + \left[\Delta + (n-1)(n-D+4)\right] A_i^{(n-1)} \\
&+ \frac{1}{\ell^2}(n+1)(n-D+4)A_i^{(n+1)} + \frac{1}{\ell^2}(D-2n-5)\widetilde{A}_i^{(n+1)} \\
&- 2\partial_u \widetilde{A}_i^{(n)} + \partial_i \widetilde{A}_u^{(n)} \,.
\end{aligned}
\tag{3.101}
$$

where we assumed that

$$
A_\mu(r,u,x^i) = \sum_n r^{-n}\left(A_\mu^{(n)}(u,x^i) + \log r\, \widetilde{A}_\mu^{(n)}(u,x^i)\right). \tag{3.102}
$$

In this gauge, the gauge parameter does not depend on the radial coordinate. The radial expansion of the gauge potential components in the even case ($D = 6$) is expressed as:

$$
\begin{aligned}
A_i &\approx A_i^{(0)} + \frac{\ell^2}{r}\left(\partial_i A_u^{(0)} - \partial_u A_i^{(0)}\right) + \frac{\ell^2}{2r^2}\left(\Delta A_i^{(0)} - \partial_i D\cdot A^{(0)}\right) \\
&+ \frac{1}{r^3}A_i^{(3)} + \mathcal{O}(r^{-4}) \,,
\end{aligned}
\tag{3.103}
$$

$$
A_u \approx A_u^{(0)} + \frac{\ell^2}{2r^2}\left(\Delta A_u^{(0)} - \partial_u D\cdot A^{(0)}\right) + \frac{1}{r^3}A_u^{(3)} + \mathcal{O}(r^{-4}) \,, \tag{3.104}
$$

and the time evolution of $A_u^{(3)}$ is restricted by

$$
\partial_u A_u^{(3)} \approx \frac{1}{\ell^2} D\cdot A^{(3)} - \frac{\ell^2}{6}(\Delta - 2)\left(\Delta A_u^{(0)} - \partial_u D\cdot A^{(0)}\right). \tag{3.105}
$$

For $D=5$, the solution space is characterized by

$$\begin{aligned} A_i \approx A_i^{(0)} + \frac{\ell^2}{r}\left(\partial_i A_u^{(0)} - \partial_u A_i^{(0)}\right) + \frac{1}{r^2}\Big[A_i^{(2)} - \frac{\ell^2}{2}\log r\Big(\ell^2\partial_u^2 A_i^{(0)} \\ - \Delta A_i^{(0)} - \ell^2\partial_i\partial_u A_u^{(0)} + \partial_i D\cdot A^{(0)}\Big)\Big] + \mathcal{O}(r^{-3})\,, \end{aligned} \tag{3.106}$$

$$A_u \approx A_u^{(0)} + \frac{1}{r^2}\left[A_u^{(2)} + \frac{\ell^2}{2}\log r\left(\Delta A_u^{(0)} - \partial_u D\cdot A^{(0)}\right)\right] + \mathcal{O}(r^{-3})\,, \tag{3.107}$$

with

$$\partial_u A_u^{(2)} \approx \frac{1}{\ell^2} D\cdot A^{(2)} + \frac{\ell^2}{2}\partial_u\left(\Delta A_u^{(0)} - \partial_u D\cdot A^{(0)}\right). \tag{3.108}$$

In both cases, the radial orders $A_i^{(0)}$, $A_i^{(D-3)}$, and $A_u^{(0)}$ remain unconstrained by the field equations, and we impose the following boundary conditions:

$$A_i \sim \mathcal{O}(1)\,, \qquad A_u \sim \mathcal{O}(1)\,. \tag{3.109}$$

For the Bondi patch, we do not discuss the Lorenz gauge since it leads to expressions too complicated for what we want to illustrate.

3.3.2 Variational principle

Poincaré coordinates

Now that we have access to the asymptotic solution space, residual symmetries, and gauge transformations, we can proceed to the next stage of calculating the asymptotic surface charges associated with this theory. In a straightforward manner, we can input the informations into the generic Iyer-Wald relation for the codimension-2 quantity (3.28).

Radial charge divergences. Considering the orientation of the AdS boundary as $n_\mu = \delta^z_\mu$ and the skew-symmetry of (3.28), the crucial component defining the charge is k^{za}_λ,

$$k^{za}_\lambda = -\frac{1}{\ell} z^{-(D-4)}\eta^{ab}\lambda\delta F_{zb}\,. \tag{3.110}$$

In the case of $D=6$, it is expressed as

$$k^{za}_\lambda = -\frac{\ell}{z}\eta^{ac}\lambda^{(0)}\partial^b\delta F^{(0)}_{bc} - \frac{1}{\ell}\eta^{ab}\lambda^{(0)}\delta F^{(2)}_{zb} + \mathcal{O}(z)\,, \tag{3.111}$$

while, in $D=5$,

$$k^{za}_\lambda = \ell\log z\,\eta^{ac}\lambda_{(0)}\partial^b\delta F^{(0)}_{bc} - \frac{1}{\ell}\eta^{ab}\lambda^{(0)}\delta F^{(1)}_{zb} + \mathcal{O}(z)\,. \tag{3.112}$$

These relationships are divergent in the limit $z \to 0$ and remain consistent in both the Lorenz and radial gauges mentioned earlier.

By segregating the boundary coordinates into time and corner coordinates, $(x^a) = (t, x^i)$, the definition of the asymptotic surface charge (2.50) should be

$$\delta H_\lambda \approx \lim_{z\to 0} \int_{\mathcal{C}} \mathrm{d}^{D-2}x \, k_\lambda^{tz} \,. \tag{3.113}$$

In the above Maxwell example, since $\delta\lambda = 0$, we notice that we can already directly integrate the charge variation as

$$H_\lambda \approx -\frac{1}{\ell} \lim_{z\to 0} \int_{\mathcal{C}} \mathrm{d}^{D-2}x \, \frac{\lambda \, F_{zt}}{z^{(D-4)}} \,. \tag{3.114}$$

In the more specific contexts dealt with in the rest of this thesis, such manipulation can only be carried out after the full calculation has been completed, since the residual gauge parameters can generally exhibit field dependence in their radial expansion. However, it is crucial to note that this charge (3.114) is not well-defined in $D = 5, 6$ due to the above highlighted radial divergences of k_λ^{tz}. Therefore, a careful examination of how to renormalize the latter becomes necessary. The first prescription we adhere to, introduced generically in section 2.3, stems from the variational principle. This approach is chosen because the variational principle itself is divergent as one approaches the boundary.

For the sake of completeness, let us acknowledge the counterpart of k_λ^{za} in the Barnich-Brandt prescription (3.29). It can be expressed as follows:

$$k^{za}_{\mathrm{BB},\lambda} = -\frac{1}{\ell} z^{-(D-4)} \Big(\lambda \partial_z \delta A^a - \lambda \partial^a \delta A_z + \delta A_z \partial^a \lambda - \delta A^a \partial_z \lambda \Big) \,. \tag{3.115}$$

This expression takes the following form in the Lorenz gauge for the same dimensional examples:

$$\begin{aligned} D = 6 : k^{za}_{\mathrm{BB},\lambda} = \frac{\ell}{9z} \Big[&\partial^a \lambda^{(0)} \delta\partial \cdot \delta A^{(0)} - 3\delta A^a_{(0)} \Box \lambda^{(0)} - \lambda^{(0)} \Big(7\partial^a \partial \cdot \delta A^{(0)} \\ &- 9 \Box \delta A^a_{(0)} \Big) \Big] + \frac{3}{\ell} \lambda^{(0)} \delta A^a_{(3)} + \mathcal{O}(z) \,, \end{aligned} \tag{3.116}$$

$$\begin{aligned} D = 5 : k^{za}_{\mathrm{BB},\lambda} = \frac{\ell}{2} \log z \, \lambda^{(0)} \Big(\ell^2 \partial^a \partial \cdot \delta A^{(0)} - 2 \Box \delta A^a_{(0)} \Big) + \frac{1}{4\ell} \Big[&\ell^4 \lambda^{(0)} \partial^a \partial \cdot \delta A^{(0)} \\ - \ell^2 \Big(2\delta A^a_{(0)} \Box \lambda^{(0)} + \lambda^{(0)} \partial^a \partial \cdot \delta A^{(0)} + 2\lambda^{(0)} \Box \delta A^a_{(0)} \\ - \partial \cdot \delta A^{(0)} \partial^a \lambda^{(0)} \Big) + 8 \lambda^{(0)} \delta A^a_{(2)} \Big] + \mathcal{O}(z) \,, \end{aligned} \tag{3.117}$$

and in the radial gauge:

$$D=6: k^{za}_{\mathrm{BB},\lambda} = \frac{\ell}{z}\lambda^{(0)}\left(\Box\delta A^{(0)}_a - \frac{2}{3}\partial_a\partial\cdot\delta A^{(0)}\right) + \frac{3}{\ell}\lambda^{(0)}\delta A^{(3)}_a + \mathcal{O}(z)\,, \tag{3.118}$$

$$\begin{aligned} D=5: k^{za}_{\mathrm{BB},\lambda} = \frac{\ell}{2}\log z\,\lambda^{(0)}\left(\ell^2\partial_a\partial\cdot\delta A^{(0)} - 2\Box\delta A^{(0)}_a\right) + \frac{1}{4\ell}\lambda^{(0)}\Big(\ell^4\partial_a\partial\cdot\delta A^{(0)} \\ - 2\ell^2\Box\delta A^{(0)}_a + 8\delta A^{(2)}_a\Big) + \mathcal{O}(z)\,. \end{aligned} \tag{3.119}$$

These expressions also exhibit radial divergences, posing challenges in defining the associated charges. However, no specific renormalization procedure is discussed in this context. Instead, the focus is on renormalizing the Iyer-Wald procedure and then transitioning to the renormalized Barnich-Brandt expression using the corner term (3.30), linking the two formalisms:

$$E^{za} = \frac{1}{\ell}z^{-(D-4)}\Big(\delta A_z\partial_a\lambda - \delta A_a\partial_z\lambda\Big)\,. \tag{3.120}$$

One can check that this term vanishes for the radial gauge, coinciding the expansions of k^{za}_{λ} and $k^{za}_{\mathrm{BB},\lambda}$. Moreover, it is important to highlight that in the Maxwell case, another aspect of the Barnich-Brandt expressions becomes then evident: they are unable to function independently of the chosen electromagnetic gauge.

Regulated action. Returning to the Iyer-Wald surface charge, let us address holographic renormalization by carefully following the steps outlined in section 2.3. The initial step, acquiring the asymptotic solution space, has been completed in the previous section. The subsequent step involves regularization of the theory. To achieve this, we introduce a regularization cut-off ε as a small parameter, ensuring $\varepsilon < z$. Consequently, we can reformulate the bulk action (3.19) as

$$S_{\mathrm{reg}} = -\frac{1}{4}\int_{z>\varepsilon} \mathrm{d}^D x\,\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu} \approx \frac{1}{2\ell\varepsilon^{D-4}}\int_{z=\varepsilon}\mathrm{d}^{D-1}x\,A^a F_{za}\,, \tag{3.121}$$

where we used the fact that, on-shell, $\sqrt{-g}\,\partial_\mu A_\nu F^{\mu\nu} \approx \partial_\mu(\sqrt{-g}A_\nu F^{\mu\nu})$ and considered the orientation of the AdS boundary given by $n_\mu = \delta^z_\mu$. Upon expanding the fields for small $z=\varepsilon$, it becomes apparent that this regulated action diverges for $D \geq 5$, allowing the isolation of a finite number of divergent terms. This is consistent with our earlier observation that the variational principle itself diverges as $z \to 0$.

In this renormalization approach, we will maintain manifest gauge invariance by radially expanding only the field strength. The justification for this lies in the following reasoning. Starting with the expression

$$S_{\text{reg}} \approx \int_{z=\varepsilon} \mathrm{d}^{D-1}x \, A^a \mathcal{F}_a \,, \qquad \mathcal{F}_a = \frac{1}{2\ell\varepsilon^{D-4}} F_{za} \,, \tag{3.122}$$

we aim to define an appropriate subtracted action. This is achieved by finding a counterterm action S_{ct} such that

$$S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}} \approx \int_{z=\varepsilon} \mathrm{d}^{D-1}x \, A^a \widetilde{\mathcal{F}}_a \,, \tag{3.123}$$

where $\widetilde{\mathcal{F}}_a$ remains finite as $\varepsilon \to 0$, given that A^a is itself finite in that limit. This can be found by inserting the asymptotic expansion of the Faraday tensor in the regulated action,

$$S_{\text{reg}} \approx \frac{1}{2\ell\varepsilon^{D-4}} \int \mathrm{d}^{D-1}x \, A^a \sum_{n=1}^{D-5} \varepsilon^n F^{(n)}_{za} + \mathcal{O}(1) \,. \tag{3.124}$$

Here, the sum formally runs from $n = 1$ to $n = (D-5)$, but only the odd terms are non-vanishing.

As previously mentioned, in this renormalization scheme, we deliberately choose not to expand $A_a = \mathcal{O}(1)$. This decision stems from our goal of obtaining the regulated action expression solely in terms of bulk fields, sufficient for deriving a finite charge. This choice does not impact the bulk-covariant form of the counterterms, except for potential scheme-dependent terms in the finite piece. It is important to note that this situation only arises in our Maxwell examples for odd-dimensional cases where $D \geq 7$. A detailed discussion can be found in (Campoleoni et al., 2023a). This choice is particularly advantageous as it allows us to rely solely on the equations of motion for the field strength. In the subsequent section, we will contrast this with a different prescription, necessitating the asymptotic expansion of the Maxwell field.

Counterterm action. In the dimensional examples we are considering, the regulated action exhibits only one divergent term. In $D = 6$, this term is given by

$$S_{\text{reg}} = \frac{1}{2\ell\varepsilon} \int_{z=\varepsilon} \mathrm{d}^5x \, A^a \partial^b F^{(0)}_{ba} + \mathcal{O}(1) \,, \tag{3.125}$$

and in $D=5$ by

$$S_{\text{reg}} = -\frac{\ell}{2}\log\varepsilon \int_{z=\varepsilon} \mathrm{d}^4x\, A^a \partial^b F^{(0)}_{ba} + \mathcal{O}(1)\,. \tag{3.126}$$

Moving on to the third step, we need to determine covariant boundary counterterms to incorporate into the bulk action. These counterterms are expressed as

$$D=6: S_{\text{ct}} = -\frac{1}{2\ell}\int_{z=\varepsilon} \mathrm{d}^5x\, \sqrt{-\gamma}\,\gamma^{ac}\gamma^{bd} A_c \partial_d F_{ba}\,, \tag{3.127}$$

$$D=5: S_{\text{ct}} = \frac{\ell}{2}\log\varepsilon \int_{z=\varepsilon} \mathrm{d}^4x\, \sqrt{-\gamma}\,\gamma^{ac}\gamma^{bd} A_c \partial_d F_{ba}\,, \tag{3.128}$$

where we have inverted the asymptotic expansions as

$$F_{ab} = F^{(0)}_{ab} + \mathcal{O}(\varepsilon^2) \quad \Rightarrow \quad F^{(0)}_{ab} = F_{ab} + \mathcal{O}(\varepsilon^2)\,. \tag{3.129}$$

This expresses the counterterm actions in terms of the fields which live at the regulating surface $z=\varepsilon$. The induced metric on this surface is denoted as $\gamma_{ab} = g_{ab}/\varepsilon$[1]. It is important to note that in the odd case, due to the logarithmic term, we cannot absorb the radial dependence. This aspect is connected to the concept of anomaly in the dual theory, as discussed in (Henningson & Skenderis, 1998; Skenderis, 2002). We insist that locality in terms of source is maintained in these counterterms.

One can improve the expression of the latter by utilizing the inverse of the Leibniz rule,

$$A^a \partial^b F_{ba} = \partial^b \left(A^a F_{ba}\right) - \frac{1}{2} F^{ab} F_{ab}\,, \tag{3.130}$$

allowing the counterterms to adopt a manifestly gauge-invariant form. Indeed, in the last equation, we can omit the first term on the right-hand side, assuming that we are working with field configurations that decay in the early past and in the far future at the boundary. This translates to imposing fall-off conditions on the boundary fields as they approach the corners,

$$F_{ab} = 0 \text{ on the boundary of } z=\varepsilon\,. \tag{3.131}$$

In the next section, we will explore an alternative method of achieving this result without imposing such corner conditions. This involves using corner terms in the symplectic structure. The above improvement leads us to the fourth step of holographic renormalization.

[1]This is not to be confused with the codimension-2 spherical metric in Bondi coordinates.

Renormalized action. The subtracted action in $D=6$ at the regularization cut-off is given by

$$
\begin{aligned}
S_{\text{sub}} &= S + S_{\text{ct}} \\
&= \int_{z>\varepsilon} \mathrm{d}^6 x \Big(-\frac{1}{4}\sqrt{-g} F_{\mu\nu}F^{\mu\nu} \Big) - \ell \int_{z=\varepsilon} \mathrm{d}^5 x \Big(-\frac{1}{4}\sqrt{-\gamma}\gamma^{ac}\gamma^{bd} F_{ab}F_{cd} \Big),
\end{aligned} \tag{3.132}
$$

and in $D=5$ by

$$
S_{\text{sub}} = \int_{z>\varepsilon} \mathrm{d}^5 x \Big(-\frac{1}{4}\sqrt{-g} F_{\mu\nu}F^{\mu\nu} \Big) + \ell \log\varepsilon \int_{z=\varepsilon} \mathrm{d}^4 x \Big(-\frac{1}{4}\sqrt{-\gamma}\gamma^{ac}\gamma^{bd} F_{ab}F_{cd} \Big). \tag{3.133}
$$

By construction, the result is finite in the asymptotic limit and yields the on-shell value of the renormalized action:

$$
D=6 : S_{\text{ren}} = \lim_{\varepsilon\to 0} S_{\text{sub}} \approx \frac{1}{\ell}\int_{z=0} \mathrm{d}^5 x\, A^a F^{(2)}_{za}\,, \tag{3.134}
$$

$$
D=5 : S_{\text{ren}} \approx \frac{1}{\ell}\int_{z=0} \mathrm{d}^4 x\, \delta A^a F^{(1)}_{za}\,. \tag{3.135}
$$

By taking the variation of the latter and considering the orientation of the boundary, we derive the radial component of the renormalized presymplectic potential:

$$
\delta S_{\text{ren}} \approx \int_{z=0} \mathrm{d}^{D-1} x\, \Theta^z_{\text{ren}}\,, \qquad \Theta^z_{\text{ren}}[A;\delta A] = \frac{1}{\ell} A^a F^{(D-4)}_{za}\,, \tag{3.136}
$$

where $F^{(D-4)}_{za} = (D-3)A^{(D-3)}_a - \partial_a A^{(D-4)}_z$ (in both gauges). The last expression (3.136) enables us to elaborate on the roles of the vacuum expectation value (VEV) and the source, which we previously assigned to $F^{(D-4)}_{za}$ and $A_a \sim A^{(0)}_a$ during our discussion of the solution space. Indeed, within the holographic interpretation, the on-shell value of the renormalized action variation yields a term expressed as the product of the VEV and the variation of the source.

Finite asymptotic charges. Following the steps of the Iyer-Wald formalism (see section 2.1), we can construct the associated asymptotic renormalized surface charge:

$$
\omega^z_{\lambda,\text{ren}} = \delta_\lambda \Theta^z_{\text{ren}}[A;\delta A] - \delta\Theta^z_{\text{ren}}[A;\delta_\lambda A] \approx \partial_a k^{za}_{\lambda,\text{ren}}\,, \quad \delta H^{\text{ren}}_\lambda \approx \int \mathrm{d}^{D-2} x\, k^{tz}_{\lambda,\text{ren}}\,. \tag{3.137}
$$

It is the fifth and last step of the holographic scheme. To provide a clearer understanding, the formal manipulations in the equation (3.137) involve examining the variation of the renormalized action along a gauge parameter λ and deducing that this can be rewritten on-shell as a corner term:

$$\delta_\lambda S_{\text{ren}} = \frac{1}{\ell} \int \mathrm{d}^{D-1}x \, \partial^a \left(\lambda F_{za}^{(D-4)} \right) , \tag{3.138}$$

where we used the divergence-free condition $\partial^a F_{za}^{(D-4)} \approx 0$ from the equations of motion (3.35). This yields the charge flux across the boundary, and the surface charges can be expressed as:

$$H_\lambda^{\text{ren}} = -\frac{1}{\ell} \int \mathrm{d}^{D-2}x \left(\lambda F_{zt}^{(D-4)} \right) . \tag{3.139}$$

The time derivative of these quantities, $\partial_t H_\lambda^{\text{ren}}$, manifestly vanishes only for the standard electric charge (i.e., when $\lambda = 1$).

Bondi coordinates

All the previous Poincaré results can be replicated in the Bondi patch of the AdS space. The holographic prescription we employed for (3.9) utilizes gauge-invariant and covariant counterterms, allowing us to straightforwardly derive the final outcomes by performing a change of coordinates between the two systems. Notably, this is achieved in (Campoleoni et al., 2023a). However, in the upcoming chapter 4, we will introduce asymptotic gauges of gravity, involving Poincaré and Bondi coordinates, without necessarily executing the gauge transformation between the two. Therefore, we refrain from demonstrating these steps. Instead, our emphasis is on reproducing the results of holographic renormalization in Bondi coordinates, following the same philosophy, to explore how one can navigate this scenario. Particularly, we adopt a manifestly gauge-invariant approach and avoid asymptotically expanding the Maxwell field.

In Bondi coordinates (3.14), introducing a regularization cut-off $r < R$, where R is a large parameter, and starting from (3.19), the on-shell regulated action reads

$$S_{\text{reg}} = -\frac{1}{2} \int_{r=R} \mathrm{d}^{D-1}x \, R^{D-2} \sqrt{-\gamma} \left[A_u F_{ur} - \frac{1}{R^2} A^i F_{ui} + \left(\frac{1}{R^2} + \frac{1}{\ell^2} \right) A^i F_{ri} \right] . \tag{3.140}$$

In this equation, the orientation of the AdS boundary in Bondi coordinates is specified by $n_\mu = \delta_\mu^r$. Let us stress that, in the Bondi patch, the γ-metric corresponds to the one of the unit sphere and is distinct from the metric induced on the boundary as previously introduced in the Poincaré case. For the example of even dimensionality ($D = 6$), the divergent part in the action is highlighted:

$$S_{\text{reg}} = -\frac{1}{2}\int_{r=R} \mathrm{d}^5x\, R\sqrt{-\gamma}\left[A^i\partial^j F_{ij} + \ell^2\left(A_u\partial^i - A^i\partial_u\right)F_{iu}\right] + \mathcal{O}(1)\,. \quad (3.141)$$

Here, we have inverted the asymptotic expansions as

$$F^{(0)}_{ij} = \left(1 + \frac{\ell^2}{r}\partial_u\right)F_{ij} + \mathcal{O}\left(r^{-2}\right), \qquad F^{(0)}_{iu} = \left(1 + \frac{\ell^2}{r}\partial_u\right)F_{iu} + \mathcal{O}\left(r^{-2}\right). \quad (3.142)$$

Adding the following local counterterm

$$S_{\text{ct}} = \frac{1}{4}\int_{r=R} \mathrm{d}^5x\, R\sqrt{-\gamma}\left(F^{ij}F_{ij} - 2\ell^2 F^i{}_u F_{iu}\right), \quad (3.143)$$

where a boundary term cancellation is achieved through a corner fall-off condition

$$F_{ij} = 0\,,\ F_{iu} = 0 \quad \text{on the boundary of } r = R\,, \quad (3.144)$$

the subtracted action $S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}}$ takes the following form:

$$S_{\text{sub}} = \int_{r\leq R} \mathrm{d}^6x \left(-\frac{\sqrt{-g}}{4}F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{4}\int_{r=R} \mathrm{d}^5x\, R\sqrt{-\gamma}\left(F^{ij}F_{ij} - 2\ell^2 F^i{}_u F_{iu}\right). \quad (3.145)$$

Note that, similarly to the Poincaré patch, it is possible to express the counterterm covariantly in terms of the regulating surface $r = R$ and the induced metric. However, for brevity, we will not delve into it here.

By varying the subtracted action on-shell and taking the asymptotic limit $R \to \infty$, the renormalized asymptotic surface charge is obtained:

$$\begin{aligned}\delta S_{\text{ren}} \approx -\int \mathrm{d}^5x\,\sqrt{-\gamma}\Big[&\delta A_u F^{(4)}_{ur} - \frac{1}{\ell^2}\delta A^i F^{(4)}_{ir} + \frac{\ell^2}{2}\delta A^i\Big(\partial^j\partial_u F^{(0)}_{ij} \\ &- 2F^{(0)}_{iu} + \partial_i\partial^j F^{(0)}_{ju}\Big)\Big]\,,\end{aligned} \quad (3.146)$$

which, when evaluated along a gauge parameter λ and exploiting (3.93), becomes:

$$\begin{aligned}\delta_\lambda S_{\text{ren}} \approx -\int \mathrm{d}^5x\,\sqrt{-\gamma}\Big\{&\partial_u\left(\lambda F^{(4)}_{ur}\right) - \frac{1}{2\ell^2}\partial^i\Big[\lambda\Big(2F^{(4)}_{ir} - \ell^4\Big(\partial^j\partial_u F^{(0)}_{ij} \\ &- 2F^{(0)}_{iu} + \partial_i\partial^j F^{(0)}_{ju}\Big)\Big)\Big]\Big\}\,.\end{aligned} \quad (3.147)$$

Since the integral of a divergence vanishes on the sphere, the square bracket in the previous equation drops out and we obtain the renormalized asymptotic surface charge:

$$H_\lambda^{\text{ren}} = -\int \mathrm{d}^4x\,\sqrt{-\gamma}\,\lambda\,F_{ur}^{(4)}\,. \tag{3.148}$$

In the case of an odd dimensionality example ($D = 5$), following the now-familiar procedure, the counterterm action is expressed as

$$S_{\text{ct}} = \frac{1}{4}\int_{r=R} \mathrm{d}^4x\,\log R\,\sqrt{-\gamma}\,\left(F^{ij}F_{ij} - 2\ell^2 F^i{}_u F_{iu}\right). \tag{3.149}$$

Notice again the locality in terms of sources. The on-shell variation of the renormalized action is then given by

$$\delta S_{\text{ren}} \approx -\int \mathrm{d}^4x\,\sqrt{-\gamma}\left(\delta A_u F_{ur}^{(3)} - \frac{1}{\ell^2}\delta A^i F_{ir}^{(3)} - \ell^2\delta A^i \partial_u F_{iu}^{(0)}\right). \tag{3.150}$$

Evaluating the above expression along a gauge parameter λ and utilizing (3.98) results in:

$$\delta_\lambda S_{\text{ren}} \approx -\int \mathrm{d}^4x\,\sqrt{-\gamma}\left\{\partial_u\left(\lambda F_{ur}^{(3)}\right) - \frac{1}{\ell^2}\partial^i\left[\lambda\left(F_{ir}^{(3)} - \ell^4\partial_u F_{iu}^{(0)}\right)\right]\right\}. \tag{3.151}$$

Once again, the square bracket simplifies on a closed surface, leading to the asymptotic renormalized corner charge:

$$H_\lambda^{\text{ren}} = -\int \mathrm{d}^3x\,\sqrt{-\gamma}\,\lambda\,F_{ur}^{(3)}\,. \tag{3.152}$$

In the Bondi coordinate system, transitioning from AdS space to flat space is feasible through a smooth flat limit, $\ell \to \infty$, which was unattainable in the Poincaré patch. Notably, the computations leading to (3.148) for $D = 6$ and (3.152) for $D = 5$ reveal that even in these more convenient coordinates, the symplectic structure itself still contains potentially troublesome terms that scale with ℓ^2. However, these problematic terms turn out to be total derivatives on the sphere and vanish entirely. Consequently, we can safely take the limit $\ell \to \infty$ and obtain the standard expression,

$$H_\lambda^{\text{flat}} = -\int \mathrm{d}^{D-2}x\,\lambda\,F_{ur}^{(D-2)}\,. \tag{3.153}$$

This serves as an illustration of a flat limit taken under a full charge level, a technique we will also apply in the more intricate context of gravity, in the next chapter, within an asymptotic gauge (specifically, the Bondi gauge) that permits such manipulations. To validate this limit (3.153), we will briefly showcase this outcome in the flat case in the final section 3.4 of this chapter.

3.3.3 Symplectic structure

Poincaré coordinates

We now transition to the renormalization of the symplectic structure. As outlined towards the conclusion of the formal holographic procedure (refer to section 2.3), one can employ the counterterm Lagrangian introduced into the bulk Lagrangian to address the Iyer-Wald ambiguities in the presymplectic potential and subsequently renormalize it (Compere & Marolf, 2008). However, as illustrated in subsection 3.3.2, this approach is not inherently systematic and can prove to be intricate. Indeed, we have previously encountered certain challenging aspects when examining the seemingly straightforward scenario of Maxwell fields propagating towards the AdS boundary. We now turn our attention to an alternative prescription proposed in section 2.4. This approach, as we have advocated previously and will delve into further here, enables the direct renormalization of the Iyer-Wald codimension-2 form (3.28) at the level of the symplectic structure (Papadimitriou & Skenderis, 2005b; Freidel et al., 2019) without discussing the addition of boundary Lagrangians.

An advantageous aspect of this framework is the derivation of a radial renormalization equation, offering a systematic approach for introducing counterterms. However, it is essential to note that, in contrast to holographic renormalization, this method necessitates not only the solution space of the field strength but also that of the gauge field. Consequently, this procedure theoretically depends on the gauge. Nevertheless, as we explore in the dimensional examples considered, we observe that although gauge invariance may not be explicitly manifest at the counterterm level, it is nonetheless effectively maintained.

We recall that, considering the AdS boundary orientation as $n_\mu = \delta^z_\mu$ and the skew-symmetry of (3.28), the computation of Iyer-Wald charges involves Θ^z and k^{za}_λ. In the $D = 6$ case, in both gauges, they are expressed as follows:

$$\begin{aligned}\Theta^z &= -\frac{z^{-(D-4)}}{\ell}\delta A^a F_{za} \\ &\approx -\frac{\ell}{z}\delta A^a_{(0)}\partial^b F^{(0)}_{ab} - \frac{1}{\ell}\delta A^a_{(0)} F^{(2)}_{za} + \mathcal{O}(z)\,,\end{aligned} \tag{3.154}$$

and

$$k^{za}_\lambda \approx -\frac{\ell}{z}\eta^{ac}\lambda^{(0)}\partial^b \delta F^{(0)}_{bc} - \frac{1}{\ell}\eta^{ab}\lambda^{(0)}\delta F^{(2)}_{zb} + \mathcal{O}(z)\,. \tag{3.155}$$

The latter expressions are divergent as $z \to 0$. The corner contribution of the presymplectic potential is given by

$$\begin{aligned}\Theta^a &= \frac{z^{-(D-4)}}{\ell}\left(\delta A_z F_z{}^a - \ell^2 \delta A_b F^{ab}\right)\\ &\approx -\frac{\ell}{z^2}\delta A_b^{(0)} F_{(0)}^{ab} + \mathcal{O}(1)\,,\end{aligned} \tag{3.156}$$

and the boundary contribution by

$$\begin{aligned}\mathscr{L} &= -\frac{z^{-(D-4)}}{4\ell}\left(2F_z{}^a F_{za} + \ell^2 F^{ab} F_{ab}\right)\\ &\approx -\frac{\ell}{4z^2} F_{(0)}^{ab} F_{ab}^{(0)} + \mathcal{O}(1)\,.\end{aligned} \tag{3.157}$$

In the $D = 5$ case, the expressions are as follows:

$$\Theta^z \approx \ell \log z \delta A^a_{(0)} \partial^b F^{(0)}_{ab} - \frac{1}{\ell}\delta A^a_{(0)} F^{(1)}_{za} + \mathcal{O}(z)\,, \tag{3.158}$$

$$\Theta^a \approx -\frac{\ell}{z}\delta A^b_{(0)} F^{(0)}_{ab} + \mathcal{O}(1)\,, \tag{3.159}$$

$$\mathscr{L} \approx -\frac{\ell}{4z} F^{ab}_{(0)} F^{(0)}_{ab} + \mathcal{O}(1)\,. \tag{3.160}$$

We observe that, in these scenarii, we are unable to employ the method of expressing the first off-shell line of the corner contribution to the bulk presymplectic potential as an analytical expression derived from radial integration; only the second on-shell line can be expressed in this manner. Therefore, the finite term prescription à la McNees-Zwikel (refer to section 2.4) cannot be applied here.

If we factor out the off-shell radial dependence of Θ^μ and $\mathscr{L}$,

$$\Theta^\mu = z^{-(D-4)}\widetilde{\Theta}^\mu\,, \qquad \mathscr{L} = z^{-(D-4)}\widetilde{\mathscr{L}}\,, \tag{3.161}$$

we can derive the asymptotic renormalization equation from (2.81) as

$$\frac{1}{z}\left(z\partial_z - (D-4)\right)\widetilde{\Theta}^z \approx \delta\widetilde{\mathscr{L}} - \partial_a \widetilde{\Theta}^a\,. \tag{3.162}$$

Upon expanding $\widetilde{\Theta}^\mu$ and $\widetilde{\mathscr{L}}$ radially,

$$\widetilde{\Theta}^\mu = \sum_n z^n\left(\widetilde{\Theta}^\mu_{(n)} + \log z\, \widetilde{\theta}^\mu_{(n)}\right), \qquad \widetilde{\mathscr{L}} = \sum_n z^n\left(\widetilde{\mathscr{L}}^{(n)} + \log z\, \widetilde{\ell}^{(n)}\right), \tag{3.163}$$

the above equation (3.162) takes the recursive form:

$$(n - D + 4)\,\widetilde{\Theta}^z_{(n)} + \widetilde{\theta}^z_{(n)} \approx \delta\widetilde{\mathscr{L}}^{(n-1)} - \partial_a \widetilde{\Theta}^a_{(n-1)}\,. \tag{3.164}$$

Consequently, it becomes evident that the orders $n < D-4$ of $\widetilde{\Theta}^z_{(n)}$, which are associated with divergent prefactors in Θ^z, are determined on-shell to be total derivatives plus total variations that can be systematically eliminated order by order. However, the term $\widetilde{\Theta}^z_{(D-4)}$, contributing to the finite order of Θ^z, remains undetermined by this equation.

The renormalized presymplectic potential thus reads (see (2.84))

$$\Theta^z_{\text{ren}} \approx \Theta^z - \delta \int \mathrm{d}z\, \mathscr{L} + \partial_a \int \mathrm{d}z\, \Theta^a\,, \tag{3.165}$$

ensuring that $\partial_z \Theta^z_{\text{ren}} \approx 0$. Defining the on-shell corner ambiguity, deduced from the asymptotic renormalization equation (3.162), as

$$C^{az}[A;\delta A] = \frac{\ell}{z}\delta A^{(0)}_b F^{ab}_{(0)} + \mathcal{O}(z) \qquad (D=6)\,, \tag{3.166}$$

$$= -\ell \log z\, \delta A^{(0)}_b F^{ab}_{(0)} + \mathcal{O}(z) \qquad (D=5)\,, \tag{3.167}$$

it can indeed be verified that its adjustment correctly renormalizes the mixed component of the Iyer-Wald codimension-2 form:

$$k^{za}_{\lambda,\text{ren}} \approx k^{za}_{\lambda} + \delta C^{az}[A;\delta_\lambda A] - \delta_\lambda C^{az}[A;\delta_A] \tag{3.168}$$

where, recalling that in $D = 6$,

$$k^{za}_{\lambda} = -\frac{\ell}{z}\eta^{ac}\lambda^{(0)}\partial^b \delta F^{(0)}_{bc} - \frac{1}{\ell}\eta^{ab}\lambda^{(0)}\delta F^{(2)}_{zb} + \mathcal{O}(z)\,, \tag{3.169}$$

and in $D = 5$,

$$k^{za}_{\lambda} = \ell \log z\, \eta^{ac}\lambda^{(0)}\partial^b \delta F^{(0)}_{bc} - \frac{1}{\ell}\eta^{ab}\lambda^{(0)}\delta F^{(1)}_{zb} + \mathcal{O}(z)\,. \tag{3.170}$$

We can then give a proper definition of the asymptotic surface charge as

$$\delta H^{\text{ren}}_{\lambda} = \int \mathrm{d}^{D-2}x\, k^{tz}_{\lambda,\text{ren}}\,, \qquad k^{tz}_{\lambda,\text{ren}} \approx -\frac{1}{\ell}\eta^{ta}\lambda^{(0)}\delta F^{(D-4)}_{za}\,, \tag{3.171}$$

since it is apparent that $\partial_z k^{tz}_{\lambda,\text{ren}} \approx 0$. Notice that in this case the present renormalization procedure yields gauge-invariant results, akin to the holographic one. As previously discussed in the holographic section, this expression for the corner charge is integrable, finite and, a priori, not time-conserved. Note that we could have arrived at the same result through an equivalent pathway by varying the radial component of the renormalized presymplectic potential, contracting it with a gauge parameter, and identifying a corner term in the resulting expression:

$$\omega^z_{\lambda,\text{ren}} = \delta_\lambda \Theta^z_{\text{ren}}[A;\delta A] - \delta\Theta^z_{\text{ren}}[A;\delta_\lambda A] \approx \partial_a k^{za}_{\lambda,\text{ren}}\,. \tag{3.172}$$

One could consolidate the Iyer-Wald ambiguities renormalizing the presymplectic potential under a symplectic counterterm which, similarly to the renormalization of the variational principle, can be expressed covariantly with respect to the regulating surface. The procedure for this has been demonstrated previously, and we will not reiterate these steps here. In this regard, even though symplectic renormalization does not explicitly involve the addition of boundary terms in the variational principle, it is noteworthy that insights can still be gained. The variation of the renormalized action, incorporating both the bulk action and the boundary terms yet to be added, precisely corresponds on-shell to the divergence of the renormalized presymplectic potential obtained. From this point onward, it is evident that we need to adjust these boundary counterterms to achieve this result by varying them on-shell.

Bondi coordinates

Just as we accomplished at the variational principle level in part 3.3.2, we can reproduce the symplectic results from 3.3.3 within the AdS Bondi patch. Once again, commencing from (2.81), if we isolate the radial off-shell dependence of the presymplectic potential and the Lagrangian,

$$\Theta^\mu = r^{D-2}\sqrt{-\gamma}\,\widetilde{\Theta}^\mu\,, \qquad \mathscr{L} = r^{D-2}\sqrt{-\gamma}\,\widetilde{\mathscr{L}}\,, \tag{3.173}$$

where

$$\widetilde{\Theta}^r = F_{ru}\delta A_u + \frac{1}{\ell^2}F_{ir}\delta A^i + \frac{1}{r^2}(F_{ui}-F_{ri})\delta A^i, \tag{3.174}$$

$$\widetilde{\Theta}^u = F_{ur}\delta A_r + \frac{1}{r^2}F_{ir}\delta A^i, \tag{3.175}$$

$$\widetilde{\Theta}^i = \frac{1}{\ell^2}F_r{}^i\delta A_r - \frac{1}{r^2}F_r{}^i\delta A_u - \frac{1}{r^2}(F_u{}^i - F_r{}^i)\delta A_r - \frac{1}{r^4}F^i{}_j\delta A^j, \tag{3.176}$$

and

$$\widetilde{\mathscr{L}} = \frac{1}{2}\left[F_{ur}F_{ur} - \frac{1}{\ell^2}F_r{}^iF_{ri} + \frac{1}{r^2}F_r{}^i\left(F_{ui}-F_{ri}\right) - \frac{1}{r^2}F_u{}^iF_{ri} - \frac{1}{r^4}F^{ij}F_{ij}\right], \tag{3.177}$$

we can deduce the asymptotic renormalization equation in the form:

$$\frac{1}{r}\left(r\,\partial_r + D - 2\right)\widetilde{\Theta}^r \approx \delta\widetilde{\mathscr{L}} - \partial_u\widetilde{\Theta}^u - \partial_i\widetilde{\Theta}^i\,. \tag{3.178}$$

We mention that the prescription of addressing finite Iyer-Wald ambiguities, as presented in (McNees & Zwikel, 2023), cannot be utilized here for the same reasons outlined in the analysis of Poincaré in subsection 3.3.3. Under the usual assumption

$$\widetilde{\Theta}^\mu = \sum_n r^{-n}\left(\widetilde{\Theta}^\mu_{(n)} + \log r\,\widetilde{\theta}^\mu_{(n)}\right), \qquad \widetilde{\mathscr{L}} = \sum_n r^{-n}\left(\widetilde{\mathscr{L}}^{(n)} + \log r\,\widetilde{\ell}^{(n)}\right), \tag{3.179}$$

the above equation (3.178) yields the recursive renormalization relation:

$$(D-2-n)\,\widetilde{\Theta}^r_{(n)} + \widetilde{\theta}^r_{(n)} \approx \delta\widetilde{\mathscr{L}}^{(n+1)} - \partial_u\widetilde{\Theta}^u_{(n+1)} - \partial_i\widetilde{\Theta}^i_{(n+1)}\,. \tag{3.180}$$

The latter establishes the divergent orders of the presymplectic potential as ambiguities, corresponding to $n < D-2$.

In the case of an even-dimensional example ($D = 6$), the asymptotic expansion of the radial presymplectic potential is obtained as follows:

$$\Theta^r = r\sqrt{-\gamma}\,\widetilde{\Theta}^r_{(3)} + \sqrt{-\gamma}\,\widetilde{\Theta}^r_{(4)} + \mathcal{O}(r^{-1})\,, \tag{3.181}$$

where the divergent and finite orders are given by

$$\widetilde{\Theta}^r_{(3)} = \partial^i F^{(0)}_{ij}\delta A^j_{(0)} + \ell^2 F^{(0)}_{ui}\delta F^i_{(0)u}\,, \tag{3.182}$$

$$\begin{aligned}\widetilde{\Theta}^r_{(4)} = \frac{1}{\ell^2}F^{(4)}_{ir}\delta A^i_{(0)} + F^{(4)}_{ru}\delta A^{(0)}_u + \frac{\ell^2}{2}\Big[\partial^j\left(\partial_u F^{(0)}_{ij} + \partial_i F^{(0)}_{uj}\right)\delta A^i_{(0)} \\ + 2F^{(0)}_{iu}\delta A^i_{(0)}\Big] + \ell^4\partial_u F^{(0)}_{iu}\delta F^i_{(0)u}\,.\end{aligned} \tag{3.183}$$

With the help of (3.180), we can renormalize the above symplectic structure by incorporating the following counterterm:

$$\Theta^r_{\text{ct}} = r\sqrt{-\gamma}\left(\partial_u\widetilde{\Theta}^u_{(4)} + \partial_i\widetilde{\Theta}^i_{(4)} - \delta\widetilde{\mathscr{L}}_{(4)}\right), \tag{3.184}$$

where

$$\widetilde{\Theta}^u_{(4)} = \ell^2 F^{(0)}_{iu}\delta A^i_{(0)}\,, \tag{3.185}$$

$$\widetilde{\Theta}^i_{(4)} = -F^i_{(0)j}\delta A^j_{(0)} + \ell^2 F^i_{(0)u}\delta A^{(0)}_u\,, \tag{3.186}$$

$$\widetilde{\mathscr{L}}_{(4)} = -\frac{1}{2}\left(F^{(0)}_{ij}F^{ij}_{(0)} + \ell^2 F^i_{(0)u}F^{(0)}_{iu}\right). \tag{3.187}$$

Indeed, one can verify that

$$\widetilde{\Theta}^r_{(3)} \approx \delta\widetilde{\mathscr{L}}_{(4)} - \partial_u\widetilde{\Theta}^u_{(4)} - \partial_i\widetilde{\Theta}^i_{(4)}\,. \tag{3.188}$$

When approaching the AdS boundary, i.e., when $r \to \infty$, we then have:

$$\Theta^r_{\text{ren}} = \sqrt{-\gamma}\,\widetilde{\Theta}^r_{(4)}\,, \tag{3.189}$$

where we emphasize the recovery of the result (3.146). Subsequently, the renormalized charge is obtained by varying this component of the presymplectic potential, evaluating the variation along the residual gauge symmetry, and identifying the corner term in the result obtained by distinguishing it from spherical divergence:

$$\omega^r_{\lambda,\text{ren}} = \delta_\lambda\Theta^r_{\text{ren}}[A;\delta_A] - \delta\Theta^r_{\text{ren}}[A;\delta_\lambda A] \approx \partial_u k^{ru}_{\lambda,\text{ren}} + \partial_i k^{ri}_{\lambda,\text{ren}}\,. \tag{3.190}$$

Proceeding in this manner leads to the same expression for the charge as in (3.148):

$$\delta H^\lambda_{\text{ren}} \approx \int \mathrm{d}^4x\, k^{ur}_{\lambda,\text{ren}} = -\int \mathrm{d}^4x\,\sqrt{-\gamma}\,\lambda\,\delta F^{(4)}_{ur}\,. \tag{3.191}$$

In the case of an odd-dimensional example ($D = 5$), the radial expansion is expressed by

$$\Theta^r = \log r\,\sqrt{-\gamma}\,\widetilde{\theta}^r_{(3)} + \sqrt{-\gamma}\,\widetilde{\Theta}^r_{(3)} + \mathcal{O}(r^{-1})\,, \tag{3.192}$$

which requires a logarithmic renormalization since, in the limit as $r \to \infty$, the following term diverges:

$$\widetilde{\theta}^r_{(3)} = \partial^i F^{(0)}_{ij}\delta A^j_{(0)} + \ell^2 F^{(0)}_{ui}\delta F^i_{(0)u}\,. \tag{3.193}$$

Utilizing (3.180), we can nullify the radial divergence through ambiguities:

$$\Theta^r_{\text{ct}} = -\log r\,\sqrt{-\gamma}\left(\delta\widetilde{\mathscr{L}}_{(4)} - \partial_u\widetilde{\Theta}^u_{(4)} - \partial_i\widetilde{\Theta}^i_{(4)}\right), \tag{3.194}$$

where the boundary, corner, and spherical terms are respectively

$$\widetilde{\mathscr{L}}_{(4)} = -\frac{1}{2}\left(F^{(0)}_{ij}F^{ij}_{(0)} + \ell^2\gamma^{ij}F^i_{(0)u}F^{(0)}_{iu}\right), \tag{3.195}$$

$$\widetilde{\Theta}^u_{(4)} = \ell^2 F^{(0)}_{iu}\delta A^i_{(0)}\,, \tag{3.196}$$

$$\widetilde{\Theta}^i_{(4)} = -F^i_{(0)j}\delta A^j_{(0)} + \ell^2 F^i_{(0)u}\delta A^{(0)}_u\,. \tag{3.197}$$

Consequently, the renormalized presymplectic potential reads

$$\Theta^r_{\text{ren}} = \lim_{r\to\infty} (\Theta^r + \Theta^r_{\text{ct}}) = \sqrt{-\gamma}\, \widetilde{\Theta}^r_{(3)}\,, \tag{3.198}$$

where

$$\widetilde{\Theta}^r_{(3)} = \frac{1}{\ell^2} F^{(3)}_{ir} \delta A^i_{(0)} + F^{(3)}_{ru} \delta A^{(0)}_u + \ell^2 \partial_u F^{(0)}_{iu} \delta A^i_{(0)}\,. \tag{3.199}$$

This aligns with the result (3.150) obtained through the renormalization of the variational principle. Ultimately, through the same steps as for $D = 6$, this leads to the corresponding asymptotic corner charge (3.152).

3.4. Flat background

In this concluding section of the chapter dedicated to the asymptotic symmetries of Maxwell theory, we extend the techniques explored in an AdS background to a flat background. In section 3.1, a coordinate patch for AdS was introduced, enabling the description of Minkowski space via a smooth flat limit. Within this coordinate system, we delved into the study of asymptotic charges in the radial gauge of the gauge potential in six and five spacetime dimensions of the AdS space. By examining corner charges, we successfully recovered the flat limit, corroborating the standard result. We briefly validate this outcome, drawing upon established literature (Tamburino & Winicour, 1966; Strominger, 2018) but employing a modern approach to their derivation (Freidel et al., 2019).

Notably, in the flat case, the boundary resides at null infinity, complicating holographic renormalization due to the absence of the parallel with the standard AdS framework. Indeed, for instance in the asymptotically flat gravitational contexts, the counterterms cannot be expressed solely as functionals of the source, necessitating a reevaluation of the analysis from its foundations. While these intricacies are further detailed in (Mann & Marolf, 2006) and related literature, we will not delve in these considerations in the general relativity chapter, opting for the presymplectic prescription. For these reasons, we adopt a parallel analysis for the flat Maxwell study.

Actually, for thoroughness and in anticipation of similar gravitational considerations, we revisit the analysis of Maxwell field propagation in a flat background from the outset. Although we can derive the flat limit from the off-shell relations obtained in the Bondi patch of AdS, we must reassess the solution space it provides. Unlike the AdS analysis, our future analyses express relations solely in terms of the Maxwell field, eschewing the use of the Faraday tensor.

This departure stems from the nature of the symplectic renormalization of the Iyer-Wald symplectic structure, where counterterms cannot be formulated a priori in a manifestly gauge-invariant manner in the Bondi setup.

Besides, a challenge surfaces in the transition from AdS to flat, a puzzle yet unresolved in asymptotic symmetry studies. Specifically, logarithmic terms emerge in the even dimensional flat analysis (Chruściel et al., 1995), while being absent in their AdS counterparts. While this aspect is singular to this particular electromagnetic example and does not recur in the rest of the thesis, we find it noteworthy due to its appearance in this simple scenario.

3.4.1 Solution space

Field equations. We utilize the same bulk Lagrangian and symplectic structure as detailed in section 3.2. The corresponding equations of motion are expressed as follows:

$$\begin{aligned} E_r &= \partial_r \left(r^{D-2} F_{ur} \right) - r^{D-4} \partial^i F_{ir} \,, && (3.200) \\ E_u &= r^2 \partial_u F_{ru} + \partial^i \left(F_{ir} - F_{iu} \right) , && (3.201) \\ E_i &= \partial_r \left(r^{D-4} \left(F_{ri} - F_{ui} \right) \right) - r^{D-6} \left(r^2 \partial_u F_{ri} + \partial^j F_{ij} \right) . && (3.202) \end{aligned}$$

A useful identity, akin to Maxwell's Noether identity (3.25), can be introduced:

$$\nabla^\mu E_\mu = \frac{1}{r^{D-2}} \left(\partial_r - \partial_u \right) \left(r^{D-2} E_r \right) - \frac{1}{r^{D-2}} \partial_r \left(r^{D-2} E_u \right) + \frac{1}{r^2} D \cdot E \,. \quad (3.203)$$

Expressing the field equations in terms of a typical arbitrary polyhomogeneous radial expansion (2.68),

$$A_\mu(r,u,x^i) = \sum_n r^{-n} \left(A_\mu^{(n)}(u,x^i) + \log r \, \widetilde{A}_\mu^{(n)}(u,x^i) \right) , \quad (3.204)$$

the aforementioned field equations, in the radial gauge $A_r = 0$, can be written as

$$\begin{aligned} E_r^{(n)} &= (n-2)D\cdot A^{(n-2)} + (n-1)(n-D+2)A_u^{(n-1)} - D\cdot a^{(n-2)} \\ &\quad + (D-2n-1)\widetilde{A}_u^{(n-1)}\,, \end{aligned} \tag{3.205}$$

$$\begin{aligned} E_u^{(n)} &= (\Delta + (n-1)(n-D+2))\,A_u^{(n-1)} + n\partial_u A_u^{(n)} - \partial_u D\cdot A^{(n-1)} \\ &\quad - \partial_u \widetilde{A}_u^{(n)} + (D-2n-1)\widetilde{A}_u^{(n-1)}\,, \end{aligned} \tag{3.206}$$

$$\begin{aligned} E_i^{(n)} &= (\Delta + (n-1)(n-D+4) - D + 3)\,A_i^{(n-1)} + (2n-D+4)\partial_u A_i^{(n)} \\ &\quad + (D-n-4)D_i A_u^{(n)} - D_i D\cdot A^{(n-1)} - 2\partial_u \widetilde{A}_i^{(n)} + D_i \widetilde{A}_u^{(n)} \\ &\quad + (D-2n-1)\widetilde{A}_i^{(n-1)}\,, \end{aligned} \tag{3.207}$$

and similarly for logarithmic terms. These equations precisely correspond to the flat limit of the equations obtained in subsection 3.3.1.

Bondi hierarchy. In the context of flat analysis, a well-established systematic procedure to solving the equations of motion (sometimes called Bondi or Tamburino-Winicour hierarchy in the literature) has been articulated for quite some time (Bondi et al., 1962; Sachs, 1962a,b; Tamburino & Winicour, 1966). This framework involves a sequential process, the initial step being the resolution of the hypersurface equation $E_r \approx 0$ (3.205). This resolves to determine the r-expansion of A_u in terms of that of $D\cdot A$. Notably, the orders $A_u^{(0)}$ and $A_u^{(D-3)}$ remain undetermined by this equation. The subsequent step involves addressing the evolution equation $E_i \approx 0$ (3.207), aiming to express the temporal evolution of the radial modes of A_i in terms of those of A_u, $D\cdot A$, and preceding terms of A_i. Significantly, the order $A_i^{(\frac{D-4}{2})}$ remains unconstrained by this equation.

Upon completing these initial two steps, the utility of the identity (3.203) becomes apparent, revealing that the only non-redundant r-order of the supplementary condition $E_u \approx 0$ (3.206) is determined by $n = D-3$. The final step involves solving this equation at this order to ascertain the temporal evolution of $A_u^{(D-3)}$, an order left indeterminate by the first two steps. To illustrate the application of this resolution procedure, we will explicitly examine two distinct dimensional examples, the same ones addressed in the AdS analysis.

Boundary conditions. It is important to note that we will enforce the boundary conditions $A_u \sim \mathcal{O}(1)$ and $A_i \sim \mathcal{O}(1)$. This choice is supported by the assumptions that $\lim_{r\to\infty} A_u = 0$ and $\lim_{r\to\infty} A_i = 0$. While one might consider overleading radiative modes with $A_i^{(n\leq 0)} \neq 0$ involving $A_u^{(n+1\leq 0)} \neq 0$, their associated gauge transformations are consistently zero. Moreover, these modes do not contribute to the surface charge, allowing us to eliminate them a posteriori without affecting the physical interpretation. In the asymptotic symmetry language, these are small or trivial gauge transformations. We will delve into this aspect further when examining the residual gauge parameters of the radial gauge and their associated charges.

The above choice of asymptotic behavior is supported by the computation of the values of specific conserved quantities, such as electromagnetic energy flux, as guiding principles. Actually, as discussed around the relation (2.30), charges associated with exact symmetries, like electromagnetic energy flux, resist renormalization through Iyer-Wald ambiguities. Consequently, the gauge choice and boundary conditions must be delicately adjusted to prevent these quantities from radially diverging in the asymptotic limit. Starting from the Maxwell Lagrangian expressed in equation (3.19), the corresponding electromagnetic stress-energy tensor takes the following form in flat space:

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha + \frac{1}{4}\eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \,. \tag{3.208}$$

Denoting $\mathrm{d}\Omega_{D-2}$ as the surface element of the $(D-2)$-dimensional sphere with unit radius and S_u as the sphere at a specific value of the retarded time u, the energy flux across S_u per unit time is given by the integral:

$$\mathscr{P}(u) = \lim_{r\to\infty} \int_{S_u} T_u{}^r r^{D-2} \mathrm{d}\Omega_{D-2} = \lim_{r\to\infty} \int_{S_u} \gamma^{ij} F_{ui} \left(F_{rj} - F_{uj}\right) r^{D-2} \mathrm{d}\Omega_{D-2} \,. \tag{3.209}$$

Preventing the last quantity from radially diverging as one approaches null infinity provides a rationale for our earlier choice in the radial gauge, namely $A_u \sim \mathcal{O}(1)$ and $A_i \sim \mathcal{O}(1)$.

Even dimensional example ($D = 6$)

The structure of the six-dimensional solution space holds particular interest due to its similarities to four-dimensional gravity. For simplicity, we eliminate logarithmic terms (further comments on this choice will follow on the next page), resulting in the following radial expansions:

$$A_r = 0\,, \tag{3.210}$$

$$A_u = A_u^{(0)} + \frac{1}{2r^2} D\cdot A^{(1)} + \frac{1}{r^3} A_u^{(3)} - \sum_{n\geq 4} \frac{n-1}{n(n-3)r^n} D\cdot A^{(n-1)}\,, \tag{3.211}$$

$$A_i = \sum_{n\geq 0} \frac{1}{r^n} A_i^{(n)}\,, \tag{3.212}$$

where n is a real integer. In the literature (Mädler & Winicour, 2016), specific orders are assigned names, to be justified later when we examine symplectic structure and asymptotic charges: $A_u^{(3)}$ is termed the charge aspect, $A_i^{(1)}$ is the Maxwell shear, and radiative modes are denoted by the orders $A_i^{(n\neq 1)}$. The Maxwell news is defined as the retarded time derivative of the shear:

$$N_i = \partial_u C_i\,, \qquad C_i = A_i^{(1)}\,. \tag{3.213}$$

This last nomenclature is justified by its role in determining the energy flux (3.209):

$$\mathscr{P}(u) = -\int_{S_u} \mathrm{d}\Omega_4 N_i N^i\,. \tag{3.214}$$

The leading order $A_u^{(0)}$ and the Maxwell shear $A_i^{(1)}$ are arbitrary functions of the boundary coordinates (u, x^i), while the charge aspect and radiative modes evolve temporally as follows:

$$\partial_u A_u^{(3)} = -\frac{1}{6}(\Delta - 2) D\cdot A^{(1)}\,, \tag{3.215}$$

$$\begin{aligned}\partial_u A_i^{(n\neq 1)} = \frac{1}{2(n-1)}\Big[&(n-2) D_i A_u^{(n)} + D_i D\cdot A^{(n-1)} \\ &- \Big(\Delta - 3 + (n-1)(n-2)\Big) A_i^{(n-1)}\Big]\,.\end{aligned} \tag{3.216}$$

Additionally, the aforementioned data are subject to constraints imposed by the free Maxwell equation on the sphere[2]:

$$(\Delta - 3) A_i^{(0)} - D_i D\cdot A^{(0)} = 0 \tag{3.217}$$

and the scalar condition:

$$D\cdot A^{(2)} = 0\,. \tag{3.218}$$

[2]Investigating the eigenvalues of the Laplacian operator on the celestial sphere at null

Commentary on logarithmic terms. If one allows for non-zero logarithmic terms in the radial expansion of the solution space, the last condition (3.218) can be relaxed. Actually, unlike the AdS case, the equations of motion in the flat case, even for even dimensions, do not set the log terms identically to zero. In this case, the asymptotic solution space becomes more intricate and reads

$$A_r = 0\,, \tag{3.219}$$

$$\begin{aligned} A_u = {} & A_u^{(0)} - \frac{1}{2r} D\cdot \widetilde{A}^{(0)} + \frac{1}{4r^2}\left(2D\cdot A^{(1)} - 3D\cdot \widetilde{A}^{(1)}\right) + \frac{\log r}{2r^2} D\cdot \widetilde{A}^{(1)} \\ & + \frac{1}{r^3} A_u^{(3)} + \frac{2\log r}{3r^3} D\cdot A^{(2)} - \sum_{n\geq 4} \frac{(n-1)\log r}{n(n-3)r^n} D\cdot A^{(n-1)} \\ & - \sum_{n\geq 4} \frac{n(n-1)(n-3)D\cdot A^{(n-1)} + (n(n-2)+3)D\cdot \widetilde{A}^{(n-1)}}{n^2(n-3)^2 r^n}\,, \end{aligned} \tag{3.220}$$

$$A_i = \sum_{n\geq 0} \frac{1}{r^n}\left(A_i^{(n)} + \log r\, \widetilde{A}_i^{(n)}\right), \tag{3.221}$$

where n is a real integer.

The leading order $A_u^{(0)}$ and the Maxwell shear $A_i^{(1)}$ are still arbitrary functions of the boundary coordinates (u, x^i), while the charge aspect and the radiative modes evolve temporally with slightly modified equations:

$$\partial_u A_u^{(3)} = \frac{1}{18}\left[(2\Delta - 1)D\cdot \widetilde{A}^{(1)} - 3(\Delta-2)D\cdot A^{(1)}\right], \tag{3.222}$$

$$\partial_u \widetilde{A}_i^{(1)} = \frac{1}{4}\left(2(\Delta-3)A_i^{(0)} - 2D_i D\cdot A^{(0)} - D_i D\cdot \widetilde{A}^{(0)} + 6\widetilde{A}_i^{(0)}\right), \tag{3.223}$$

$$\begin{aligned} \partial_u \widetilde{A}_i^{(n\neq 1)} = {} & \frac{1}{2(n-1)}\Big[(n-2)D_i A_u^{(n)} + D_i D\cdot A^{(n-1)} \\ & - \Big(\Delta - 3 + (n-1)(n-2)\Big)A_i^{(n-1)}\Big], \end{aligned} \tag{3.224}$$

$$\begin{aligned} \partial_u A_i^{(n\neq 1)} = {} & \frac{1}{2(n-1)^2}\Big[(n-1)(n-2)D_i A_u^{(n)} - (n-1)(\Delta + n(n-3) \\ & - 1)A_i^{(n-1)} + (n-1)D_i D\cdot A^{(n-1)} + D_i D\cdot \widetilde{A}^{(n-1)} \\ & - (\Delta - 3 - (n-1)(n-3))\widetilde{A}_i^{(n-1)} - D_i \widetilde{A}_u^{(n)}\Big]\,, \end{aligned} \tag{3.225}$$

infinity, it was shown in (Campoleoni et al., 2020) that this constraint reduces $A_i^{(0)}$ to a pure gauge term of the form $\partial_i \lambda^{(0)}$.

once again subject to the following additional constraint but now at the level of the log leading term:

$$(\Delta - 3)\widetilde{A}_i^{(0)} - D_i D \cdot \widetilde{A}^{(0)} = 0\,. \tag{3.226}$$

Odd dimensional example ($D = 5$)

In the five-dimensional case, our focus does not dwell on integrating logarithmic terms, given their similarity to the $D = 6$ case, which we have previously addressed. However, this particular example, characterized by its odd dimensionality, introduces its own unique feature. Actually, it necessitates the incorporation of two separate radial expansions to adequately account for both radiation and Coulombic contributions (Campoleoni et al., 2018a).

In this scenario, the asymptotic solution space can be described as follows (where n is a real integer):

$$A_r = 0\,, \tag{3.227}$$

$$A_u = A_u^{(0)} + \frac{2}{3r^{3/2}} D \cdot A^{(1/2)} + \frac{1}{r^2} A_u^{(2)} - \sum_{n \geq 5/2} \frac{n-1}{n(n-2)r^n} D \cdot A^{(n-1)}\,, \tag{3.228}$$

$$A_i = \sum_{n \geq 0} \frac{1}{r^n} A_i^{(n)}\,. \tag{3.229}$$

This is a typical case where it is necessary to refine the usual radial expansion in integer powers to also include half-integer ones in order to capture the VEV $A_i^{(1/2)}$ associated with the source $A_i^{(0)}$. Otherwise, for example, the associated energy flux (3.209) would be zero. Indeed, the latter reads as follows:

$$\mathscr{P}(u) = -\int_{S_u} \mathrm{d}\Omega_3 N_i N^i\,, \tag{3.230}$$

where, similarly to the previous instance, we define the Maxwell shear and news as:

$$N_i = \partial_u C_i\,, \qquad C_i = A_i^{(1/2)}\,. \tag{3.231}$$

A novelty with respect to the even example arises since the charge aspect $A_u^{(2)}$ remains independent of u, while the retarded temporal evolution of the radiative modes adheres to the constraints:

$$\begin{aligned}\partial_u A_i^{(n \neq 1/2)} = \frac{1}{2n-1)} \Big[&(n-1) D_i A_u^{(n)} + D_i D \cdot A^{(n-1)} \\ &- \Big(\Delta - 1 + n(n-2)\Big) A_i^{(n-1)} \Big]\,,\end{aligned} \tag{3.232}$$

where the leading order $A_u^{(0)}$ and the Maxwell shear C_i are again not determined by the equations of motion. Additionally, we also observe the presence of the scalar condition:

$$D \cdot A^{(1)} = 0\,. \tag{3.233}$$

3.4.2 Symplectic structure

After determining the solution space, the next step involves examining the associated symplectic structure, residual symmetries, and surface charges. In general, these charges may exhibit divergence as one approaches the boundary, particularly in the limit $r \to \infty$. We will demonstrate how to address and renormalize these divergences using the approach outlined in section 2.4 and in the same vein as the analogous AdS analysis in section 3.3. Moreover, analyzing the presymplectic potential proves advantageous for two key reasons. Firstly, its finite component provides insights into conjugate pairs at future null infinity and details regarding the sources of flux and non-integrability in the charges. Secondly, its divergent portion becomes instrumental in renormalizing charge divergences through the incorporation of a corner term.

If we separate the radial off-shell dependence of the presymplectic potential and the Maxwell Lagrangian in Bondi coordinates, expressed as

$$\Theta^\mu = r^{D-2}\sqrt{-\gamma}\,\widetilde{\Theta}^\mu\,, \qquad \mathscr{L} = r^{D-2}\sqrt{-\gamma}\,\widetilde{\mathscr{L}}\,, \tag{3.234}$$

where in the radial gauge

$$\begin{aligned} \widetilde{\Theta}^r &= F_{ru}\delta A_u + \frac{1}{r^2}(F_{ui} - F_{ri})\delta A^i\,, \qquad \widetilde{\Theta}^u = \frac{1}{r^2}F_{ir}\delta A^i\,, \\ \widetilde{\Theta}^i &= -\frac{1}{r^2}F_r{}^i\delta A_u - \frac{1}{r^4}F^i{}_j\delta A^j\,, \end{aligned} \tag{3.235}$$

and

$$\widetilde{\mathscr{L}} = \frac{1}{2}\left(F_{ur}F_{ur} + \frac{1}{r^2}F_r{}^i\,(F_{ui} - F_{ri}) - \frac{1}{r^2}F_u{}^iF_{ri} - \frac{1}{r^4}F^{ij}F_{ij}\right), \tag{3.236}$$

we can formulate the asymptotic renormalization equation as

$$\frac{1}{r}\,(r\,\partial_r + D - 2)\,\widetilde{\Theta}^r \approx \delta\widetilde{\mathscr{L}} - \partial_u\widetilde{\Theta}^u - \partial_i\widetilde{\Theta}^i\,. \tag{3.237}$$

Assuming

$$\widetilde{\Theta}^\mu = \sum_n r^{-n}\left(\widetilde{\Theta}^\mu_{(n)} + \log r\,\widetilde{\theta}^\mu_{(n)}\right), \qquad \widetilde{\mathscr{L}} = \sum_n r^{-n}\left(\widetilde{\mathscr{L}}^{(n)} + \log r\,\widetilde{\ell}^{(n)}\right), \tag{3.238}$$

the equation (3.237) delivers the recursive renormalization relation:

$$(D-2-n)\,\widetilde{\Theta}^r_{(n)}+\widetilde{\theta}^r_{(n)}\approx\delta\widetilde{\mathscr{L}}^{(n+1)}-\partial_u\widetilde{\Theta}^u_{(n+1)}-\partial_i\widetilde{\Theta}^i_{(n+1)}\,. \tag{3.239}$$

This fixes the divergent orders of the presymplectic potential as definition ambiguities, corresponding to $n<D-2$. The finite order is not constrained by this equation, allowing the choice of a prescription for $n=D-2$. These relationships are evidently the flat limit of the off-shell ones addressed in the AdS case 3.3.3.

Even dimensional example ($D=6$)

For simplicity, we eliminate logarithmic terms (a discussion on this follows at the end of the paragraph), resulting in a linear radial divergence for the codimension-2 quantity k^{ur}_λ:

$$k^{ur}_\lambda\approx -r\sqrt{-\gamma}\,\lambda^{(0)}D\cdot\delta A^{(1)}-3\sqrt{-\gamma}\,\lambda^{(0)}\delta A^{(3)}_u+\mathcal{O}(r^{-1})\,. \tag{3.240}$$

Subsequently, we analyze the associated symplectic potential to reveal the activated sources of flux in the radial gauge and identify the divergent ambiguities available for renormalizing the surface charge:

$$\begin{aligned}\Theta^r\approx{}& r\sqrt{-\gamma}\Big(\partial_u A^{(1)}_i\delta A^i_{(0)}-D\cdot A^{(1)}\delta A^{(0)}_u\Big)+\frac{1}{2}\sqrt{-\gamma}\Big(5A^{(1)}_i\delta A^i_{(0)}\\&+2\partial_u A^{(1)}_i\delta A^i_{(1)}-\Delta A^{(1)}_i\delta A^i_{(0)}-6A^{(3)}_u\delta A^{(0)}_u\Big)+\mathcal{O}(r^{-1})\,.\end{aligned} \tag{3.241}$$

The relevant divergent ambiguities are then given by

$$C^{ur}=r\sqrt{-\gamma}\,A^{(1)}_i\delta A^i_{(0)}+\mathcal{O}(r^{-1})\,, \tag{3.242}$$

$$C^{ir}=-r\sqrt{-\gamma}\,\Big(F^{ij}_{(0)}\delta A^{(0)}_j-A^i_{(1)}\delta A^{(0)}_u\Big)+\mathcal{O}(r^{-1})\,, \tag{3.243}$$

$$B=-\frac{r}{2}\sqrt{-\gamma}\,F^{ij}_{(0)}F^{(0)}_{ij}+\mathcal{O}(r^{-1})\,. \tag{3.244}$$

Thus, the renormalized potential can be expressed on-shell as

$$\begin{aligned}\Theta^r_{\text{ren}}&=\Theta^r+\partial_u C^{ur}-\delta B+\partial_i C^{ir}\\&\approx\frac{\sqrt{-\gamma}}{2}\Big(5A^{(1)}_i\delta A^i_{(0)}+2\partial_u A^{(1)}_i\delta A^i_{(1)}-\Delta A^{(1)}_i\delta A^i_{(0)}-6A^{(3)}_u\delta A^{(0)}_u\Big)\,,\end{aligned} \tag{3.245}$$

ensuring that $\partial_r \Theta^r_{\text{ren}} \approx 0$. Consequently, the renormalized charge is given by

$$\begin{aligned} k^{ur}_{\lambda,\text{ren}} &= k^{ur}_{\lambda} - \delta C^{ur}[A;\delta_\lambda A] + \delta_\lambda C^{ur}[A;\delta A] \\ &\approx -\partial^i \left(r\sqrt{-\gamma}\, A^{(1)}_i \lambda^{(0)} \right) - 3\sqrt{-\gamma}\,\lambda^{(0)}\delta A^{(3)}_u + \mathcal{O}(r^{-1})\,, \end{aligned} \tag{3.246}$$

$$\delta H^{\text{ren}}_{\lambda} = \lim_{r\to\infty} \int \mathrm{d}^4x\, k^{ur}_{\lambda,\text{ren}} \approx -3\int \mathrm{d}^4x\,\sqrt{-\gamma}\,\lambda^{(0)}\delta A^{(3)}_u\,. \tag{3.247}$$

This expression is finite, integrable,

$$H^{\text{ren}}_{\lambda} = -3\int \mathrm{d}^4x\,\sqrt{-\gamma}\,\lambda^{(0)} A^{(3)}_u\,, \tag{3.248}$$

yet it is non-conserved in the retarded temporal evolution:

$$\partial_u H^{\text{ren}}_{\lambda} = -3\int \mathrm{d}^4x\,\sqrt{-\gamma}\left(\partial_u\lambda^{(0)} A^{(3)}_u - \frac{1}{6}\lambda^{(0)}(\Delta-2)D\cdot C\right). \tag{3.249}$$

It is noteworthy that the above charge (3.248) matches the one obtained for the flat limit (3.153) derived from the AdS analysis (3.148), providing solid justification for taking this limit at the charge level without encountering any issues.

Commentary on logarithmic terms. If we incorporate the logarithmic terms, a similar analysis as above yields the following Iyer-Wald codimension-2 quantity:

$$k^{ur}_{\lambda,\text{ren}} \approx -\frac{\sqrt{-\gamma}}{3}\lambda^{(0)}\left(9\delta A^{(3)}_u - 2D\cdot\delta A^{(2)}\right), \tag{3.250}$$

where we observe the emergence of a new finite term that was previously zero due to (3.218). However, it can be rewritten as a pure boundary term so that if we fix the finite prescription as follows,

$$\Theta^r_{\text{ren}} \to \Theta^r_{\text{ren}} - \delta\left(\frac{2}{3}\sqrt{-\gamma}D\cdot A^{(2)}A^{(0)}_u\right), \tag{3.251}$$

we recover the standard value of the asymptotic corner charge as before:

$$H^{\lambda}_{\text{ren}} = -\int \mathrm{d}^4x\,\sqrt{-\gamma}\,\lambda\, F^{(4)}_{ur}\,. \tag{3.252}$$

Investigating further the implications at the dual level of such a prescription can be interesting for future endeavors. For instance, one could holographically interpret $D\cdot A^{(2)}$ in (3.250) as an enhancement of the VEV, while in (3.251) as a new source (in order to preserve locality) which can be gauged away by a small gauge transformation since it would not appear in the finite value of the charge. We shall discuss these peculiar aspects further in the next chapter at the level of the Einstein-Hilbert theory. We refer also to, e.g., (Compère et al., 2020; Geiller et al., 2021; Geiller & Zwikel, 2022; Campoleoni et al., 2022, 2023a; Ciambelli et al., 2023; Geiller & Zwikel, 2024) for more information on these aspects.

Discussion. We can conclude with a brief remark on gauge relaxation. In the above analysis, we have been operating within the radial gauge, $A_r = 0$. The presymplectic potential can be expressed in terms of a gauge-invariant potential, provided we make the substitution:

$$A_\mu \to A_\mu - \partial_r \int_0^r A_r \,. \tag{3.253}$$

One might inquire about the constraints imposed by the radial gauge fixation on the symplectic structure. In fact, it has been shown in (Freidel et al., 2019) that the absent term can be identified to a corner term in the radial component of the renormalized presymplectic potential using the Gauss law.

Within the AdS context using Bondi coordinates, we expect that the additional finite term arising from gauge relaxation in the charge follows the expression $\lambda^{(0)}\partial_u A_r^{(D-2)}$. This is based on our explicitly gauge-invariant analysis outlined in equations (3.148), (3.152), and (3.153) via the holographic renormalization. It would be intriguing to demonstrate in the future that this newly introduced term can be effectively expressed as a corner ambiguity at the level of the presymplectic potential, mirroring the possibility (3.253) observed in the flat case. Furthermore, investigating the analogous term in Poincaré coordinates, as presented in (3.136), i.e., $\lambda^{(0)}\partial_a A_z^{(D-4)}$, holds potential interest. To maintain coherence with the AdS analysis, we refrain from delving deeper into this matter at present. A more detailed examination of such phenomena will be proposed within the gravitational context in chapter 4.

Odd dimensional example ($D = 5$)

To complete the illustration of Maxwell's asymptotic symmetries, we delve into the evaluation of the symplectic renormalization prescription within the framework of a five-dimensional flat background.

Prescribing the following diverging Iyer-Wald ambiguities to the radial component of the bulk presymplectic potential,

$$C^{ur} = r^{1/2}\sqrt{-\gamma}\, A^i_{(1/2)}\delta A^{(0)}_i + \mathcal{O}(r^{-1/2})\,, \tag{3.254}$$
$$C^{ir} = -r^{1/2}\sqrt{-\gamma}\, A^i_{(1/2)}\delta A^{(0)}_u + \mathcal{O}(r^{-1/2})\,, \tag{3.255}$$
$$B = \mathcal{O}(r^{-1/2})\,, \tag{3.256}$$

we obtain the following finite expression:

$$\begin{aligned}\Theta^r_{\text{ren}} \approx \sqrt{-\gamma}\Big(& 2A^i_{(0)}\delta A^{(0)}_i + D^i D\cdot A^{(0)}\delta A^{(0)}_i + \partial_u A^i_{(1/2)}\delta A^{(1/2)}_i \\ & - \delta A^i_{(0)}\Delta A^{(0)}_i - 2A^{(2)}_u \delta A^{(0)}_u\Big)\,.\end{aligned} \tag{3.257}$$

When evaluated along a gauge parameter in the radial gauge, this expression leads to the renormalized integrable surface charge:

$$H^{\text{ren}}_\lambda \approx -2\int \mathrm{d}^3x\,\sqrt{-\gamma}\,\lambda^{(0)}A^{(2)}_u\,, \tag{3.258}$$

which, assuming that the leading order $A^{(0)}_u$ is non-zero and thus the gauge parameter is arbitrary in the boundary coordinates, is non-conserved with respect to the retarded temporal evolution.

CHAPTER 4

Gravitation

"Quella forza simile alla gravità,
che ci spinge al nostro ben essere,
non si trattiene che a misura degli
ostacoli che gli sono opposti."

Cesare Beccaria Bonesana

We now employ the covariant phase space formality and acquired electromagnetic skills to explore gravitational theory in this chapter. More concretely, in the previous one, we examined the asymptotic behavior of the Maxwell field propagating in an AdS and flat background. In the present chapter, our focus shifts to asymptotically considering spacetime itself. Within the body of the text, we will provide a more precise definition of what we mean by asymptotically AdS and flat spacetime. Adding to the inherent difficulty of gravitational theory due to the non-linearity of its equations of motion, this theory can also present complexities related to gravitational fluxes, as outlined in section 2.1.4. These complexities may arise in spacetime dimensions equal to or greater than four. While these are interesting for the study of associated gravitational waves, we choose to concentrate mainly on asymptotically AdS and asymptotically flat three-dimensional spacetimes for the reasons given in the introduction.

This chapter is organized as follows. In section 4.1, we delineate the considered theory by specifying the bulk Lagrangian. This enables us to derive the equations of motion, symplectic structure, and Iyer-Wald codimension-2 form. We review these steps for the Einstein-Hilbert action for gravity generically in any spacetime dimensions and the Chern-Simons action for three-dimensional gravity. Moving on to sections 4.2 and 4.3, we elucidate the concept of asymptotically AdS spacetimes by defining the two most standard gauges in the literature: the Fefferman-Graham (Fefferman & Graham, 1985, 2011) and the Bondi gauges (Bondi et al., 1962; Sachs, 1962a,b). In the Fefferman-Graham setup, particularly useful in the holographic context of AdS/CFT correspondence (Balasubramanian & Kraus, 1999; Skenderis, 2001), the Weyl symmetry is spontaneously broken (Henningson & Skenderis, 1998). Since the asymptotic boundary sits at conformal infinity, one could expect from holographic considerations to maintain Weyl covariance. We shall explicit this aspect more concretely in the second section 4.2. We then relax this gauge to restore this symmetry, leading to the Weyl-Fefferman-Graham gauge (Ciambelli & Leigh, 2020; Jia & Karydas, 2021; Jia et al., 2023). We explore the finite charges of this new gauge and establish its relation to the Weyl geometry of the boundary. We base this discussion on (Ciambelli et al., 2023) and detail the dual Weyl aspects in appendix B.1.

In the following section 4.3, we delve into the Bondi gauge, tailored for studying gravitational waves and facilitating a smooth flat boundary to describe asymptotically flat spacetimes. This last fact will be examined in the final section 4.4. The Bondi gauge has seen renewed interest in recent years due to its connections with the symmetries of asymptotically flat spacetimes, soft theorems and memory effects (Barnich & Compere, 2007; Barnich & Troessaert, 2010; Campiglia & Laddha, 2014; Strominger, 2018; Ashtekar et al., 2018; Compère et al., 2018; Donnay et al., 2019). To covariantize it with respect to the pseudo-Riemannian boundary, akin to the Fefferman-Graham gauge, we introduce its relaxation known as the covariant Bondi gauge (Ciambelli et al., 2018b). Similarly, we investigate charges using symplectic renormalization, following the approach in (Campoleoni et al., 2022), and propose an interpretation in terms of boundary geometry, featuring relativistic hydrogeometry in the AdS case and Carrollian in the flat case (see Appendix B.2). These various analyses are supported by the Chern-Simons formulation of three-dimensional gravity (Achucarro & Townsend, 1986; Witten, 1988; Banados, 1996), making it possible to justify the symplectic prescription by providing the adequate boundary counterterms to add to the bulk action.

4.1. Lagrangian

We define in this section the bulk field theory by specifying the action outlined in this chapter. In the realm of general relativity, the fundamental action employed is the Einstein-Hilbert action:

$$S = \int_{\mathcal{M}} \mathrm{d}^D x \, \mathscr{L} \,, \qquad \mathscr{L} = \frac{\sqrt{-g}}{16\pi G} \left(R - 2\Lambda\right), \tag{4.1}$$

where the metric $g_{\mu\nu}$ of the D-dimensional manifold $\mathcal{M}$ serves as the dynamic field. In the equation (4.1), G denotes Newton's gravitational constant, Λ stands for the cosmological constant, g and R respectively represent the determinant and the Ricci scalar associated with the spacetime metric. In case of an asymptotically AdS spacetime, the cosmological constant can be expressed in terms of the AdS radius ℓ as

$$\Lambda = -\frac{(D-1)(D-2)}{\ell^2} \,. \tag{4.2}$$

In adherence to the fundamental principle of covariance, the Lagrangian density governing gravitation (4.1) remains invariant up to a total derivative under the infinitesimal transformation induced by the action of a diffeomorphism $\xi^\mu(x^\nu)$, achieved through a Lie derivative along this vector $\xi = \xi^\mu \partial_\mu$,

$$\delta_\xi g_{\mu\nu} = \mathscr{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho(\mu} \partial_{\nu)} \xi^\rho \,, \tag{4.3}$$

where we recall that we symmetrize the indices without applying an overall factor. The vector ξ^μ hence serves as the gauge symmetry parameter within the context of the Einstein-Hilbert action. This symmetry neatly aligns with the relations (2.19) and (2.20), thus leading to the derivation of conserved quantities via the second Noether theorem (2.22). Additionally, given the present nature of gauge invariance under diffeomorphism, we naturally fit into the original Iyer-Wald prescription as outlined in the subsection 2.1.4, resulting in the generic form of the codimension-2 form in (2.41).

More specifically, the symplectic structure can be obtained by varying the Lagrangian (4.1) with respect to an arbitrary change in the metric, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$,

$$\delta \mathscr{L} = E^{\mu\nu} \delta g_{\mu\nu} + \partial_\mu \Theta^\mu \,, \tag{4.4}$$

where the Einstein field equations (2.11) read

$$E^{\mu\nu}[g] = -\frac{\sqrt{-g}}{16\pi G} \left(G^{\mu\nu} + \Lambda g^{\mu\nu}\right), \qquad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \,, \tag{4.5}$$

and the presymplectic potential (2.16) is given by

$$\Theta^\mu[g;\delta g] = \frac{\sqrt{-g}}{16\pi G}\Big(\nabla_\nu(\delta g)^{\mu\nu} - \nabla^\mu(\delta g)^\nu{}_\nu\Big)\,. \tag{4.6}$$

Here, $(\delta g)^{\mu\nu}$ should be understood[1] as $(\delta g)^{\mu\nu} = -\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$, and similarly for $(\delta g)^\nu{}_\nu$. The Noether identities (2.24) in this context correspond to the contracted Bianchi identities for the Einstein tensor $G_{\mu\nu}$:

$$N^\mu = \nabla_\nu G^{\mu\nu} = 0\,. \tag{4.7}$$

The (Lee-Wald) symplectic form (2.12) arises from the antisymmetrized second variation of the action:

$$\begin{aligned}\omega^\mu[g;\delta_1 g;\delta_2 g] = \frac{\sqrt{-g}}{16\pi G}\bigg[&\frac{1}{2}\,g^{\mu\lambda}\Big(g^{\alpha\beta}g^{\sigma\nu} - g^{\alpha\sigma}g^{\beta\nu}\Big) + \frac{1}{2}\,g^{\beta\nu}\Big(g^{\alpha\lambda}g^{\sigma\mu} - g^{\alpha\sigma}g^{\lambda\mu}\Big)\\ &+ \frac{1}{2}\,g^{\lambda\nu}\Big(g^{\mu\alpha}g^{\beta\sigma} - g^{\alpha\beta}g^{\sigma\mu}\Big)\bigg]\Big(\delta_2 g_{\alpha\beta}\,\nabla_\sigma\delta_1 g_{\lambda\nu} - (1\leftrightarrow 2)\Big)\,.\end{aligned} \tag{4.8}$$

According to the second Noether theorem (2.22) and using the inverse Leibniz rule, one can deduce the following expression for the Iyer-Wald codimension-2 form $k^{\mu\nu}_\xi$ (Iyer & Wald, 1994),

$$\begin{aligned}k^{\mu\nu}_\xi = \frac{\sqrt{-g}}{16\pi G}\bigg[&\xi^\mu\nabla_\lambda(\delta g)^{\nu\lambda} - \xi^\mu\nabla^\nu(\delta g)^\lambda{}_\lambda + \xi_\lambda\nabla^\nu(\delta g)^{\mu\lambda}\\ &+ \frac{1}{2}(\delta g)^\lambda{}_\lambda\nabla^\nu\xi^\mu - \delta g^{\lambda\nu}\nabla_\lambda\xi^\mu - (\mu\leftrightarrow\nu)\bigg]\,.\end{aligned} \tag{4.9}$$

By comparison, the analogous expression in the Barnich-Brandt formalism is given by (Barnich & Brandt, 2002; Barnich, 2003):

$$\begin{aligned}k^{\mu\nu}_{\mathrm{BB},\xi} = \frac{\sqrt{-g}}{8\pi G}\bigg[&\xi^\mu\nabla^\lambda(\delta g)^\nu{}_\lambda - \xi^\mu\nabla^\nu(\delta g)^\lambda{}_\lambda + \xi^\lambda\nabla^\nu(\delta g)^\mu{}_\lambda + \frac{1}{2}(\delta g)^\lambda{}_\lambda\nabla^\nu\xi^\mu\\ &+ \frac{1}{2}(\delta g)^\nu{}_\lambda\Big(\nabla^\mu\xi^\lambda - \nabla^\lambda\xi^\mu\Big)\bigg]\,.\end{aligned} \tag{4.10}$$

The latter aligns with the result obtained from the integration by parts procedure à la Abbott-Deser (Abbott & Deser, 1982a,b; Deser & Tekin, 2002, 2003)

[1]This means that we vary the metric before raising the indices with the inverse of the

and differs from (4.9) due to the following corner term (see (2.63)):

$$E^{\mu\nu}[g;\delta g;\delta g] = \frac{1}{32\pi G}(\delta g)^{\mu}{}_{\lambda} \wedge (\delta g)^{\lambda\nu} \,. \tag{4.11}$$

Up to this point, our discussions regarding the gravitational context can be applied to spacetime manifolds of any dimension in the presence of a boundary. For the rest of this section, we narrow our focus to three-dimensional gravity. As previously mentioned, one of the advantages of this number of dimensions lies in the associated topological nature of general relativity. This property enables us to reinterpret the Einstein-Hilbert theory as a Chern-Simons theory (Achucarro & Townsend, 1986; Witten, 1988; Banados, 1996; Henneaux et al., 2000; Rooman & Spindel, 2001a; Allemandi et al., 2003).

Actually, to see this more concretely, let us consider what is the gauge group of such a theory when adapted to gravitation. In the upcoming sections, we will delve into the analyses of asymptotically AdS and flat spaces. The isometry algebra of AdS_3, denoted $\mathfrak{so}(2,2)$, can be expressed as follows:

$$\begin{aligned} [M_B, M_C] &= \varepsilon_{BCD} M^D \,, \\ [M_B, P_C] &= \varepsilon_{BCD} P^D \,, \\ [P_B, P_C] &= \left(\frac{G}{\ell}\right)^2 \varepsilon_{BCD} M^D \,, \end{aligned} \tag{4.12}$$

where P_B and M_B represent the transvection and Lorentz generators, respectively, with the algebra basis indices denoted by upper-case letters of the beginning of the Latin alphabet. The generators M_B are related to the conventional Lorentz M_{BC} generators by

$$M_B = \frac{1}{2}\varepsilon_{BCD} M^{CD} \,, \tag{4.13}$$

where the Levi-Civita symbol convention is chosen as $\varepsilon^{012} = 1$. In the flat limit $\ell \to \infty$ of the cosmological constant Λ, we recover the isometry Poincaré algebra $\mathfrak{iso}(1,2)$ of three-dimensional Minkowski space.

Then, we introduce a differential one-form valued in the algebra (4.12), termed Chern-Simons connection:

$$\mathscr{A} = \left(\frac{1}{G} e_{\mu}{}^{B} P_B + \omega_{\mu}{}^{B}(e) M_B\right) \mathrm{d}x^{\mu} \,, \tag{4.14}$$

metric.

where $e_\mu{}^B$ denotes the bulk dreibein and $\omega_\mu{}^B(e)$ represents its associated dualized spin connection, which can be obtained from the Cartan structure equation:

$$\mathrm{d}e^B + \varepsilon^{BCD}\omega_C(e) \wedge e_D = 0\,. \tag{4.15}$$

By rearranging terms, the 3D Einstein-Hilbert action (4.1) can thus be reformulated as the following Chern-Simons action (Achucarro & Townsend, 1986; Witten, 1988)

$$S = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(\mathscr{A} \wedge \mathrm{d}\mathscr{A} + \frac{2}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A}\right). \tag{4.16}$$

In the last equation, we introduced the following Killing metric:

$$\mathrm{Tr}\,(M_B M_C) = \mathrm{Tr}\,(P_B P_C) = 0\,, \qquad \mathrm{Tr}\,(M_B P_C) = \eta_{BC}\,, \tag{4.17}$$

with η_{BC} the Minkowski metric in the algebra basis. In the AdS case, i.e., for $\frac{1}{\ell} \neq 0$, one can exploit the isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ to rewrite the action (4.16) as the difference of two $\mathfrak{sl}(2,\mathbb{R})$ Chern-Simons actions:

$$S = S_{CS}[\mathrm{A}] - S_{CS}[\widetilde{\mathrm{A}}]\,, \tag{4.18}$$

with

$$S_{CS}[\mathrm{A}] = \frac{\ell}{16\pi G}\int_{\mathcal{M}} \mathrm{tr}\left(\mathrm{A} \wedge \mathrm{dA} + \frac{2}{3}\,\mathrm{A} \wedge \mathrm{A} \wedge \mathrm{A}\right). \tag{4.19}$$

Here, A and $\widetilde{\mathrm{A}}$ represent the gauge connections, taking values in the algebra $\mathfrak{sl}(2,\mathbb{R})$, such that the corresponding generators satisfy

$$[J_B, J_C] = \varepsilon_{BC}{}^D\, J_D\,, \qquad \mathrm{tr}(J_B J_C) = \frac{1}{2}\,\eta_{BC}\,. \tag{4.20}$$

One can rewrite the forms A and $\widetilde{A}$ in terms of the bulk dreibein and the spin connection as

$$\mathrm{A}^B = \omega^B(e) + \frac{1}{\ell}e^B\,, \qquad \widetilde{\mathrm{A}}^B = \omega^B(e) - \frac{1}{\ell}e^B\,. \tag{4.21}$$

To be totally accurate, it is noteworthy that the action (4.16), or equivalently (4.18), corresponds to the Einstein-Hilbert action (4.1) up to boundary terms:

$$S_{\mathrm{bdy}} = -\frac{\ell}{16\pi G}\int_{\partial\mathcal{M}} \mathrm{tr}\left(\mathrm{A} \wedge \widetilde{\mathrm{A}}\right). \tag{4.22}$$

These are responsible for some notable finite ambiguities at the level of the asymptotic corner charges (Compere & Marolf, 2008; Geiller, 2017). We will look at their importance in the next sections.

Starting from the Chern-Simons reformulation of gravitational theory, we can once again employ the formal covariant phase space procedure outlined in section 2.1 to deduce the form of the surface charges. In this context, the Lagrangian density associated to the action principle (4.19) reads

$$S_{CS} = \int_{\mathcal{M}} L\,, \qquad L = \frac{\ell}{16\pi G}\,\mathrm{tr}\left(\mathrm{A}\wedge \mathrm{dA} + \frac{2}{3}\,\mathrm{A}\wedge\mathrm{A}\wedge\mathrm{A}\right). \tag{4.23}$$

An arbitrary field variation $\mathrm{A}\to\mathrm{A}+\delta\mathrm{A}$ of the latter yields (see (2.10)), after iterative applications of the inverse Leibniz rule,

$$\delta L = E\wedge\delta\mathrm{A} - \mathrm{d}\Theta\,, \tag{4.24}$$

where E denotes the equations of motion of the theory:

$$E = \mathrm{dA} + \mathrm{A}\wedge\mathrm{A} \approx 0\,. \tag{4.25}$$

The latter flatness condition imposed on the Chern-Simons connections is a crucial aspect to consider: all solutions to the equations (4.25) are pure gauge. Essentially, these equations serve as a reinterpretation of the three-dimensional Einstein field equations. These are the conditions of vanishing torsion, constant Ricci scalar and vanishing trace-free Ricci tensor. This highlights a significant aspect of the theory: its lack of local degrees of freedom. Consequently, the theory exhibits no gravitational radiation. In essence, this implies that the entirety of dynamics within the system is encapsulated within the boundary degrees of freedom.

The local presymplectic potential form Θ is given by

$$\Theta = -\frac{\ell}{4\pi G}\mathrm{tr}\left(\mathrm{A}\wedge\delta\mathrm{A}\right). \tag{4.26}$$

We define the local (Lee-Wald) presymplectic two-form (2.12) as

$$\omega = \delta\Theta = -\frac{\ell}{16\pi G}\mathrm{tr}\left(\delta\mathrm{A}\wedge\delta\mathrm{A}\right). \tag{4.27}$$

This local expression can be integrated over an arbitrary Cauchy slice $\Sigma\subset\mathcal{M}$ to yield the global presymplectic two-form:

$$\Omega = \int_{\Sigma}\omega\,. \tag{4.28}$$

Once again, this theory is gauge-invariant, so the second Noether theorem (2.22) applies. The gauge symmetry of the Chern-Simons field is expressed as

$$\delta_\lambda \mathrm{A} = I_\lambda \delta \mathrm{A} = \mathrm{d}\lambda + [\mathrm{A}, \lambda] \,, \tag{4.29}$$

where $\lambda \in \mathfrak{sl}(2, \mathbb{R})$. It stems from the fact that the Chern-Simons equations of motion (4.25) are invariant under the following gauge transformations:

$$\mathrm{A} \to U^{-1} \mathrm{A}\, U + U^{-1} \mathrm{d}U \,, \tag{4.30}$$

where $U = \exp(\lambda) \in SL(2, \mathbb{R})$. The diffeomorphisms of metric formulation can be linked to these parameters through the equation (Witten, 1988):

$$\xi^\mu = \frac{1}{2} e_B{}^\mu (\lambda^B - \widetilde{\lambda}^B) \,, \tag{4.31}$$

where $e_B{}^\mu$ is the inverse of the dreibein. Applying Noether's second theorem and the fundamental theorem of covariant phase space to this gauge symmetry,

$$I_{V_\lambda} \Omega = -\delta H_\lambda \,, \tag{4.32}$$

one can deduce the associated on-shell corner charges (Regge & Teitelboim, 1974; Banados, 1996; Coussaert et al., 1995; Banados, 1999; Henneaux et al., 2000; Bunster et al., 2014):

$$\delta H_\lambda \approx \int_{\mathcal{C}} k_\lambda \,, \qquad k_\lambda = -\frac{\ell}{8\pi G} \mathrm{tr} \left(\lambda \, \delta \mathrm{A} \right) . \tag{4.33}$$

We shall typically see that in this gauge theory, unlike electromagnetism, we can not directly integrate this expression of the variation of the charges, since the gauge parameter can exhibit field dependence in its expansion. Moreover, in three dimensions, it is always possible to find integrable slicings for the charges (Adami et al., 2020b; Alessio et al., 2021; Adami et al., 2020a; Ruzziconi & Zwikel, 2021; Geiller et al., 2021; Adami et al., 2021a). This is why we have always used δ instead of $\not\delta$ in this section, without loss of generality. This is related to the absence of propagation of local degrees of freedom in this case. We will reintroduce this symbol in the Bondi section, but for this argument, we will see that this non-integrability is only apparent and can be resolved by deftly field-dependent redefining the gauge parameters.

4.2. Fefferman-Graham gauge

In this second section, we delve into the discussion of one of the two mainstream gauges for asymptotically AdS spaces in three dimensions, focusing on asymptotic symmetries and corner charges with a modern perspective. This gauge, called Fefferman-Graham, is best suited to the holographic context of the AdS/CFT correspondence. This is the content of the first subsection 4.2.1. As we go along, we will introduce a need to release this gauge and explore one possibility in detail in the second subsection 4.2.2.

4.2.1 Fefferman-Graham

Solution space

Let us in a first step delineate the context of our analysis concerning asymptotic symmetries. We are dealing with general relativity, as discussed earlier, within the framework of an asymptotically Anti de Sitter three-dimensional spacetime. We shall be more precise by what we mean exactly by the last aspect in the next page.

In this setup, the Fefferman-Graham gauge entails selecting bulk coordinates $x^\mu = (z, x^a)$, where $z \geq 0$ serves as a radial coordinate, and $x^a = (t, \theta)$ represent the boundary coordinates with the boundary positioned at $z = 0$. Notably, these coordinates align with the ones utilized in the Poincaré patch (3.10). From the figure 3.2b, we understand that t is the time coordinate and $\theta \sim \theta + 2\pi$ is the angular coordinate on the circle at infinity. The line element can be expressed as (Starobinsky, 1983; Fefferman & Graham, 1985, 2011; Skenderis, 2002; Papadimitriou, 2010):

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\mathrm{d}z^2}{z^2} + h_{ab}(z,x)\mathrm{d}x^a \mathrm{d}x^b \,, \tag{4.34}$$

where, for the sake of notation simplicity, we have set $\ell = 1$ in this section. The gauge fixing (4.34) represents the second step in our examination of asymptotic symmetries. Besides, we adhere to conventions similar to the ones found in references such as (Ruzziconi, 2020; Ciambelli et al., 2020a,b).

We complete the second step, which involves imposing boundary conditions on the structure h_{ab}. These can be selected as follows:

$$h_{ab}(z,x) = \frac{1}{z^2} h^{(0)}_{ab}(x) + \mathcal{O}(z^{-1}) \,, \tag{4.35}$$

where the primary contribution constitutes the metric $h^{(0)}_{ab}(x)$ of the boundary. Subsequently, we will utilize this metric and its inverse to lower and raise boundary indices. In the asymptotic scenario, where we apply the additional Brown-Henneaux condition (Brown & Henneaux, 1986),

$$h^{(0)}_{ab} = \eta_{ab} \,, \tag{4.36}$$

we recover the AdS metric (3.10) in Poincaré coordinates. This validates our reference to (4.34) as a gauge for asymptotically AdS spacetimes. In the subsequent discussion, we will relax the Dirichlet boundary condition (4.36)[2], as proposed in (Troessaert, 2013; Alessio et al., 2021).

The third step is to determine the space of asymptotic solutions. Essentially, we assume a polyhomogeneous expansion (2.68) of the dynamical field h_{ab} and solve the Einstein equations (4.5) radially order by order. In the three-dimensional context, these equations allow us to express this asymptotic expansion as a finite sum:

$$h_{ab}(z,x) = z^{-2} h^{(0)}_{ab}(x) + h^{(2)}_{ab}(x) + z^2 h^{(4)}_{ab}(x)\,, \tag{4.37}$$

where

$$h^{(4)}_{ab} = \frac{1}{4} h^{(2)}_{ac} h^{cd}_{(0)} h^{(2)}_{db}\,, \qquad h^{ab}_{(0)} h^{(2)}_{ab} = -\frac{1}{2} R^{(0)}\,, \qquad D^{a}_{(0)} h^{(2)}_{ab} = -\frac{1}{2} D^{(0)}_{b} R^{(0)}\,. \tag{4.38}$$

The symbol $R^{(0)}$ represents the Ricci scalar and $D^{(0)}_a$ denotes the covariant derivative, both associated with the boundary metric $h^{(0)}_{ab}$. Notably, in this scenario, the logarithmic terms just like the subleading terms with respect to the fourth order are on-shell fixed to zero. Given that every two-dimensional metric conforms to being flat, we can enforce the subsequent condition on the boundary metric (Troessaert, 2013; Alessio et al., 2021):

$$h^{(0)}_{ab}(x) = \mathrm{e}^{2\phi(x)} \eta_{ab}\,, \tag{4.39}$$

where we introduce the conformal factor $\phi = \phi(x^a)$ as an arbitrary function of the boundary coordinates. These boundary conditions (4.39) are motivated by the geometric approach à la Penrose (Penrose, 1963, 1964). Specifically, when applied to AdS_3, the bulk metric induces not a particular metric but rather a conformal class of metrics at the boundary.

Actually, in the proposed conformal completion, the boundary data of AdS should be situated at infinite distance. This challenge arises because of the second-order pole structure of AdS. However, it can be addressed through a specific operation applied to the bulk metric:

$$g \to \Omega^2 g\,. \tag{4.40}$$

[2]We refer to it as a Dirichlet boundary condition since we shall see a posteriori that it corresponds to the adequate constraint to impose on the variational principle so that it is well-defined.

Here, Ω represents a positive function with a simple zero on the boundary, enabling the definition of an induced metric on this geometry. Nevertheless, it is worth noting that the selection of the function Ω is not unique. An alternative modification involves transforming Ω as follows:

$$\Omega \to \mathrm{e}^{\sigma}\Omega\,, \tag{4.41}$$

where $\sigma(x)$ is a smooth function independent of z. Consequently, this transformation induces the following change on the boundary metric:

$$h^{(0)} \to \mathrm{e}^{2\sigma} h^{(0)}\,. \tag{4.42}$$

This adjustment corresponds to a Weyl transformation, revealing that the ambiguity in defining Ω only permits the delineation of a conformal class of the boundary metric at best. Furthermore, it prompts consideration that this Weyl symmetry might ensure a manifest covariance of the theory. However, as we shall gradually see with, for instance, the presence of a Weyl anomaly in the associated dual theory, this symmetry is broken in the Fefferman-Graham framework. The quest for the restoration of this covariance will lead us in the next subsection to relax this gauge.

Expressed in terms of the light-cone coordinates,

$$x^{\pm} = \theta \pm t\,, \tag{4.43}$$

the boundary metric (4.39) adopts the form:

$$\mathrm{d}s^2_{\text{bdy}} = h^{(0)}_{ab}\mathrm{d}x^a \mathrm{d}x^b = \mathrm{e}^{2\phi(x^+,x^-)}\mathrm{d}x^+\mathrm{d}x^-\,. \tag{4.44}$$

Thus, the solution space (4.38) is delineated as follows:

$$h^{(0)}_{\pm\pm} = 0\,, \qquad h^{(2)}_{\pm\pm} = \ell_{\pm} - (\partial_{\pm}\phi)^2 + \partial^2_{\pm}\phi\,, \qquad h^{(4)}_{\pm\pm} = \mathrm{e}^{-2\phi}\,\partial^2_{\pm}\phi\, h^{(2)}_{\pm\pm}\,, \tag{4.45}$$

and

$$h^{(0)}_{+-} = \frac{1}{2}\mathrm{e}^{2\phi}\,, \qquad h^{(2)}_{+-} = \partial_+\partial_-\phi\,, \qquad h^{(4)}_{+-} = \frac{1}{2}\mathrm{e}^{-2\phi}\Big[(\partial_+\partial_-\phi)^2 + h^{(2)}_{++}h^{(2)}_{--}\Big]\,, \tag{4.46}$$

subject to the (anti-)holomorphic condition:

$$\partial_{\pm}\ell_{\mp} = 0\,. \tag{4.47}$$

Residual symmetries and algebra

The fourth step involves identifying the residual symmetries that maintain the gauge fixations (4.34) and the boundary conditions (4.39) we have imposed on the metric field. In the context of diffeomorphism invariant theory, this entails examining the following Lie derivatives along the vector field $\xi = \xi^\mu \partial_\mu$:

$$\mathscr{L}_\xi g_{zz} = 0\,, \qquad \mathscr{L}_\xi g_{za} = 0\,, \qquad \mathscr{L}_\xi h_{ab} = \mathcal{O}(z^{-2})\,. \tag{4.48}$$

This yields the subsequent radial expansion of the residual diffeomorphisms:

$$\xi^z = z\,\omega + \mathcal{O}(z^3), \qquad \xi^a = Y^a + z^2\zeta^a + \mathcal{O}(z^4)\,, \tag{4.49}$$

where

$$\omega(x^+, x^-) = -\sigma + \frac{1}{2}\partial_a Y^a + Y^a \partial_a \phi\,, \tag{4.50}$$

$$\zeta^\pm(x^+, x^-) = \mathrm{e}^{-2\phi}\left[\partial_\mp \sigma - \left(Y^a\partial_a + \partial_\mp Y^\mp\right)\partial_\mp\phi - \frac{1}{2}\partial_\mp\partial_a Y^a\right], \tag{4.51}$$

with the gauge parameters $Y^\pm = Y^\pm(x^\pm)$ and $\sigma = \sigma(x^+, x^-)$ representing (anti-)chiral and arbitrary functions of the boundary coordinates, respectively. In existing literature, these residual diffeomorphisms (4.49) are commonly referred to as asymptotic Killing vectors, compared to exact analogues satisfying $\mathscr{L}_\xi g_{\mu\nu} = 0$.

Furthermore, under these gauge transformations, the boundary metric undergoes the following variation:

$$\delta_\xi h^{(0)}_{ab} = 2\sigma h^{(0)}_{ab}\,. \tag{4.52}$$

This illustrates, in comparison with (4.42), why we introduced the same notation for this parameter $\sigma(x)$: it triggers a Weyl transformation (Boulanger, 1999; Imbimbo et al., 2000). Besides, it reveals the role of the vectors Y^a as diffeomorphisms of the boundary. Meanwhile, the physical fields $\ell^\pm$ fluctuate akin to the components of an anomalous two-dimensional CFT stress tensor,

$$\delta_\xi \ell_\pm = Y^\pm \partial_\pm \ell_\pm + 2\ell_\pm \partial_\pm Y^\pm - \frac{1}{2}\partial^3_\pm Y^\pm\,. \tag{4.53}$$

Utilizing the modified Lie bracket (2.55) (Schwimmer & Theisen, 2008; Barnich & Troessaert, 2010), owing to the field dependence of the generators,

$$\left[\xi_1, \xi_2\right]_\star := \left[\xi_1, \xi_2\right] - \delta_{\xi_1}\xi_2 + \delta_{\xi_2}\xi_1\,, \tag{4.54}$$

we deduce the ensuing residual algebra: it corresponds to a double copy of a Witt algebra engendered by the boundary diffeomorphisms Y^a, in direct sum with a $\mathfrak{u}(1)$ algebra induced by the Weyl rescalings σ. More precisely, we obtain the following brackets:

$$[\xi_1, \xi_2]_\star = \hat{\xi}\,, \tag{4.55}$$

where the modified gauge parameters read

$$\hat{Y}^\pm = Y_1^\pm \partial_\pm Y_2^\pm - Y_2^\pm \partial_\pm Y_1^\pm\,, \qquad \hat{\sigma} = 0\,. \tag{4.56}$$

This structure becomes more evident by introducing the subsequent Fourier mode expansions:

$$Y^\pm \sim \mathrm{e}^{inx^\pm}\,, \qquad \sigma \sim \mathrm{e}^{ipx^+}\mathrm{e}^{iqx^-}\,, \tag{4.57}$$

where $n, p, q \in \mathbb{Z}$. In fact, the commutation relations of the modified Lie bracket then take the form:

$$[\xi_n^\pm, \xi_m^\pm]_\star = \mathrm{i}(n-m)\xi_{n+m}^\pm\,, \qquad [\xi_{pq}^\sigma, \xi_{rs}^\sigma]_\star = 0\,, \tag{4.58}$$

where $\xi_n^\pm$ denotes the gauge parameters in which only $Y^\pm$ expanded as in (4.57) are activated, and likewise for ξ_{pq}^σ.

Holographic renormalization

While our focus in this chapter is not to delve into a detailed holographic renormalization of the variational principle of general relativity, it is worth noting that extensive studies on this topic exist in the literature (Henningson & Skenderis, 1998; Skenderis, 2002; Compere & Marolf, 2008), following a formal procedure similar to that discussed in section 2.3 and applied to electromagnetism in subsection 3.3.2

In three-dimensional Fefferman-Graham gauge (4.34), the renormalized action can be expressed as

$$S_{\text{ren}} = \int_{\mathcal{M}} L + \int_{\mathcal{B}} L_{\text{B}}\,, \qquad L_B = L_{\text{GHY}} + L_{\text{ct}}\,, \tag{4.59}$$

where the bulk Lagrangian corresponds to the Einstein-Hilbert Lagrangian (4.1), and the boundary Lagrangian encompasses a finite component and a divergent component in the limit $z \to 0$.

The finite part is given by the Gibbons-Hawking-York boundary term (York, 1972; Gibbons & Hawking, 1977):

$$L_{\text{GHY}} = \frac{\sqrt{-\gamma}}{8\pi G}\,(K-1)\,\mathrm{d}^2x\,, \tag{4.60}$$

where n_a denotes the normal to radial constant hypersurfaces, γ_{ab} represents the induced metric and $K = g^{ab}\nabla_a n_b$ signifies the extrinsic curvature, both on these surfaces. The inclusion of (4.60) is crucial for correctly defining the variational principle when all induced fields on the boundary are fixed (Papadimitriou & Skenderis, 2005b), ensuring that $\delta S_{\text{ren}} \approx 0$. The divergent part is introduced to offset the radial divergences initially present in the Einstein-Hilbert action (4.1):

$$L_{\text{ct}} = \frac{\log z}{16\pi G}\sqrt{-\gamma}\, R^{(0)} \mathrm{d}^2 x\,. \tag{4.61}$$

These holographic considerations regarding the variational principle will prove beneficial later in this section as we delve into gauge relaxation, specifically focusing on the Weyl-Fefferman-Graham gauge. Indeed, we will observe that incorporating boundary terms à la Compère-Marolf (Compere & Marolf, 2008) at the level of the presymplectic potential in this gauge fails to correctly (in the sense of a covariant procedure) reveal the new finite charge found in the Chern-Simons formulation. Instead, what emerges is the addition of a suitable covariant corner term, which finds significance in symplectic renormalization (McNees & Zwikel, 2023).

Symplectic renormalization

The last considerations of the last paragraph prompt us to revisit renormalization through the symplectic structure,

$$\delta S_{\text{ren}} \approx \int \mathrm{d}^2 x\, \Theta^z_{\text{ren}}\,, \tag{4.62}$$

with the radial component of the presymplectic potential given by

$$\Theta^z_{\text{ren}} \approx -\frac{1}{2}\sqrt{-h^{(0)}}\, T^{ab}\delta h^{(0)}_{ab}\,. \tag{4.63}$$

Actually, the contribution from the corner to the presymplectic potential is as follows:

$$\Theta^a = \frac{1}{16\pi G}\frac{\sqrt{-h}}{z}\left(D_b(\delta h)^{ab} - D^a(\delta h)^b{}_b\right) \tag{4.64}$$

$$\approx \frac{1}{16\pi G}\frac{\sqrt{-h^{(0)}}}{z}\left(D^{(0)}_b(\delta h^{(0)})^{ab} - D^a_{(0)}(\delta h^{(0)})^b{}_b\right) + \mathcal{O}(z)\,. \tag{4.65}$$

Notice that the McNees-Zwikel prescription for the finite ambiguity cannot be applied here. The diverging ambiguity in renormalizing charges can be expressed as

$$C^{az} = \frac{\sqrt{-h^{(0)}}}{16\pi G} \log z \left(D_b^{(0)} (\delta h^{(0)})^{ab} - D^a_{(0)} (\delta h^{(0)})^b{}_b \right) + \mathcal{O}(z^2) \,. \tag{4.66}$$

With the divergence of the latter concerning the transverse space to the holographic direction satisfying the following on-shell relationship:

$$\partial_a C^{az} \approx \delta \left(\frac{\sqrt{-h^{(0)}}}{16\pi G} \log z \, R^{(0)} \right) + \mathcal{O}(z^2) \,, \tag{4.67}$$

one can infer the form (4.63) of the renormalized presymplectic potential, albeit with a diverging pure boundary ambiguity $\delta(\dots)$. Once again, through this systematic approach, insights into the boundary counterterm actions utilized in holographic renormalization can be recovered, as seen in the equations (4.61) and (4.67).

In the expression (4.63), we encounter the conventional holographic interpretation in terms of the vacuum expectation value (VEV) multiplied by the source variation. This leads us to interpret the boundary metric as the source and the tensor T_{ab} as the associated VEV, which is known as the holographic stress-energy tensor (Brown & York, 1993; Balasubramanian & Kraus, 1999; Skenderis, 2001). It is defined as

$$T_{ab} = \frac{1}{8\pi G} \left(h^{(2)}_{ab} + \frac{1}{2} h^{(0)}_{ab} \, R^{(0)} \right) , \tag{4.68}$$

and satisfies the following on-shell relationships:

$$T_a{}^a = \frac{c}{24\pi} R^{(0)} \,, \qquad D^{(0)}_a T^{ab} = 0 \,, \tag{4.69}$$

where we define the three-dimensional Brown-Henneaux central charge (Brown & Henneaux, 1986; Henningson & Skenderis, 1998)

$$c = \frac{3}{2G} \,. \tag{4.70}$$

We would like to take a moment to make a remarque en passant. One notable advantage of holographic renormalization is its ability to derive the equations for the energy-momentum tensor trace and its conservation from Ward holographic identities associated with the renormalized variational principle (de Boer et al., 2000; Corley, 2000; Kalkkinen et al., 2001). The first equation in (4.69) arises from evaluating the renormalized action along the Weyl rescaling parameter $\sigma(x)$, and it can be reformulated as:

$$\delta_\sigma S_{\rm ren} \approx \int \mathrm{d}^2x \sqrt{-h^{(0)}}\mathcal{A}\,\sigma\,, \qquad \mathcal{A} = \frac{c}{24\pi}R^{(0)}\,. \tag{4.71}$$

In this case, one encounter difficulties in defining the variational principle, which has been associated with the presence of a Weyl anomaly in the dual theory (Deser & Schwimmer, 1993; Henningson & Skenderis, 1998). As already mentioned, we shall see in the following how to associate it with the fact that the Weyl symmetry is broken in the Fefferman-Graham context. The second equation in (4.69) stems from a similar analysis conducted along the diffeomorphisms $Y^\pm(x^\pm)$ of the boundary.

Surface charges and algebra

Once we have computed the on-shell value of the renormalized presymplectic potential, we can apply the Iyer-Wald procedure to derive the asymptotic corner charges. This marks the completion of the fourth step in determining asymptotic symmetries. The renormalized (Lee-Wald) presymplectic form (2.12) takes the form:

$$\omega^r_{\rm ren}[g;\delta_1 g;\delta_2 g] = -\frac{1}{2}\delta_1\left(\sqrt{-h^{(0)}}T^{ab}\right)\wedge\delta_2 h^{(0)}_{ab}\,, \tag{4.72}$$

and the associated surface charges, when evaluated along the residual diffeomorphisms (4.49), are

$$H_\xi \approx -\frac{1}{8\pi G}\int_0^{2\pi}\mathrm{d}\theta\Big(\ell_+Y^+ - \ell_-Y^- + \phi\,\partial_t\sigma - \sigma\,\partial_t\phi\Big)\,, \tag{4.73}$$

assuming $\delta\sigma = \delta Y = 0$. These charges are finite, integrable, and non-conserved in temporal evolution due to the arbitrariness of the conformal factor and its associated gauge parameter in terms of the boundary coordinates. Notice that, in the Fefferman-Graham gauge (4.34), the Barnich-Brandt and the canonical Iyer-Wald procedures coincide, yielding the same surface charge (4.73) as described above.

The fifth and final step involves determining the algebra associated with the asymptotic charges. As clarified towards the end of subsection 2.1.4, we can show that these charges constitute a projective representation of the residual symmetry algebra (4.58) under the Poisson bracket (2.54). Specifically, we observe a double copy structure, featuring the Virasoro algebra, corresponding to the part of the algebra satisfying Brown-Henneaux conditions (Brown & Henneaux, 1986), alongside an affine algebra:

$$\{H_{\xi_n^\pm}, H_{\xi_m^\pm}\} = \mathrm{i}(n-m)H_{\xi_{n+m}^\pm} - \mathrm{i}m^3\frac{c}{12}\delta_{n+m,0}, \tag{4.74}$$

$$\{H_{\xi_{pq}^\sigma}, H_{\xi_{rs}^\sigma}\} = -\mathrm{i}(r-q)\frac{c}{3}\mathrm{e}^{2\mathrm{i}(q+s)t}\delta_{p+r,q+s}\,. \tag{4.75}$$

This algebraic structure mirrors findings from asymptotic symmetry analyses in three dimensions conducted in the Bondi-Weyl gauge (Geiller et al., 2021), as well as in studies involving generic hypersurfaces (Adami et al., 2020b, 2022, 2023). Further insights on this topic will be provided in subsection 4.3, which is dedicated to the Bondi gauge.

One can remark that the central extension of the Weyl rescaling component (4.75) explicitly depends on the time coordinate t, an unusual characteristic indicating its dependence on the specific point within the solution space being considered. The aforementioned asymptotic symmetry algebra represents more of a one-parameter (the value of t being the parameter) family of algebras, called algebroids.

4.2.2 Weyl-Fefferman-Graham

From the unveiling presence of the Weyl anomaly (4.71) in the dual theory, as elucidated by (Henningson & Skenderis, 1998), we understand that the associated symmetry is broken. It becomes apparent through the Fefferman-Graham gauge (4.34), where it is relatively easy to observe the explicit Weyl covariance breakdown of the boundary. The underlying geometric rationale lies in the fact that the induced connection from the bulk is the Levi-Civita connection, not the Weyl one. Considering the finite version of the Weyl transformations (4.52), represented by the Penrose-Brown-Henneaux transformation (Penrose & Rindler, 1985; Brown & Henneaux, 1986; Boulanger, 1999; Imbimbo et al., 2000; Rooman & Spindel, 2001a,b; Bautier et al., 2000), as expressed in

$$z \to z' = \frac{z}{\mathfrak{B}(x)}\,, \qquad x^a \to x'^a = x^a + \xi^a(z,x)\,, \tag{4.76}$$

we discern that within this framework, a diffeomorphism in transverse space of the radial direction automatically accompanies a rescaling in this direction. This diffeomorphism vanishes at the boundary $z=0$. Examining the asymptotic solution space (4.37)-(4.38), the transformation (4.76) affects the subleading terms, causing them to lose Weyl covariance under (4.52). To counteract this effect, one can relax the Fefferman-Graham gauge (4.34).

Solution space

In the fixation (4.34), the following gauge conditions are imposed:

$$g_{zz} = \frac{1}{z^2}\,, \qquad g_{za} = 0\,. \tag{4.77}$$

According to (Ciambelli & Leigh, 2020), we suggest relaxing the second condition, altering the line element to the form:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \Big(\frac{\mathrm{d}z}{z} - k_a(z,x)\mathrm{d}x^a\Big)^2 + h_{ab}(z,x)\mathrm{d}x^a\mathrm{d}x^b\,. \tag{4.78}$$

This modified ansatz (4.78) addresses the aforementioned issue since Weyl rescalings (4.52) now induce a purely radial transformation, as expressed by

$$z \to z' = \frac{z}{\mathfrak{B}(x)}\,, \qquad x^a \to x'^a = x^a\,. \tag{4.79}$$

Moreover, this formulation allows the radial subleading orders to maintain covariance under these transformations, where:

$$k_a(z,x) \quad \to \quad k'_a(z',x') = k_a(\mathfrak{B}(x)z',x) - \partial_a \ln\mathfrak{B}(x)\,, \tag{4.80}$$

$$h_{ab}(z,x) \quad \to \quad h'_{ab}(z',x') = h_{ab}(\mathfrak{B}(x)z',x)\,. \tag{4.81}$$

A clearer perspective on this matter emerges when we delve into the asymptotic radial polyhomogeneous expansions of the quantities h_{ab} and k_a, akin to (2.68). By setting specific boundary conditions that dictate the leading order of these expansions,

$$h_{ab} \sim \frac{1}{z^2}h^{(0)}_{ab} + \mathcal{O}(z^{-1})\,, \qquad k_a \sim k^{(0)}_a + \mathcal{O}(z)\,, \tag{4.82}$$

we can express the above transformations in terms of radial orders as follows:

$$k^{(2n)}_a(x) \quad \to \quad k^{(2n)}_a(x)\,\mathfrak{B}(x)^{2n} - \delta_{n,0}\,\partial_a \ln\mathfrak{B}(x)\,, \tag{4.83}$$

$$h^{(2n)}_{ab}(x) \quad \to \quad h^{(2n)}_{ab}(x)\,\mathfrak{B}(x)^{2n-2}\,, \tag{4.84}$$

where n is any positive integer. Particularly noteworthy is the conventional interpretation of h_{ab} as the boundary metric, as it transforms under Penrose-Brown-Henneaux (4.76) according to:

$$h^{(0)}_{ab} \to \mathfrak{B}^{-2}h^{(0)}_{ab}\,. \tag{4.85}$$

In general, we observe that all subleading orders undergo Weyl-covariantly transformations, in line with transformation laws of Weyl tensors with a defined Weyl weight determined by the power of $\mathfrak{B}(x)$. However, the leading order of k_a stands as an exception, following an inhomogeneous transformation just like a Weyl connection:

$$k_a^{(0)} \to k_a^{(0)} - \partial_a \ln\mathfrak{B} \,. \tag{4.86}$$

We substantiate this geometric interpretation in appendix B.1 by revisiting fundamental concepts of Weyl geometry, which, when adapted to the gauge (4.78), elucidates how this relaxation reinstates the Weyl covariance of the boundary by inducing the connection (4.86) from the bulk. This rationale justifies the nomenclature "Weyl-Fefferman-Graham" attributed to this fresh ansatz. In the subsequent subsections, we delve into the asymptotic symmetries and corner charges associated with this gauge, aiming to uncover any physical implications behind the restored Weyl connection and consequently behind the diffeomorphism between the two gauges (4.34) and (4.78).

In a similar manner to the analysis conducted in the Fefferman-Graham scenario, we proceed by assuming the boundary metric is expressed as detailed in the equation (4.39), while also introducing the light-cone coordinates outlined in (4.43) on the boundary. In the Weyl-Fefferman-Graham case, inspired by the expansion of h_{ab} in the conventional setup, we assume the following polynomial expansion for the quantity k_a:

$$k_a(z,x) = \sum_{n\geq 0} z^{2n} k_a^{(2n)}(x) \,. \tag{4.87}$$

This expansion leads to an asymptotic solution space that cannot be reduced to a finite sum, without imposing any preconditions further restricting the physical content, even in three dimensions. This is a peculiarity of this relaxed gauge. Instead, it takes the form:

$$h_{ab}(z,x) = \frac{1}{z^2} h_{ab}^{(0)}(x) + h_{ab}^{(2)}(x) + z^2 h_{ab}^{(4)}(x) + \mathcal{O}(z^4) \,, \tag{4.88}$$

with specific expressions for its components. For instance:

$$h_{\pm\pm}^{(0)} = 0 \,, \tag{4.89}$$

$$h_{\pm\pm}^{(2)} = \ell_\pm - \left(K_\pm^{(0)}\right)^2 - \partial_\pm K_\pm^{(0)}, \tag{4.90}$$

$$h_{\pm\pm}^{(4)} = -\mathrm{e}^{-2\phi}\, \partial_\pm K_\mp^{(0)} h_{\pm\pm}^{(2)} - k_\pm^{(2)} \left(\partial_\pm \phi + 2K_\pm^{(0)}\right) - \frac{1}{2} \partial_\pm k_\pm^{(2)} \,, \tag{4.91}$$

and

$$h^{(0)}_{+-} = \frac{1}{2}\mathrm{e}^{2\phi}\,, \tag{4.92}$$

$$h^{(2)}_{+-} = -\frac{1}{2}\Big(\partial_- K^{(0)}_+ + \partial_+ K^{(0)}_-\Big)\,, \tag{4.93}$$

$$\begin{aligned} h^{(4)}_{+-} = \frac{1}{4}\mathrm{e}^{-2\phi}\Big[&2\partial_+ K^{(0)}_- \partial_- K^{(0)}_+ - \mathrm{e}^{2\phi}\Big(\partial_- k^{(2)}_+ + \partial_+ k^{(2)}_- + 2k^{(0)}_+ k^{(2)}_- \\ &+ 2k^{(0)}_- k^{(2)}_+\Big) + 2h^{(2)}_{++} h^{(2)}_{--}\Big]\,, \end{aligned} \tag{4.94}$$

and so forth. In particular, the higher orders $h^{(2n)}_{ab}$ rely on $k^{(2n)}_a$. We define the adjustment of the Weyl connection by a pure gauge factor, as depicted in (4.86):

$$K^{(0)}_\pm = k^{(0)}_\pm - \partial_\pm \phi\,. \tag{4.95}$$

Additionally, the equations of motion necessitate the conditions:

$$\partial_\pm \ell_\mp = 0\,. \tag{4.96}$$

In summary, the solution space comprises one independent function that characterizes the boundary metric $\phi(x^\pm)$, the Weyl structure $k_a(z, x^\pm)$, and two chiral functions $\ell^\pm(x^\pm)$. The zero mode of these last functions encapsulates a blend of the mass and angular momentum, as we have seen from the Fefferman-Graham stress-energy tensor (4.68).

Residual symmetries

In the subsequent step, while maintaining the Weyl-Fefferman-Graham asymptotic behavior and gauge fixings, the asymptotic Killing vectors are expressed as follows:

$$\xi^z = z\,\omega + \mathcal{O}(z^3)\,, \qquad \xi^\pm = Y^\pm + z^2\zeta^\pm + \mathcal{O}(z^4)\,. \tag{4.97}$$

Here, we have relabeled:

$$\omega(x^+, x^-) = -\sigma + \frac{1}{2}\partial_a Y^a + Y^a \partial_a \phi\,, \tag{4.98}$$

$$\zeta^\pm(x^+, x^-) = \mathrm{e}^{-2\phi}\left(K^{(0)}_\mp \partial_\mp Y^\mp - H^{(0)}_\mp - \frac{1}{2}\partial_\mp \partial_a Y^a + Y^a \partial_a K^{(0)}_\mp\right), \tag{4.99}$$

ensuring that ξ^z and $\xi^\pm$ correspond to gauge parameters ω and $H^{(0)}_\pm$ respectively, without mixing. To mimic (4.95), we redefine the gauge parameters $h^{(0)}_\pm$:

$$H^{(0)}_\pm = h^{(0)}_\pm - \partial_\pm \sigma\,. \tag{4.100}$$

This is because $\delta_\xi k_\pm^{(0)} = h_\pm^{(0)}$ under these asymptotic Killing vectors (4.97). In the same way, we deduce that the remaining physical fields follow the changes:

$$\delta_\xi \ell_\pm = Y^\pm \partial_\pm \ell_\pm + 2\ell_\pm \partial_\pm Y^\pm - \frac{1}{2}\partial_\pm^3 Y^\pm \,, \quad \delta_\xi \phi = \sigma \,, \quad \delta_\xi k_\pm^{(2n)} = h_\pm^{(2n)} \,, \tag{4.101}$$

where $n \in \mathbb{N}$. Notably, we observe that

$$\delta_\xi K_\pm^{(0)} = H_\pm^{(0)} \,. \tag{4.102}$$

We introduce Fourier mode expansions for the symmetry parameters, similar to the ones in the equations (4.57)-(4.58):

$$Y^\pm \sim \mathrm{e}^{\mathrm{i}nx^\pm} \,, \quad \sigma \sim \mathrm{e}^{\mathrm{i}px^+}\mathrm{e}^{\mathrm{i}qx^-} \,, \quad H_\pm^{(0)} \sim \mathrm{e}^{\mathrm{i}px^+}\mathrm{e}^{\mathrm{i}qx^-} \,, \tag{4.103}$$

where n, p, and q are arbitrary integers. We then derive the residual symmetry algebra using the modified Lie bracket (2.55) (Schwimmer & Theisen, 2008; Barnich & Troessaert, 2010):

$$[\xi_n^\pm, \xi_m^\pm]_\star = \mathrm{i}(n-m)\xi_{n+m}^\pm \,, \tag{4.104}$$

where we have only listed the non-zero commutators for clarity. We recall that $\xi_n^\pm$ denotes diffeomorphisms where only the gauge parameters $Y^\pm$ are switched on. This algebra thus consists of two Witt algebras generated by the boundary diffeomorphisms Y^a, an Abelian sector generated by the Weyl rescalings σ, and another Abelian sector generated by the boundary vector $H_a^{(0)}$.

Holographic renormalization

In the Weyl-Fefferman-Graham gauge, adapting holographic renormalization as discussed in the earlier part of this section remains straightforward. Here, the renormalized action retains the same structure as in the equation (4.59), utilizing the familiar bulk Einstein-Hilbert Lagrangian (4.1) and the boundary Gibbons-Hawking-York Lagrangian (4.60). However, the counterterm now takes the form (Ciambelli et al., 2023):

$$L_{\mathrm{ct}} = \frac{\sqrt{-\gamma}}{16\pi G}\left(k_a \gamma^{ab} k_b + \log z\, \hat{R}^{(0)}\right) \mathrm{d}^2 x \,, \tag{4.105}$$

where we recall that γ_{ab} denotes the induced metric on the boundary. Notice the presence of the radial expansion of k_a and the Weyl-covariantization of the Ricci scalar with respect to the analogous relation (4.61) in the Fefferman-

Graham gauge[3]. By variating on-shell the resulting renormalized action according to the prescription (Compere & Marolf, 2008; Papadimitriou & Skenderis, 2005b; Freidel et al., 2020; Compère et al., 2020), we access the renormalized presymplectic potential:

$$\delta S_{\text{ren}} \approx \int \mathrm{d}^2 x \, \Theta^z_{\text{ren}} \,, \qquad \Theta^z_{\text{ren}} = -\sqrt{-h^{(0)}} \left(\frac{1}{2} T^{ab} \delta h^{(0)}_{ab} - J^a \delta k^{(0)}_a \right) . \quad (4.106)$$

Interpreting this holographically, we define the holographic stress tensor T^{ab} and the new holographic Weyl current J^a as follows:

$$T^{ab} = -\frac{2}{\sqrt{-h^{(0)}}} \frac{\delta S_{\text{ren}}}{\delta h^{(0)}_{ab}} \approx \frac{1}{8\pi G} \left(h^{ab}_{(2)} + \frac{1}{2} h^{ab}_{(0)} R^{(0)} + \frac{1}{2} \hat{\nabla}^{(a}_{(0)} k^{b)}_{(0)} \right) , \quad (4.107)$$

$$J^a = \frac{1}{\sqrt{-h^{(0)}}} \frac{\delta S_{\text{ren}}}{\delta k^{(0)}_a} \approx \frac{1}{8\pi G} k^a_{(0)} \,. \quad (4.108)$$

These represent the VEVs in the holographic dictionary. Notably, the holographic stress tensor includes the term $\hat{\nabla}^{(a}_{(0)} k^{b)}_{(0)}$, which differs from the usual Brown-York expression (4.68). The notation $\hat{\nabla}^{(0)}$ is elucidated in the appendix B.1 as the Weyl-covariant derivative with respect to $h^{(0)}_{ab}$ and $k^{(0)}_a$, see the equation (B.19). The sources associated with these VEVs correspond to the boundary geometry, namely the conformal class of boundary metric $h^{(0)}_{ab}$ and the Weyl connection $k^{(0)}_a$.

In this scenario, deriving relations on the trace and divergence of the stress-energy tensor is more straightforward through the application of Ward holographic identities. Their adaptations from (4.69) to the relaxed Fefferman-Graham gauge are as follows:

$$\hat{\nabla}^{(0)}_a T^a{}_b = J^a f^{(0)}_{ab} + \frac{c}{24\pi} \hat{R}^{(0)} k^{(0)}_b \,, \qquad T^a{}_a + \hat{\nabla}^{(0)}_a J^a = \frac{c}{24\pi} \hat{R}^{(0)} \,, \quad (4.109)$$

where these expressions pertain to variations of the renormalized action along boundary diffeomorphisms and Weyl rescalings, respectively. In the last equation, we defined the leading order of the Weyl curvature (B.4) as

$$f^{(0)}_{ab} = \nabla^{(0)}_a k^{(0)}_b - \nabla^{(0)}_b k^{(0)}_a \,. \quad (4.110)$$

In the final step, we compute the corner charges related to both the renormalized presymplectic potential and the residual symmetries. Assuming again

[3] We refer to the appendix B.1 for further insights.

$\delta\sigma = \delta Y = 0$, these charges coincide exactly with the standard ones outlined in (4.73). Consequently, the charge linked to the Weyl connection vanishes, indicating that the corresponding residual diffeomorphism is pure gauge within the phase space and can thus be factored out. This elucidates why we have not addressed the Ward identities associated with this symmetry just above. The same applies to all radial orders $k_a^{(2n)}$, with $n \in \mathbb{N}$. This outcome holds irrespective of the boundary metric, not solely confined to the conformal gauge (4.39). The holographic interpretation of this outcome is that there are no discernible observables sensitive to the Weyl connection. In this setup, where we have utilized the conventional holographic renormalization, transitioning from the Weyl-Fefferman-Graham gauge to the Fefferman-Graham gauge can be achieved without sacrificing any physical content and the Weyl covariantization of the boundary geometry can be obtained for free. However, we will refine this assertion by examining Chern-Simons formulation and metric symplectic renormalization in the subsequent part of this subsection.

Chern-Simons formulation

Indeed, first and foremost, when examining the Chern-Simons formulation of three-dimensional gravity (see the end of section 4.1), it becomes apparent that the resulting outcome differs due to a finite corner term. By scrutinizing the Chern-Simons connections, as defined under the relaxed gauge (4.78), we derive the following expressions:

$$A_z = -\frac{1}{z} L_0 + 2\sqrt{2}\, z^2 \mathrm{e}^{-\phi} \left(k_-^{(2)} L_1 - k_+^{(2)} L_{-1} \right) + \mathcal{O}(z^3)\,, \tag{4.111}$$

$$\begin{aligned} A_+ = {} & \frac{\sqrt{2}}{z} \mathrm{e}^{\phi} L_1 + \left(2\, k_+^{(0)} - \partial_+ \phi \right) L_0 + \sqrt{2}\, z\, \mathrm{e}^{-\phi} h_{++}^{(2)} L_{-1} \\ & + 2\, z^2 k_+^{(2)} L_0 + \mathcal{O}(z^3)\,, \end{aligned} \tag{4.112}$$

$$A_- = \partial_- \phi L_0 - \sqrt{2}\, z\, \mathrm{e}^{-\phi} \partial_- K_+^{(0)} L_{-1} + \mathcal{O}(z^3)\,, \tag{4.113}$$

and similarly for the second copy. It yields the associated residual gauge symmetries:

$$\lambda(z, x^+, x^-) = \epsilon^B(z, x^+, x^-) L_B\,, \tag{4.114}$$

where

$$\epsilon^1 = \frac{\sqrt{2}}{z} \mathrm{e}^{\phi} Y^+ + \mathcal{O}(z^3)\,, \tag{4.115}$$

$$\begin{aligned}\epsilon^{-1} = \frac{z}{\sqrt{2}} \mathrm{e}^{-\phi}\Big[2\Big(\ell_+ Y^+ - (K_+^{(0)})^2 Y^+ + K_+^{(0)} \partial_+ Y^+ - H_+^{(0)}\Big) \\ + \partial_+^2 Y^+\Big] + \mathcal{O}(z^3)\,,\end{aligned} \tag{4.116}$$

$$\epsilon^0 = \sigma - \partial_+ Y^+ + 2Y^+ K_+^{(0)}\,. \tag{4.117}$$

The $\mathfrak{sl}(2,\mathbb{R})$ basis used here is defined as

$$[L_1, L_{-1}] = -L_0\,, \qquad [L_1, L_0] = L_1\,, \qquad [L_{-1}, L_0] = -L_{-1}\,. \tag{4.118}$$

The computation of surface charges (4.33) yields the following result:

$$H_\Lambda = \lim_{z\to 0}\Big(H_\lambda - \widetilde{H}_{\widetilde{\lambda}}\Big) = -\frac{1}{8\pi G}\int_0^{2\pi} \mathrm{d}\theta \Big[\ell_+ Y^+ - \ell_- Y^- - \phi H_t^{(0)} + \sigma K_t^{(0)}\Big]\,, \tag{4.119}$$

where, following (Banados, 1996),

$$H_\lambda = -\frac{1}{8\pi G}\int_0^{2\pi} \mathrm{d}\theta\, \mathrm{tr}\Big(\lambda\, A_\theta\Big)\,, \qquad \widetilde{H}_{\widetilde{\lambda}} = -\frac{1}{8\pi G}\int_0^{2\pi} \mathrm{d}\theta\, \mathrm{tr}\Big(\widetilde{\lambda}\, \widetilde{A}_\theta\Big)\,. \tag{4.120}$$

Unlike the previous standard metric formulation, in this context, the Chern-Simons symplectic structure remains finite as one approaches the asymptotic boundary ($z \to 0$), due to the presence of the boundary term (4.22). This term, following the prescription (Compere & Marolf, 2008), introduces a corner term in the resulting Iyer-Wald codimension-2 form, which crucially cancels out radially divergent terms and introduces a new finite charge.

Let us get more specific. The computation reveals an interesting outcome: while the fields associated with higher radial orders of the Weyl structure $k_a^{(2p)}$, for $p \in \mathbb{N}_0$ and the component $K_\phi^{(0)}$ do not contribute to the asymptotic charges just as in (4.73), there is now a dependence on $K_t^{(0)}$ and its gauge parameter. This indicates that we can reduce the asymptotic expansion of k_a to the leading order only, rendering the metric expansion finite, as is usual in three dimensions in the Fefferman-Graham gauge. However, setting this order to zero cancels a physical charge when trying to fix it from the relaxed Weyl-Fefferman-Graham gauge. It is noteworthy that the charge associated with Y^a is conserved, unlike the ones associated with σ and $H_t^{(0)}$ since we have imposed any condition on the boundary conformal factor and the Weyl connection.

This result remains consistent with the Fefferman-Graham theorem (Fefferman & Graham, 1985, 2011), as this gauge can always be attained, but then at the expense of constraining the physical content. It is interesting to note that this theorem was adapted to the Weyl-Fefferman-Graham setup in (Jia et al., 2023), utilizing the ambient construction for Weyl manifolds, thus generalizing the original construction based on conformal manifolds. This adaptation aligns well with our descriptions detailed in appendix B.1 in the sense that this gauge relaxation induces the entire Weyl geometry at the boundary, where such geometry serves as a natural extension of conformal geometry mediated by a Weyl connection.

Symplectic renormalization

We have arrived at a pivotal juncture through the exploration of the Chern-Simons formulation, shedding light on the absence of a holographic interpretation within the second-order framework of general relativity. Previously, our recourse to holographic renormalization, as detailed in (Alessio et al., 2021), provided a familiar outcome. However, we now pivot in a second place towards symplectic renormalization, yielding an identical novel result (4.119) akin to Chern-Simons. While in (Ciambelli et al., 2023), it was also suggested to refine the holographic procedure, we abstain from delving into it here, as it differs from the standard formalization established in section 2.3. Actually, this proposal entails elevating the Weyl connection to a physical entity at the expense of introducing a non-covariant boundary Lagrangian. Instead, our focus lies on incorporating a finite covariant corner counterterm, facilitating the recovery of the phase space observed in the Chern-Simons formulation.

Drawing from (Papadimitriou & Skenderis, 2005b; Freidel et al., 2019; McNees & Zwikel, 2023; Campoleoni et al., 2023a; Geiller & Zwikel, 2024; Riello & Freidel, 2024), we refine the renormalized action (4.106) by introducing a finite corner term:

$$\widetilde{S}_{\text{ren}} = S_{\text{ren}} + S_C, \qquad S_C = \int \mathrm{d}^2x\, \partial_a L^a_C[h^{(0)}_{bc}, k^{(0)}_d]\,, \tag{4.121}$$

where $L^a_C[h^{(0)}_{bc}, k^{(0)}_d]$ represents a corner Lagrangian exhibiting covariance with respect to boundary diffeomorphisms. We insist that this cannot be due to McNees-Zwikel finite prescription since it is not applicable in this context. This requires another choice for the finite ambiguities. We select it in such a way as to respect the above requirements and reproduce the Chern-Simons result (4.119). Then, our choice for the corner Lagrangian, as the boundary geometry is approached, adopts the following form:

$$L^a_C = \lim_{z\to 0}\left[-\frac{\sqrt{-\gamma}}{16\pi G}\gamma^{ab}k_b\right] \approx -\frac{\sqrt{-h^{(0)}}}{16\pi G}h^{ab}_{(0)}k^{(0)}_b\,. \tag{4.122}$$

Consequently, the renormalized presymplectic potential, refined via the systematic Iyer-Wald ambiguity fixation, is expressed as:

$$\begin{aligned}\widetilde{\Theta}_{\text{ren}} = \lim_{z\to 0}\Bigg[\Theta^z + \partial_a\int \mathrm{d}z\,\Theta^a + \frac{1}{8\pi G}\delta\Bigg(&\sqrt{-\gamma}\,(K+1) + \frac{1}{2}\sqrt{-\gamma}\,k_a\gamma^{ab}k_b\\ &+ \frac{\log z}{2}\int \mathrm{d}^2x\sqrt{-\gamma}\,\hat{R}^{(0)} - \frac{1}{2}\partial_a(\sqrt{-\gamma}\,\gamma^{ab}k_b)\Bigg)\Bigg]\,,\end{aligned} \tag{4.123}$$

where

$$\lim_{z\to 0}\partial_a\Theta^a \approx \delta\left(-\frac{1}{16\pi G}\log\rho\sqrt{-h^{(0)}}\hat{R}^{(0)}\right). \tag{4.124}$$

Upon realization of the asymptotic limit, we derive, on-shell:

$$\widetilde{\Theta}_{\text{ren}} \approx \sqrt{-h^{(0)}}\left(-\frac{1}{2}\widetilde{T}^{ab}\delta h^{(0)}_{ab} + J^a\delta K^{(0)}_a\right). \tag{4.125}$$

Here, we introduce refinements of the stress-energy tensor (Belinfante, 1940; Rosenfeld, 1940) and of the Weyl connection as expressed in the equation (4.95):

$$\widetilde{T}^{ab} = T^{ab} + \frac{1}{2}h^{ab}_{(0)}\hat{\nabla}^{(0)}_c J^c\,, \qquad K^{(0)}_a = k^{(0)}_a - \frac{1}{2}\partial_a\ln\sqrt{-h^{(0)}}\,. \tag{4.126}$$

These symplectic considerations lead us to a main outcome: upon contraction of the renormalized presymplectic potential along the residual gauge symmetries, we retrieve the same result as in (4.119) within the first-order formulation, unveiling the emergence of the new finite charge. Thanks to the metric formulation, we can ascribe a holographic interpretation to these charges via the symplectic potential. According to the standard dictionary, in terms of (VEV) $\times$ δ(sources), the boundary metric $h^{(0)}_{ab}$ serves as a source, with the notable addition that $K^{(0)}_a$ assumes a similar role. Unlike in the variational principle renormalization procedure, this secondary source is not stricto sensu a Weyl connection; instead, it has been imbued with a Weyl pure-gauge shift, rendering it Weyl invariant. Investigating the holographic ramifications of this choice presents an intriguing avenue for future inquiry.

4.3. Bondi gauge

In this section, we transition to discussing a gravitational gauge tailored for managing asymptotically AdS and flat spacetimes. This approach is similar to the examination of electromagnetic theory in chapter 3, where we analyzed AdS and Minkowski backgrounds in Bondi coordinates through the possibility of a smooth flat limit. We shall examine the flat limit of this framework in the next section 4.4. In the forthcoming subsection 4.3.1, similarly to our approach in the preceding subsection 4.2.1, we will commence with a concise overview of asymptotic symmetries in the Bondi gauge for three-dimensional gravity in asymptotically AdS spaces. Our focus will be on modern perspectives, adhering to the conventions established in works such as (Ruzziconi, 2020; Ciambelli et al., 2020a; Ruzziconi & Zwikel, 2021).

In the second subsection 4.3.2, we delve into a specific relaxation, aiming to amalgamate the strengths of two standard gauges discussed earlier in this chapter: the Bondi and the Fefferman-Graham gauges. The latter is universally applicable but lacks precision in describing asymptotically flat spaces, similar to how Poincaré coordinates fall short in characterizing exact equivalent Minkowski space. As demonstrated in the preceding subsection, one of its key merits in the context of the AdS/CFT correspondence lies in its covariance concerning the pseudo-Riemannian boundary. Conversely, the Bondi gauge, as we will explicitly illustrate in this subsection, lacks this property. Nevertheless, it offers the advantage of validity regardless of the cosmological constant value.

4.3.1 Bondi

Solution space

The first step in analyzing asymptotic symmetries involves specifying the theory at hand: Einstein gravitation in three dimensions within an asymptotically AdS space. The second step is to enforce the gauge and boundary conditions. The Bondi gauge is defined by selecting Bondi coordinates $x^\mu = (r, u, \theta)$ (see (3.14)), where $r \geq 0$ represents a null radial coordinate, u denotes the retarded time, and $\theta \sim \theta + 2\pi$ signifies the angular coordinate on the circle at infinity. This asymptotic boundary is situated at $r \to \infty$.

Additionally, we impose the following three gauge-fixing conditions (Bondi et al., 1962; Sachs, 1962a,b):

$$g_{rr} = 0\,, \qquad g_{r\theta} = 0\,, \qquad g_{\theta\theta} = r^2 \mathrm{e}^{2\phi}\,, \tag{4.127}$$

where we introduce the conformal factor $\phi = \phi(u, \theta)$. Such gauge fixations can always be imposed utilizing the degrees of freedom originating from coordinate transformations. However, as we will explore in the subsequent part of this subsection, similarly to the Fefferman-Graham gauge, these conditions may constrain the physical content of the theory. The gauge conditions (4.127) render the bulk metric in the form:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{V}{r}\mathrm{e}^{2\beta}\mathrm{d}u^2 - 2\mathrm{e}^{2\beta}\mathrm{d}u\,\mathrm{d}r + r^2\mathrm{e}^{2\phi}(\mathrm{d}\theta - U\mathrm{d}u)^2\,, \tag{4.128}$$

where V, β, and U represent arbitrary functions of the bulk coordinates. The boundary conditions are fixed to ensure that the AdS space in Bondi coordinates (3.14) can be recovered in the asymptotic limit:

$$\beta \sim \mathcal{O}(1)\,, \qquad U \sim \mathcal{O}(1)\,, \qquad \frac{V}{r} \sim \mathcal{O}(r^2)\,. \tag{4.129}$$

The third step involves solving the Einstein field equations (4.5). Assuming a polyhomogeneous expansion of the form (2.68), we find that the logarithmic terms are fixed to zero on-shell, and infinite expansions reduce to finite sums. This yields respectively for the (rr)-, $(r\theta)$- and (ur)-component of the equations of motion:

$$\beta = \beta_0(u,\theta)\,, \tag{4.130}$$

$$U = U_0(u,\theta) + \frac{2}{r}\mathrm{e}^{2\beta_0-2\phi}\partial_\theta\beta_0 - \frac{1}{r^2}\mathrm{e}^{2\beta_0-2\phi}N(u,\theta)\,, \tag{4.131}$$

$$\begin{aligned}\frac{V}{r} = &-\frac{r^2}{\ell^2}\mathrm{e}^{2\beta_0} - 2r(\partial_u\phi + \partial_\theta U_0 + U_0\partial_\theta\phi) + M(u,\theta)\\ &+\frac{4}{r}\mathrm{e}^{2\beta_0-2\phi}N\partial_\theta\beta_0 - \frac{1}{r^2}\mathrm{e}^{2\beta_0-2\phi}N^2\,.\end{aligned} \tag{4.132}$$

Here, to account for the flat limit (as discussed in section 4.4), we reintroduce the AdS radius ℓ. In the literature, similar to the electromagnetic terminology introduced in Bondi coordinates, $M = M(u,\theta)$ is referred to as the Bondi mass aspect, and $N = N(u,\theta)$ is termed the angular momentum aspect. Their temporal evolutions are governed by the $(u\theta)$-equation:

$$\begin{aligned}(\partial_u + \partial_u\phi)N = &\frac{1}{2}\left(\partial_\theta + 2\partial_\theta\beta_0\right)M - 2N\partial_\theta U_0 - U_0(\partial_\theta N + N\partial_\theta\phi)\\ &+ 4\mathrm{e}^{2\beta_0-2\phi}\left[2(\partial_\theta\beta_0)^3 - (\partial_\theta\phi)(\partial_\theta\beta_0)^2 + (\partial_\theta\beta_0)(\partial_\theta^2\beta_0)\right],\end{aligned} \tag{4.133}$$

and the (uu)-component:

$$\begin{aligned}\partial_u M = {} & (2\partial_u\beta_0 - 2\partial_u\phi - 2\partial_\theta U_0 + U_0 2\partial_\theta\beta_0 - U_0 2\partial_\theta\phi - U_0\partial_\theta)\, M \\ & + \frac{2}{\ell^2}\mathrm{e}^{4\beta_0-2\phi}[\partial_\theta N + N(4\partial_\theta\beta_0 - \partial_\theta\phi)] - 2\mathrm{e}^{2\beta_0-2\phi}\{\partial_\theta U_0[8(\partial_\theta\beta_0)^2 \\ & - 4\partial_\theta\beta_0\partial_\theta\phi + (\partial_\theta\phi)^2 + 4\partial_\theta^2\beta_0 - 2\partial_\theta^2\phi] - \partial_\theta^3 U_0 + U_0[\partial_\theta\beta_0(8\partial_\theta^2\beta_0 \\ & - 2\partial_\theta^2\phi) + \partial_\theta\phi(-2\partial_\theta^2\beta_0 + \partial_\theta^2\phi) + 2\partial_\theta^3\beta_0 - \partial_\theta^3\phi] + 2\partial_u\partial_\theta\beta_0(4\partial_\theta\beta_0 \\ & - \partial_\theta\phi) + \partial_u\partial_\theta\phi(-2\partial_\theta\beta_0 + \partial_\theta\phi) + 2\partial_u\partial_\theta^2\beta_0 - \partial_u\partial_\theta^2\phi\}\,.\end{aligned} \tag{4.134}$$

The induced boundary metric on the asymptotic boundary is characterized by the remaining three arbitrary boundary coordinate functions:

$$\lim_{r\to\infty}\left(\frac{1}{r^2}\mathrm{d}s^2\right) = \left(-\frac{\mathrm{e}^{4\beta_0}}{\ell^2} + \mathrm{e}^{2\phi}U_0^2\right)\mathrm{d}u^2 - 2\mathrm{e}^{2\phi}U_0\,\mathrm{d}u\,\mathrm{d}\theta + \mathrm{e}^{2\phi}\mathrm{d}\theta^2\,. \tag{4.135}$$

Residual symmetries

Determining the residual diffeomorphisms that uphold the asymptotic solution space is the fourth step. The components of these vectors are specified as follows:

$$\xi^u = f\,, \tag{4.136}$$

$$\xi^\theta = Y - \frac{1}{r}\partial_\theta f\,\mathrm{e}^{2\beta_0-2\phi}\,, \tag{4.137}$$

$$\begin{aligned}\xi^r = {} & -r[\partial_\theta Y - \sigma - U_0\partial_\phi f + Y\partial_\theta\phi + f\partial_u\phi] \\ & + \mathrm{e}^{2\beta_0-2\phi}\left(\partial_\theta^2 f - \partial_\theta f\partial_\theta\phi + 4\partial_\theta f\partial_\theta\beta_0\right) - \frac{1}{r}\mathrm{e}^{2\beta_0-2\phi}N\partial_\theta f\,,\end{aligned} \tag{4.138}$$

where f, Y, and σ are arbitrary functions of (u,θ), which may depend on the fields. The gauge transformation of physical fields is as follows:

$$\delta_\xi\phi = \sigma\,, \tag{4.139}$$

$$\delta_\xi\beta_0 = (f\partial_u + Y\partial_\theta)\beta_0 + \frac{1}{2}\left(\partial_u - \partial_u\phi + 2U_0\partial_\theta\right)f - \frac{1}{2}(\partial_\theta Y + Y\partial_\theta\phi - \sigma)\,, \tag{4.140}$$

$$\begin{aligned}\delta_\xi U_0 = {} & (f\partial_u + Y\partial_\theta - \partial_\theta Y)U_0 - \frac{1}{\ell^2}\left(\ell^2\partial_u Y - \mathrm{e}^{4\beta_0}\mathrm{e}^{-2\phi}\partial_\theta f\right) \\ & + U_0(\partial_u f + U_0\partial_\theta f)\,,\end{aligned} \tag{4.141}$$

$$\delta_\xi N = (f\partial_u + Y\partial_\theta + 2\partial_\theta Y + f\partial_u\phi + Y\partial_\theta\phi - \sigma - 2U_0\partial_\theta f)N \\ + M\partial_\theta f - \mathrm{e}^{2\beta_0 - 2\phi}[3\partial_\theta^2 f(2\partial_\theta\beta_0 - \partial_\theta\phi) + \partial_\theta f(4(\partial_\theta\beta_0)^2 \\ - 8\partial_\theta\beta_0\partial_\theta\phi + 2(\partial_\theta\phi)^2 + 2\partial_\theta^2\beta_0 - \partial_\theta^2\phi) + \partial_\theta^3 f]\,, \tag{4.142}$$

$$\delta_\xi M = \frac{4}{\ell^2}\partial_\theta f \mathrm{e}^{4\beta_0 - 2\phi}N + (\partial_u f + f\partial_u\phi + \partial_\theta Y + Y\partial_\theta\phi - \sigma)M \\ - 2\mathrm{e}^{2\beta_0 - 2\phi}\Big[2\partial_\theta^2 f\partial_u\beta_0 + 4\partial_u\partial_\theta f\partial_\theta\beta_0 + \partial_u\partial_\theta^2 f + \partial_\theta^2 f\partial_\theta U_0 \\ + 8\partial_\theta^2 f\partial_\theta\beta_0 U_0 + \partial_\theta f\Big((4\partial_\theta\beta_0 - \partial_\theta\phi)(2\partial_u\beta_0 - \partial_u\phi) + 4\partial_u\partial_\theta\beta_0 \\ + \partial_\theta U_0(8\partial_\theta\beta_0 - 2\partial_\theta\phi) - \partial_\theta^2 U_0 - 2\partial_u\partial_\theta\phi + U_0(-4\partial_\theta\beta_0\partial_\theta\phi \\ + 8(\partial_\theta\beta_0)^2 + 4\partial_\theta^2\beta_0 + (\partial_\theta\phi)^2 - 2\partial_\theta^2\phi)\Big) - 2\partial_\theta^2 f U_0\partial_\theta\phi + \partial_\theta^3 f U_0 \\ - \partial_u\partial_\theta f\partial_\theta\phi - \partial_\theta^2 f\partial_u\phi - 2f\partial_\theta\beta_0\partial_u\partial_\theta\phi + 2\partial_\theta\beta_0\partial_\theta\sigma - 2\partial_\theta\beta_0\partial_\theta Y\partial_\theta\phi \\ - 2\partial_\theta\beta_0\partial_\theta^2 Y - 2\partial_\theta\beta_0 Y\partial_\theta^2\phi\Big] + f\partial_u M + \partial_\theta M Y\,. \tag{4.143}$$

The roles of the parameters f and Y are noteworthy: f and Y act as boundary diffeomorphisms, whereas σ functions as Weyl transformations. Under the modified Lie bracket (2.55), the residual symmetry algebra is closed, given that we define

$$\hat{f} = f_1\partial_u f_2 + Y\partial_\phi f_2 - \delta_{\xi_1} f_2 - (1 \leftrightarrow 2)\,, \tag{4.144}$$

$$\hat{Y} = f_1\partial_u Y_2 + Y_1\partial_\phi Y_2 - \delta_{\xi_1} Y_2 - (1 \leftrightarrow 2)\,, \tag{4.145}$$

$$\hat{\sigma} = \delta_{\xi_2}\sigma_1 - (1 \leftrightarrow 2)\,. \tag{4.146}$$

This algebra represents a double copy of the diffeomorphism algebra on the boundary cylinder spanned, in direct sum with the smooth functions defined over this cylinder.

In the Bondi framework, we express the Dirichlet boundary conditions of Brown-Henneaux (Brown & Henneaux, 1986) as

$$\beta_0 = 0\,, \qquad U_0 = 0\,, \qquad \phi = 0\,. \tag{4.147}$$

These constraints render the induced boundary metric (4.135) flat. Consequently, the residual gauge parameters must adhere to the following conditions:

$$\partial_u f = \partial_\theta Y\,, \qquad \partial_u Y = \frac{1}{\ell^2}\partial_\theta f\,, \qquad \sigma = 0\,. \tag{4.148}$$

It is straightforward to show that these equations typically correspond to the conditions satisfied by conformal Killing vectors induced on the boundary.

Renormalization and surface charges

The continuation of the fourth step involves computing the asymptotic surface charges. It can be demonstrated that the associated variational problem, and consequently, the radial component of the presymplectic potential, exhibit radial divergences of the order $\mathcal{O}(r^2)$, along with $\mathcal{O}(\ell^2)$ terms. These last terms pose an obstacle to taking the flat limit $\ell \to \infty$. Although this presents a greater challenge, it is feasible to explore the holographic renormalization within this gauge. For further insights, see particularly (Ruzziconi & Zwikel, 2021). However, for the various reasons we discussed so far in this manuscript, we will focus on the symplectic prescription moving forward in the Bondi setup, especially due to the fact that we want to treat the asymptotically flat spacetimes thanks to the flat limit of this gauge.

By following steps akin to the ones achieved in this prescription thus far, specifically addressing the diverging Iyer-Wald ambiguities from the corner contribution of the bulk presymplectic potential, we derive:

$$\begin{aligned}\Theta^r_{\text{ren}} = \frac{1}{16\pi G}\big[&\mathrm{e}^{\phi} M\delta(\phi - 2\beta_0) + 2\mathrm{e}^{\phi} N\delta U_0 + 2\mathrm{e}^{2\beta_0-\phi}\big(6\partial_\theta\beta_0\partial_\theta\delta\beta_0 \\ &- \partial_\theta\phi\partial_\theta\delta\beta_0 + \partial_\theta^2\delta\beta_0\big)\big]\,.\end{aligned} \tag{4.149}$$

In this scenario, one can apply the McNees-Zwikel prescription (McNees & Zwikel, 2023) to address the finite corner ambiguity, which is the one affecting the charge. This method is applicable since Θ^u inherently becomes a complete radial derivative in the Bondi gauge (4.128), obviating the need to enforce the equations of motion. Let us delve a bit deeper into this explanation. Due to the gauge fixings, specifically $\partial_r\beta = \partial_r\phi = 0$, we find that:

$$\Theta^u = -2\mathrm{e}^{\phi}\delta\left(\beta - \phi\right) = \partial_r C^{ur}\,, \tag{4.150}$$

where

$$C^{ur} = -2r\mathrm{e}^{\phi}\delta\left(\beta - \phi\right). \tag{4.151}$$

Upon going on-shell through these steps,

$$C^{ur} \approx -2r\mathrm{e}^{\phi}\delta\left(\beta_0 - \phi\right), \tag{4.152}$$

we can observe directly that it nullifies the arbitrary function of (u, θ) that would have arisen had we conducted such a radial integration directly at the on-shell level from scratch:

$$\Theta^u \approx -2\mathrm{e}^{\phi}\delta\left(\beta_0 - \phi\right) = \partial_r C^{ur}\,, \qquad C^{ur} \approx -2r\mathrm{e}^{\phi}\delta\left(\beta_0 - \phi\right) + \text{function}(u,\theta)\,. \tag{4.153}$$

Although the result vanishes, this is still a finite prescription setting. This maneuvering results in a non-zero finite corner ambiguity in the relaxed Bondi-Weyl gauge (Geiller et al., 2021), rendering the charges integrable.

Ultimately, in the present Bondi setup, (4.149) leads us to the following surface charges:

$$\begin{aligned}\delta\!\!\!/ H_\xi = \frac{1}{8\pi G}\int_0^{2\pi} d\theta \Big[& Y\delta\left(\mathrm{e}^{\phi}N\right) + \partial_\theta\left(\mathrm{e}^{2\beta_0-\phi}\partial_\theta f\right)\delta\left(\beta_0-\phi\right) \\ & + f\Big(\frac{1}{2}\mathrm{e}^{\phi}\delta M - \mathrm{e}^{\phi}M\delta\left(\beta_0-\phi\right) - U_0\delta(\mathrm{e}^{\phi}N) \\ & + \mathrm{e}^{2\beta_0-\phi}(6\partial_\theta\beta_0\partial_\theta\delta\beta_0 - \partial_\theta\phi\partial_\theta\delta\beta_0 + \partial_\theta^2\delta\beta_0)\Big)\Big]\,. \end{aligned} \tag{4.154}$$

We would have arrived at precisely the same expression using Barnich-Brandt's prescription of the codimension-2 form $k_\xi^{\mu\nu}$. One might observe the absence of a charge associated with the function ϕ and its parameter σ, which induce Weyl rescalings similarly to the Fefferman-Graham context. Unlike the latter, we lack a finite charge sensitive to the presence of such a symmetry. Nevertheless, in the subsequent part of this subsection, we will explore the possibility of revealing this presence by manipulating the finite ambiguities of the relaxed covariant Bondi gauge.

Although operating within three dimensions with no local degree-of-freedom propagation, we observe that the charges (4.154) lack integrability, thus necessitating the reappearance of the symbol $\delta\!\!\!/$. However, this obstruction is merely apparent and can be resolved through a clever redefinition of the gauge parameters, amounting to solving Pfaff's problem[4]. We could have avoided this type of peculiarity by selecting the conformal gauge of the boundary (4.39) as we have done for Fefferman-Graham and will do again later in the manuscript. However, we find this example interesting and meaningful on the notion of integrability, and then wish to address it.

By proposing the following redefinitions:

$$\widetilde{f} = f\,\mathrm{e}^{2\beta_0-\phi}, \qquad \widetilde{Y} = Y - U_0\,f, \qquad \widetilde{\sigma} = \sigma\,, \tag{4.155}$$

such that they are field-independent, i.e., $\delta\tilde{f} = \delta\tilde{Y} = \delta\tilde{\sigma} = 0$, the surface charges (4.154) become integrable and read

$$H_\xi = \frac{1}{16\pi G}\int_0^{2\pi} d\theta\Big[2\widetilde{Y}\widetilde{N} + \widetilde{f}\widetilde{M}\Big]\,, \tag{4.156}$$

[4]For further details, we refer to, e.g., (Barnich & Compère, 2008; Grumiller et al., 2020a; Adami et al., 2020b; Ruzziconi & Zwikel, 2021).

where we rename:

$$\widetilde{N} = \mathrm{e}^{\phi} N \,, \tag{4.157}$$
$$\widetilde{M} = \mathrm{e}^{2\phi - 2\beta_0} M + 8(\partial_\theta \beta_0)^2 - 4\partial_\theta \beta_0 \partial_\theta \phi + (\partial_\theta \phi)^2 + 2\partial_\theta^2 (2\beta_0 - \phi) \,. \tag{4.158}$$

Consequently, the algebra of residual symmetries transforms into

$$\hat{\widetilde{f}} = \widetilde{Y}_1 \partial_\theta \widetilde{f}_2 + \widetilde{f}_1 \partial_\theta \widetilde{Y}_2 - (1 \leftrightarrow 2) \,, \tag{4.159}$$
$$\hat{\widetilde{Y}} = \widetilde{Y}_1 \partial_\theta \widetilde{Y}_2 + \frac{1}{\ell^2} \widetilde{f}_1 \partial_\theta \widetilde{f}_2 - (1 \leftrightarrow 2) \,, \tag{4.160}$$
$$\hat{\widetilde{\sigma}} = 0 \,, \tag{4.161}$$

whose projective representation is the following charge algebra (see (2.54)):

$$\{H_{\xi_1}, H_{\xi_2}\} = H_{[\xi_1, \xi_2]_\star} + \frac{1}{8\pi G} \int_0^{2\pi} d\theta \left(\partial_\theta^2 \widetilde{f}_1 \, \partial_\theta \widetilde{Y}_2 - \partial_\theta^2 \widetilde{f}_2 \, \partial_\theta \widetilde{Y}_1 \right) . \tag{4.162}$$

As witnessed in the Fefferman-Graham context, the dependence on the value of the retarded time u at which it is evaluated corresponds to a double copy of the Virasoro algebroid, with its one-dimensional base space parametrized by the temporal coordinate. The structure of the charge algebra can be better understood by considering the Dirichlet boundary conditions (4.147). In this scenario, we find the first part of the previous Fefferman-Graham algebra (4.74) without the Weyl sector. Here, the chiral parameters $Y^{\pm}(x^{\pm})$ are defined in the lightcone coordinates $x^{\pm} = \theta \pm \frac{u}{\ell}$ as follows:[5]

$$\widetilde{f} = \frac{\ell}{2} \left(Y^+ + Y^- \right) , \qquad \widetilde{Y} = \frac{1}{2} \left(Y^+ - Y^- \right) . \tag{4.163}$$

The integrable charges (4.156) do not encompass the maximum number of independent charges, which, in this scenario, is two. Specifically, as previously mentioned, the conformal factor and its gauge parameter are absent from these charges. This clearly indicates that the Bondi gauge setting is not the most comprehensive framework for exploring the theory maximal phase space. Consequently, we are prompted to reconsider the gauge conditions we impose in the latter part of this subsection.

In the relaxation we are about to examine, we propose relaxing the condition on the component $g_{r\theta}$, which allows a new physical field to appear in the finite charge, thereby influencing the Lorentz symmetry associated with

[5] This is because, with this redefinition of the gauge parameters (4.163), the conformal

the boundary zweibein. In the literature, to enable the function ϕ to retain its role associated with Weyl rescalings, an alternative approach has been proposed. Instead of relaxing the mixed component of the bulk metric, the equivalent of the Bondi-Sachs determinant condition – i.e., the component $g_{\theta\theta}$ in three dimensions – is relaxed. This proposition led to the establishment of the Bondi-Weyl gauge (Geiller et al., 2021). While we will not delve into the latter approach here, opting instead to concentrate on a different perspective, it would certainly be intriguing to establish connections with the Weyl-Fefferman-Graham and covariant Bondi relaxations.

4.3.2 Covariant Bondi

Solution space

For the reasons mentioned in the general introduction and the previous subsection 4.3.1 dedicated to the Bondi gauge, we introduce a relaxation of the Bondi gauge inspired by the fluid/gravitational picture. As previously stated, compared with the standard gauge (4.128), we relax the condition on the component $g_{r\theta}$ of the bulk metric in Bondi coordinates, $(x^\mu) = (r, x^a)$ and $(x^a) = (u, \theta)$.

The resulting line element can be expressed conveniently in terms of the boundary zweibein (Campoleoni et al., 2019a) as follows:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{2}{k^2}\mathrm{u}\,(\mathrm{d}r + r\,\mathrm{A}) + r^2 q_{ab}\,\mathrm{d}x^a\mathrm{d}x^b + \frac{8\pi G}{k^4}\,\mathrm{u}\,(\varepsilon\,\mathrm{u} + \chi\star\mathrm{u})\,, \tag{4.164}$$

where k represents the inverse of the AdS radius, denoted as $k = \frac{1}{\ell}$. In section B.2, we shall provide justification for this ansatz within the bulk reconstruction framework, based on a two-dimensional relativistic fluid situated on the asymptotic boundary, where the constant k plays the role of the speed of light on this host geometry. It is noteworthy that all radial dependencies are explicitly manifested within the bulk metric (4.164).

Primarily, the solution space is defined in terms of the boundary zweibein, comprising two independent one-forms, being functions of the boundary coordinates, expressed as

$$\mathrm{u} = u_a(x^b)\mathrm{d}x^a\,, \qquad \star\mathrm{u} = \star u_a(x^b)\mathrm{d}x^a\,. \tag{4.165}$$

This pair $(\mathrm{u}, \star\mathrm{u})$ is orthogonal, normalized to $-k^2$ and k^2, thus, timelike and spacelike, respectively. Consequently, the metric of the two-dimensional

Killing equations (4.148) result in the chirality condition $\partial_\mp Y^\pm = 0$.

boundary in the Cartan frame (which is therefore not orthonormal) can be written as

$$\mathrm{d}s^2_{\text{bdy}} = q_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{1}{k^2}\left(-u_a u_b + \star u_a \star u_b\right)\mathrm{d}x^a\mathrm{d}x^b\,. \tag{4.166}$$

We use this metric and its inverse to lower and raise the boundary indices, respectively. Secondly, we also introduce the Weyl connection (Loganayagam, 2008; Ciambelli & Leigh, 2020)[6]:

$$\mathrm{A} = A_a(x^b)\mathrm{d}x^a = \frac{1}{k^2}\left(\Theta^\star \star\mathrm{u} - \Theta\,\mathrm{u}\right), \tag{4.167}$$

where the expansions $\Theta = \nabla_a u^a$ and $\Theta^\star = \nabla_a \star u^a$ are derived from the Cartan structure equations of the boundary zweibein:

$$\mathrm{d}\mathrm{u} = \frac{\Theta^\star}{k^2}\star\mathrm{u}\wedge\mathrm{u}\,, \qquad \mathrm{d}\star\mathrm{u} = \frac{\Theta}{k^2}\star\mathrm{u}\wedge\mathrm{u}\,. \tag{4.168}$$

Lastly, the scalars ε and χ, constituting components of the boundary stress-energy tensor, are present in the ansatz (4.164). Further elucidation on these shall be provided towards the conclusion of this first part on the asymptotic solution space analysis.

Similar to our considerations for the (Weyl-)Fefferman-Graham gauge (see (4.52) and (B.12)), under Weyl rescalings of the boundary metric (4.166), denoted by

$$(\mathrm{u}, \star\mathrm{u}) \to (\mathrm{u}, \star\mathrm{u})/\mathfrak{B}\,, \tag{4.169}$$

the one-form A transforms as a connection

$$\mathrm{A} \to \mathrm{A} - \mathrm{d}\ln\mathfrak{B}\,. \tag{4.170}$$

This transformation justifies its name. Furthermore, as detailed in appendix B.2, the construction of the bulk line element (4.164) relies on ensuring its invariance under boundary Weyl transformations induced by $r \to \mathfrak{B}r$. Under these transformations, the scalars ε and χ adjust as follows:

$$(\varepsilon, \chi) \to \mathfrak{B}^2(\varepsilon, \chi)\,. \tag{4.171}$$

This situation mirrors that encountered in the Weyl-Fefferman-Graham gauge (see subsection 4.2.2), where boundary Weyl transformations are governed by

[6] The context will always distinctly differentiate between this and the Chern-Simons $\mathfrak{sl}(2,\mathbb{R})$ connection, ensuring there is no confusion between the two notations.

straightforward bulk diffeomorphisms, thus instigating a complete Weyl geometry at the boundary as reviewed in appendix B.1. Consequently, for greater precision in our analysis, we find in appendix B.2 that the covariant Bondi gauge (4.164) engenders a Weyl-hydrogeometry at the boundary. The curvature of the Weyl connection is defined as (refer to (B.4))

$$F_{ab} = \partial_a A_b - \partial_b A_a = \frac{1}{k^2}\left(\partial_a \Theta^\star \star u_b - \partial_b \Theta^\star \star u_a - \partial_a \Theta \, u_b + \partial_b \Theta \, u_a\right), \quad (4.172)$$

while its Hodge dual reads

$$F = \star \mathrm{dA} = \frac{1}{k^2}\left(u^a \partial_a \Theta^\star - \star u^a \partial_a \Theta\right). \quad (4.173)$$

Upon prescribing the gauge fixings and boundary conditions, we proceed to solve the Einstein equations. Consistency of these equations relies on the satisfaction of suitable differential conditions by the six independent boundary functions $\{\mathrm{u}, \star\mathrm{u}, \varepsilon, \chi\}$:

$$u^a\left(\partial_a + 2A_a\right)\varepsilon = -\star u^a\left(\partial_a + 2A_a\right)\left(\chi - \frac{F}{4\pi G}\right), \quad (4.174)$$

$$u^a\left(\partial_a + 2A_a\right)\chi = -\star u^a\left(\partial_a + 2A_a\right)\varepsilon. \quad (4.175)$$

These equations resemble the retarded time constraints on mass and angular momentum in the standard Bondi gauge (4.128), albeit in a more compact form due to the use of the covariant Bondi gauge. Notably, the derivatives within these equations are Weyl covariant, rendering both equations fully Weyl covariant. To further streamline the expressions, we introduce the symmetric Brown-York stress tensor (Brown & York, 1993; de Haro et al., 2001), which, in this context, reads (Campoleoni et al., 2019a)

$$T_{ab} = \frac{1}{2k}\left(\widetilde{T}_{ab} + \widehat{T}_{ab}\right), \quad (4.176)$$

where

$$\widetilde{\mathrm{T}} = \frac{\varepsilon}{k^2}\left(\mathrm{u}^2 + \star\mathrm{u}^2\right) + \frac{\chi}{k^2}\left(\mathrm{u}\star\mathrm{u} + \star\mathrm{u}\,\mathrm{u}\right) + \frac{R^{(0)}}{8\pi G k^2}\star\mathrm{u}^2, \quad (4.177)$$

$$\widehat{\mathrm{T}} = \frac{1}{8\pi G k^4}\left(u^a\partial_a\Theta + \star u^a \partial_a \Theta^\star - \frac{k^2}{2}R^{(0)}\right)\left(\mathrm{u}^2 + \star\mathrm{u}^2\right) - \frac{1}{4\pi G k^4}\star u^a \partial_a \Theta\left(\mathrm{u}\star\mathrm{u} + \star\mathrm{u}\,\mathrm{u}\right). \quad (4.178)$$

As usual, the Ricci scalar of the boundary metric is denoted by $R^{(0)}$. Subsequently, Einstein's equations can be rewritten as

$$\nabla_a T^{ab} = 0 \,, \qquad T^a{}_a = \frac{R^{(0)}}{16\pi G k} \,. \tag{4.179}$$

In the fluid/gravitational image, the normalized vector congruence u^a is understood as the velocity of a two-dimensional fluid living on a curved background q_{ab}, with local energy density ε and heat density χ. This interpretation finds justification in examining the Brown-York stress tensor (4.176) within the hydrodynamic holographic framework. In this context, the first term $\widetilde{T}_{ab}$ corresponds to the stress-energy tensor of a perfect relativistic fluid. Actually, this term includes a viscous stress tensor that depends on both the energy density and the scalar curvature of the boundary. On the other hand, the second term $\widehat{T}_{ab}$ represents the external force density acting on this fluid. The entirety of the stress tensor (4.176) can be rationalized within the fluid/gravity correspondence by reconstructing the bulk metric holographically from a relativistic fluid sitting at the asymptotic boundary.

It is worth noting that in this dual analysis, the background metric and the normalized vector u^a are typically regarded as independent boundary data (Campoleoni et al., 2019a; Ciambelli et al., 2020a). In this latter case, $\star$u serves as the Hodge dual of u, following the convention such that $\varepsilon_{01} = +1$:

$$\star u_a = \sqrt{-q}\, \varepsilon_{ab} u^b \,, \qquad u_a = \sqrt{-q}\, \varepsilon_{ab} \star u^b \,. \tag{4.180}$$

Nevertheless, as previously noted, in the main body of this thesis, in alignment with (Ciambelli et al., 2020b; Campoleoni et al., 2022), we treat u and $\star$u as the two independent one-forms composing the boundary zweibein and express the boundary metric in their terms within the Cartan frame (4.166).

Residual symmetries and algebra

Once we determine the asymptotic solution space (4.164), we can characterize the residual diffeomorphisms that preserve it. This allows us to identify the components of the asymptotic Killing vectors $\xi = \xi^\mu \partial_\mu$, which are as follows:

$$\xi^r = r\,\omega + \frac{1}{k^2}\left(\star u^a\, \partial_a \eta + \Theta^\star \eta\right) + \frac{4\pi G}{k^2 r}\, \chi\, \eta \,, \tag{4.181}$$

$$\xi^a = Y^a - \frac{1}{k^2 r}\, \eta \star u^a \,. \tag{4.182}$$

The significance of the four gauge parameters becomes evident when examining the gauge transformations of the boundary metric (4.166) and its associated zweibein. Notably, for the penultimate one, we observe that it varies as

$$\delta_\xi q_{ab} = \mathscr{L}_Y q_{ab} + 2\,\omega\, q_{ab}\,. \tag{4.183}$$

This elucidates that the vector $Y^a(x^b)$ generates boundary diffeomorphisms, while the parameter $\omega(x^a)$ acts as a Weyl rescaling (refer to (4.52) in the Fefferman-Graham framework).

The novelty introduced by Bondi's covariant gauge relaxation is captured in the parameter $\eta(x^a)$. Indeed, since we consider the boundary zweibein in this context, an additional symmetry emerges: the rotations of the Cartan frame $(\mathrm{u}, \star\mathrm{u})$. This symmetry becomes evident when observing the variation of the latter under residual bulk diffeomorphisms:

$$\delta_\xi \mathrm{u} = \mathscr{L}_Y \mathrm{u} + \omega\,\mathrm{u} + \eta\star\mathrm{u}\,, \tag{4.184}$$

$$\delta_\xi \star\mathrm{u} = \mathscr{L}_Y \star\mathrm{u} + \omega\star\mathrm{u} + \eta\,\mathrm{u}\,. \tag{4.185}$$

The function η induces infinitesimal transformations typical of a two-dimensional local Lorentz boost:

$$\begin{pmatrix} \mathrm{u}' \\ \star\mathrm{u}' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \star\mathrm{u} \end{pmatrix}. \tag{4.186}$$

In the Weyl-hydrogeometric image, this relates to the covariance of the hydrodynamic frame.

We can elaborate a little further on this Lorentz symmetry. In fact, whereas the form A (4.167) transforms as a connection under Weyl rescalings (compare with the finite version in (4.170) with $\mathfrak{B} \simeq 1-\omega$)

$$\mathrm{A} \to \mathrm{A} + \mathrm{d}\omega\,, \tag{4.187}$$

the local Lorentz symmetry is also accompanied by a gauge connection given by the dual connection of A:

$$\star\mathrm{A} = \frac{1}{k^2}\left(\Theta^\star\mathrm{u} - \Theta\star\mathrm{u}\right). \tag{4.188}$$

In fact, it transforms as follows under an infinitesimal Lorentz boost:

$$\star\mathrm{A} \to \star\mathrm{A} - \mathrm{d}\eta\,. \tag{4.189}$$

In comparison with its Weyl analogue (4.173), its curvature is given by the Ricci scalar of the boundary,

$$R^{(0)} = \frac{1}{2} \star \mathrm{d}{\star}\mathrm{A} \,. \tag{4.190}$$

This is because the Lorentz connection is the sole independent component of the two-dimensional spin connection associated with the zweibein $(\mathrm{u}, \star\mathrm{u})$.

After examining the boundary zweibein at the physical field level, it is necessary to account for the gauge transformation induced by the asymptotic Killing vectors on the remaining physical fields, namely the scalars ε and χ. Similar to the constraints they impose on the Einstein field equations, these variations can be succinctly expressed using the Brown-York stress tensor (4.176):

$$\delta_\xi T_{ab} = \mathscr{L}_Y T_{ab} + \frac{1}{16\pi G k}\Big(\mathscr{L}_{\partial\omega} q_{ab} - \left(q^{cd}\,\mathscr{L}_{\partial\omega} q_{cd}\right) q_{ab}\Big)\,, \tag{4.191}$$

where $\mathscr{L}_{\partial\omega}$ denotes the Lie derivative along the vector $\partial\omega \equiv q^{ab}\partial_b\omega\partial_a$. Consequently, (4.176) behaves as a tensor under diffeomorphisms, exhibits non-linear behavior under a Weyl transformation, and remains unaffected by Lorentz boosts.

Before delving into the analysis of the symplectic structure and the associated corner charges, we compute the residual symmetry algebra using the modified Lie bracket (2.55). This calculation reveals that the algebra closes on a semi-direct sum structure, combining boundary diffeomorphisms with two Abelian sub-algebras corresponding to Weyl and Lorentz transformations,

$$\left[(Y_1, \omega_1, \eta_1)\,, (Y_2, \omega_2, \eta_2)\right]_\star = (Y_{12}, \omega_{12}, \eta_{12})\,. \tag{4.192}$$

This holds true under the condition that we define the modified parameters as follows (assuming $\delta_\xi Y = \delta_\xi\omega = \delta_\xi\eta = 0$):

$$Y_{12} = [Y_1, Y_2]\,, \tag{4.193}$$

$$\omega_{12} = Y_1^a\,\partial_a\omega_2 - Y_2^a\,\partial_a\omega_1\,, \tag{4.194}$$

$$\eta_{12} = Y_1^a\,\partial_a\eta_2 - Y_2^a\,\partial_a\eta_1\,. \tag{4.195}$$

Symplectic renormalization

In the Bondi covariant gauge, the radial component of the presymplectic potential in the Einstein-Hilbert formalism (4.1) is represented by[7]:

$$\Theta^r = r^2\,\Theta^r_{(2)} + r\,\Theta^r_{(1)} + \Theta^r_{(0)} + \mathcal{O}(r^{-1})\,, \tag{4.196}$$

where, on-shell, ($\text{Vol}_{\partial\mathcal{M}} = \frac{\sqrt{-q}}{2}\,\varepsilon_{rab}\text{d}x^a\wedge\text{d}x^b$ is the boundary volume form)

$$\Theta^r_{(2)} \approx -\frac{k}{8\pi G}\Big(\delta\ln\sqrt{-q}\Big)\,\text{Vol}_{\partial\mathcal{M}}\,, \tag{4.197}$$

$$\Theta^r_{(1)} \approx \frac{1}{16\pi Gk}\left[-2\,\frac{\delta(\Theta\sqrt{-q})}{\sqrt{-q}} - \nabla_a\delta u^a\right]\text{Vol}_{\partial\mathcal{M}}\,, \tag{4.198}$$

$$\begin{aligned}\Theta^r_{(0)} \approx \bigg(&\frac{1}{2}T^{ab}\delta q_{ab} + \frac{1}{2k\sqrt{-q}}\,\delta(\sqrt{-q}\,\varepsilon) - \frac{1}{16\pi Gk\sqrt{-q}}\,\delta(\sqrt{-q}R^{(0)})\\ &+\frac{1}{16\pi Gk^3\sqrt{-q}}\,\delta[\sqrt{-q}(\Theta^2-\Theta^{\star 2})] + \frac{1}{8\pi Gk^3}\,\nabla_a(\delta\Theta\,u^a)\\ &-\frac{1}{16\pi Gk^3}\,\nabla_a[\delta(\Theta^\star\star u^a)]\bigg)\text{Vol}_{\partial\mathcal{M}}\,.\end{aligned} \tag{4.199}$$

Thus, it diverges near the conformal boundary ($r\to\infty$). Employing the symplectic renormalization procedure for divergent terms (see section 2.4), we resolve the Iyer-Wald ambiguities (2.13) as

$$B = r^2\,B_{(2)} + r\,B_{(1)} + B_{(0)}\,,\qquad C = r\,C_{(1)} + C_{(0)}\,, \tag{4.200}$$

where the divergent pieces are

$$B_{(2)} = \frac{k}{8\pi G}\,\text{Vol}_{\partial\mathcal{M}}\,, \tag{4.201}$$

$$B_{(1)} = \frac{1}{8\pi Gk}\,\Theta\,\text{Vol}_{\partial\mathcal{M}}\,, \tag{4.202}$$

$$C_{(1)} = \frac{1}{16\pi Gk}\,\sqrt{-q}\,\varepsilon_{ab}\delta u^b\text{d}x^a\,. \tag{4.203}$$

so that the renormalized presymplectic potential reads

$$\Theta^r_{\text{ren}} \approx \lim_{r\to\infty}\,(\Theta^r + \delta B - \text{d}C)\,. \tag{4.204}$$

[7] This is not to be confused with the Θ and $\Theta^\star$ expansions of the u congruence and its $\star$u dual. The context will always clearly distinguish the interpretation associated with the same notation.

After this process of renormalization, we find logically that the renormalized portion (4.204) corresponds to the finite term in (4.199). The issue with this outcome is its failure to nullify under Dirichlet boundary conditions, a natural expectation in holography. As a result, we are confronted with a decision regarding the approach to handling the finite component of the Iyer-Wald ambiguities while ensuring resolution of this issue. This necessitates that the renormalized presymplectic potential takes the form of a linear combination of $\delta\mathrm{u}$ and $\delta\star\mathrm{u}$. Note that, similar to the Bondi gauge, the McNees-Zwikel prescription can also be employed here, but it yields a zero result for the corner ambiguity. Therefore, we need to select an appropriate prescription based on the criteria mentioned above. Given the composition of the finite part (4.199) of Θ^r, an initial reasonable choice is as follows:

$$B_{(0)} = \left(-\frac{\varepsilon}{2k} - \frac{\Theta^2 - \Theta^{\star 2} + k^2 R^{(0)}}{16\pi G k^3}\right)\mathrm{Vol}_{\partial\mathcal{M}}\,, \tag{4.205}$$

$$C_{(0)} = -\frac{\sqrt{-q}\,\varepsilon_{ab}}{8\pi G k^3}\left(u^b\delta\Theta - \frac{\delta(\star u^b\Theta^\star)}{2}\right)\mathrm{d}x^a\,, \tag{4.206}$$

resulting in a renormalized expression similar to that derived in the Fefferman-Graham framework (de Haro et al., 2001; Skenderis, 2002):

$$\Theta^{r(\mathrm{W})}_{\mathrm{ren}} \approx \frac{1}{2}T^{ab}\delta q_{ab}\mathrm{Vol}_{\partial\mathcal{M}} = \frac{1}{k^2}T^{ab}\big(-u_a\,\delta u_b + \star u_a\,\delta\star u_b\big)\mathrm{Vol}_{\partial\mathcal{M}}\,. \tag{4.207}$$

Consequently, the charges derived from this expression (4.207) coincide with the ones in (4.73), giving rise to a Weyl contribution (Alessio et al., 2021). This addition was absent in the conventional Bondi framework, and its inclusion was one of the objectives of covariant relaxation. It would be interesting to compare this outcome with the Bondi-Weyl gauge (Geiller et al., 2021), since one its aims is also to unveil the presence of Weyl rescalings in the finite charge but using a different relaxation path, focusing more on the $g_{\theta\theta}$-component. However, since the gauge transformations revealed that $\delta_\eta T_{ab} = \delta_\eta q_{ab} = 0$, it implies the absence of a Lorentz contribution. Thus we are not finished yet, and there is another step we can take.

To address this last fact, we consider another prescription for the finite term, which stems from the main concept of the symplectic renormalization method: focusing more on the corner contribution to the bulk presymplectic potential, akin to what we have done in the Weyl-Fefferman-Graham gauge. A posteriori, we shall also see that it is suggested by the Chern-Simons formulation. With this approach, we define:

$$B_{(0)} = -\frac{\varepsilon}{2k}\,\mathrm{Vol}_{\partial\mathcal{M}}\,, \qquad C_{(0)} = \frac{\sqrt{-q}\,\varepsilon_{ab}}{16\pi G k^3}\left(\delta\star u^b\,\Theta^\star - \star u^b\,\delta\Theta^\star\right)\mathrm{d}x^a\,, \tag{4.208}$$

leading to

$$\Theta^{r(\mathrm{L})}_{\mathrm{ren}} \approx \left(J^a\,\delta u_a + J^a_\star\,\delta\star u_a\right)\mathrm{Vol}_{\partial\mathcal{M}}\,, \tag{4.209}$$

where we introduce the currents:

$$J^a = -\frac{1}{k^2}\,T^{ab}u_b + \frac{1}{16\pi G k^5}\,u^a(\Theta^2 - \Theta^{\star 2}) - \frac{\varepsilon^{ab}}{8\pi G\sqrt{-q}k^3}\,\partial_b\Theta^\star\,, \tag{4.210}$$

$$J^a_\star = \frac{1}{k^2}\,T^{ab}\star u_b - \frac{1}{16\pi G k^5}\,\star u^a(\Theta^2 - \Theta^{\star 2}) + \frac{\varepsilon^{ab}}{8\pi G\sqrt{-q}k^3}\,\partial_b\Theta\,. \tag{4.211}$$

These current definitions stem from the standard holographic interpretation, involving the product of the form (VEV) multiplied by δ(source), derived from the renormalized potential (4.209). Consequently, we can group these currents to enhance the Brown-York stress tensor (Belinfante, 1940; Rosenfeld, 1940):

$$\begin{aligned}\mathscr{T}^a{}_b &= J^a u_b + J^a_\star\star u_b\\ &= T^a{}_b - \frac{\delta^a_b}{16\pi G k^3}\left(\Theta^2 - \Theta^{\star 2}\right) - \frac{\varepsilon^{ac}}{8\pi G\sqrt{-q}k^3}\left(\partial_c\Theta^\star u_b - \partial_c\Theta\star u_b\right).\end{aligned} \tag{4.212}$$

This tensor satisfies the following Ward holographic identities derived from Einstein field equations:

$$\nabla_a\mathscr{T}^{ab} = -\frac{1}{8\pi G k}\,F^{ab}A_a\,,\qquad \mathscr{T}^a{}_a = 0\,,\qquad \mathscr{T}_{[ab]} = \frac{1}{16\pi G k}\,F_{ab}\,, \tag{4.213}$$

where we observe the Weyl connection (4.167) and its curvature (4.173) alter the standard identities (4.179). However, the trace is no longer associated with the Lorentz curvature (which, as a reminder, corresponds to the Ricci curvature of the boundary (4.190)).

This type of relationship, as expressed in (4.213), has been recognized in the literature as indicative of a Lorentz anomaly in the dual theory (see, for instance, section 12.5 of (Bertlmann, 1996)), analogous to the Weyl anomaly observed in the Fefferman-Graham prescription (4.207) (Henningson & Skenderis, 1998). Verification of this aspect will occur as we delve into the variational principle associated with the symplectic structure (4.209) outlined in the symplectic renormalization prescription. As reiterated multiple times throughout this thesis, such a prescription necessitates corresponding boundary terms to be added to the bulk action. Due to the intricacies involved in holographic considerations within the Bondi framework, in particular in the flat case, as discussed and revisited in preceding chapters, we defer this discussion to a few pages from here, where we approach the topic of the Chern-Simons formulation. In that context, deriving these boundary terms becomes significantly more straightforward compared to the metric formulation. For the time being, we assume that it is possible to justify the finite prescription of Iyer-Wald ambiguities in such a way, and thus make progress under this hypothesis.

Surface charges and conformal gauge

We can proceed to determine the surface charges to discern which residual symmetries carry charge and which do not. For the previous reasons, our starting point is the symplectic form associated with the prescription (4.209):

$$\begin{aligned}\omega^{r(\mathrm{L})}_{\mathrm{ren}} &= \frac{1}{\sqrt{-q}}\Big(\delta(\sqrt{-q}\,J^a)\wedge\delta u_a + \delta(\sqrt{-q}\,J^a_\star)\wedge\delta\star u_a\Big)\mathrm{Vol}_{\partial\mathcal{M}}\\ &= \omega^{r(\mathrm{W})}_{\mathrm{ren}} + \frac{1}{8\pi G k^3}\nabla_a\left[\frac{\delta(\sqrt{-q}\,u^a)}{\sqrt{-q}}\wedge\delta\Theta - \frac{\delta(\sqrt{-q}\,\star u^a)}{\sqrt{-q}}\wedge\delta\Theta^\star\right]\mathrm{Vol}_{\partial\mathcal{M}}\,.\end{aligned} \tag{4.214}$$

We observe that this is connected by a corner term to the form associated with (4.207),

$$\omega^{r(\mathrm{W})}_{\mathrm{ren}} = \frac{1}{2\sqrt{-q}}\,\delta\big(\sqrt{-q}\,T^{ab}\big)\wedge\delta q_{ab}\mathrm{Vol}_{\partial\mathcal{M}}\,. \tag{4.215}$$

The ambiguous corner term (4.214) is notably crucial for what ensues, as it permits transitioning between prescriptions, selecting either the Weyl or Lorentz anomaly. Moreover, it facilitates achieving the flat limit as dictated in the Bondi framework, a departure from the Fefferman-Graham approach. Additionally, we will observe its natural emergence in the Chern-Simons formulation, originating from the boundary terms that connect it to the second-order one.

In the earlier calculation within the standard Bondi gauge (refer to the preceding subsection 4.3.1), we encountered complexities due to apparent non-integrabilities when computing charges. To address this and to concentrate on the new physical insights emerging from the relaxation of the covariant Bondi gauge (4.164), we simplify these discussions by opting for a suitable gauge of the boundary metric, a choice previously employed in section 4.2, as shown in the equation (4.39). This gauge selection will also enhance the transparency of interpreting the holographic anomaly when we shall handle boundary terms in the Chern-Simons bulk action.

Given that the boundary is two-dimensional, it is always feasible to locally express the boundary metric in a conformally flat form using light-cone coordinates $x^\pm = \theta \pm k\,u$:

$$\mathrm{d}s^2_{\text{bdy}} = q_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{e}^{2\phi}\mathrm{d}x^+\mathrm{d}x^-\,, \tag{4.216}$$

where the function $\phi = \phi(x^+, x^-)$ represents the conformal factor. This formulation leads to the following parametrization of the boundary zweibein $(\mathrm{u}, \star\mathrm{u})$ in the Cartan frame (4.166):

$$\mathrm{u} = -\frac{k}{2}\,\mathrm{e}^{\phi}\Big(\mathrm{e}^{\zeta}\,\mathrm{d}x^+ - \mathrm{e}^{-\zeta}\,\mathrm{d}x^-\Big)\,, \qquad \star\mathrm{u} = \frac{k}{2}\,\mathrm{e}^{\phi}\Big(\mathrm{e}^{\zeta}\,\mathrm{d}x^+ + \mathrm{e}^{-\zeta}\,\mathrm{d}x^-\Big)\,, \tag{4.217}$$

and where $\zeta(x^+, x^-)$ is another arbitrary boundary function. We shall see that ζ is linked with the Lorentz symmetry of the theory, while the conformal factor ϕ is associated with the Weyl symmetry. The Brown-York stress tensor (4.176) is represented by the following components:

$$T_{+-} = -\frac{1}{8\pi Gk}\,\partial_+\partial_-\phi\,, \qquad T_{\pm\pm} = \frac{1}{8\pi Gk}\Big(\ell_\pm + \partial_\pm^2\phi - (\partial_\pm\phi)^2\Big)\,. \tag{4.218}$$

The holomorphic and anti-holomorphic functions $\ell_\pm(x^\pm)$ are derived from ε and χ through the resolution of the associated Ward constraints (4.179) by

$$\varepsilon + \chi = \frac{\mathrm{e}^{-2(\phi-\zeta)}}{2\pi G}\Big(\ell_-(x^-) - (\partial_-\zeta)^2 - \partial_-^2\zeta - \mathrm{e}^{-2\zeta}\partial_-\partial_+\zeta\Big)\,, \tag{4.219}$$

$$\varepsilon - \chi = \frac{\mathrm{e}^{-2(\phi+\zeta)}}{2\pi G}\Big(\ell_+(x^+) - (\partial_+\zeta)^2 + \partial_+^2\zeta + \mathrm{e}^{2\zeta}\partial_-\partial_+\zeta\Big)\,. \tag{4.220}$$

The conformal gauge simplifies the residual bulk diffeomorphism gauge parameters to the following expressions:

$$Y^a\partial_a = Y^+\partial_+ + Y^-\partial_-, \tag{4.221}$$

$$\eta = -h + \frac{1}{2}\left(\partial_+Y^+ - \partial_-Y^-\right) + Y^+\partial_+\zeta + Y^-\partial_-\zeta\,, \tag{4.222}$$

$$\omega = \sigma - \frac{1}{2}\left(\partial_+Y^+ + \partial_-Y^-\right) - Y^+\partial_+\phi - Y^-\partial_-\phi\,. \tag{4.223}$$

We observe that the boundary diffeomorphisms are reduced to the set of conformal transformations parameterized infinitesimally by $Y^\pm(x^\pm)$. We intentionally introduce a field-dependent shift of the Lorentz and Weyl symmetry transformations to factorize the gauge transformations, a manipulation previously applied in subsections 4.2.1 and 4.2.2. Under these residual symmetries, the physical fields undergo gauge transformations such as

$$\delta_\xi\ell_\pm = Y^\pm\partial_\pm\ell_\pm + 2\,\partial_\pm Y^\pm\ell_\pm - \frac{1}{2}\,\partial_\pm^3 Y^\pm\,, \tag{4.224}$$

$$\delta_\xi\phi = \sigma\,, \tag{4.225}$$

$$\delta_\xi\zeta = h\,. \tag{4.226}$$

The algebra of residual symmetries (4.192) through the modified Lie bracket (2.55) is defined by the following modified parameters:

$$Y^{\pm}_{12} = Y^{\pm}_2 \partial_{\pm} Y^{\pm}_1 - Y^{\pm}_1 \partial_{\pm} Y^{\pm}_2 \,, \qquad \sigma_{12} = 0 \,, \qquad h_{12} = 0 \,, \tag{4.227}$$

which is equivalent to a double copy of the Witt algebra in direct sum with the Abelian parts of the Weyl rescalings and Lorentz boosts, $(\text{Witt} \oplus \overline{\text{Witt}}) \oplus \text{Weyl} \oplus \mathfrak{so}(1,1)$.

In the prescription (4.209), the renormalized presymplectic potential and its associated current can be simplified thanks to the conformal gauge:

$$\Theta^{r(\text{L})}_{\text{ren}} = \frac{\mathrm{e}^{-2\phi}}{8\pi G k} \, \zeta \, \delta(\mathrm{e}^{2\phi} \, F) \, \text{Vol}_{\partial\mathcal{M}} \,, \tag{4.228}$$

$$\omega^{r(\text{L})}_{\text{ren}} = \frac{\mathrm{e}^{-2\phi}}{8\pi G k} \left[\delta\zeta \wedge \delta(\mathrm{e}^{2\phi} \, F) \right] \text{Vol}_{\partial\mathcal{M}} \,, \tag{4.229}$$

where the Weyl curvature (4.173) is given by

$$F = -4 \, \mathrm{e}^{-2\phi} \partial_+ \partial_- \zeta \,. \tag{4.230}$$

This leads to the following finite, integrable but non-conserved surface charges:

$$H_\xi = \frac{1}{8\pi G k} \int_0^{2\pi} \mathrm{d}\theta \left[Y^+ \ell_+ - Y^- \ell_- + h \, (\partial_- - \partial_+) \, \zeta - \zeta \, (\partial_- - \partial_+) \, h \right] . \tag{4.231}$$

The first two terms resemble the usual Brown-Henneaux expression (Brown & Henneaux, 1986), while the last two are similar to the Fefferman-Graham Weyl sector (4.73), albeit with the Weyl symmetry replaced by the Lorentz one. Notably, charges associated with Weyl symmetries vanish identically, indicating their pure gauge nature in the covariant Bondi gauge, while a new finite charge linked with Lorentz boosts emerges compared to the standard Bondi gauge. This implies that the entire Weyl covariantization of boundary hydrogeometry discussed in appendix B.2 can be undertaken without expanding the physical content, unlike in the covariant symplectic prescription of the Weyl-Fefferman-Graham gauge discussed in subsection 4.2.2.

Since the charge mirrors (4.73), we draw similar conclusions for the charge algebra (4.74)-(4.75), with the Lorentz sector replacing the Weyl sector. Thus, the algebra remains consistent with the ones obtained in (Adami et al., 2020b; Alessio et al., 2021; Geiller et al., 2021; Adami et al., 2022, 2023), but derived from a different solution space. For further details, we refer to the discussions surrounding these equations, and to (Campoleoni et al., 2022) for explicit expressions of Poisson brackets.

Chern-Simons formulation and boundary terms

As discussed in the context of the symplectic renormalization prescription leading to (4.209), it is essential to justify the introduced Iyer-Wald ambiguities by incorporating appropriate boundary terms into the bulk action. The Chern-Simons formulation offers a more convenient approach for this purpose. A natural choice, which is manifestly Weyl-invariant, for the bulk dreibein associated with the covariant Bondi ansatz (4.164) can be expressed as follows:

$$e^1 = \frac{r\,\mathrm{u}}{k}\,, \tag{4.232}$$

$$e^{-1} = -\frac{1}{4rk}\left(-r^2\,\mathrm{u} + 2\,\mathrm{d}r + 2\,r\,\mathrm{A} + \frac{8\pi G}{k^2}\,(\varepsilon\,\mathrm{u} + \chi\,\star\mathrm{u})\right), \tag{4.233}$$

$$e^0 = \frac{r\,\star\mathrm{u}}{k}\,, \tag{4.234}$$

where we employed the following Minkowski metric (with $B, C \in \{-1, 0, 1\}$)

$$\eta_{BC} = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix}. \tag{4.235}$$

Studying the solution space and residual gauge transformations in the conformal gauge yields the same expression for charges as in (4.231). This indicates that the Chern-Simons formulation naturally selects the Lorentz prescription (4.209) through the boundary term (4.22), establishing a connection with the metric formulation.

The variation of the Chern-Simons action (4.19) on-shell simplifies to the following boundary term, expressed in Bondi coordinates:

$$\delta S = \delta S_{CS}[A] - \delta S_{CS}[\widetilde{A}] = -\frac{1}{8\pi G k}\int \mathrm{d}^2x\,\mathrm{tr}\left(A_u\,\delta A_\theta - \widetilde{A}_u\,\delta\widetilde{A}_\theta\right). \tag{4.236}$$

For the Brown-Henneaux boundary conditions (which, in the Bondi covariant context, entail Dirichlet conditions on the forms $(\mathrm{u}, \star\mathrm{u})$), ensuring a well-defined variational problem involves augmenting the bulk action with the Coussaert-Henneaux-Van Driel boundary term (Coussaert et al., 1995):

$$S_{CHVD} = -\frac{1}{16\pi G}\int \mathrm{d}^2x\,\mathrm{tr}\left(A_\theta^2 + \widetilde{A}_\theta^2\right). \tag{4.237}$$

However, in the specific case at hand, including this boundary term alone is not adequate to derive the Lorentz potential (4.209) from the on-shell variation of the action. Hence, we need to introduce an additional term:

$$S_{\text{tot}}[A, \widetilde{A}] = S_{EH} + \frac{1}{16\pi G k} \int \mathrm{d}^2x \, \mathrm{tr} \left(A_u \, A_\theta - \widetilde{A}_u \, \widetilde{A}_\theta \right). \tag{4.238}$$

Indeed, its on-shell variation yields the boundary integral of the pull-back of $\Theta^{r(\text{L})}_{\text{ren}}$:

$$\delta S_{\text{tot}}[A, \widetilde{A}] = \frac{1}{2\pi G k} \int \delta\zeta \, \mathrm{e}^{-2\phi} \, \partial_+ \partial_- \zeta \, \mathrm{Vol}_{\partial\mathcal{M}} \,. \tag{4.239}$$

This rationale supports the symplectic prescription utilized for finite ambiguities. Nevertheless, the last equation cannot be made integrable without imposing additional constraints, thereby preventing access to a well-defined variational principle, similarly to the Weyl anomaly (4.71). In terms of the enhanced stress tensor (4.212), the Weyl symmetry appears to be non-anomalous:

$$\delta_\omega S_{\text{tot}} = \int \omega \, \mathscr{T}^a{}_a \mathrm{Vol}_{\partial\mathcal{M}} = 0 \,. \tag{4.240}$$

Meanwhile, it is the Lorentz symmetry that behaves anomalously:

$$\begin{aligned} \delta_\xi S_{\text{tot}} &= \int \left[-Y^b \left(\nabla_a \mathscr{T}^a{}_b + \frac{1}{8\pi G k} F_{ab} A^a \right) + \omega \mathscr{T}^a{}_a + \frac{\eta}{\sqrt{-q}} \, \varepsilon^{ab} \mathscr{T}_{[ab]} \right] \mathrm{Vol}_{\partial\mathcal{M}} \\ &= \frac{1}{8\pi G k} \int \eta \, F \, \mathrm{Vol}_{\partial\mathcal{M}} \,. \end{aligned} \tag{4.241}$$

These observations align with the conclusions drawn from the holographic Ward identities (4.213) and the charge calculation (4.231).

In essence, the covariant Bondi gauge facilitates shifting the anomaly from the Weyl to the Lorentz sector relative to the Fefferman-Graham gauge, within a Bondi-style framework conducive to the possibility of realizing a smooth flat limit. This shift can be comprehended from a cohomological perspective, and further insights can be found in the appendix of (Campoleoni et al., 2022).

4.4. Flat limit

In this last section, we finally delve into the flat limit of the Bondi framework, as explored in the preceding section 4.3 for the asymptotically AdS case. This approach enables us to effectively address the asymptotic symmetries of asymptotically flat spaces, a topic that has garnered significant interest in recent years due to its connections with soft theorems, memory effects (Barnich &

Compere, 2007; Barnich & Troessaert, 2010; Campiglia & Laddha, 2014; Strominger, 2018), as well as celestial holography (Pasterski, 2019; Raclariu, 2021; Pasterski et al., 2021), and Carrollian theories (Duval et al., 2014b; Hartong, 2015; Bagchi et al., 2016; Ciambelli et al., 2018b). We will explore the latter aspect further in appendix B.2.2, dedicated to interpreting the flat covariant Bondi gauge boundary in terms of conformal Carrollian fluid data.

Starting with subsection 4.4.1, we initially address the flat limit within the AdS Bondi gauge, an extension of the discussion in part 4.3.1. Subsequently, in subsection 4.4.2, we transition to the covariant relaxation, building upon the ultrarelativistic limit of the quantities introduced in 4.3.2.

4.4.1 Bondi

Solution space

The Bondi gauge setting (4.128) is valid regardless of the sign of the cosmological constant. In this subsection, we consider the Ricci-flat limit, i.e. for $\ell \to \infty$, describing an asymptotically flat space. The boundary conditions (4.129) are tailored and adapted to describe a Minkowski space perturbation in Bondi coordinates (3.14),

$$\beta \sim \mathcal{O}(1)\,, \qquad U \sim \mathcal{O}(1)\,, \qquad \frac{V}{r} \sim \mathcal{O}(r)\,. \tag{4.242}$$

To determine the asymptotic solutions, we can either repeat the steps outlined in subsection 4.3.1, or more efficiently, derive them directly from the relations (4.130) to (4.134), explicitly stated as

$$\beta = \beta_0(u,\theta)\,, \tag{4.243}$$

$$U = U_0(u,\theta) + \frac{2}{r}\mathrm{e}^{2\beta_0-2\phi}\partial_\theta\beta_0 - \frac{1}{r^2}\mathrm{e}^{2\beta_0-2\phi}N(u,\theta)\,, \tag{4.244}$$

$$\begin{aligned}\frac{V}{r} &= -2r(\partial_u\phi + \partial_\theta U_0 + U_0\partial_\theta\phi) + M(u,\theta) \\ &\quad + \frac{4}{r}\mathrm{e}^{2\beta_0-2\phi}N\partial_\theta\beta_0 - \frac{1}{r^2}\mathrm{e}^{2\beta_0-2\phi}N^2\,,\end{aligned} \tag{4.245}$$

and

$$(\partial_u + \partial_u\phi)N = \frac{1}{2}\left(\partial_\theta + 2\partial_\theta\beta_0\right)M - 2N\partial_\theta U_0 - U_0(\partial_\theta N + N\partial_\theta\phi) \\ + 4\mathrm{e}^{2\beta_0-2\phi}\left[2(\partial_\theta\beta_0)^3 - (\partial_\theta\phi)(\partial_\theta\beta_0)^2 + (\partial_\theta\beta_0)(\partial_\theta^2\beta_0)\right], \tag{4.246}$$

$$\partial_u M = \left(2\partial_u\beta_0 - 2\partial_u\phi - 2\partial_\theta U_0 + U_0 2\partial_\theta\beta_0 - U_0 2\partial_\theta\phi - U_0\partial_\theta\right)M \\ - 2\mathrm{e}^{2\beta_0-2\phi}\{\partial_\theta U_0[8(\partial_\theta\beta_0)^2 - 4\partial_\theta\beta_0\partial_\theta\phi + (\partial_\theta\phi)^2 + 4\partial_\theta^2\beta_0 \\ - 2\partial_\theta^2\phi] - \partial_\theta^3 U_0 + U_0[\partial_\theta\beta_0(8\partial_\theta^2\beta_0 - 2\partial_\theta^2\phi) + \partial_\theta\phi(-2\partial_\theta^2\beta_0 \\ + \partial_\theta^2\phi) + 2\partial_\theta^3\beta_0 - \partial_\theta^3\phi] + 2\partial_u\partial_\theta\beta_0(4\partial_\theta\beta_0 - \partial_\theta\phi) \\ + \partial_u\partial_\theta\phi(-2\partial_\theta\beta_0 + \partial_\theta\phi) + 2\partial_u\partial_\theta^2\beta_0 - \partial_u\partial_\theta^2\phi\}\,. \tag{4.247}$$

While these expressions are derived similarly, it is essential to highlight the distinct nature of the underlying physics. We can already see that this distinction is evident from the Bondi mass and Bondi momentum time constraints, which differ in the flat limit. As we delve deeper into investigating the residual diffeomorphisms, we shall observe this disparity more clearly. Actually, the boundary metric (4.135) degenerates as $\ell \to \infty$, indicating that the timelike asymptotic boundary becomes a null infinity. Consequently, on such a degenerate space, the geometry assumes a Carrollian nature, a concept we will elaborate on in subsection 4.4.2 and its associated Carrollian hydrogeometry in appendix B.2.2.

Residual symmetries and algebra

Determining residual diffeomorphisms involves taking the limit $\Lambda \to 0$ of AdS analogues, (4.136) to (4.138). Since the latter expressions are independent of the cosmological constant, we obtain identical expressions for the vector $\xi = \xi^\mu\partial_\mu$ along with the same modified algebra. Regarding the gauge transformations (4.139)-(4.143) of the physical fields, although they are functions of ℓ, their flat limit can be taken smoothly and reads

$$\delta_\xi\phi = \sigma\,, \tag{4.248}$$

$$\delta_\xi\beta_0 = (f\partial_u + Y\partial_\theta)\beta_0 + \frac{1}{2}\left(\partial_u - \partial_u\phi + 2U_0\partial_\theta\right)f - \frac{1}{2}(\partial_\theta Y + Y\partial_\theta\phi - \sigma)\,, \tag{4.249}$$

$$\delta_\xi U_0 = (f\partial_u + Y\partial_\theta - \partial_\theta Y)U_0 + U_0(\partial_u f + U_0\partial_\theta f)\,, \tag{4.250}$$

and

$$\begin{aligned}\delta_\xi N = &(f\partial_u + Y\partial_\theta + 2\partial_\theta Y + f\partial_u\phi + Y\partial_\theta\phi - \sigma - 2U_0\partial_\theta f)N \\ &+ M\partial_\theta f - \mathrm{e}^{2\beta_0-2\phi}[3\partial_\theta^2 f(2\partial_\theta\beta_0 - \partial_\theta\phi) + \partial_\theta f(4(\partial_\theta\beta_0)^2 \\ &- 8\partial_\theta\beta_0\partial_\theta\phi + 2(\partial_\theta\phi)^2 + 2\partial_\theta^2\beta_0 - \partial_\theta^2\phi) + \partial_\theta^3 f]\,,\end{aligned} \tag{4.251}$$

$$\begin{aligned}\delta_\xi M = &f\partial_u M + \partial_\theta M Y + (\partial_u f + f\partial_u\phi + \partial_\theta Y + Y\partial_\theta\phi - \sigma)M \\ &- 2\mathrm{e}^{2\beta_0-2\phi}\Big[2\partial_\theta^2 f\partial_u\beta_0 + 4\partial_u\partial_\theta f\partial_\theta\beta_0 + \partial_u\partial_\theta^2 f + \partial_\theta^2 f\partial_\theta U_0 \\ &+ 8\partial_\theta^2 f\partial_\theta\beta_0 U_0 + \partial_\theta f\Big((4\partial_\theta\beta_0 - \partial_\theta\phi)(2\partial_u\beta_0 - \partial_u\phi) + 4\partial_u\partial_\theta\beta_0 \\ &+ \partial_\theta U_0(8\partial_\theta\beta_0 - 2\partial_\theta\phi) - \partial_\theta^2 U_0 - 2\partial_u\partial_\theta\phi + U_0(-4\partial_\theta\beta_0\partial_\theta\phi \\ &+ 8(\partial_\theta\beta_0)^2 + 4\partial_\theta^2\beta_0 + (\partial_\theta\phi)^2 - 2\partial_\theta^2\phi)\Big) - 2\partial_\theta^2 f U_0\partial_\theta\phi + \partial_\theta^3 f U_0 \\ &- \partial_u\partial_\theta f\partial_\theta\phi - \partial_\theta^2 f\partial_u\phi - 2f\partial_\theta\beta_0\partial_u\partial_\theta\phi + 2\partial_\theta\beta_0\partial_\theta\sigma - 2\partial_\theta\beta_0\partial_\theta Y\partial_\theta\phi \\ &- 2\partial_\theta\beta_0\partial_\theta^2 Y - 2\partial_\theta\beta_0 Y\partial_\theta^2\phi\Big]\,.\end{aligned} \tag{4.252}$$

While the diffeomorphisms exhibit the same form, their physical implications differ. Indeed, when imposing the Dirichlet conditions (4.147), the residual gauge parameters are further constrained by:

$$\partial_u f = \partial_\theta Y\,, \qquad \partial_u Y = 0\,, \qquad \sigma = 0\,, \tag{4.253}$$

which are the flat limit counterparts of their AdS analogues. Notably, these constraints offer a unique interpretation of the conformal Killing vectors induced at the boundary in the flat scenario. This interpretation arises from directly solving these constraints:

$$Y = Y(\theta)\,, \qquad f = T(\theta) + u\,\partial_\theta Y\,. \tag{4.254}$$

where one can therefore understand that $T(\theta)$ represents the generators of supertranslations and $Y(\theta)$ of superrotations. These solutions characterize the residual symmetries, known as the BMS_3 algebra. This is the subtlety of asymptotically flat spaces, in the sense that one would naively expect to recover the group of isometries of Minkowski space, i.e. the Poincaré group, but this is not the case. Indeed, in addition to the latter, we obtain generalized translations and rotations (called supertranslations and superrotations, respectively) whose direction depends on the particular point from which they are viewed on the circle at infinity.

Surface charges and algebra

In subsection 4.3.1, the symplectic renormalization procedure we explored yields a renormalized presymplectic potential (4.149) and its associated non-integrable charge (4.154), both independent of the AdS radius. Taking the limit $\ell \to \infty$ at these levels thus becomes straightforward. Additionally, redefining the gauge parameters as in (4.155) removes the apparent non-integrability, resulting in (4.156). It is worth emphasizing once more that the underlying physics differs from the AdS case. Specifically, in the flat case, the charge algebra (4.162) yields a centrally extended BMS_3 group algebra under Dirichlet conditions (4.147), instead of a double copy of the Virasoro algebra.

Moving forward, in the next subsection, we will delve into a relaxation of this Bondi flat gauge. This relaxation arises from the flat limit of the covariant Bondi gauge designed for asymptotically AdS spacetimes. It involves a similar relaxation of the condition on the component $g_{r\theta}$.

4.4.2 Covariant Bondi

The Ricci-flat limit of the AdS covariant Bondi gauge can be taken to describe an equivalent for asymptotically flat spaces. Similar to the case in AdS, the described flat gauge is a relaxation of the Bondi gauge, where the $g_{r\theta}$-component is relaxed. The appendix B.2.1 demonstrates that the AdS ansatz (4.164) can be interpreted from the dual point of view to a relativistic Weyl hydrogeometry at the timelike boundary. Additionally, in appendix B.2.2, it will be shown that its limit as $\ell \to \infty$ (i.e., $k \to 0$) corresponds to the description of a conformal Carrollian fluid living at null infinity.

Solution space

The foundation of the covariant Bondi gauge rests on the fluid/gravitational approach. Accordingly, we establish the prescribed fall-offs in k for various quantities present in the bulk metric (4.164) of AdS (Campoleoni et al., 2019a; Ciambelli et al., 2020a,b; Campoleoni et al., 2022):

$$\mu = \lim_{k\to 0} \frac{\mathrm{u}}{k^2}\,, \qquad \mu^\star = \lim_{k\to 0} \frac{\star\mathrm{u}}{k}\,, \qquad \upsilon = \lim_{k\to 0} u\,, \qquad \upsilon_\star = \lim_{k\to 0} \frac{\star u}{k}\,, \qquad (4.255)$$

where $\mathrm{u} = u_a \mathrm{d}x^a$ denotes forms and $u = u^a\partial_a$ denotes vectors associated to the fluid velocity, with analogous notation for the dual $\star\mathrm{u}$ and $\star u$. Scalars exhibit the following behavior for small values of k:

$$\alpha = \lim_{k\to 0} \frac{\chi}{k}\,, \qquad \varepsilon = \lim_{k\to 0} \varepsilon\,. \qquad (4.256)$$

This will be justified and substantiated in the appendix B.2.2.

In this scenario, the boundary metric in the Cartan frame (4.166) becomes degenerate in the limit, akin to what we observed with the standard Bondi gauge:

$$\mathrm{d}s^2_{\mathscr{C},\mathrm{bdy}} = \lim_{k\to 0} \mathrm{d}s^2_{\mathrm{bdy}} = (\upmu^\star)^2 \,. \tag{4.257}$$

This degeneracy induces a Carrollian structure (Duval et al., 2014a), as previously indicated, and as we shall elucidate further from a hydrodynamic perspective. Within this framework, the pair $(\upmu, \upmu^\star)$ forms a zweibein. With the degeneracy in mind, we have chosen in (4.255) distinct notations for forms and vectors due to their lack of direct connection under the boundary metric (4.257) application. Specifically, we should rather consider them as linked as follows:

$$\upmu(\upsilon) = -1\,, \qquad \upmu^\star(\upsilon_\star) = 1\,, \qquad \upmu(\upsilon_\star) = 0\,, \qquad \upmu^\star(\upsilon) = 0\,. \tag{4.258}$$

Applying Cartan's structure equations to the Carrollian zweibein yields

$$\mathrm{d}\upmu = \uptheta^\star \star \upmu \wedge \upmu\,, \qquad \mathrm{d} \star \upmu = \uptheta \star \upmu \wedge \upmu\,. \tag{4.259}$$

In analogy to the AdS equivalent (4.168), we denote the Carrollian expansions as $\uptheta$ and $\uptheta^\star$, derived from the latter via the following ultrarelativistic limit according to the scalings (4.255):

$$\uptheta = \lim_{k\to 0} \Theta\,, \qquad \uptheta^\star = \lim_{k\to 0} \frac{\Theta^\star}{k}\,. \tag{4.260}$$

This collective treatment implies the following small-k behaviors and expressions for the Carrollian counterparts of the Weyl connection (4.167) and its curvature (4.173):

$$\mathcal{A} = \lim_{k\to 0} \mathrm{A} = \upmu^\star \uptheta^\star - \upmu\uptheta\,, \qquad \mathcal{F}_{\mu\nu} = \lim_{k\to 0} F_{\mu\nu}, \qquad \mathcal{F} = \lim_{k\to 0} k\, F\,. \tag{4.261}$$

In appendix B.2.2, we will further demonstrate that this form transforms as a connection under the Weyl rescalings of the Carroll structure.

Injecting these relations into the AdS ansatz (4.164) and taking the flat limit $k \to 0$ yields a finite line element describing the covariant Bondi gauge of asymptotically flat spacetimes:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 2\upmu\,(\mathrm{d}r + r\mathcal{A}) + r^2\,(\upmu^\star)^2 + 8\pi G\,\upmu\,(\epsilon\,\upmu + \alpha\,\upmu^\star)\,. \tag{4.262}$$

One can explore the asymptotic solution space by solving the Einstein equations (4.5) based on this metric. Alternatively, similarly to the approach for

Bondi, we can take the ultrarelativistic limit of the Brown-York stress tensor conservation equations (4.174)-(4.175) for a more efficient analysis. This result is already finite in k and imposes constraints on the scalars ϵ and α:

$$\upsilon^a(\partial_a + 2\mathcal{A}_a)\,\epsilon = \frac{1}{4\pi G}\,\upsilon^a_\star(\partial_a + 2\mathcal{A}_a)\,\mathcal{F}\,, \tag{4.263}$$

$$\upsilon^a(\partial_a + 2\mathcal{A}_a)\,\alpha = -\upsilon^a_\star(\partial_a + 2\mathcal{A}_a)\,\epsilon\,. \tag{4.264}$$

Residual symmetries and algebra

Identifying the diffeomorphisms of the bulk metric (4.262) or equivalently realizing the Ricci-flat limit of the AdS Killing vectors (4.181)-(4.182), we determine the following components of the residual symmetries $\xi = \xi^\mu\partial_\mu$ of the flat covariant Bondi gauge:

$$\xi^r = r\,\omega + \upsilon^a_\star\partial_a\lambda + \theta^\star\lambda + \frac{4\pi G}{r}\alpha\,\lambda\,, \tag{4.265}$$

$$\xi^a = Y^a - \frac{1}{r}\lambda\,\upsilon^a_\star\,. \tag{4.266}$$

To understand the physical interpretation of the above four gauge parameters (Y^a, ω, λ), we consider the Carrollian zweibein transformation under these residual bulk diffeomorphisms:

$$\delta_\xi \mu = \mathscr{L}_Y \mu + \omega\,\mu + \lambda\,\mu^\star\,, \tag{4.267}$$

$$\delta_\xi \mu^\star = \mathscr{L}_Y \mu^\star + \omega\,\mu^\star\,. \tag{4.268}$$

Here, the vector Y^a remains unchanged from the AdS perspective and still represents boundary diffeomorphisms, as does the function ω, which parameterizes boundary Weyl rescalings.

We also note the presence of the function λ, which plays a special role in relation to the other symmetries: it solely affects the form μ and not its dual part. In fact, this function serves as the Carrollian counterpart of Lorentz transformations:

$$\lambda(x) = \lim_{k\to 0}\frac{\eta(x)}{k}\,. \tag{4.269}$$

Let us take a look at how to better understand it. Firstly, similarly to Lorentz transformations, this symmetry arises because the boundary metric (4.257) is formulated in terms of the Carrollian zweibein in two dimensions. As only the component $\mu^\star$ is involved, variations in λ do not affect this expression. Secondly, understanding the transformation of μ under this infinitesimal local Carroll boost involves considering its finite version. This transformation entails a constant spatial vector $\vec{\lambda}$, representing a coordinate shift denoted by $u \to u+\vec{\lambda}\cdot\vec{x}$[8]. In two dimensions, the infinitesimal version materializes as $\delta u = \lambda\delta\theta$, leading to $\delta_\lambda \upmu = \lambda \upmu^\star$.

We proceed to derive the gauge transformations for the remaining physical fields of the asymptotic solution space (see (4.191) and (4.256)):

$$\delta_\xi \epsilon = \mathscr{L}_Y \epsilon - 2\,\omega\,\epsilon - \frac{1}{4\pi G}\left(\uptheta\, \upsilon^a_\star\, \partial_a\lambda + \upsilon^a\, \partial_a(\upsilon^b_\star\, \partial_b\lambda) - \mathcal{F}\,\lambda\right), \tag{4.270}$$

$$\delta_\xi \alpha = \mathscr{L}_Y \alpha - 2\,\omega\,\alpha - 2\,\lambda\,\epsilon + \frac{1}{4\pi G}\left(\uptheta^\star\, \upsilon^a_\star\, \partial_a\lambda + \upsilon^a_\star\, \partial_a(\upsilon^b_\star\, \partial_b\lambda)\right). \tag{4.271}$$

Additionally, we present the algebra of residual symmetries, which retains the same expression as in the relativistic case (4.192):

$$Y_{12} = [Y_1, Y_2]\,, \tag{4.272}$$

$$\omega_{12} = Y_1^a\, \partial_a\omega_2 - Y_2^a\, \partial_a\omega_1\,, \tag{4.273}$$

$$\lambda_{12} = Y_1^a\, \partial_a\lambda_2 - Y_2^a\, \partial_a\lambda_1\,, \tag{4.274}$$

where we assumed again that $\delta_\xi Y = \delta_\xi\omega = \delta_\xi\lambda = 0$. However, it is essential to recall that, although the algebra form remains identical, the Bondi flat analysis has provided insights into the different interpretations we must understand from it.

Symplectic renormalization

Having detailed the Ricci-flat limit prescriptions concerning the solution space and residual diffeomorphisms, we can extend this limit to the symplectic renormalization in the covariant Bondi formalism of AdS, mirroring our approach for the standard Bondi gauge. This results in a renormalized expression of the presymplectic potential (4.209), with a finite Iyer-Wald ambiguity fixation induced by the presymplectic relativistic procedure. This facilitates a smooth transition to the flat limit, unlike the Fefferman-Graham-like prescription (4.207).

Thus, for asymptotically flat spaces, the renormalized presymplectic potential takes the form ($\mathrm{vol}_{\partial\mathcal{M}} = \lim_{k\to 0}\frac{\mathrm{Vol}_{\partial\mathcal{M}}}{k}$ is the boundary volume form on null infinity)

$$\Theta^{r(\mathrm{C})}_{\mathrm{ren}} = \lim_{k\to 0}\Theta^{r(\mathrm{L})}_{\mathrm{ren}} \approx (j^a\delta\mu_a + j^a_\star\delta\mu^\star_a)\,\mathrm{vol}_{\partial\mathcal{M}}, \tag{4.275}$$

[8] We emphasize that the notation u in this relation pertains to the Bondi retarded time coordinate, not the congruence vector.

where, in line with the holographic dictionary, we establish the flat analogs of the AdS holographic currents (4.210)-(4.211):

$$j = \lim_{k\to 0} k^3 J = \frac{1}{2}\,\epsilon\,\upsilon + \frac{1}{8\pi G}\,\upsilon_\star\,\mathcal{F}\,, \tag{4.276}$$

$$j_\star = \lim_{k\to 0} k^2 J_\star = \frac{1}{2}\,\epsilon\,\upsilon_\star + \frac{1}{2}\,\alpha\,\upsilon\,. \tag{4.277}$$

We can consolidate these currents into a Carrollian stress tensor, which serves as the ultrarelativistic counterpart to the enhancement of the Brown-York holographic energy-momentum tensor (4.212) à la Belinfante-Rosenfeld. This tensor is defined by the following expression, where the appropriate scaling in k is determined by the earlier specified prescriptions for various quantities:

$$t^a{}_b = \lim_{k\to 0} k\mathscr{T}^a{}_b = j^a \mu_b + j^a_\star \mu^\star_b\,. \tag{4.278}$$

According to Einstein field equations, the Carrollian tensor must satisfy the holographic Ward identities outlined below. These identities can also be derived as Ricci-flat limits of their AdS counterparts (4.213):

$$D_a t^a{}_b = -\frac{1}{8\pi G}\,\mathcal{F}_{ab}\,\mathcal{A}^a\,, \qquad t^a{}_a = 0\,, \qquad t^a{}_b \mu^\star_a\,\upsilon^b = -\frac{\mathcal{F}}{8\pi G}\,, \tag{4.279}$$

where

$$D_a t^a{}_b = \lim_{k\to 0} \nabla_a \mathscr{T}^a{}_b\,. \tag{4.280}$$

Surface charges and conformal gauge

In the symplectic renormalization procedure outlined in (4.209), the associated symplectic form also possesses a smooth, flat limit. Specifically, in the covariant Bondi flat gauge, it reads

$$\omega^{r(\mathrm{C})}_{\mathrm{ren}} = \lim_{k\to 0} \omega^{r(\mathrm{L})}_{\mathrm{ren}} = \mathcal{D}^{-1}\Big(\delta(\mathcal{D}\, j^a) \wedge \delta\mu^\star_a + \delta(\mathcal{D}\, j^a_\star) \wedge \delta\mu_a\Big)\mathrm{vol}_{\partial\mathcal{M}}\,, \tag{4.281}$$

where the density $\mathcal{D}$ is defined as

$$\mathcal{D} = \lim_{k\to 0} \frac{\sqrt{-q}}{k}\,. \tag{4.282}$$

In continuation of steps akin to the subsection 4.3, we proceed with determining asymptotic surface charges within the conformal gauge of the boundary metric. For establishing the Carrollian analogue, we choose appropriate boundary coordinates $(x^a) = (u, \theta)$, aligning the dual part of the zweibein with the angular coordinate. Subsequently, we consider the following expressions for the pair of forms $(\mu, \mu^\star)$:

$$\mu = -\mathrm{e}^{\phi}\,(\mathrm{d}u + \beta\,\mathrm{d}\theta)\,, \qquad \mu^\star = \mathrm{e}^{\phi}\,\mathrm{d}\theta\,, \tag{4.283}$$

where $\phi(u,\theta)$ and $\beta(u,\theta)$[9] represent arbitrary boundary functions. The former corresponds exactly to the conformal factor as in AdS, while the latter is associated with symmetry under Carroll boosts. These expressions (4.283) are equivalent to specifying the behavior $\zeta = k\beta$ in the AdS conformal parameterization of the zweibein (4.217) and then realizing the limit $k \to 0$ following the scalings (4.255). In this gauge, solving the equations constraining the scalars ϵ and α yields the following analytical expressions:

$$\epsilon = \frac{\mathrm{e}^{-2\phi}}{8\pi G}\Big(8\pi G\,\varepsilon_0 - (\partial_u\beta)^2 + 2\,\partial_u\partial_\theta\beta - 2\,\beta\,\partial_u^2\beta\Big)\,, \tag{4.284}$$

$$\begin{aligned}\alpha = \frac{\mathrm{e}^{-2\phi}}{4\pi G}\Big(&4\pi G(\alpha_0 - u\,\partial_\theta\varepsilon_0 + 2\,\beta\varepsilon_0) - \beta\left[(\partial_u\beta)^2 - 2\,\partial_u\partial_\theta\beta + \beta\,\partial_u^2\beta\right]\\ &- \partial_\theta^2\beta + \partial_u\beta\,\partial_\theta\beta\Big)\,.\end{aligned} \tag{4.285}$$

Here, we defined the fields $\varepsilon_0 = \varepsilon_0(\phi)$ and $\alpha_0 = \alpha_0(\phi)$, being obtained from the (anti-)holomorphic functions $\ell_\pm(x^\pm)$ in AdS as

$$\varepsilon_0(\theta) = \frac{1}{4\pi G}\lim_{k\to 0}\left(\ell_+ + \ell_-\right)\,, \tag{4.286}$$

$$\alpha_0(\theta) - u\,\partial_\theta\varepsilon_0(\theta) = -\frac{1}{4\pi G}\lim_{k\to 0}\frac{\ell_+ - \ell_-}{k}\,. \tag{4.287}$$

Similarly, the conformal gauge further simplifies the residual gauge parameters, derived from the scalings of their AdS counterparts, as follows:

$$Y^\pm(x^\pm) = Y(\theta) \pm k\,(H(\theta) + u\,\partial_\theta Y(\theta))\,, \qquad h(x^+,x^-) = k\,\tilde{h}(u,\theta)\,, \tag{4.288}$$

where the parameters (ω,λ) have been redefined in a field-dependent manner to $(\sigma,\tilde{h})$, ensuring the following transformations of physical fields under bulk diffeomorphisms:

$$\delta_\xi\varepsilon_0 = Y\,\partial_\theta\varepsilon_0 + 2\,\varepsilon_0\,\partial_\theta Y - \frac{1}{4\pi G}\,\partial_\theta^3\,Y\,, \tag{4.289}$$

$$\delta_\xi\alpha_0 = Y\,\partial_\theta\alpha_0 + 2\,\alpha_0\,\partial_\theta Y - H\,\partial_\theta\varepsilon_0 - 2\,\varepsilon_0\,\partial_\theta H + \frac{1}{4\pi G}\,\partial_\theta^3\,H\,, \tag{4.290}$$

$$\delta_\xi\phi = \sigma\,, \tag{4.291}$$

$$\delta_\xi\beta = \tilde{h}\,. \tag{4.292}$$

[9] Although this uses the same notation, it has nothing to do with the function appearing

In fact, the various sectors do not mix. The above reduction leads the algebra of residual symmetries to a direct sum of three-dimensional BMS transformations, Weyl rescalings, and Carroll boosts, which are essentially local $\mathbb{R}$-transformations:

$$Y_{12} = Y_2\,\partial_\theta Y_1 - Y_1\,\partial_\theta Y_2\,, \tag{4.293}$$
$$H_{12} = H_2\,\partial_\theta Y_1 - H_1\,\partial_\theta Y_2 + Y_2\,\partial_\theta H_1 - Y_1\,\partial_\theta H_2\,, \tag{4.294}$$
$$\sigma_{12} = 0\,, \tag{4.295}$$
$$\tilde{h}_{12} = 0\,. \tag{4.296}$$

Using the results derived from the renormalized presymplectic potential in the Carrollian prescription and the diffeomorphisms in the conformal gauge, we obtain the following finite, integrable, and non-conserved surface charges:

$$H_\xi = \frac{1}{2}\int_0^{2\pi} \mathrm{d}\theta \left[\frac{1}{2}\Big(H\,\varepsilon_0 - Y\,\alpha_0\Big) + \frac{1}{8\pi G}\Big(\partial_u\tilde{h}\,\beta - \tilde{h}\,\partial_u\beta\Big)\right]. \tag{4.297}$$

The initial two terms align with the three-dimensional BMS charges (Barnich & Compere, 2007), derived from the Ricci-flat limit of AdS Virasoro charges. Conversely, the latter two terms serve as the Carrollian counterpart to the Lorentzian boost charges depicted in (4.231). This introduces a fresh perspective within the covariant Bondi gauge for asymptotically flat spaces. Furthermore, it is noteworthy to mention once more the omission of both the conformal factor and its Weyl rescaling parameter from (4.297).

Expanding the physical fields within this gauge into Fourier series,

$$\varepsilon_0(\theta) = -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} T_n\,\mathrm{e}^{-\mathrm{i}n\theta}\,, \tag{4.298}$$
$$\alpha_0(\theta) = -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} Y_n\,\mathrm{e}^{-\mathrm{i}n\theta}\,, \tag{4.299}$$
$$B(u,\theta) = -\frac{1}{2\pi}\sum_{p,q\in\mathbb{Z}} B_{pq}\,\mathrm{e}^{-\mathrm{i}(p-q)\theta}\,\mathrm{e}^{-\mathrm{i}(p+q)u}\,, \tag{4.300}$$

we redefine the β-function of Carroll local boosts as follows:

$$B(u,\theta) = \frac{1}{4\pi G}\,\partial_u\beta(u,\theta)\,. \tag{4.301}$$

in the standard Bondi gauge setting.

This expansion leads to the following charge algebra:

$$\mathrm{i}\,\{Y_n, Y_m\} = (n-m)\, Y_{n+m}\,, \tag{4.302}$$

$$\mathrm{i}\,\{Y_n, T_m\} = (n-m)\, T_{n+m} + \frac{\tilde{c}}{12}\, n^3\, \delta_{n+m,0}\,, \tag{4.303}$$

$$\mathrm{i}\,\{B_{pq}, B_{rs}\} = -\frac{\tilde{c}}{6}\,(r-q)\, \mathrm{e}^{2\mathrm{i}(q+s)u}\, \delta_{p+r,q+s}\,. \tag{4.304}$$

The algebra formed by Y_n and T_n embodies a $\mathfrak{bms}_3$ structure with a central charge denoted as $\tilde{c}$, given by

$$\tilde{c} = \frac{3}{G}\,. \tag{4.305}$$

Remarkably, the central extension in the Carroll boosts sector exhibits an explicit dependence on the temporal coordinate u, echoing a pattern seen in the AdS context. As a result, the asymptotic symmetries algebra described earlier appears as a one-parameter array of algebras, a trait attributed to similar underlying principles elucidated in the AdS analysis (4.74)-(4.75).

Chern-Simons formulation and boundary terms

Finally, we support the symplectic renormalization prescription, which leads to Carroll boosts, via the Chern-Simons formulation. This step is crucial for determining the correct boundary counterterms to supplement the bulk action. To achieve this, we utilize the $\mathfrak{iso}(1,2)$ Chern-Simons connection:

$$\mathscr{A} = b^{-1}\left[\widetilde{\mathscr{A}} + \mathrm{d}\right] b\,, \tag{4.306}$$

where $\widetilde{\mathscr{A}} = \widetilde{\mathscr{A}_a}(u,\theta)\mathrm{d}x^a$ and b represents the $ISO(1,2)$ group element:

$$b(r) = \exp\left(\frac{r}{2}\, P_{-1}\right). \tag{4.307}$$

We expressed the latter using a convenient basis of the Poincaré algebra:

$$\begin{aligned} [M_B, M_C] &= (B-C)\, M_{B+C}\,, \\ [M_B, P_C] &= (B-C)\, P_{B+C}\,, \\ [P_B, P_C] &= 0\,, \end{aligned} \tag{4.308}$$

with $B, C \in \{-1, 0, 1\}$. In this same basis, the components of the boundary connection can be chosen as follows, in the conformal gauge (4.217):

$$\begin{aligned}\widetilde{\mathscr{A}_\theta} &= \frac{\mathrm{e}^{-\phi}}{\sqrt{2}}\Big(4\pi G\,\varepsilon_0 - \frac{1}{2}\,(\partial_u\beta)^2 + \partial_u\partial_\theta\beta\Big)\,M_1 - \Big(\partial_\theta\phi - \partial_u\beta\Big)\,M_0 \\ &\quad - \frac{\mathrm{e}^{\phi}}{\sqrt{2}}\,M_{-1} + \frac{\mathrm{e}^{\phi}\,\beta}{\sqrt{2}}\,P_{-1} + \frac{\mathrm{e}^{-\phi}}{\sqrt{2}}\Big(4\pi G\,(\alpha_0 - u\,\partial_\theta\varepsilon_0) - \partial_\theta^2\beta \\ &\quad + \partial_u\beta\,\partial_\theta\beta + \frac{\beta}{2}\,\big(8\pi G\,\varepsilon_0 - (\partial_u\beta)^2 + 2\,\partial_u\partial_\theta\beta\big)\Big)\,P_1\,,\end{aligned} \tag{4.309}$$

$$\begin{aligned}\widetilde{\mathscr{A}_u} &= \frac{\mathrm{e}^{-\phi}}{\sqrt{2}}\Big[\partial_u^2\beta\,M_1 - \Big(4\pi G\,\varepsilon_0 - \frac{1}{2}\,(\partial_u\beta)^2 + \partial_u\partial_\theta\beta - \beta\,\partial_u^2\beta\Big)\,P_1\Big] \\ &\quad - \partial_u\phi\,M_0 + \frac{\mathrm{e}^{\phi}}{\sqrt{2}}\,P_{-1}\,.\end{aligned} \tag{4.310}$$

Studying the solution space (4.25) and residual gauge transformations (4.29) from these Chern-Simons connections yields a charge identical to (4.297). This indicates that the boundary term (4.22), which connects the Chern-Simons formulation to the metric one, naturally favors the Carroll-Lorentz prescription (4.275). This prescription is obtained by taking the $k \to 0$ limit of (4.209). Following a similar approach to the analysis of the variational principle in the first-order formalism as described above, one arrives at the correct boundary term to add to the bulk action:

$$S_{\text{bdy}}[\mathscr{A}] = \frac{1}{8\pi G}\int \mathrm{d}^2x\,\mathrm{Tr}\big(\mathscr{A}_\theta\,\mathscr{A}_u\big)\,. \tag{4.311}$$

Consequently, the on-shell variation of the total action justifies the symplectic prescription by reproducing $\Theta^{r(\mathrm{C})}_{\text{ren}}$ via its pull-back integrated on the boundary:

$$\delta S_{\text{tot}}[\mathscr{A}] = \delta S_{EH}[\mathscr{A}] + \delta S_{\text{bdy}}[\mathscr{A}] \approx -\frac{1}{8\pi G}\int \delta\beta\,\mathrm{e}^{-2\phi}\,\partial_u^2\beta\,\mathrm{vol}_{\partial\mathcal{M}}\,. \tag{4.312}$$

Once again, this term remains non-integrable, precluding a well-defined variational principle. It becomes evident that the anomaly lies not in Weyl-Carroll symmetry but in the Carroll-Lorentz one:

$$\begin{aligned}\delta_\xi S_{\text{tot}} &= \int \Big[-\xi^b(D_b\,t^a{}_a + \frac{1}{8\pi G}\mathcal{F}_{ab}\,\mathcal{A}^a) + \omega\,t^a{}_a - \lambda\,t^a{}_b\,\mu^\star_a\,\upsilon^b\Big]\,\mathrm{vol}_{\partial\mathcal{M}} \\ &= \int \Big(\lambda\,\frac{\mathcal{F}}{8\pi G}\Big)\mathrm{vol}_{\partial\mathcal{M}}\,.\end{aligned} \tag{4.313}$$

This outcome is not unexpected, as it stems from the fact that the underlying Carrollian hydrodynamic equations (B.80)-(B.83) are typically not invariant under Carrollian boosts. While such anomal results are well understood in holography for asymptotically AdS spaces (Henningson & Skenderis, 1998; de Haro et al., 2001), the scenario differs for asymptotically flat spaces. Therefore, it is worth emphasizing that (4.313) constitutes a novel prediction concerning the characteristics of the conjectured conformal Carrollian field theory existing at null infinity.

CHAPTER 5

Summary and future perspectives

> "Il piacere è sempre o passato o futuro, e non è mai presente."
>
> Giacomo Leopardi

In conclusion to this thesis, let us first offer a concise overview of our journey and highlight the diverse findings we have uncovered. Concurrently, we will delve into the intriguing inquiries this work prompts for future exploration and research.

Our main objective was to deepen our understanding of holography in asymptotically flat spaces by examining asymptotically AdS features within the classical framework of the AdS/CFT correspondence via a smooth flat limit. This involved analyzing asymptotic symmetries in general relativity. These symmetries are discerned from pure gauge transformations by the presence of a non-zero value in the associated asymptotic Noether charge. However, as we approach the boundaries of these asymptotic spaces, such as in the method pioneered by Abbott and Deser (Abbott & Deser, 1982a,b) and generalized to any gauge theory by Barnich and Brandt covariant phase space (Barnich & Brandt, 2002), charge divergences may occur in the holographic coordinate of radial evolution. The crux of this divergence lies in the potential manifestation of similar traits within the underlying variational principle. Consequently, it becomes imperative to comprehend and address this phenomenon through renormalization schemes. This manuscript has adopted this approach instead of the conventional belief that encountering such divergences necessitates refining the theory, its boundary conditions, and its gauge choices to avoid them.

In this regard, our methodology aligned with the paradigm shift introduced in the corner proposal (Donnelly & Freidel, 2016; Speranza, 2018; Geiller, 2017, 2018), which seeks liberation from such constraints by meticulously considering the codimension-2 support (referred to as corners) where Noether charges are defined for gauge theories.

In the two approaches mentioned earlier, while it should theoretically be possible to prescribe effective counterterms for renormalization, this process typically involves manual intervention with no underlying systematism. For instance, in the Barnich-Brandt scenario (Barnich & Brandt, 2002), the determination of charges relies not on the bulk action but on the structure of the equations of motion. Seeking to overcome such limitations and establish a more robust framework, we opted in the main body of the text for the Iyer-Wald formalization of the covariant phase space (Lee & Wald, 1990; Wald, 1993; Wald & Zoupas, 2000). In the latter, the presymplectic potential emerges from the variational principle, accompanied by its own set of ambiguities (2.13). These ambiguities afford us the opportunity to establish connections between the asymptotic corner charges of Iyer-Wald and the ones of Barnich-Brandt through judicious adjustments. However, throughout the manuscript, we chose to resolve them in a manner that effectively mitigates radial divergences.

Over the past two decades, two prescriptions have surfaced regarding the resolution of divergent Iyer-Wald ambiguities. The first, termed holographic renormalization, prioritizes the renormalization of the bulk action through the addition of boundary counterterms (Henningson & Skenderis, 1998; de Haro et al., 2001; Bianchi et al., 2002), thereby renormalizing the associated symplectic structure (Compere & Marolf, 2008) allowing to define finite surface or corner charges in the asymptotic limit. Conversely, the second approach eschews boundary terms, focusing instead on incorporating corner contributions directly into the bulk presymplectic potential (Freidel et al., 2019; McNees & Zwikel, 2023), earning it the label symplectic renormalization. In every part of the text and the various chapters, we have extensively confronted these two methodologies, particularly elucidating them through a comprehensive examination of asymptotic symmetries in Maxwell theory. This investigation aimed to get sufficient insight and familiarity for subsequent gravitational inquiries. Notably, we analyzed photon propagation in both AdS and flat backgrounds (Campoleoni et al., 2023a), utilizing coordinate systems tailored to the gauge frameworks commonly employed in the literature on general relativity for asymptotically AdS and flat spaces.

These considerations prompted us to explore these asymptotic spaces in three-dimensional Einstein-Hilbert theory, given its simpler characteristics compared to higher dimensions, yet still offering considerable interest such as the BTZ black hole (Banados et al., 1992, 1993; Carlip, 1995). In pursuit of a deeper comprehension of the AdS/CFT correspondence through the examination of asymptotic charges, we investigated a potential modification to the standard Fefferman-Graham gauge, known as the Weyl-Fefferman-Graham gauge (Ciambelli & Leigh, 2020). The latter (4.78) proves to be more adapted to holography than the former (4.34) due to its capability to induce Weyl rescalings of the conformal boundary solely through radial bulk diffeomorphisms (4.79). Indeed, the Weyl covariance, which is compromised in the Fefferman-Graham gauge (Henningson & Skenderis, 1998), is reinstated in the Weyl-Fefferman-Graham gauge. It is a natural holographic expectation for the dual theory since the asymptotic boundary sits at conformal infinity. This modification facilitates the realization of the complete Weyl geometry, comprising a conformal metric class and a Weyl connection (4.86), at the boundary. The contribution of this thesis lies in scrutinizing the resulting expanded set of independent residual gauge symmetries, along with computing the charges associated with these asymptotic Killing vectors.

We have demonstrated that with a suitable choice of symplectic structure, these extra residual symmetries may possess non-zero values (4.119), thus encoding physical information (Ciambelli et al., 2023). Specifically, we have revealed their existence through a finite covariant corner contribution (4.122) to the bulk presymplectic potential. This assertion finds support in our analysis of the variational principle within the Chern-Simons formulation, facilitated by the topological nature of three-dimensional gravity (Achucarro & Townsend, 1986; Witten, 1988), by prescribing the appropriate finite boundary Lagrangians to be added to the bulk action. Notably, this rewriting offers also the advantage of obviating the necessity for renormalization. These findings affirm previous studies suggesting that complete gauge-fixing in the presence of boundaries could lead to the elimination of potentially significant physical degrees of freedom (see, for example, (Grumiller & Riegler, 2016; Adami et al., 2020b; Geiller et al., 2021; Adami et al., 2023)). Particularly, this underscores that while the Fefferman-Graham gauge may still be always achievable, it could impose constraints on the physical content. Exploring the ramifications of these additional physical symmetries for the corresponding field theory counterpart presents an intriguing avenue for future investigation.

Actually, we have already gained initial insights into this putative dual field theory: employing a covariant approach, the new source emerges as a Weyl-invariant amalgamation of the Weyl connection and the metric[1]. Significantly, we noticed that the transformation laws governing the Weyl connection and the Weyl-source are typical of a one-form symmetry. A non-zero charge associated with this symmetry indicates the presence of physical states at the boundary that respond to this operator, akin to a potential Weyl Wilson line. This concept lays a groundwork for realizing higher form symmetries through a bottom-up holographic approach. This avenue holds considerable promise, especially in crafting a novel holographic framework for AdS/CFT utilizing Weyl covariant quantities within the Weyl-Fefferman-Graham gauge. We note also that leveraging on the analogy of one-form symmetry offers opportunities for insights when applying this approach to Einstein-Maxwell bulk systems (see, for example, (Taylor, 2000; Barnich et al., 2015; Bosma et al., 2024)).

While the Fefferman-Graham framework is universal applicable, it falls short in describing flat space through a smooth flat limit, a key objective of this thesis. Conversely, the Bondi gauge remains valid for any value of the cosmological constant. However, it lacks covariance concerning the pseudo-Riemannian boundary, unlike Fefferman-Graham (Ruzziconi & Zwikel, 2021). This discrepancy prompts a potential relaxation of the Bondi gauge, termed covariant Bondi (Ciambelli et al., 2020b), introduced to explore new finite charges and deepen our understanding of the dual theory, both in AdS and in the flat spacetime regime where a smooth flat limit is then achievable. This progress has been skillfully realized by employing boundary Cartan zweibein, which results in an incomplete bulk gauge fixing (4.164). This appears to introduce supplementary boundary degrees of freedom and novel residual symmetries related to local frame boosts, whether Lorentzian or Carrollian, in both AdS and flat spacetimes. Another benefit of this gauge lies in its origin from fluid/gravity correspondence, facilitating discussions of the dual theory in hydrogeometric terms (Bhattacharyya et al., 2008a; Haack & Yarom, 2008; Bhattacharyya et al., 2008b; Hubeny et al., 2012).

Once more, our contribution involved determining the asymptotic corner charges within this framework (Campoleoni et al., 2022). Following holographic and symplectic renormalization, the task arose again to resolve finite ambiguities. Notably, two distinct prescriptions surfaced. The first approach (4.207) leads to outcomes similar to the Fefferman-Graham results, where Weyl rescalings become part of asymptotic symmetries, while Lorentz boosts remain entirely gauge-dependent. In contrast, the second approach (4.209) elevates hyperbolic rotations of the Cartan frame to asymptotic symmetries, albeit at the

[1] Refer to (Ciambelli et al., 2023) for a non-covariant approach where the Weyl connection directly serves as the new source.

expense of Weyl rescalings becoming purely gauge-related. Unlike the former, this approach allows for a finite flat limit (4.262), wherein Carrollian boosts of the frame merge into asymptotically flat symmetries. This last result is particularly intriguing as it suggests that the covariant gauge is well-suited for resolving the ambiguities of the presymplectic potential, especially when ensuring the smoothness of the flat limit applies not only to the solution space but also to its symplectic structure. Moreover, this computation aligns more closely with symplectic renormalization, as it addresses the corner Iyer-Wald ambiguity over the boundary counterpart. This analysis was also supported by the inclusion of counterterms into the bulk action via the Chern-Simons formulation, naturally favoring this second finite Lorentz/Carroll prescription.

Furthermore, employing the covariant Bondi gauge enhances comprehension of the boundary conformal anomaly, whether approached from Weyl or Lorentz perspectives, due to inherent cohomology properties. Nevertheless, the existence of a gauge ensuring a finite flat limit for the presymplectic potential that respects Weyl symmetry without a priori excluding Lorentz symmetry, as undertaken in (Detournay & Riegler, 2017) or its potential generalization outlined in (Alessio et al., 2021), remains uncertain and is an interesting perspective for the future. Additionally, we obtained a suggestion that a novel anomaly (4.313) might exist in Carrollian conformal field theories, which presently eludes genuine quantum computation. While some anomalies in such field theories have been discussed in (Bagchi et al., 2021), they differ from our findings, constituting a holographic prediction warranting further exploration. Besides, connecting these results on Carrollian holography within the covariant Bondi gauge with the celestial holography proposal (Strominger, 2018) is also intriguing, following the approach outlined in (Donnay et al., 2022; Bagchi et al., 2022; Donnay et al., 2023).

To conclude, we would like to highlight certain additional aspects of our outcomes that we believe merit further investigation. One pivotal area for exploration involves exploring the implications of our results in higher dimensions, which hold greater cosmological significance. For instance, incorporating the Weyl-Fefferman-Graham gauge into the systematic phase space analysis outlined in (Fiorucci & Ruzziconi, 2021) could yield valuable insights. Notably, in three-dimensional bulk spaces, we have seen that the boundary consistently exhibits conformal flatness, thereby enabling the realization of the complete conformal isometries group. However, instances may occur in higher dimensions where the boundary does not maintain conformal flatness as, e.g., 3D boundaries with a non-vanishing Cotton tensor, resulting in a diminished conformal isometries group. Nevertheless, given the boundary location at conformal infinity, the dual theory retains Weyl covariance. Hence, in such scenarii, we are confronted with a conformal field theory in curved spaces possessing Weyl symmetry, commonly referred to as a Weyl field theory. Expanding our analysis to higher dimensions holds the potential to shed light on these theories, which remain relatively unexplored.

Moreover, in higher dimensions, Bondi's covariant relaxation leads to a asymetry between the Weyl and Lorentz groups, ceasing their isomorphism. Specifically, in four dimensions, the former is characterized by one function of the boundary coordinates, whereas the latter comprises one rotation and two boosts. This disrupts the precise parallelism drawn in our analysis in the covariant Bondi gauge between Weyl and Lorentz. While we anticipate the continuation of this gauge relaxation, resulting in a tangible Cartan frame on the boundary, the absence of a Weyl anomaly introduces complexity to the three-dimensional framework. Additionally, comprehending all anomaly issues in higher-dimensional theories, where boundary frames undergo transformation with the Lorentz or Carroll groups, or more broadly with the general linear group, demands further exploration. Some preliminary inspections can be found in, for example, (Petkou et al., 2022; Mittal et al., 2023; Campoleoni et al., 2023b).

Lastly, an intriguing avenue of inquiry pertains to our demonstration that new charges could stem from both finite boundary and corner Lagrangians. Because of that, this prompts the crucial question of categorizing these novel charges linked to selections of symplectic spaces. As a fundamental principle, one could posit that a greater number of physical charges would be advantageous, as it would result in larger algebras capable of more effectively organizing the theory observables. The specific examples discussed herein thus serves as a gateway to a more fundamental issue: the classification of charges arising from partial gauge fixings. Besides, considering that we have not yet reached the maximum number of charges possible in three dimensions (Grumiller et al., 2020b; Adami et al., 2020b), it would be intriguing to explore methods for their emergence. One avenue worth investigating could involve performing gauge transformations between Weyl-Fefferman-Graham and covariant Bondi relaxations, as well as with the Bondi-Weyl gauge, which might shed light on this endeavor. Additionally, it would be of broad interest for the asymptotic-symmetry program to address the conditions under which no additional degrees of freedom arise from a gauge relaxation. These conditions may entail the necessity of a finite presymplectic potential that vanishes under Dirichlet boundary conditions. Insights into this matter could potentially be gleaned from analogous analyses in linearized gauge theories, such as Fronsdal's theory of massless higher spin bosonic fields (Fronsdal, 1978), as discussed in (Campoleoni et al., 2017, 2018a, 2020) for instance.

To summarize the key points of this thesis, the main idea is that by investigating asymptotic symmetries using covariant phase space methods, we explored various aspects of the AdS/CFT correspondence. Specifically, we looked at how partial gauge fixations can limit the physical content of the theory by preventing the manifestation of associated asymptotic charges. This concept is intriguing within the AdS framework, as it helps develop a gauge-fixation-free theory better suited for transitioning to quantum gravity. However, it becomes even more compelling in the flat limit. In the latter, the dual theory remains largely unknown, and this approach allows for its investigation based on physical symmetries. These considerations and inquiries introduce an exciting direction in the theory of asymptotic symmetries, with significant implications for both AdS and flat holography that have yet to be fully explored.

Appendices

APPENDIX **A**

Conventions and notations

In this appendix, we compile the conventions and notations essential for efficiently navigating through the manuscript. Where necessary, these are also reiterated directly within the main text.

Unless specified otherwise, we operate under the assumption that the bulk theory is defined on a differentiable Lorentzian $D-$dimensional manifold $\mathcal{M}$ with coordinates represented as $(x^\mu) = (r, x^a)$, where r signifies a radial coordinate and the boundary resides at $r \to \infty$. Assuming that $\mathcal{M}$ encompasses a regulating boundary $\partial\mathcal{M}$ featuring a radial isosurface component, we denote this surface as $\mathcal{B}$ with coordinates labeled by x^a. We work at large r-values with the asymptotic limit taken at the end. The theory is formalized by providing a Lagrangian form $L = \mathscr{L}\mathrm{d}^D x$, where the Lagrangian density is denoted $\mathscr{L}$. This density equals $\sqrt{-g}$ multiplied by the corresponding scalar, where $g_{\mu\nu}$ is the metric (with Lorentzian signature following the mostly plus convention) on the manifold $\mathcal{M}$ and g represents its determinant. We denote the covariant derivative with respect to this form $g_{\mu\nu}$ by the operator ∇. The convention for expressing the Riemann tensor in terms of the Levi-Civita connection is given by:

$$R^\alpha{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\alpha_{\sigma\nu} - \partial_\sigma \Gamma^\alpha_{\rho\nu} - \Gamma^\beta_{\rho\nu}\Gamma^\alpha_{\sigma\beta} + \Gamma^\beta_{\sigma\nu}\Gamma^\alpha_{\rho\beta}\,, \tag{A.1}$$

such that the Ricci tensor and the scalar curvature read respectively:

$$R_{\nu\sigma} = R^\rho{}_{\nu\rho\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma}\,, \qquad R = g^{\nu\sigma} R_{\nu\sigma}\,. \tag{A.2}$$

In the Einstein equations governing the metric $g_{\mu\nu}$, G represents Newton's gravitational constant, and Λ stands for the cosmological constant.

Occasionally, for example to compute the charge, we break down the coordinates of the boundary into timelike and spacelike components, $(x^a) = (t, x^i)$, where i corresponds to the spacelike coordinates on $\mathcal{B}$. We assume the existence of an isosurface with respect to the time coordinate t on a neighborhood of $\mathcal{B}$ and which intersects $\mathcal{B}$ on a closed codimension-2 surface $\mathcal{C}$, called corner. Specifically, the coordinates x^i refer to the ones along the corner $\mathcal{C}$. We employ the same subscript to indicate the collection of fields for the theory, $\varphi = (\varphi^i)$, with the context allowing for clear differentiation. In the Fefferman-Graham framework, where the above decomposition is not utilized, the boundary indices are lowered and raised using the codimension-1 boundary metric and its inverse, respectively. In the case of an asymptotically AdS spacetime, the cosmological constant can be expressed in terms of the AdS radius ℓ as $\Lambda = -\ell^{-2}$. Conversely, in the Bondi setup, effectively using the breaking for the spherical indices, similar operations are performed but employing the codimension-2 spherical metric. Otherwise, it involves the bulk metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. These points are nuanced in the Carrollian approach, where the host metric is degenerate and hence not invertible. Continuing on the topic of indices, we label the algebra basis indices using uppercase letters from the start of the Latin alphabet.

When varying the Lagrangian with respect to the fields, the symbol $\approx$ signifies that equality is evaluated on-shell of the equations of motion. In this context, $(\delta g)^{\mu\nu}$ should be interpreted as follows:

$$(\delta g)^{\mu\nu} = -\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} \,, \tag{A.3}$$

and similarly for $(\delta g)^{\nu}{}_{\nu}$. The Lagrangian form, and more generally any $(D-p)$-spacetime form (where $p \in \mathbb{N}$ and $p \leq D$), are expressed as

$$A = A^{\mu_1 \dots \mu_p} (\mathrm{d}^{D-p} x)_{\mu_1 \dots \mu_p} \,, \tag{A.4}$$

in terms of the following basis:

$$(\mathrm{d}^{D-p} x)_{\mu_1 \dots \mu_p} = \frac{1}{p!(D-p)!} \varepsilon_{\mu_1 \dots \mu_p \nu_{p+1} \dots \nu_D} \mathrm{d}x^{\nu_{p+1}} \wedge \dots \wedge \mathrm{d}x^{\nu_D} \,. \tag{A.5}$$

The action of the exterior derivative on such a differential form is given by

$$\mathrm{d}A = \partial_\nu A^{\mu_1 \dots \mu_{p-1} \nu} (\mathrm{d}^{D-p+1} x)_{\mu_1 \dots \mu_{p-1}} \,. \tag{A.6}$$

In the equation (A.5), $\varepsilon_{\mu_1 \dots \mu_D}$ represents the Levi-Civita density. Specifically, in the main body of the text, we adhere to the following two conventions regarding its sign: $\varepsilon_{01} = 1$ and $\varepsilon^{012} = 1$. When round and square brackets are utilized around indices on the components of forms, it indicates symmetrization and antisymmetrization, respectively, on these indices, without any overall factor. As an illustration, for a 2-form:

$$A_{(\mu\nu)} = A_{\mu\nu} + A_{\nu\mu}\,, \qquad A_{[\mu\nu]} = A_{\mu\nu} - A_{\nu\mu}\,. \tag{A.7}$$

APPENDIX B

Geometry review

B.1. Weyl geometry

This first section of the appendix serves to provide an overview of the Weyl geometry, inspired by (Folland, 1970; Hall, 1992), laying the groundwork for the Weyl application within the Weyl-Fefferman-Graham gauge context of asymptotically Anti de Sitter spacetimes (see subsection 4.2.1). Although this appendix primarily focuses on this specific gauge, the concepts of Weyl geometry also find utility in discussing the covariant Bondi gauge in subsection 4.3, as well as in the subsequent appendix B.2 concerning holographic hydrogeometry.

To grasp the underlying geometry of the gauge (4.78), we introduce the following dual form basis on the bulk manifold $\mathcal{M}$:

$$e^z = \frac{\mathrm{d}z}{z} - k_a(z,x)\mathrm{d}x^a\,, \qquad e^a = \mathrm{d}x^a\,, \tag{B.1}$$

along with its corresponding vector basis:

$$e_z = z\partial_z \equiv D_z\,, \qquad e_a = \partial_a + zk_a(z,x)\partial_z \equiv D_a\,. \tag{B.2}$$

These basis vectors $\{D_z, D_a\}$ constitute the tangent space at any point of $\mathcal{M}$, with the spatial vectors $\{D_a\}$ forming a 2-dimensional distribution $\mathcal{D}$ on $\mathcal{M}$. Notably, this distribution belongs to the kernel of e^z. The Lie brackets of the basis vectors take the form:

$$[D_z, D_a] = D_z k_a D_z\,, \qquad [D_a, D_b] = f_{ab} D_z\,, \tag{B.3}$$

where

$$f_{ab} \equiv D_a k_b - D_b k_a \tag{B.4}$$

represents the curvature associated with k_a. If this curvature vanishes, such that $[D_a, D_b] = 0$, the distribution proves to be integrable as per the Frobenius theorem (Morita, 2001). In the Fefferman-Graham gauge, where $\mathcal{D}$ is defined by $\{\partial_a\}$ and divides $\mathcal{M}$ into surfaces with constant z, this holds true, unlike in Weyl-Fefferman-Graham relaxation.

In the realm of generalized Riemannian manifolds $(\bar{\mathcal{M}}, \bar{g})$ endowed with a connection $\bar{\nabla}$, the associated coefficients $\bar{\Gamma}$ are conventionally defined by:

$$\bar{\nabla}_{e_A} e_B = \bar{\Gamma}^C_{AB} e_C \,. \tag{B.5}$$

where $\{e_A\}$ represents an arbitrary basis, with $(A, B, \dots)$ serving as internal Lorentz indices. When considering the manifold $\mathcal{M}$ of the Weyl-Fefferman-Graham setup equipped with the Levi-Civita connection ∇, the connection coefficients of ∇ within the frame $\{D_z, D_a\}$ can be expressed as:

$$\nabla_{D_a} D_b = \Gamma^c_{ab} D_c + \Gamma^z_{ab} D_z \,. \tag{B.6}$$

Here, the coefficients Γ^c_{ab} delineate the induced connection on the distribution $\mathcal{D}$ over $\mathcal{M}$ (refer to, for instance, (Muñoz-Lecanda, 2018)). By substituting the polyhomogeneous asymptotic radial expansions (2.68) of the quantities h_{ab} and k_a from the ansatz (4.78) into these coefficients, we derive, at the leading order (corresponding to the zero order), the following expression:

$$\Gamma^{(0)}{}^c{}_{ab} = \frac{1}{2} h^{cd}_{(0)} \left(\partial_a h^{(0)}_{bd} + \partial_b h^{(0)}_{ad} - \partial_d h^{(0)}_{ab} \right) - \left(k^{(0)}_a \delta^c_b + k^{(0)}_b \delta^c_a + k^{(0)}_d h^{cd}_{(0)} h^{(0)}_{ab} \right) . \tag{B.7}$$

Our aim now is to demonstrate that the latter adhere to a torsion-free connection with Weyl metricity, as discussed in (Ciambelli & Leigh, 2020; Jia & Karydas, 2021; Jia et al., 2023).

To address this matter, let us revisit the general discussion of an arbitrary Riemannian manifold $(\bar{\mathcal{M}}, \bar{g})$ as presented in equation (B.5). The torsion tensor and Riemann curvature tensor of the connection are defined as follows:

$$\bar{T}^C_{AB} e_C \equiv \bar{\nabla}_{e_A} e_B - \bar{\nabla}_{e_B} e_A - [e_A, e_B] \,, \tag{B.8}$$

$$\bar{R}^A_{BCD} e_A \equiv \bar{\nabla}_{e_C} \bar{\nabla}_{e_D} e_B - \bar{\nabla}_{e_D} \bar{\nabla}_{e_C} e_B - \bar{\nabla}_{[e_C, e_D]} e_B \,, \tag{B.9}$$

where the commutation coefficients $[e_A, e_B] = C_{AB}{}^C e_C$ are denoted by the structure constants $C_{AB}{}^C$. In the preceding paragraph, we mentioned the Levi-Civita connection, a special case of these connections where we assume zero torsion. By definition, this connection, denoted by $\mathring{\nabla}$, has the following properties: compatibility metricity and torsion-free conditions, which can be written respectively as

$$0 = (\mathring{\nabla}\bar{g})(e_A, e_B, e_C) = \mathring{\nabla}_{e_C}\bar{g}(e_A, e_B) - \mathring{\Gamma}^D_{CA}\bar{g}(e_D, e_B) - \mathring{\Gamma}^D_{CB}\bar{g}(e_D, e_A)\,, \tag{B.10}$$

$$0 = \mathring{T}^C_{AB} = \mathring{\Gamma}^C_{AB} - \mathring{\Gamma}^B_{BA} - C_{AB}{}^C\,. \tag{B.11}$$

In these relations, we denoted the components of the metric $\bar{g}$ in the frame $\{e_A\}$ by $\bar{g}_{AB} \equiv \bar{g}(e_A, e_B)$.

To make further progress, we opt for a coordinate basis $\{\partial_a\}$ and its corresponding dual basis $\{\mathrm{d}x^a\}$[1]. Subsequently, we consider that the metric undergoes a Weyl transformation of the type

$$\bar{g} \to \mathfrak{B}^{-2}\bar{g}\,. \tag{B.12}$$

In these transformations, ∂_a and $\{\mathrm{d}x^a\}$ remain weightless, while $e_A \equiv e_A{}^a\partial_a$ and $e^A \equiv e_a{}^A\mathrm{d}x^a$ possess weights of $+1$ and -1, respectively. It is worth noting that the metricity tensor $\mathring{\nabla}\bar{g}$ does not transform covariantly under (B.12). Hence, we introduce a Weyl connection, $\bar{A} = \bar{A}_a\mathrm{d}x^a$, to reinstate this covariance. This connection follows a typical Weyl-form law transformation:

$$\bar{A}_a \to \bar{A}_a - \mathring{\nabla}_a \ln\mathfrak{B}\,. \tag{B.13}$$

This leads us to define the following connection $\hat{\nabla}$ by its action on a generic tensor T of any type (with suppressed indices) with a Weyl weight ω_T:

$$\hat{\nabla}_a T := \nabla_a T + \omega_T \bar{A}_a T\,. \tag{B.14}$$

This connection exhibits Weyl covariance, meaning that

$$\hat{\nabla}_a T \to \mathfrak{B}^{\omega_T}\hat{\nabla}_a T\,. \tag{B.15}$$

In addition to the torsion-free condition of the Levi-Civita connection, we can impose the metricity condition in a Weyl manner as follows:

$$\hat{\nabla}_a\bar{g}_{bc} = \mathring{\nabla}_a\bar{g}_{bc} - 2\bar{A}_a\bar{g}_{bc} = 0\,. \tag{B.16}$$

This condition implies the following relation on the relative connection coefficients:

$$\hat{\Gamma}^c_{ab} = \frac{1}{2}\bar{g}^{cd}\Big(\partial_a\bar{g}_{db} + \partial_b\bar{g}_{ad} - \partial_d\bar{g}_{ab}\Big) - \Big(\bar{A}_a\delta^c_b + \bar{A}_b\delta^c_a - \bar{g}^{cd}\bar{A}_d\bar{g}_{ab}\Big)\,, \tag{B.17}$$

[1] We intentionally use the same indices as the ones on the boundary $\mathcal{B}$ of the Weyl-

which differs from the usual Levi-Civita Christoffel symbols by an additional term involving the Weyl connection. Notably, these coefficients mirror the ones in the equation (B.7), where $\bar{A}_a$ and $\bar{g}_{ab}$ correspond respectively to the Weyl-Fefferman-Graham leading orders $k_a^{(0)}$ and $h_{ab}^{(0)}$. This validates the interpretation of the Weyl connection associated with $k_a^{(0)}$, as well as explains why we utilized the same indices as along the asymptotic boundary $\mathcal{B}$.

In summary, within the framework of Weyl-Fefferman-Graham, this implies that the ansatz (4.78) reinstates Weyl geometry at the conformal boundary of the asymptotically AdS space, encompassing both the boundary metric $h_{ab}^{(0)}$ and the Weyl connection $k_a^{(0)}$. The induced connection $\nabla^{(0)}$ operates as follows:

$$\nabla_a^{(0)} h_{bc}^{(0)} = 2k_a^{(0)} h_{bc}^{(0)} \,. \tag{B.18}$$

One can then formulate a Weyl-covariant connection for a generic tensor T of arbitrary type with a Weyl weight ω_T, as shown in the following equation (to be compared with (B.14)):

$$\hat{\nabla}_a^{(0)} T := \nabla_a^{(0)} T + \omega_T k_a^{(0)} T \,. \tag{B.19}$$

This ensures that the connection $\hat{\nabla}^{(0)}$ maintains metricity, and $\hat{\nabla}_a^{(0)} T$ exhibits Weyl covariance. Consequently, all standard geometric quantities can be elevated to Weyl quantities. Specifically, any geometric quantity originally derived using the boundary metric $h_{ab}^{(0)}$ and the Levi-Civita connection in the Fefferman-Graham setup (4.34) now possesses a Weyl-covariant counterpart, constructed using $h_{ab}^{(0)}$, $k_a^{(0)}$, and $\hat{\nabla}^{(0)}$ in the relaxed gauge (4.78).

B.2. Hydrogeometry

In this section, our focus shifts to examining the boundary geometry induced by the covariant Bondi gauge of the bulk metric for asymptotically AdS and flat spacetimes. We coin the term "hydrogeometry" to denote this geometry due to its connections with fluid propagation on this asymptotic boundary. Specifically, we demonstrate that the covariant Bondi gauge (4.164) of the subsection 4.3.2 aligns with an Eddington-Finkelstein type of gauge, which naturally arises in the fluid/gravity correspondence when reconstructing the bulk spacetime from the boundary data. In the latter context, this gauge is commonly known as derivative expansion. This designation stems from the

Fefferman-Graham manifold setup.

fact that fluid dynamics in this scenario involves expressing various dissipative and non-dissipative quantities constituting the fluid stress tensor as expansions in increasing derivative order of the fluid velocity, temperature, and chemical potentials (if additional currents are present). We shall see explicitly such a feature in the following subsections.

This same rationale extends to reconstructing the bulk metric associated with this fluid, achieved through an order-by-order expansion in inverse powers of the holographic coordinate r, which here serves as a null radial coordinate. In this regard, this construction bears resemblance to the Fefferman-Graham gauge philosophy, albeit with a distinct nature pertaining to the holographic direction. Additionally, the coefficients of this expansion comprise derivatives of the fluid fundamental fields (velocity, temperature, and chemical potentials) of increasing order, tailored to ensure the invariance of the line element concerning boundary Weyl transformations. Consequently, this approach facilitates the establishment of a connection with the Weyl-Fefferman-Graham gauge and its Weyl geometry at the boundary (see appendix B.1), leading instead to a Weyl-hydrogeometry at the boundary in the context of the covariant Bondi gauge.

While we will employ in this appendix a similar approach to illustrate the hydrogeometric connection between the covariant Bondi gauge and fluid/-gravity, we do not pursue this restrictive path in the analysis of asymptotic symmetries in subsection 4.3.2. Actually, in the latter, we do not directly apply a derivative expansion to demonstrate this connection. Instead, to elevate the fluid/gravity correspondence to a genuine generating procedure for arbitrary Einstein spacetimes, we need to emancipate every quantity present in the energy-momentum tensor from constraints imposed by constitutive relations. This approach allows for the inclusion of non-hydrodynamic modes. Consequently, the fluid velocity and energy density, the heat current, the stress tensor, and the boundary metric all become arbitrary functions. These components form the building blocks for the expansion in inverse powers of the radial light-like coordinate, which is governed by Weyl covariance, akin to the Weyl-Fefferman-Graham gauge.

In particular, we divide this section in two subsections B.2.1 and B.2.2 in order to delve in the hydrodynamic interpretation of the boundary data of the asymptotically AdS and flat spacetimes parametrized in the covariant Bondi gauge. More concretely, these are mapped respectively to relativistic and more exotic so-called Carrollian fluids. The latter, also referred to as ultrarelativistic fluids, emerge from relativistic fluids through the vanishing limit of the light velocity, denoted as $k \to 0$. This limit is tied to the AdS radius through an inverse power law, $k = \frac{1}{\ell}$. In such scenarii, the asymptotic boundary of the bulk space transitions into null infinity, ushering us into the realm of Carrollian physics.

B.2.1 Relativistic fluid

Two-dimensional hydrodynamic aspects

We begin by considering a relativistic fluid, exhibiting flow accompanied by dissipative processes similar to the ones commonly studied in standard textbooks of fluid mechanics (we refer particularly to (Landau & Lifshitz, 1987)). Let us imagine this fluid propagating over an arbitrary two-dimensional Riemannian geometry whose metric is denoted by q_{ab}. The choice of this notation is deliberate as it pertains to the boundary metric of the covariant Bondi gauge (4.164). The dynamics of this fluid are encapsulated by its energy-momentum tensor $\widetilde{T}_{ab}$, which adheres to the relativistic hydrodynamic equation, and the continuity equation:

$$\nabla^a \widetilde{T}_{ab} = f_b \,, \qquad \nabla^a n_a = 0 \,, \tag{B.20}$$

where f_b signifies an external force density, and n_a represents the particle current. The covariant derivative employed here pertains to the background metric q_{ab}.

In scenarii involving dissipative processes like momentum exchange due to viscosity or energy transfer through thermal conduction, the stress tensor can generally be decomposed as follows:

$$\widetilde{T}_{ab} = (\varepsilon + p)\frac{u^a u^b}{k^2} + p\, q_{ab} + \tau_{ab} + \frac{u_a q_b}{k^2} + \frac{u_b q_a}{k^2} \,. \tag{B.21}$$

Here, k denotes the constant light velocity, u^a represents the fluid velocity (timelike and normalized to $-k^2$), ε is the internal energy density and p signifies the fluid pressure. In addition to the terms typical of a perfect fluid, there are additional contributions, including the symmetrical viscosity tensor τ_{ab} and the heat conduction vector q_a. The particle flow is expressed as

$$n^a = \frac{1}{k}\left(n u^a + q^a\right), \tag{B.22}$$

where n is the proper density of the number of particles in the fluid.

When assuming an equation of state in the form of $\varepsilon = p$, the expression (B.21) precisely corresponds to the initial part of the holographic Brown-York stress tensor, leading to (4.176) in the covariant Bondi gauge. To illustrate this connection, we need to delve a bit into the expression of (B.21) by leveraging a characteristic of relativistic fluids: they can be locally described using two distinct velocity fields. Classically, the hydrodynamic velocity is defined with respect to mass flux. However, in the relativistic framework, particularly in the presence of heat flux, this notion becomes less clear. Following principles outlined in (Landau & Lifshitz, 1987), we redefine fluid velocity by ensuring its momentum vanishes in the proper reference frame and that energy is determined using the same relations as in situations devoid of dissipative processes.

In such a frame, where the velocity components are by definition given by $(u^0, u^1) = (k, 0)$, with $(x^a) = (x^0, x^1)$, and $n = n^0$ in this frame, we derive the following transversality relations:

$$\tau_{ab} u^b = 0 \,, \qquad q_a u^a = 0 \,. \tag{B.23}$$

Since the described fluid remains invariant under changes in hydrodynamic velocity, we extend these relations to all hydrodynamic frames. Consequently, we can express the energy density and heat flow as

$$\varepsilon = \frac{1}{k^2} \widetilde{T}_{ab} u^a u^b \,, \qquad q_a = -\varepsilon u_a - u^b \widetilde{T}_{ab} \,. \tag{B.24}$$

Utilizing the two-dimensional nature of the background, we introduce the dual congruence $\star u^a$ (spacelike, normalized to k^2) via Hodge duality, following the conventions used in subsection 4.3. This leads to the rewriting of the dissipative tensors,

$$\tau_{ab} = \tau h_{ab} \,, \qquad q_a = \chi \star u_a \,, \tag{B.25}$$

where τ represents the unique component of τ_{ab} referred to as the viscosity scalar, h_{ab} acts as the projector onto the space transverse to the velocity field:

$$h_{ab} = \frac{1}{k^2} \star u_a \star u_b \,, \tag{B.26}$$

and χ denotes the heat density:

$$\chi = -\frac{1}{k^2} \star u^a \widetilde{T}_{ab} u^b \,. \tag{B.27}$$

Additionally, defining the expansion of the velocity and its dual congruence as

$$\Theta = \nabla_a u^a \,, \qquad \Theta^\star = \nabla_a \star u^a \,, \tag{B.28}$$

we can express the fluid acceleration by the following equation:

$$a_b = u^c \nabla_c u_b = \Theta^\star \star u_b \,. \tag{B.29}$$

Furthermore, it proves convenient to introduce the Cartan hydrodynamic frame $\{\mathrm{u}/k, \star\mathrm{u}/k\}$, where the metric of the host geometry takes the form (see (4.166)):

$$\mathrm{d}s^2_{\mathrm{bdy}} = q_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{1}{k^2}\left(-\mathrm{u}^2 + \star\mathrm{u}^2\right), \tag{B.30}$$

and where $\mathrm{u} = u_a\mathrm{d}x^a$ and $\star\mathrm{u} = \star u_a\mathrm{d}x^a$.

These relations allow us to express the stress tensor (B.21) in a concise form:

$$\widetilde{\mathrm{T}} = \widetilde{T}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{\varepsilon}{k^2}\left(\mathrm{u}^2 + \star\mathrm{u}^2\right) + \frac{\chi}{k^2}\left(\mathrm{u}\star\mathrm{u} + \star\mathrm{u}\,\mathrm{u}\right) + \frac{\tau}{k^2}\star\mathrm{u}^2\,, \tag{B.31}$$

which matches (4.176) if we express the viscous pressure τ as proportional to the curvature $R^{(0)}$ of the fluid host geometry:

$$\tau = \frac{R^{(0)}}{8\pi G} = \frac{1}{4\pi G k^2}\left(\Theta^2 - \Theta^{\star 2} + u^a\nabla_a\Theta - \star u^a\nabla_a\Theta^\star\right). \tag{B.32}$$

This characteristic emerges from the holographic fluid present in the bulk gravity reconstruction, a concept we will briefly elaborate on at the conclusion of this subsection. It is worth noting that in the preceding discussion, we introduced all the hydrodynamic quantities describing a relativistic two-dimensional fluid. Other quantities, such as the shear and vorticity tensors, cancel out identically in two dimensions. This cancellation can be understood algebraically: since these tensors have a zero trace and are transverse to u^a (Israel & Stewart, 1979; Landau & Lifshitz, 1987), the associated Young tables contain more than one box in the first two columns (see, for example, (Hamermesh, 2012)).

Conformal aspects

In this initial segment, we have elucidated the hydrodynamic foundation of the first portion of the Brown-York tensor (4.176) within the covariant Bondi gauge, along with several associated quantities outlined in the ansatz (4.164). However, there is still more to discover. Specifically, we have observed the inclusion of a Weyl connection (4.167) within this ansatz.

In the context of the fluid/gravity correspondence, which substantiates this approach, the boundary where we articulate the hydrodynamic equations possesses notable conformal attributes in holographic setups. As we have previously explored in the Fefferman-Graham gauge within section 4.2, the AdS duality confines us to induce a conformal class rather than a specific metric at the boundary, where each variant is interconnected through a Weyl transformation akin to (B.12). Consequently, this gives rise to the concept of a two-dimensional relativistic conformal fluid and the idea of Weyl-hydrogeometry.

Given our prior discussion on the broad contours of Weyl geometry in section B.1, we will provide a concise overview in this subsection.

To covariantize the hydrodynamic theory with respect to the Weyl rescalings (B.12) of the host boundary geometry, we first need to introduce a spacetime one-form that transforms as a connection under these symmetries. This involves examining the Weyl variations and weights of the various quantities defined earlier in the fluid description. For instance, by considering the fluid in the Cartan hydrodynamic frame so that its background geometry aligns with (4.166), we can infer that the form u has a conformal weight equal to -1, while both the energy density ε and the heat density χ possess a weight of 2.

The only quantity transforming as a connection (4.170) is then provided by (4.167) (Loganayagam, 2008):

$$\mathrm{A} = A_a \mathrm{d}x^a = \frac{1}{k^2}\left(\mathrm{a} - \Theta \mathrm{u}\right), \tag{B.33}$$

where $\mathrm{a} = a_b \mathrm{d}x^b$. We can then introduce a covariant derivative, as in (B.14), that incorporates this Weyl connection. This derivative is both metric compatible and possesses an effective torsion, expressed as:

$$\left(\hat{\nabla}_a \hat{\nabla}_b - \hat{\nabla}_b \hat{\nabla}_a\right) \psi = w\, \psi\, F_{ab}\,, \tag{B.34}$$

where ψ is a scalar function of Weyl-weight w, and F_{ab} denotes the field strength associated with the connection A, as defined in (4.173).

Having this Weyl geometry available, we can proceed to covariantize the entire theory and establish Weyl counterparts for all geometric tensors defined on the fluid host background. This process clarifies the Weyl covariance present in the equations of motion (4.174)-(4.175). It is worth noting that in relativistic hydrodynamics, we necessitate the Weyl covariant derivative of the fluid velocity to be transverse and have zero trace (Campoleoni et al., 2019a):

$$u^b \hat{\nabla}_b u_a = 0\,, \qquad \hat{\nabla}_a u^a = 0\,. \tag{B.35}$$

This requirement uniquely determines the form of the Weyl connection (B.33).

Frame covariance aspects

We now turn our attention to the understanding of the hydrodynamic aspect of the Lorentz symmetry within the covariant Bondi gauge (4.164), which emerges during the examination of residual diffeomorphisms in this gauge relaxation.

This essence is rooted in the characteristic of relativistic fluids mentioned earlier in this subsection: locally, we can depict identically such a fluid with various velocity fields. These fields are interconnected through local Lorentz transformations, which, for instance, transition the fluid from the Cartan frame $\{\mathrm{u}/k, \star\mathrm{u}/k\}$ to the subsequent one (refer to equations (4.186) and (B.30)):

$$\begin{pmatrix} \mathrm{u}' \\ \star\mathrm{u}' \end{pmatrix} = \begin{pmatrix} \cosh\eta(x) & \sinh\eta(x) \\ \sinh\eta(x) & \cosh\eta(x) \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \star\mathrm{u} \end{pmatrix}. \tag{B.36}$$

Under these Lorentz boosts, the Weyl connection transforms as follows:

$$\mathrm{A}' = \mathrm{A} - \star\mathrm{d}\eta\,, \tag{B.37}$$

whereas we have previously observed in (4.188) that the dual of this form plays the role of the Lorentz connection (4.189).

Given the covariance property of the hydrodynamic frames stated above, the transformation (B.36) must represent a symmetry of the theory, ensuring $\widehat{\mathrm{T}}' = \widehat{\mathrm{T}}$. Consequently, the scalars ε and χ must undergo appropriate transformations under this velocity change:

$$\begin{pmatrix} \varepsilon' \\ \chi' \end{pmatrix} = \begin{pmatrix} \cosh 2\eta & -\sinh 2\eta \\ -\sinh 2\eta & \cosh 2\eta \end{pmatrix} \begin{pmatrix} \varepsilon \\ \chi \end{pmatrix} + \tau \sinh\eta \begin{pmatrix} \sinh\eta \\ -\cosh\eta \end{pmatrix}, \tag{B.38}$$

where $\tau' = \tau$ and we have assumed that the equation of state of the type $\varepsilon' = p'$ still holds in the new frame. We thus recover the gauge transformations of these energy and heat densities studied in the analysis of the asymptotic symmetries of the subsection 4.3.

Fluid/gravity correspondence

Let us briefly review how we can interpret the ansatz (4.164) in reconstructing the bulk holographically through the fluid/gravity correspondence.

Starting from the hydrodynamic data and equations of the relativistic conformal fluid discussed in various paragraphs of this subsection, which we consider from now on located at the boundary of the bulk Einstein spacetime, we follow the original approach outlined in (Bhattacharyya et al., 2008a; Haack & Yarom, 2008; Bhattacharyya et al., 2008b; Hubeny et al., 2012). This involves an inverse expansion of the radial component r of the bulk, where the Weyl transformations of the boundary (B.12) are compensated at each order by a radial rescaling of the form $r \to \mathfrak{B}(x^a)r$. Such a construction has been implemented in four space-time dimensions (Petkou et al., 2022; Mittal et al., 2023; Campoleoni et al., 2023b), resulting in an infinite polynomial expansion.

However, in three dimensions, where we observe that most of the geometric and hydrodynamic tensors of the dual two-dimensional fluid are identically zero, this expansion reduces to a finite sum, akin to the (Weyl-)Fefferman-Graham and Bondi expansions in this dimensionality. The approach highlighted above for solving Einstein's radial evolution equation leads to the ansatz (4.164) through purely Weyl-hydrogeometry holographic considerations. These resolutions of the radial equations, conceptualized analogously to solutions in classical mechanics, are obtained by evolving the equations of motion from initial conditions. Additionally, they identify the holographic viscous pressure as anomalously dependent on the Ricci of the boundary geometry (B.32).

The remaining Einstein equations further constrain the boundary data. As previously demonstrated in the analysis of the asymptotic solution space in the subsection 4.3, the bulk reconstructed metric (4.164) is indeed an exact solution of the asymptotically Anti de Sitter spacetime Einstein equations with $R = -6k^2$, if and only if the stress tensor (B.31) of the conformal relativistic fluid satisfies the constraint:

$$\nabla^a \widetilde{T}_{ab} = f_b \,, \tag{B.39}$$

where the external holographic force constraining fluid motion is defined as

$$f_b = -\nabla^a \widehat{T}_{ab} \tag{B.40}$$

such that

$$\begin{aligned}\widehat{T}_{ab}\mathrm{d}x^a\mathrm{d}x^b = {}& \frac{1}{8\pi G k^4}\left(u^a\partial_a\Theta + \star u^a\partial_a\Theta^\star - \frac{k^2}{2}R^{(0)}\right)(\mathrm{u}^2 + \star\mathrm{u}^2) \\ & - \frac{1}{4\pi G k^4}\star u^a\partial_a\Theta\,(\mathrm{u}\star\mathrm{u} + \star\mathrm{u}\,\mathrm{u})\,.\end{aligned} \tag{B.41}$$

This corresponds to the second part of the Brown-York tensor (4.176) of the covariant Bondi gauge, providing a holographic explanation for its appearance from fluid/gravitational duality. These constrained hydrodynamic equations can be reformulated as Weyl-covariant evolution equations for energy (4.174) and heat (4.175) densities:

$$(u^a + \star u^a)\widehat{\nabla}_a(\varepsilon + \chi) = \frac{1}{4\pi G}\big(\star u^a\partial_a F + 2\Theta^\star F\big) = \frac{1}{4\pi G}\star u^a\widehat{\nabla}_a F\,, \tag{B.42}$$

$$(u^a - \star u^a)\widehat{\nabla}_a(\varepsilon - \chi) = \frac{1}{4\pi G}\big(\star u^a\partial_a F + 2\Theta^\star F\big) = \frac{1}{4\pi G}\star u^a\widehat{\nabla}_a F\,. \tag{B.43}$$

B.2.2 Carrollian fluid

Randers-Papapetrou frame

We wish to probe the characteristics of what is known as a Carrollian fluid. A convenient approach for describing such exotic fluids within relativistic setups is to consider an ultrarelativistic limit using the hydrodynamic frame outlined in (Randers, 1941; Papapetrou, 1966)[2], rather than the Cartan frame (B.30):[3]

$$\mathrm{d}s^2_{\text{bdy}} = -k^2\left(\Omega \mathrm{d}u - b_\theta d\theta\right)^2 + \mathfrak{a}\mathrm{d}\theta^2\,, \tag{B.44}$$

where we introduce the arbitrary functions $\Omega(u,\theta)$, $b_\theta(u,\theta)$, and $\mathfrak{a}(u,\theta)$. It is assumed that all dependence on k is explicit in (B.44) (see (Freidel & Jai-akson, 2023) for relaxation of this assumption concerning the quantity $\mathfrak{a}$). This hydrodynamic frame proves to be efficient, as we shall observe in the following, since the metric (B.44) is well-suited for Carrollian diffeomorphisms.

In the so-called Randers-Papapetrou frame (B.44), we can generally decompose the velocity vector field u as:

$$u = u^a\partial_a = \widetilde{\gamma}(u,\theta)\left(\partial_u + v^\theta(u,\theta)\partial_\theta\right). \tag{B.45}$$

Keeping Carroll geometry concepts in mind, we parameterize the component $v^\theta = u^\theta/\widetilde{\gamma}$ using a quantity $\beta^\theta(u,\theta)$:

$$v^\theta = \frac{k^2\Omega\beta^\theta}{1+k^2\beta^\theta b_\theta}\,, \tag{B.46}$$

as well as

$$\widetilde{\gamma} = \frac{1+k^2\beta^\theta b_\theta}{\Omega\sqrt{1-\mathfrak{a}k^2(\beta^\theta)^2}}\,. \tag{B.47}$$

This parameterization will be justified below by the characteristic feature of a genuine Carrollian vector field attributed to $\beta = \beta^\theta\partial_\theta$[4] in the limit $k \to 0$.

Using the metric (B.44) and Hodge duality, we derive the following forms:

$$\mathrm{u} = -\frac{k^2}{\sqrt{1-\mathfrak{a}k^2(\beta^\theta)^2}}\left(\Omega \mathrm{d}u - (b_\theta + \mathfrak{a}\beta^\theta)\mathrm{d}\theta\right), \tag{B.48}$$

$$\star\mathrm{u} = k\sqrt{\mathfrak{a}}\Omega\widetilde{\gamma}\left(\mathrm{d}\theta - v^\theta \mathrm{d}u\right). \tag{B.49}$$

By reversing the process from the last expression, we deduce that the associated vector reads

$$\star u = \frac{k}{\sqrt{\mathfrak{a}}\sqrt{1-\mathfrak{a}k^2(\beta^\theta)^2}}\left(\frac{b_\theta+\mathfrak{a}\beta^\theta}{\Omega}\partial_u + \partial_\theta\right). \tag{B.50}$$

[2] It is also discussed in, e.g., (Gibbons et al., 2009; Leigh et al., 2012b,a).

[3] We recall that the time coordinate u should not be confused with the fluid vector velocity.

[4] It is important not to confuse this quantity with the scalar function of Carrollian boosts

Similarly, comparing with the expressions (B.25) and (B.26), we arrive at the analogous parametrization for the q_θ-component of the heat current and the $\tau_{\theta\theta}$-component of the viscosity tensor:

$$\chi = \frac{q_\theta}{k\sqrt{\mathfrak{a}}\Omega\tilde{\gamma}}\,, \qquad \tau = \frac{\tau_{\theta\theta}}{\mathfrak{a}\Omega^2\tilde{\gamma}^2}\,. \tag{B.51}$$

Carrollian geometry

After establishing these definitions and parametrizations, we can explore the ultrarelativistic limit at the level of the host geometry $\mathbb{R}\times\mathbb{L}$ of the fluid (B.44), where $u \in \mathbb{R}$ now functions as Carrollian time:

$$\mathrm{d}s^2_{\mathscr{C},\mathrm{bdy}} = \lim_{k\to 0}\mathrm{d}s^2_{\mathrm{bdy}} = 0\,\mathrm{d}u^2 + \mathfrak{a}\,\mathrm{d}\theta^2\,. \tag{B.52}$$

In this limit, we indeed encounter a degenerate Riemannian metric, indicative of reaching a Carroll geometry. Here, the Randers-Papapetrou quantity $\mathfrak{a}$ becomes the metric of the one-dimensional space $\mathbb{L}$:

$$\mathrm{d}s^2_{\mathbb{L}} = \mathfrak{a}\,\mathrm{d}\theta^2\,, \qquad \mathfrak{a} := \mathfrak{a}_{\theta\theta} = \frac{1}{\mathfrak{a}^{\theta\theta}}\,. \tag{B.53}$$

This one-dimensional metric is responsible for raising and lowering angular indices, for instance: $b^\theta = b\theta/\mathfrak{a}$ and $b_\theta = \mathfrak{a}\,b^\theta$. This highlights one of the advantages of the Carrollian Randers-Papapetrou frame (B.53). Notably, in the limit $k \to 0$, both the form $\mathrm{b} = b_\theta \mathrm{d}\theta$ and the scalar Ω persist[5].

The Carrollian diffeomorphisms (Lévy-Leblond, 1965; Sen Gupta, 1966), defined as,

$$u' = u'(u,\theta)\,, \qquad \theta' = \theta'(\theta)\,, \tag{B.54}$$

have Jacobian functions given by

$$J_u(u,\theta) = \frac{\partial u'}{\partial u}\,, \quad j_{u\theta}(u,\theta) = \frac{\partial u'}{\partial \theta}\,, \quad j_{\theta u}(u,\theta) = \frac{\partial \theta'}{\partial u} = 0\,, \quad J_\theta(u,\theta) = \frac{\partial \theta'}{\partial \theta}\,. \tag{B.55}$$

Consequently, the transformation of derivatives of a scalar function read

$$\partial_u{}' = J_u{}^{-1}\partial_u\,, \qquad \partial_\theta{}' = J_\theta{}^{-1}\left(\partial_\theta - j_{u\theta}J_u{}^{-1}\partial_u\right). \tag{B.56}$$

described in the parameterization in the conformal gauge (4.283) of the covariant Bondi gauge, nor the gauge condition in the standard Bondi ansatz (4.128).

[5] We assume Ω is non-zero. One should carefully revisit the scenario where it equals zero at isolated points, as it leads to what are known as Carrollian black holes (Ecker et al., 2023).

Observing that the spatial derivative does not follow a typical form transformation law, we introduce the Carrollian derivative:

$$\widehat{\partial}_u = \partial_u \,, \qquad \widehat{\partial}_\theta = \partial_\theta + \frac{b_\theta}{\Omega}\partial_u \,, \tag{B.57}$$

so that

$$\widehat{\partial}'_\theta = {j_{u\theta}}^{-1}\widehat{\partial}_\theta \,. \tag{B.58}$$

By adopting this definition, the transformations of other tensor quantities are rendered covariant by introducing a covariant Carroll derivative $\widehat{\nabla}_\theta$, equipped with a Levi-Civita-Carroll connection (Ciambelli et al., 2018b; Ciambelli & Marteau, 2019),

$$\widehat{\Gamma}^\theta_{\theta\theta} = \widehat{\partial}_\theta \ln\sqrt{\mathfrak{a}} \,. \tag{B.59}$$

In this manner, we recognize another advantage of (B.53) through its explicit accommodation of Carrollian diffeomorphisms (B.54). Following the terminology of (Duval et al., 2014a), the Carroll geometry described above is characterized as having weak Carrollian structure, where the form $\mathrm{b} = b_\theta \mathrm{d}\theta$ is analogous to the Ehresmann connection in manifold fiber bundle perception (Ciambelli et al., 2019). Besides, it is clear from the aforementioned transformations that the scalar Ω varies as a density.

Conformal aspects

We recall an intriguing aspect of the fluid/gravity correspondence: the boundary where the hydrodynamic equations are formulated is a conformal boundary. This feature enables the hydrogeometry to incorporate Weyl covariance, akin to the Weyl-Fefferman-Graham (4.78), even in the Carrollian limit (B.53). To align the standard Randers-Papapetrou frame (B.44) with the conventional Weyl rescalings (B.12), we impose the following Weyl weights on the Carrollian geometry objects:

$$\mathfrak{a} \to \frac{\mathfrak{a}}{\mathfrak{B}^2} \,, \quad b_\theta \to \frac{b_\theta}{\mathfrak{B}} \,, \quad \Omega \to \frac{\Omega}{\mathfrak{B}} \,, \quad \beta_\beta \to \frac{\beta_\theta}{\mathfrak{B}} \,. \tag{B.60}$$

Once more, the Levi-Civita-Carroll derivative fails to transform covariantly under these transformations. Thus, we still aim to achieve covariance with respect to these rescalings. To this end, we introduce the following quantities, which transform as connections under Weyl:

$$\varphi_\theta = \frac{1}{\Omega}(\partial_u b_\theta + \partial_\theta \Omega) = \partial_u \frac{b_\theta}{\Omega} + \widehat{\partial}_\theta \ln \Omega \,, \tag{B.61}$$

$$\theta = \frac{1}{\Omega}\partial_u \ln\sqrt{\mathfrak{a}} \,. \tag{B.62}$$

Indeed, these undergo the following transformations:

$$\varphi_\theta \to \varphi_\theta - \widehat{\partial}_\theta \ln \mathfrak{B}\,, \qquad \uptheta \to \mathfrak{B}\uptheta - \frac{1}{\Omega}\partial_u \mathfrak{B}\,. \tag{B.63}$$

We refer to these quantities as Carrollian acceleration and expansion, respectively. While their significance will become clearer in the subsequent paragraph, we already note the association of the second quantity with the equation (4.260). Similarly, in the forthcoming discussion, we will elucidate how these expressions relate to the Carrollian Weyl connection, as defined by the flat limit in the equation (4.261) of the covariant Bondi gauge (4.262) for asymptotically flat spaces.

According to these definitions, let us introduce the Weyl-Carroll spatial and temporal covariant derivatives (Ciambelli et al., 2018a) symbolically as:

$$\widehat{\mathfrak{D}}_\theta = \widehat{\partial}_\theta + w\varphi_\theta\,, \qquad \widehat{\mathfrak{D}}_u = \partial_u + \Omega w \theta\,. \tag{B.64}$$

While applying $\widehat{\mathfrak{D}}_\theta$ does not alter the conformal weight w of the tensor it operates on, $\widehat{\mathfrak{D}}_u$ increases it by one unit upon its action. Indeed, for instance,

$$\widehat{\mathfrak{D}}_\theta V^\theta = \widehat{\nabla}_\theta V^\theta + (w-1)\varphi_\theta V^\theta\,, \tag{B.65}$$

$$\frac{1}{\Omega}\widehat{\mathfrak{D}}_u V^\theta = \frac{1}{\Omega}\partial_u V^\theta + w\uptheta V^\theta\,. \tag{B.66}$$

Just like in the relativistic case, this connection maintains metric compatibility and exhibits effective torsion, as illustrated by the sole non-zero component (coming from the commutation of $\widehat{\mathfrak{D}}_\theta$ with $\frac{1}{\Omega}\widehat{\mathfrak{D}}_u$) of the following curvature (of Weyl weight equal to 1):

$$\mathscr{R}_\theta = \frac{1}{\Omega}\partial_u \varphi_\theta - \uptheta\varphi_\theta - \widehat{\partial}_\theta \uptheta\,. \tag{B.67}$$

Carrollian hydrodynamics

We can now establish the equations governing a conformal Carrollian fluid. These equations are derived from taking the limit $k \to 0$ of the relativistic relation $\nabla^a \widetilde{T}_{ab} = f_b$ (B.20). It is essential to construct these Carrollian expressions in a manner that ensures a smooth ultrarelativistic limit, ensuring Weyl covariance.

Considering this second aspect, we need to categorize the various Weyl-covariant quantities derived from the definitions in the preceding paragraphs based on their Weyl weight. Firstly, we note the residual presence of the kinematic variable $\upbeta = \beta_\theta \mathrm{d}\theta$ from the limit of the relativistic velocity parameterization (B.48) in the Randers-Papapetrou frame (B.44). This indicates that the Carrollian fluid under consideration cannot be considered rest but with a velocity parameterized by $\upbeta$. Nevertheless, strictly speaking, we should not interpret it as a velocity since motion is prohibited in Carrollian spacetimes. Instead, it is better understood as an inverse velocity, describing a temporal frame and serving a dual role. This leads us to define the acceleration observable $\upgamma = \gamma_\theta \mathrm{d}\theta$ as:

$$\gamma_\theta = \frac{1}{\Omega}\partial_u \beta_\theta\,. \tag{B.68}$$

However, since this quantity is not Weyl-covariant, we refine it into a quantity with zero conformal weight as follows:

$$\delta_\theta = \frac{1}{\Omega}\hat{\mathfrak{D}}_u \beta_\theta = \gamma_\theta - \uptheta\beta_\theta = \frac{\sqrt{\mathfrak{a}}}{\Omega}\partial_u \frac{\beta_\theta}{\sqrt{\mathfrak{a}}}\,. \tag{B.69}$$

Let us proceed to discuss quantities with unit Weyl weight. We define the Carrollian hyperacceleration as:

$$\mathscr{A}_\theta = \frac{1}{\Omega}\hat{\mathfrak{D}}_u \frac{1}{\Omega}\hat{\mathfrak{D}}_u \beta_\theta = \frac{1}{\Omega}\partial_u\left(\frac{1}{\Omega}\partial_u\beta_\theta - \uptheta\beta_\theta\right), \tag{B.70}$$

which is a Carroll one-form with a conformal weight of 1. As they share the same weight, we can combine this quantity with the curvature one-form (B.67):

$$s_\theta = \mathscr{A}_\theta + \mathscr{R}_\theta = \frac{1}{\Omega}\partial_u\left(\frac{1}{\Omega}\partial_u\beta_\theta - \uptheta\beta_\theta\right) + \frac{1}{\Omega}\partial_u\varphi_\theta - \uptheta\varphi_\theta - \hat{\partial}_\theta\theta\,. \tag{B.71}$$

With the contraction brought by the line metric $\mathfrak{a}$, we can define a Carrollian scalar with a conformal weight of 2:

$$s = \frac{s_\theta}{\sqrt{\mathfrak{a}}}\,. \tag{B.72}$$

This particular definition will become more significant in a few pages from now on since we will reveal that it corresponds to the $k \to 0$ limit of relativistic Weyl curvature:[6]

$$s = -\mathcal{F} = -\lim_{k\to 0} k\,F\,. \tag{B.73}$$

This leads us to consider the first aspect discussed at the beginning of this point: ensuring the smoothness of the ultrarelativistic limit. To achieve this,

[6]Note the opposite sign compared to the definition of $\mathcal{F}$ in (4.261).

a straightforward approach is proposed, suggesting a scaling of all relativistic quantities by powers of k (Ciambelli et al., 2018b). The Carrollian energy density $\varepsilon = \epsilon + \mathcal{O}(k^2)$ and pressure $p = p + \mathcal{O}(k^2)$, which are still linked by the equation of state $\epsilon = p$ for simplicity, are obtained through the Carroll limit ($k \to 0$) and consistently have a conformal weight of 2. Introducing the ultrarelativistic heat current $\pi_\theta(u, \theta)$ from its relativistic counterpart as:

$$q^\theta = k^2 \pi^\theta + O(k^4)\,, \tag{B.74}$$

it then carries a conformal weight of 1. We have omitted the possibility of a term of order $\mathcal{O}(1)$ in the speed of light since, unlike possible scenarii in higher dimensions, this term diverges in the reconstruction of the three-dimensional holographic bulk by the fluid/gravity correspondence. Comparing with the expression (B.51) of the Randers-Papapetrou frame (B.44), we infer the following behavior of the local relativistic heat density χ:

$$\chi = k\alpha + O(k^3)\,, \tag{B.75}$$

where

$$\alpha = \pi^\theta \sqrt{\mathfrak{a}}\,. \tag{B.76}$$

Similarly, using the relations (B.25) and (B.26), we define the zero-weight Carrollian viscosity tensors $\Sigma = \Sigma_{\theta\theta}\mathrm{d}\theta^2$ and $\Xi = \Xi_{\theta\theta}\mathrm{d}\theta^2$ as:

$$\tau^{\theta\theta} = -\frac{\Sigma^{\theta\theta}}{k^2} - \Xi^{\theta\theta} + O(k^2)\,. \tag{B.77}$$

Therefore, comparing with (B.51), we deduce for the viscosity scalar that:

$$\tau = \frac{\tau_\Sigma}{k^2} + \tau_\Xi + O(k^2)\,, \tag{B.78}$$

where we read (in this expression, we must understand the notation such that $\upbeta^2 = \beta^\theta \beta_\theta$)

$$\Sigma^\theta{}_\theta = -\tau_\Sigma\,, \qquad \Xi^\theta{}_\theta = -\tau_\Xi - \upbeta^2 \tau_\Sigma\,. \tag{B.79}$$

We can now derive the hydrodynamic equations describing such a peculiar fluid, enabling a Weyl-Carroll-covariant parametrization suited for holographic study, by smoothly transitioning from the relativistic framework (Campoleoni et al., 2019a). The resulting equations take the following form:

$$-\left(\frac{1}{\Omega}\partial_u + 2\uptheta\right)\left(\epsilon - \upbeta^2 \Sigma^\theta{}_\theta\right) + (\widehat{\nabla}^\theta + 2\varphi^\theta)(\beta_\theta \Sigma^\theta{}_\theta) + \uptheta(\Xi^\theta{}_\theta - \upbeta^2 \Sigma^\theta{}_\theta) = e\,, \tag{B.80}$$

$$\theta\Sigma^{\theta}{}_{\theta} = f\,, \tag{B.81}$$

$$(\widehat{\nabla}_{\theta} + \varphi_{\theta})(\epsilon - \Xi^{\theta}{}_{\theta}) + \varphi_{\theta}(\epsilon - \beta^2\Sigma^{\theta}{}_{\theta}) + \left(\frac{1}{\Omega}\partial_u + \theta\right)(\pi_{\theta} + \beta_{\theta}(2\epsilon - \Xi^{\theta}{}_{\theta})) = g_{\theta}\,, \tag{B.82}$$

$$-(\widehat{\nabla}_{\theta} + \varphi_{\theta})\Sigma^{\theta}{}_{\theta} - \left(\frac{1}{\Omega}\partial_u + \theta\right)(\beta_{\theta}\Sigma^{\theta}{}_{\theta}) = h_{\theta}\,. \tag{B.83}$$

These equations are obtained under the assumption of specific scalings for the external force density constraining the fluid:

$$\frac{k}{\Omega}f_u = \frac{f}{k^2} + e + O(k^2)\,, \qquad f^{\theta} = \frac{h^{\theta}}{k^2} + g^{\theta} + O(k^2)\,. \tag{B.84}$$

Fluid/gravity correspondence

We now arrive at an understanding of the ansatz (4.262) for asymptotically flat spacetimes within the covariant Bondi gauge as a reconstruction of the bulk via the Carrollian fluid/gravity correspondence. In this context, the vanishing of the cosmological constant in the bulk implies a similar limit of the speed of light on the boundary. Consequently, we anticipate describing not a relativistic fluid but its ultrarelativistic counterpart. Similar to the analysis in AdS, we can envision the two-dimensional Weyl-Carroll hydrodynamics, established in the preceding paragraphs, as a null infinity dataset of a three-dimensional Ricci-flat space. These data allow us to holographically reconstruct the gravity confined within this conformal asymptotic boundary.

A more efficient approach involves starting from the bulk metric (4.262), obtained by taking the flat limit of (4.164), whose Weyl-hydrogeometric interpretation has been established in previous analyses. We then proceed to take the equivalent Carrollian limit at the level of the hydrodynamic equations and verify that they align with those we have derived for a Carroll conformal fluid. To achieve this, we first consider the fluid in the relativistic Randers-Papapetrou frame (B.44), adopt the prescriptions for the behavior of hydrodynamic quantities for low values of k outlined in the previous paragraphs, and incorporate the expressions for kinematic variables within this framework. By utilizing the parametrizations (B.48) and (B.49) of congruences, we derive the following behaviors:

$$\mathrm{u} = -k^2\left(\Omega \mathrm{d}u - (b_{\theta} + \beta_{\theta})\mathrm{d}\theta\right) + \mathcal{O}(k^4)\,, \qquad \star\mathrm{u} = k\sqrt{\mathfrak{a}}\mathrm{d}\theta + \mathcal{O}(k^3)\,, \tag{B.85}$$

which imply the subsequent relationships for relativistic expansion, acceleration, and Weyl connection:

$$\Theta = \theta + \mathcal{O}(k^2)\,, \tag{B.86}$$

$$\mathrm{a} = k^2(\varphi_\theta + \gamma_\theta)\mathrm{d}\theta + \mathcal{O}(k^4)\,, \tag{B.87}$$

$$\mathrm{A} = \theta\Omega\mathrm{d}u + (\alpha_\theta + \delta_\theta)\mathrm{d}\theta + \mathcal{O}(k^2)\,. \tag{B.88}$$

In the ansatz (4.164), the viscous pressure takes a particular holographic expression in terms of the curvature $R^{(0)}$ of the fluid host geometry, given by the equation (B.32). In the parametrization (B.44), the Ricci scalar of the boundary is expressed as:

$$R^{(0)} = \frac{2}{k^2}\left(\theta^2 + \frac{1}{\Omega}\partial_u\theta\right) - 2\left(\widehat{\nabla}_\theta + \varphi_\theta\right)\varphi^\theta\,. \tag{B.89}$$

Utilizing the relations (B.78) and (B.79), we derive the Carrollian counterpart of (B.32):

$$\tau_\Sigma = \frac{1}{4\pi G}\left(\theta^2 + \frac{\partial_u\theta}{\Omega}\right), \qquad \tau_\Xi = -\frac{1}{4\pi G}\left(\widehat{\nabla}_\theta + \varphi_\theta\right)\varphi^\theta\,, \tag{B.90}$$

where

$$\Sigma^\theta{}_\theta = -\frac{1}{4\pi G}\left(\theta^2 + \frac{\partial_u\theta}{\Omega}\right), \tag{B.91}$$

$$\Xi^\theta{}_\theta = \frac{1}{4\pi G}\left(\left(\widehat{\nabla}_\theta + \varphi_\theta\right)\varphi^\theta - \beta^2\left(\theta^2 + \frac{\partial_u\theta}{\Omega}\right)\right). \tag{B.92}$$

Combining all the components, we arrive at the following smooth flat limit of the bulk metric (4.164) (Campoleoni et al., 2019a):

$$\begin{aligned} \mathrm{d}s^2 = &-2(\Omega\mathrm{d}u - \mathrm{b} - \beta)(\mathrm{d}r + r(\varphi + \gamma + \theta(\Omega\mathrm{d}u - \mathrm{b} - \beta))) \\ &+ r^2\mathrm{d}s^2_{\mathbb{L}} + 8\pi G(\Omega\mathrm{d}u - \mathrm{b} - \beta)(\epsilon(\Omega\mathrm{d}u - \mathrm{b} - \beta) - \pi_\theta\mathrm{d}\theta)\,. \end{aligned} \tag{B.93}$$

Employing the redefinitions of the Randers-Papapetrou velocity forms in terms of μ and $\star\mu$ (see (4.255)), we observe that this metric (B.93), along with (B.90), precisely corresponds to the line element (4.262) explored in Bondi's asymptotically flat covariant gauge (Ciambelli et al., 2020a,b) with $R = 0$. This correspondence validates the earlier mentioned relations for the Weyl-Carroll connection and its curvature:

$$\mathcal{A} = \theta\Omega\mathrm{d}u + (\alpha_\theta + \delta_\theta)\mathrm{d}\theta\,, \qquad s = -\mathcal{F}\,. \tag{B.94}$$

Similar to its AdS predecessor, the asymptotically flat metric (B.93) remains invariant under Weyl rescalings $r \to \mathfrak{B} r$ and satisfies the Einstein radial equation, with the remaining equations further constraining its defining quantities. These additional equations stem from the Carrollian limit of (B.42) and (B.43), namely (4.263) and (4.264), which can be expressed as follows:

$$\frac{1}{\Omega}\widehat{\mathfrak{D}}_u \epsilon + \frac{1}{4\pi G}\left(\frac{2s_\theta}{\Omega}\widehat{\mathfrak{D}}_u \beta^\theta + \frac{\beta_\theta}{\Omega}\widehat{\mathfrak{D}}_u s^\theta + \widehat{\mathfrak{D}}^\theta s_\theta\right) = 0\,, \tag{B.95}$$

$$\widehat{\mathfrak{D}}_\theta \epsilon - \frac{\beta_\theta}{\Omega}\widehat{\mathfrak{D}}_u \epsilon + \frac{1}{\Omega}\widehat{\mathfrak{D}}_u(\pi_\theta + 2\epsilon\beta_\theta) = 0\,. \tag{B.96}$$

These equations precisely correspond to the Carrollian hydrodynamic equations derived in (B.80)-(B.83). Notably, upon considering the Carrollian force and power densities derived from the limits of (B.40) and (B.41), the equations (B.81) and (B.83) are automatically satisfied. This leaves us with only the ultrarelativistic fluid equations of motion (B.80) and (B.82), which align with the last two equations above. Thus, this appendix concludes our exploration of the Weyl-Carroll-hydrogeometric interpretation of the asymptotic boundary theory described through the flat covariant Bondi gauge.

Bibliography

Abbott, L. F., & Deser, S. (1982a). Charge Definition in Nonabelian Gauge Theories. *Phys. Lett. B*, *116*, 259–263.

Abbott, L. F., & Deser, S. (1982b). Stability of Gravity with a Cosmological Constant. *Nucl. Phys. B*, *195*, 76–96.

Achucarro, A., & Townsend, P. K. (1986). A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories. *Phys. Lett. B*, *180*, 89.

Adami, H., Grumiller, D., Sadeghian, S., Sheikh-Jabbari, M. M., & Zwikel, C. (2020a). T-Witts from the horizon. *JHEP*, *04*, 128.

Adami, H., Grumiller, D., Sheikh-Jabbari, M. M., Taghiloo, V., Yavartanoo, H., & Zwikel, C. (2021a). Null boundary phase space: slicings, news & memory. *JHEP*, *11*, 155.

Adami, H., Mao, P., Sheikh-Jabbari, M. M., Taghiloo, V., & Yavartanoo, H. (2022). Symmetries at causal boundaries in 2D and 3D gravity. *JHEP*, *05*, 189.

Adami, H., Parvizi, A., Sheikh-Jabbari, M. M., Taghiloo, V., & Yavartanoo, H. (2023). Hydro & thermo dynamics at causal boundaries, examples in 3d gravity. *JHEP*, *07*, 038.

Adami, H., Sheikh-Jabbari, M. M., Taghiloo, V., Yavartanoo, H., & Zwikel, C. (2020b). Symmetries at null boundaries: two and three dimensional gravity cases. *JHEP*, *10*, 107.

Adami, H., Sheikh-Jabbari, M. M., Taghiloo, V., Yavartanoo, H., & Zwikel, C. (2021b). Chiral Massive News: Null Boundary Symmetries in Topologically Massive Gravity. *JHEP*, *05*, 261.

Adamo, T., Mason, L., & Sharma, A. (2019). Celestial amplitudes and conformal soft theorems. *Class. Quant. Grav.*, *36*(20), 205018.

Aharony, O., Gubser, S. S., Maldacena, J. M., Ooguri, H., & Oz, Y. (2000). Large N field theories, string theory and gravity. *Phys. Rept.*, *323*, 183–386.

Alessio, F., Barnich, G., Ciambelli, L., Mao, P., & Ruzziconi, R. (2021). Weyl charges in asymptotically locally AdS_3 spacetimes. *Phys. Rev. D*, *103*(4), 046003.

Allemandi, G., Francaviglia, M., & Raiteri, M. (2003). Covariant charges in Chern-Simons AdS(3) gravity. *Class. Quant. Grav.*, *20*, 483–506.

Anastasiou, G., Miskovic, O., Olea, R., & Papadimitriou, I. (2020). Counterterms, Kounterterms, and the variational problem in AdS gravity. *JHEP*, *08*, 061.

Anderson, I. M. (1989). The variational bicomplex. Tech. rep., Utah State Technical Report, 1989, http://math. usu. edu/ fg mp.

Anninos, D., Hartman, T., & Strominger, A. (2017). Higher Spin Realization of the dS/CFT Correspondence. *Class. Quant. Grav.*, *34*(1), 015009.

Arnowitt, R. L., Deser, S., & Misner, C. W. (1962). The Dynamics of general relativity. *Gen. Rel. Grav.*, *40*, 1997–2027.

Ashtekar, A. (2014). Geometry and Physics of Null Infinity.

Ashtekar, A., Bonga, B., & Kesavan, A. (2015). Asymptotics with a positive cosmological constant: I. Basic framework. *Class. Quant. Grav.*, *32*(2), 025004.

Ashtekar, A., Campiglia, M., & Laddha, A. (2018). Null infinity, the BMS group and infrared issues. *Gen. Rel. Grav.*, *50*(11), 140–163.

Ashtekar, A., & Streubel, M. (1981). Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity. *Proc. Roy. Soc. Lond.*, *A376*, 585–607.

Bagchi, A., Banerjee, S., Basu, R., & Dutta, S. (2022). Scattering Amplitudes: Celestial and Carrollian. *Phys. Rev. Lett.*, *128*(24), 241601.

Bagchi, A., Basu, R., Kakkar, A., & Mehra, A. (2016). Flat Holography: Aspects of the dual field theory. *JHEP*, *12*, 147.

Bagchi, A., Dutta, S., Kolekar, K. S., & Sharma, P. (2021). BMS field theories and Weyl anomaly. *JHEP*, *07*, 101.

Balasubramanian, V., & Kraus, P. (1999). A Stress tensor for Anti-de Sitter gravity. *Commun. Math. Phys.*, *208*, 413–428.

Ball, A., Himwich, E., Narayanan, S. A., Pasterski, S., & Strominger, A. (2019). Uplifting AdS_3/CFT_2 to flat space holography. *JHEP*, *08*, 168.

Banados, M. (1996). Global charges in Chern-Simons field theory and the (2+1) black hole. *Phys. Rev. D*, *52*, 5816–5825.

Banados, M. (1999). Three-dimensional quantum geometry and black holes. *AIP Conf. Proc.*, *484*(1), 147–169.

Banados, M., Henneaux, M., Teitelboim, C., & Zanelli, J. (1993). Geometry of the (2+1) black hole. *Phys. Rev. D*, *48*, 1506–1525. [Erratum: Phys.Rev.D 88, 069902 (2013)].

Banados, M., Teitelboim, C., & Zanelli, J. (1992). The Black hole in three-dimensional space-time. *Phys. Rev. Lett.*, *69*, 1849–1851.

Barnich, G. (2003). Boundary charges in gauge theories: Using Stokes theorem in the bulk. *Class. Quant. Grav.*, *20*, 3685–3698.

Barnich, G., & Brandt, F. (2002). Covariant theory of asymptotic symmetries, conservation laws and central charges. *Nucl. Phys.*, *B633*, 3–82.

Barnich, G., & Compere, G. (2007). Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions. *Class. Quant. Grav.*, *24*, F15–F23.

Barnich, G., & Compère, G. (2008). Surface charge algebra in gauge theories and thermodynamic integrability. *J. Math. Phys.*, *49*, 042901.

Barnich, G., & Del Monte, F. (2018). Introduction to Classical Gauge Field Theory and to Batalin-Vilkovisky Quantization.

Barnich, G., Gomberoff, A., & Gonzalez, H. A. (2012). The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes. *Phys. Rev. D*, *86*, 024020.

Barnich, G., & Lambert, P.-H. (2013). Einstein-Yang-Mills theory: Asymptotic symmetries. *Phys. Rev. D*, *88*, 103006.

Barnich, G., Lambert, P.-H., & Mao, P. (2015). Three-dimensional asymptotically flat Einstein–Maxwell theory. *Class. Quant. Grav.*, *32*(24), 245001.

Barnich, G., & Troessaert, C. (2010). Aspects of the BMS/CFT correspondence. *JHEP*, *05*, 062.

Barnich, G., & Troessaert, C. (2011). BMS charge algebra. *JHEP*, *12*, 105.

Bautier, K., Englert, F., Rooman, M., & Spindel, P. (2000). The Fefferman-Graham ambiguity and AdS black holes. *Phys. Lett. B*, *479*, 291–298.

Bekenstein, J. D. (1972). Black holes and the second law. *Lett. Nuovo Cim.*, *4*, 737–740.

Belinfante, F. (1940). On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields. *Physica*, *7*(5), 449–474.

Benguria, R., Cordero, P., & Teitelboim, C. (1977). Aspects of the Hamiltonian Dynamics of Interacting Gravitational Gauge and Higgs Fields with Applications to Spherical Symmetry. *Nucl. Phys. B*, *122*, 61–99.

Bergshoeff, E., Gomis, J., Rollier, B., Rosseel, J., & ter Veldhuis, T. (2017). Carroll versus Galilei Gravity. *JHEP*, *03*, 165.

Bertlmann, R. A. (1996). *Anomalies in quantum field theory*. The International series of monographs on physics 91 Oxford science publications. Clarendon Press.

Bhattacharyya, S., Hubeny, V. E., Minwalla, S., & Rangamani, M. (2008a). Nonlinear Fluid Dynamics from Gravity. *JHEP*, *02*, 045.

Bhattacharyya, S., Loganayagam, R., Mandal, I., Minwalla, S., & Sharma, A. (2008b). Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions. *JHEP*, *12*, 116.

Bianchi, M., Freedman, D. Z., & Skenderis, K. (2002). Holographic renormalization. *Nucl. Phys. B*, *631*, 159–194.

Bondi, H., van der Burg, M. G. J., & Metzner, A. W. K. (1962). Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems. *Proc. Roy. Soc. Lond.*, *A269*, 21.

Bosma, J., Geiller, M., Majumdar, S., & Oblak, B. (2024). Radiative Asymptotic Symmetries of 3D Einstein-Maxwell Theory. *SciPost Phys.*, *16*, 092.

Boulanger, N. (1999). *L'anomalie conforme en dimensions 4 et 6; Dérivation par la méthode de DeWitt-Schwinger et via la correspondance AdS/CFT*. Master's thesis, ULB - Université Libre de Bruxelles [Faculté des Sciences], Brussels, Belgium. Mémoire de licence.

Brown, J. D., & Henneaux, M. (1986). Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity. *Commun. Math. Phys.*, *104*, 207–226.

Brown, J. D., & York, J. W., Jr. (1993). Quasilocal energy and conserved charges derived from the gravitational action. *Phys. Rev. D*, *47*, 1407–1419.

Bunster, C., Henneaux, M., Perez, A., Tempo, D., & Troncoso, R. (2014). Generalized Black Holes in Three-dimensional Spacetime. *JHEP*, *05*, 031.

Cachazo, F., Spradlin, M., & Volovich, A. (2006). Hidden beauty in multiloop amplitudes. *JHEP*, *07*, 007.

Campiglia, M., & Laddha, A. (2014). Asymptotic symmetries and subleading soft graviton theorem. *Phys. Rev.*, *D90*(12), 124028.

Campoleoni, A., Ciambelli, L., Delfante, A., Marteau, C., Petropoulos, P. M., & Ruzziconi, R. (2022). Holographic Lorentz and Carroll frames. *JHEP*, *12*, 007.

Campoleoni, A., Ciambelli, L., Marteau, C., Petropoulos, P. M., & Siampos, K. (2019a). Two-dimensional fluids and their holographic duals. *Nucl. Phys. B*, *946*, 114692.

Campoleoni, A., Delfante, A., Francia, D., & Heissenberg, C. (2023a). Renormalization of spin-one asymptotic charges in AdS_D. *JHEP*, *12*, 061. [Erratum: JHEP 03, 038 (2024)].

Campoleoni, A., Delfante, A., Pekar, S., Petropoulos, P. M., Rivera-Betancour, D., & Vilatte, M. (2023b). Flat from anti de Sitter. *JHEP*, *12*, 078.

Campoleoni, A., Francia, D., & Heissenberg, C. (2017). On higher-spin supertranslations and superrotations. *JHEP*, *05*, 120.

Campoleoni, A., Francia, D., & Heissenberg, C. (2018a). Asymptotic Charges at Null Infinity in Any Dimension. *Universe*, *4*(3), 47.

Campoleoni, A., Francia, D., & Heissenberg, C. (2018b). Asymptotic symmetries and charges at null infinity: from low to high spins. *EPJ Web Conf.*, *191*, 06011.

Campoleoni, A., Francia, D., & Heissenberg, C. (2019b). Electromagnetic and color memory in even dimensions. *Phys. Rev. D*, *100*(8), 085015.

Campoleoni, A., Francia, D., & Heissenberg, C. (2020). On asymptotic symmetries in higher dimensions for any spin. *JHEP*, *12*, 129.

Campoleoni, A., & Fredenhagen, S. (2024). Higher-spin gauge theories in three spacetime dimensions, arXiv:2403.16567.

Campoleoni, A., Fredenhagen, S., Pfenninger, S., & Theisen, S. (2010). Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields. *JHEP*, *11*, 007.

Capone, F., Mitra, P., Poole, A., & Tomova, B. (2023). Phase space renormalization and finite BMS charges in six dimensions. *JHEP*, *11*, 034.

Carlip, S. (1995). The (2+1)-Dimensional black hole. *Class. Quant. Grav.*, *12*, 2853–2880.

Carlip, S. (2005). Dynamics of asymptotic diffeomorphisms in (2+1)-dimensional gravity. *Class. Quant. Grav.*, *22*, 3055–3060.

Chandrasekaran, V., Flanagan, E. E., Shehzad, I., & Speranza, A. J. (2022). A general framework for gravitational charges and holographic renormalization. *Int. J. Mod. Phys. A*, *37*(17), 2250105.

Chruściel, P. T., MacCallum, M. A., & Singleton, D. B. (1995). Gravitational waves in general relativity xiv. bondi expansions and the 'polyhomogeneity'of scri. *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences*, *350*(1692), 113–141.

Ciambelli, L. (2023). From Asymptotic Symmetries to the Corner Proposal. *PoS*, *Modave2022*, 002.

Ciambelli, L., Delfante, A., Ruzziconi, R., & Zwikel, C. (2023). Symmetries and charges in Weyl-Fefferman-Graham gauge. *Phys. Rev. D*, *108*(12), 126003.

Ciambelli, L., & Leigh, R. G. (2020). Weyl Connections and their Role in Holography. *Phys. Rev. D*, *101*(8), 086020.

Ciambelli, L., & Leigh, R. G. (2021). Isolated surfaces and symmetries of gravity. *Phys. Rev. D*, *104*(4), 046005.

Ciambelli, L., & Leigh, R. G. (2023). Universal corner symmetry and the orbit method for gravity. *Nucl. Phys. B*, *986*, 116053.

Ciambelli, L., Leigh, R. G., Marteau, C., & Petropoulos, P. M. (2019). Carroll Structures, Null Geometry and Conformal Isometries. *Phys. Rev. D*, *100*(4), 046010.

Ciambelli, L., Leigh, R. G., & Pai, P.-C. (2022). Embeddings and Integrable Charges for Extended Corner Symmetry. *Phys. Rev. Lett.*, *128*.

Ciambelli, L., & Marteau, C. (2019). Carrollian conservation laws and Ricci-flat gravity. *Class. Quant. Grav.*, *36*(8), 085004.

Ciambelli, L., Marteau, C., Petkou, A. C., Petropoulos, P. M., & Siampos, K. (2018a). Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids. *Class. Quant. Grav.*, *35*(16), 165001.

Ciambelli, L., Marteau, C., Petkou, A. C., Petropoulos, P. M., & Siampos, K. (2018b). Flat holography and Carrollian fluids. *JHEP*, *07*, 165.

Ciambelli, L., Marteau, C., Petropoulos, P. M., & Ruzziconi, R. (2020a). Fefferman-Graham and Bondi Gauges in the Fluid/Gravity Correspondence. *PoS*, *CORFU2019*, 154.

Ciambelli, L., Marteau, C., Petropoulos, P. M., & Ruzziconi, R. (2020b). Gauges in Three-Dimensional Gravity and Holographic Fluids. *JHEP*, *11*, 092.

Compère, G., & Fiorucci, A. (2018). Advanced Lectures on General Relativity.

Compère, G., Fiorucci, A., & Ruzziconi, R. (2018). Superboost transitions, refraction memory and super-Lorentz charge algebra. *JHEP*, *11*, 200.

Compère, G., Fiorucci, A., & Ruzziconi, R. (2020). The Λ-BMS$_4$ charge algebra. *JHEP*, *10*, 205.

Compere, G., & Marolf, D. (2008). Setting the boundary free in AdS/CFT. *Class. Quant. Grav.*, *25*, 195014.

Compère, G., Song, W., & Strominger, A. (2013). New Boundary Conditions for AdS3. *JHEP*, *05*, 152.

Corley, S. (2000). A Note on holographic Ward identities. *Phys. Lett. B*, *484*, 141–148.

Coussaert, O., Henneaux, M., & van Driel, P. (1995). The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant. *Class. Quant. Grav.*, *12*, 2961–2966.

Crnkovic, C. (1988). Symplectic Geometry of the Covariant Phase Space, Superstrings and Superspace. *Class. Quant. Grav.*, *5*, 1557–1575.

Crnkovic, C., & Witten, E. (1987). Covariant description of canonical formalism in geometrical theories. *Three hundred years of gravitation*, (pp. 676–684).

Darboux, G. (1882). Sur le probleme de pfaff. *Bulletin des sciences mathématiques et astronomiques*, *6*(1), 14–36.

Dautcourt, G. (1998). On the ultrarelativistic limit of general relativity. *Acta Phys. Polon. B*, *29*, 1047–1055.

de Boer, J., Verlinde, E. P., & Verlinde, H. L. (2000). On the holographic renormalization group. *JHEP*, *08*, 003.

de Haro, S., Solodukhin, S. N., & Skenderis, K. (2001). Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence. *Commun. Math. Phys.*, *217*, 595–622.

De Rham, G. (1955). *Variétés différentiables: formes, courants, formes harmoniques*. FeniXX.

Deser, S., & Jackiw, R. (1984). Three-Dimensional Cosmological Gravity: Dynamics of Constant Curvature. *Annals Phys.*, *153*, 405–416.

Deser, S., Jackiw, R., & 't Hooft, G. (1984). Three-Dimensional Einstein Gravity: Dynamics of Flat Space. *Annals Phys.*, *152*, 220.

Deser, S., & Schwimmer, A. (1993). Geometric classification of conformal anomalies in arbitrary dimensions. *Phys. Lett. B*, *309*, 279–284.

Deser, S., & Tekin, B. (2002). Gravitational energy in quadratic curvature gravities. *Phys. Rev. Lett.*, *89*, 101101.

Deser, S., & Tekin, B. (2003). Energy in generic higher curvature gravity theories. *Phys. Rev. D*, *67*, 084009.

Detournay, S., & Riegler, M. (2017). Enhanced Asymptotic Symmetry Algebra of 2+1 Dimensional Flat Space. *Phys. Rev. D*, *95*(4), 046008.

D'Hoker, E., & Freedman, D. Z. (2002). Supersymmetric gauge theories and the AdS / CFT correspondence. In *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions*, (pp. 3–158).

Donnay, L. (2024). Celestial holography: An asymptotic symmetry perspective. *Phys. Rept.*, *1073*, 1–41.

Donnay, L., Fiorucci, A., Herfray, Y., & Ruzziconi, R. (2022). Carrollian Perspective on Celestial Holography. *Phys. Rev. Lett.*, *129*(7), 071602.

Donnay, L., Fiorucci, A., Herfray, Y., & Ruzziconi, R. (2023). Bridging Carrollian and celestial holography. *Phys. Rev. D*, *107*(12), 126027.

Donnay, L., & Marteau, C. (2019). Carrollian Physics at the Black Hole Horizon. *Class. Quant. Grav.*, *36*(16), 165002.

Donnay, L., Puhm, A., & Strominger, A. (2019). Conformally Soft Photons and Gravitons. *JHEP*, *01*, 184.

Donnelly, W., & Freidel, L. (2016). Local subsystems in gauge theory and gravity. *JHEP*, *09*, 102.

Donnelly, W., Freidel, L., Moosavian, S. F., & Speranza, A. J. (2021). Gravitational edge modes, coadjoint orbits, and hydrodynamics. *JHEP*, *09*, 008.

Dray, T., & Streubel, M. (1984). Angular momentum at null infinity. *Class. Quant. Grav.*, *1*(1), 15–26.

Duval, C., Gibbons, G. W., & Horvathy, P. A. (2014a). Conformal Carroll groups. *J. Phys. A*, *47*(33), 335204.

Duval, C., Gibbons, G. W., & Horvathy, P. A. (2014b). Conformal Carroll groups and BMS symmetry. *Class. Quant. Grav.*, *31*, 092001.

Ecker, F., Grumiller, D., Hartong, J., Pérez, A., Prohazka, S., & Troncoso, R. (2023). Carroll black holes. *SciPost Phys.*, *15*(6), 245.

Esmaeili, E., & Hosseinzadeh, V. (2021). p-form surface charges on AdS: renormalization and conservation. *JHEP*, *11*, 062.

Esmaeili, E., Hosseinzadeh, V., & Sheikh-Jabbari, M. M. (2019). Source and response soft charges for Maxwell theory on AdS_d. *JHEP*, *12*, 071.

Fefferman, C., & Graham, C. R. (1985). Conformal invariants. In *Élie Cartan et les mathématiques d' aujourd'hui - Lyon, 25-29 juin 1984*, no. S131 in Astérisque, (pp. 95–116). Société mathématique de France.

Fefferman, C., & Graham, C. R. (2011). The ambient metric. *Ann. Math. Stud.*, *178*, 1–128.

Fiorucci, A. (2021). *Leaky covariant phase spaces: Theory and application to Λ-BMS symmetry*. Ph.D. thesis, Brussels U., Intl. Solvay Inst., Brussels.

Fiorucci, A., & Ruzziconi, R. (2021). Charge algebra in Al(A)dS$_n$ spacetimes. *JHEP*, *05*, 210.

Folland, G. B. (1970). Weyl manifolds. *Journal of differential geometry*, *4*(2), 145–153.

François, J., Parrini, N., & Boulanger, N. (2021). Note on the bundle geometry of field space, variational connections, the dressing field method, & presymplectic structures of gauge theories over bounded regions. *JHEP*, *12*, 186.

Freidel, L. (2021). A canonical bracket for open gravitational system.

Freidel, L., Geiller, M., & Pranzetti, D. (2020). Edge modes of gravity. Part I. Corner potentials and charges. *JHEP*, *11*, 026.

Freidel, L., Hopfmüller, F., & Riello, A. (2019). Asymptotic Renormalization in Flat Space: Symplectic Potential and Charges of Electromagnetism. *JHEP*, *10*, 126.

Freidel, L., & Jai-akson, P. (2023). Carrollian hydrodynamics from symmetries. *Class. Quant. Grav.*, *40*(5), 055009.

Fronsdal, C. (1978). Massless Fields with Integer Spin. *Phys. Rev. D*, *18*, 3624.

Fuentealba, O., Henneaux, M., & Troessaert, C. (2023). Logarithmic supertranslations and supertranslation-invariant Lorentz charges. *JHEP*, *02*, 248.

Gawędzki, K. (1972). On the geometrization of the canonical formalism in the classical field theory. *Reports on Mathematical Physics*, *3*(4), 307–326.

Gawedzki, K. (1991). Classical origin of quantum group symmetries in Wess-Zumino-Witten conformal field theory. *Commun. Math. Phys.*, *139*, 201–214.

Geiller, M. (2017). Edge modes and corner ambiguities in 3d Chern–Simons theory and gravity. *Nucl. Phys. B*, *924*, 312–365.

Geiller, M. (2018). Lorentz-diffeomorphism edge modes in 3d gravity. *JHEP*, *02*, 029.

Geiller, M., Goeller, C., & Zwikel, C. (2021). 3d gravity in Bondi-Weyl gauge: charges, corners, and integrability. *JHEP*, *09*, 029.

Geiller, M., & Zwikel, C. (2022). The partial Bondi gauge: Further enlarging the asymptotic structure of gravity. *SciPost Phys.*, *13*, 108.

Geiller, M., & Zwikel, C. (2024). The partial Bondi gauge: Gauge fixings and asymptotic charges. *SciPost Phys.*, *16*, 076.

Gibbons, G. W., & Hawking, S. W. (1977). Action Integrals and Partition Functions in Quantum Gravity. *Phys. Rev. D*, *15*, 2752–2756.

Gibbons, G. W., Herdeiro, C. A. R., Warnick, C. M., & Werner, M. C. (2009). Stationary Metrics and Optical Zermelo-Randers-Finsler Geometry. *Phys. Rev. D*, *79*, 044022.

Grumiller, D., Merbis, W., & Riegler, M. (2017). Most general flat space boundary conditions in three-dimensional Einstein gravity. *Class. Quant. Grav.*, *34*(18), 184001.

Grumiller, D., Pérez, A., Sheikh-Jabbari, M., Troncoso, R., & Zwikel, C. (2020a). Spacetime structure near generic horizons and soft hair. *Phys. Rev. Lett.*, *124*(4), 041601.

Grumiller, D., & Riegler, M. (2016). Most general AdS_3 boundary conditions. *JHEP*, *10*, 023.

Grumiller, D., Sheikh-Jabbari, M. M., & Zwikel, C. (2020b). Horizons 2020. *Int. J. Mod. Phys. D*, *29*(14), 2043006.

Gubser, S. S., Klebanov, I. R., & Polyakov, A. M. (1998). Gauge theory correlators from noncritical string theory. *Phys. Lett. B*, *428*, 105–114.

Haack, M., & Yarom, A. (2008). Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT. *JHEP*, *10*, 063.

Hall, G. (1992). Weyl manifolds and connections. *Journal of mathematical physics*, *33*(7), 2633–2638.

Hamermesh, M. (2012). *Group theory and its application to physical problems*. Courier Corporation.

Hansen, R. O., Newman, E. T., Penrose, R., & Tod, K. P. (1978). The Metric and Curvature Properties of H Space. *Proc. Roy. Soc. Lond. A*, *363*, 445–468.

Hartong, J. (2015). Gauging the Carroll Algebra and Ultra-Relativistic Gravity. *JHEP*, *08*, 069.

Hawking, S. W. (1975). Particle Creation by Black Holes. *Commun. Math. Phys.*, *43*, 199–220. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

He, T., Lysov, V., Mitra, P., & Strominger, A. (2015). BMS supertranslations and Weinberg's soft graviton theorem. *JHEP*, *05*, 151.

Henneaux, M. (1979). Geometry of Zero Signature Space-times. *Bull. Soc. Math. Belg.*, *31*, 47–63.

Henneaux, M., Maoz, L., & Schwimmer, A. (2000). Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity. *Annals Phys.*, *282*, 31–66.

Henneaux, M., & Rey, S.-J. (2010). Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity. *JHEP*, *12*, 007.

Henneaux, M., & Teitelboim, C. (1985). Asymptotically anti-De Sitter Spaces. *Commun. Math. Phys.*, *98*, 391–424.

Henningson, M., & Skenderis, K. (1998). The Holographic Weyl anomaly. *JHEP*, *07*, 023.

Herfray, Y. (2020). Asymptotic shear and the intrinsic conformal geometry of null-infinity. *J. Math. Phys.*, *61*(7), 072502.

Hollands, S., Ishibashi, A., & Marolf, D. (2005). Counter-term charges generate bulk symmetries. *Phys. Rev. D*, *72*, 104025.

Hubeny, V. E., Minwalla, S., & Rangamani, M. (2012). The fluid/gravity correspondence. In *Theoretical Advanced Study Institute in Elementary Particle Physics: String theory and its Applications: From meV to the Planck Scale*, (pp. 348–383).

Imbimbo, C., Schwimmer, A., Theisen, S., & Yankielowicz, S. (2000). Diffeomorphisms and holographic anomalies. *Class. Quant. Grav.*, *17*, 1129–1138.

Isham, C. J. (1976). Some Quantum Field Theory Aspects of the Superspace Quantization of General Relativity. *Proc. Roy. Soc. Lond. A*, *351*, 209–232.

Israel, W., & Stewart, J. M. (1979). Transient relativistic thermodynamics and kinetic theory. *Annals Phys.*, *118*, 341–372.

Iyer, V., & Wald, R. M. (1994). Some properties of Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D*, *50*, 846–864.

Jankiewicz, C. (1954). Sur les espaces riemanniens dégénérés. *Bull. Acad. Polon. Sci. Cl. III*, *2*, 301.

Jia, W., & Karydas, M. (2021). Obstruction tensors in Weyl geometry and holographic Weyl anomaly. *Phys. Rev. D*, *104*(12), 126031.

Jia, W., Karydas, M., & Leigh, R. G. (2023). Weyl-ambient geometries. *Nucl. Phys. B*, *991*, 116224.

Kalkkinen, J., Martelli, D., & Mueck, W. (2001). Holographic renormalization and anomalies. *JHEP*, *04*, 036.

Kapec, D., Mitra, P., Raclariu, A.-M., & Strominger, A. (2017). 2D Stress Tensor for 4D Gravity. *Phys. Rev. Lett.*, *119*(12), 121601.

Kijowski, J. (1973). A finite-dimensional canonical formalism in the classical field theory. *Commun. Math. Phys.*, *30*, 99–128.

Kijowski, J., & Szczyrba, W. (1976). A Canonical Structure for Classical Field Theories. *Commun. Math. Phys.*, *46*, 183–206.

Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics, Second Edition: Volume 6 (Course of Theoretical Physics)*. Course of theoretical physics / by L. D. Landau and E. M. Lifshitz, Vol. 6. Butterworth-Heinemann, 2 ed.

Lee, J., & Wald, R. M. (1990). Local symmetries and constraints. *J. Math. Phys.*, *31*, 725–743.

Leigh, R. G., Petkou, A. C., & Petropoulos, P. M. (2012a). Holographic Fluids with Vorticity and Analogue Gravity. *JHEP*, *11*, 121.

Leigh, R. G., Petkou, A. C., & Petropoulos, P. M. (2012b). Holographic Three-Dimensional Fluids with Nontrivial Vorticity. *Phys. Rev. D*, *85*, 086010.

Lévy-Leblond, J.-M. (1965). Une nouvelle limite non-relativiste du groupe de poincaré. In *Annales de l'institut Henri Poincaré. Section A, Physique Théorique*, vol. 3, (pp. 1–12).

Loganayagam, R. (2008). Entropy Current in Conformal Hydrodynamics. *JHEP*, *05*, 087.

Mädler, T., & Winicour, J. (2016). Bondi-Sachs Formalism. *Scholarpedia*, *11*, 33528.

Maldacena, J. M. (1998). The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, *2*, 231–252.

Mann, R. B., & Marolf, D. (2006). Holographic renormalization of asymptotically flat spacetimes. *Class. Quant. Grav.*, *23*, 2927–2950.

McLoughlin, T., Puhm, A., & Raclariu, A.-M. (2022). The SAGEX review on scattering amplitudes chapter 11: soft theorems and celestial amplitudes. *J. Phys. A*, *55*(44), 443012.

McNees, R., & Zwikel, C. (2023). Finite charges from the bulk action. *JHEP*, *08*, 154.

McNees, R., & Zwikel, C. (2024). The symplectic potential for leaky boundaries.

Mittal, N., Petropoulos, P. M., Rivera-Betancour, D., & Vilatte, M. (2023). Ehlers, Carroll, charges and dual charges. *JHEP*, *07*, 065.

Morita, S. (2001). *Geometry of differential forms*. 201. American Mathematical Soc.

Muñoz-Lecanda, M.-C. (2018). On some aspects of the geometry of non integrable distributions and applications. *Journal of Geometric Mechanics*, *10*(4), 445–465.

Ojeda, E., & Pérez, A. (2019). Boundary conditions for General Relativity in three-dimensional spacetimes, integrable systems and the KdV/mKdV hierarchies. *JHEP*, *08*, 079.

Papadimitriou, I. (2010). Holographic renormalization as a canonical transformation. *JHEP*, *11*, 014.

Papadimitriou, I., & Skenderis, K. (2005a). AdS / CFT correspondence and geometry. *IRMA Lect. Math. Theor. Phys.*, *8*, 73–101.

Papadimitriou, I., & Skenderis, K. (2005b). Thermodynamics of asymptotically locally AdS spacetimes. *JHEP*, *08*, 004.

Papapetrou, A. (1966). Champs gravitationnels stationnaires à symétrie axiale. In *Annales de l'institut Henri Poincaré. Section A, Physique Théorique*, vol. 4, (pp. 83–105).

Pasterski, S. (2019). Implications of Superrotations. *Phys. Rept.*, *829*, 1–35.

Pasterski, S. (2021). Lectures on celestial amplitudes. *Eur. Phys. J. C*, *81*(12), 1062.

Pasterski, S., Pate, M., & Raclariu, A.-M. (2021). Celestial Holography. In *2022 Snowmass Summer Study*.

Penrose, R. (1963). Asymptotic properties of fields and space-times. *Phys. Rev. Lett.*, *10*, 66–68.

Penrose, R. (1964). Republication of: Conformal treatment of infinity. *General Relativity and Gravitation*, *43*, 901–922.

Penrose, R., & Rindler, W. (1985). *Spinors and Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge, UK: Cambridge Univ. Press.

Perez, A., Tempo, D., & Troncoso, R. (2015). Higher Spin Black Holes. *Lect. Notes Phys.*, *892*, 265–288.

Pérez, A., Tempo, D., & Troncoso, R. (2016). Boundary conditions for General Relativity on AdS_3 and the KdV hierarchy. *JHEP*, *06*, 103.

Petkou, A. C., Petropoulos, P. M., Betancour, D. R., & Siampos, K. (2022). Relativistic fluids, hydrodynamic frames and their Galilean versus Carrollian avatars. *JHEP*, *09*, 162.

Raclariu, A.-M. (2021). Lectures on Celestial Holography.

Randers, G. (1941). On an asymmetrical metric in the four-space of general relativity. *Physical Review*, *59*(2), 195.

Regge, T., & Teitelboim, C. (1974). Role of Surface Integrals in the Hamiltonian Formulation of General Relativity. *Annals Phys.*, *88*, 286.

Riegler, M., & Zwikel, C. (2018). Canonical Charges in Flatland. *PoS*, *Modave2017*, 004.

Riello, A., & Freidel, L. (2024). Renormalization of conformal infinity as a stretched horizon. *Class. Quant. Grav.*, *41*(17), 175013.

Rooman, M., & Spindel, P. (2001a). Holonomies, anomalies and the Fefferman-Graham ambiguity in AdS(3) gravity. *Nucl. Phys. B*, *594*, 329–353.

Rooman, M., & Spindel, P. (2001b). Uniqueness of the asymptotic AdS(3) geometry. *Class. Quant. Grav.*, *18*, 2117–2124.

Rosenfeld, L. (1940). On the energy-momentum tensor. *Mem. Acad. Royal Belg*, *18*, N6.

Ruzziconi, R. (2020). Asymptotic Symmetries in the Gauge Fixing Approach and the BMS Group. *PoS*, *Modave2019*, 003.

Ruzziconi, R., & Zwikel, C. (2021). Conservation and Integrability in Lower-Dimensional Gravity. *JHEP*, *04*, 034.

Sachs, R. (1962a). Asymptotic symmetries in gravitational theory. *Phys. Rev.*, *128*, 2851–2864.

Sachs, R. K. (1962b). Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times. *Proc. Roy. Soc. Lond.*, *A270*, 103–126.

Schwimmer, A., & Theisen, S. (2008). Entanglement Entropy, Trace Anomalies and Holography. *Nucl. Phys. B*, *801*, 1–24.

Sen Gupta, N. D. (1966). On an analogue of the Galilei group. *Nuovo Cim. A*, *44*(2), 512–517.

Skenderis, K. (2001). Asymptotically Anti-de Sitter space-times and their stress energy tensor. *Int. J. Mod. Phys. A*, *16*, 740–749.

Skenderis, K. (2002). Lecture notes on holographic renormalization. *Class. Quant. Grav.*, *19*, 5849–5876.

Speranza, A. J. (2018). Local phase space and edge modes for diffeomorphism-invariant theories. *JHEP*, *02*, 021.

Starobinsky, A. A. (1983). Isotropization of arbitrary cosmological expansion given an effective cosmological constant. *JETP Lett.*, *37*, 66–69.

Staruszkiewicz, A. (1963). Gravitation Theory in Three-Dimensional Space. *Acta Phys. Polon.*, *24*, 735–740.

Strominger, A. (1998). Black hole entropy from near horizon microstates. *JHEP*, *02*, 009.

Strominger, A. (2001). The dS / CFT correspondence. *JHEP*, *10*, 034.

Strominger, A. (2018). *Lectures on the Infrared Structure of Gravity and Gauge Theory*. Princeton University Press.

Susskind, L. (1995). The World as a hologram. *J. Math. Phys.*, *36*, 6377–6396.

Susskind, L., & Witten, E. (1998). The Holographic bound in anti-de Sitter space.

't Hooft, G. (1993). Dimensional reduction in quantum gravity. *Conf. Proc. C*, *930308*, 284–296.

Tamburino, L. A., & Winicour, J. H. (1966). Gravitational Fields in Finite and Conformal Bondi Frames. *Phys. Rev.*, *150*, 1039–1053.

Taylor, M. (2000). More on counterterms in the gravitational action and anomalies.

Troessaert, C. (2013). Enhanced asymptotic symmetry algebra of AdS_3. *JHEP*, *08*, 044.

Troessaert, C. (2016). Hamiltonian surface charges using external sources. *J. Math. Phys.*, *57*(5), 053507.

Tulczyjew, W. (2006). The euler-lagrange resolution. In *Differential Geometrical Methods in Mathematical Physics: Proceedings of the Conferences Held at Aix-en-Provence, September 3–7, 1979 and Salamanca, September 10–14, 1979*, (pp. 22–48). Springer.

Vogel, W. O. (1965). Über lineare zusammenhänge in singulären riemannschen räumen. *Archiv der Mathematik*, *16*(1), 106–116.

Wald, R. M. (1993). Black hole entropy is the Noether charge. *Phys. Rev. D*, *48*(8), R3427–R3431.

Wald, R. M., & Zoupas, A. (2000). A General definition of 'conserved quantities' in general relativity and other theories of gravity. *Phys. Rev.*, *D61*, 084027.

Wieland, W. (2022). Barnich–Troessaert bracket as a Dirac bracket on the covariant phase space. *Class. Quant. Grav.*, *39*(2), 025016.

Witten, E. (1988). (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nucl. Phys. B*, *311*, 46.

Witten, E. (1998). Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, *2*, 253–291.

York, J. W., Jr. (1972). Role of conformal three geometry in the dynamics of gravitation. *Phys. Rev. Lett.*, *28*, 1082–1085.