

Higher-spin gauge theories in three spacetime dimensions

Andrea Campoleoni and Stefan Fredenhagen

Abstract These lecture notes provide an introduction to higher-spin gauge theories in three spacetime dimensions, with a focus on their asymptotic symmetries, their holographic description in terms of conformal field theories with \mathscr{W} -symmetries as well as on their couplings to scalar matter.

1	Introduction	2
2	Higher spins and Chern–Simons theory	4
	2.1 Three-dimensional gravity as a Chern–Simons theory	5
	2.2 Including higher-spin fields	9
3	Asymptotic symmetries	19
	3.1 Chern–Simons theory with boundary	19
	3.2 Asymptotically AdS solutions: the Drinfeld–Sokolov reduction	26
	3.3 The $\mathfrak{sl}(2)$ example: asymptotic symmetries of gravity	29
	3.4 The $\mathfrak{sl}(3)$ case	33
	3.5 Beyond $\mathfrak{sl}(3)$	38
4	Quantum \mathscr{W} -algebras and minimal-model holography	42
	4.1 Motivation	42
	4.2 \mathscr{W} -algebras and operator product expansion	44
	4.3 The quantum \mathscr{W}_3 -algebra	48
	4.4 \mathscr{W}_N and the quantum Miura transform	51
	4.5 The $\mathscr{W}_\infty[\lambda]$ quantum algebra and its triality	54
	4.6 Conformal field theories with \mathscr{W} -symmetry	56
	4.7 Minimal-model holography	60
5	Coupling to matter	64
	5.1 Free scalar field in the unfolded formulation	64
	5.2 Non-Linear Theory	79
6	Summary and further developments	87
A	Frame-like formulation in arbitrary dimension	92
B	More on the classical Miura transformation	96
C	Constrained Hamiltonian systems	100
D	Solutions to selected exercises	102
	References	123

A. Campoleoni,
Université de Mons, Service de Physique de l’Univers, Champs et Gravitation, Place du Parc 20,
7000 Mons, Belgium, e-mail: andrea.campoleoni@umons.ac.be

S. Fredenhagen,
University of Vienna, Faculty of Physics, Boltzmannngasse 5, 1090 Vienna, Austria, e-mail:
stefan.fredenhagen@univie.ac.at

1 Introduction

Elementary particles are characterised by their mass and spin. For free particles, any value of the spin is allowed, while the structure of interactions strongly depends on the spin. The interactions of massless particles with spin greater than two, also known as higher-spin particles, are severely constrained by various no-go results; see, e.g., [1, 2] for a review. In spite of this, gauge theories involving higher-spin fields have been investigated over the years,¹ mainly in view of their possible impact in the study of quantum gravity. Indeed, on the one hand, building consistent interactions for massless higher-spin fields typically also requires the presence of a massless spin-two field in the spectrum. This leads to field theories including a putative graviton and displaying a huge gauge symmetry, that is expected to improve the UV behaviour of Einstein's gravity [4]. On the other hand, string theory does involve massive excitations of arbitrary spin, and it has been conjectured that it might actually describe a symmetry-broken phase of a higher-spin gauge theory [5] (see also [6, 7] for a review of the ensuing developments). More recently, these suggestions have been reinforced by the natural emergence of higher-spin theories in the AdS/CFT correspondence, as gravity duals of certain weakly coupled conformal field theories; see, e.g., [8] for a review.

In spacetimes of dimension four or higher, there are strong indications that higher-spin gauge theories must involve an infinite number of fields. An explicit example of a complete interacting theory of this sort has been provided by Vasiliev and collaborators on constant curvature backgrounds [9, 10] (see also [11, 12] for a review). Higher-spin generalisations of self-dual Yang-Mills and gravity have also been obtained on flat four-dimensional manifolds with Euclidean or split signature [13, 14], and the status of higher-spin theories on Minkowski backgrounds was recently reconsidered in [15]. Conformal gravity admits a higher-spin extension too [16], and we refer to [17] for an overview of higher-spin theories and their applications. All previous examples entail various peculiarities – allowing them to evade the plethora of no-go results – together with technical difficulties brought by the unbounded spectra of fields. Studying higher-spin theories on three-dimensional spacetimes gives the unique opportunity to explore some of these peculiarities in a highly simplified context, where, for instance, interacting theories involving only a finite number of higher-spin fields can be built.

Strictly speaking, there are no massless higher-spin particles in three dimensions: the little group does not admit massless representations of arbitrary helicity, so that only scalar and spin-1/2 degrees of freedom can propagate [18]. On the other hand, one can still consider field theories similar to those describing higher-spin degrees of freedom in four dimensions and use them as toy models for higher-spin theories, in analogy with the role played by three-dimensional gravity. One can indeed consider vacuum Einstein's equations also in three dimensions, although they do not allow any local degrees of freedom [19, 20]. Equivalently, no gravitational, spin-two waves propagate in three dimensions. Still, black hole solutions do exist [21, 22] and three-

¹ See, e.g., [3] for an historical overview.

dimensional gravity has been a powerful testing ground for various ideas about their quantum properties as well as a source of insights that found applications in higher dimensions too; see, e.g., [23–25].

Analogous considerations extend to arbitrary spin, since handy non-linear completions of the free equations of motion can be defined in three dimensions for any value of the spin, with or without a cosmological constant [26, 27]. They profit from the same simplifications that allow the rewriting of the Einstein–Hilbert action as a Chern–Simons action [28, 29] and, as anticipated, they do not require to consider an infinite number of fields as in four dimensions.² The resulting interacting higher-spin gauge theories have been explored in various directions in recent years and, since they typically involve also spin-two fields, they are often collectively denoted as higher-spin gravity. A notable research avenue revolves around a holographic duality, dubbed minimal-model holography, that links higher-spin gauge theories coupled to matter on anti-de Sitter (AdS) spacetime to a specific large- N limit of \mathscr{W}_N -minimal models [31] (see also [32] for a review). Such relation elaborates upon the observation that the asymptotic symmetries of three-dimensional higher-spin gauge theories are given by non-linear \mathscr{W} -algebras, rather than by Lie algebras [33, 34]. As a result, their putative boundary duals must admit global symmetries of the same type. In spite of this peculiarity, minimal-model holography can be considered as a three-dimensional counterpart of the holographic duality between Vasiliev’s higher-spin theories in four dimensions and the large- N limit of the three-dimensional $O(N)$ vector model [35] (see also [36–40, 8]). It provides a privileged setup where to test quantitatively the AdS₃/CFT₂ correspondence, thanks to its virtue of combining the good analytic control one has over \mathscr{W}_N -minimal models with the option to access the semiclassical regime via a large- N limit. It also triggered interesting complementary developments as, e.g., the study of black holes in theories in which the higher-spin gauge symmetry makes their usual geometric characterisations meaningless [41–43], as it might be expected in quantum gravity (see also [44–46] for a review).

These lecture notes aim at providing an introduction to three-dimensional higher-spin theories allowing to access research papers or specialised reviews on minimal-model holography, like [32]. To this end, three main topics are discussed: sections 2 and 3 set the stage by presenting the Chern–Simons formulation of higher-spin gauge theories in three dimensions and their asymptotic symmetries. The focus is on higher-spin theories on anti-de Sitter space, but we stress that the same techniques apply to any value of the cosmological constant. Section 4 begins with an introduction to quantum \mathscr{W} -algebras and then reviews the bases of minimal-model holography, mainly from the viewpoint of the boundary conformal field theory. Section 5 discusses the subtleties encountered when coupling higher-spin fields to matter, and reviews the Prokushkin–Vasiliev model [47]. This provides an example of such a coupling and has been considered as a natural candidate bulk dual of the conformal field theories introduced in the previous section. Additional research directions on three-dimensional higher-spin theories and their links with the material here re-

² The option to define interacting higher-spin theories with field spectra that are not consistent in four dimensions was already noticed in [30].

viewed are sketched in section 6. We also refer to this section for a more detailed overview of the material presented in these lecture notes.

Each topic is approached assuming only a textbook knowledge of gravity and conformal field theory, but more advanced developments and applications are discussed towards the end of each section. This allows for different reading paths: a reader with experience in three-dimensional higher-spin theories might still profit, e.g., from section 3.5, reviewing more sophisticated techniques to characterise the asymptotic symmetries of Chern–Simons theories, or from sections 4.4–4.7 that delve into the general properties of quantum \mathscr{W} -algebras and their applications in minimal-model holography. Section 5 is meant to provide an introduction to the Prokushkin–Vasiliev model, accessible also to readers without any previous exposure to Vasiliev’s unfolded formulation of higher-spin gauge theories. On the other hand, it introduces a number of techniques that might be useful also in other contexts, like the oscillator realisations of higher-spin algebras discussed in section 5.1.2. Several exercises complement the text. They suggest some computations that might be useful to consolidate the understanding of the material here reviewed, and we collected a number of solutions in section D. Exercises with a solution are marked with an asterisk.

2 Higher spins and Chern–Simons theory

In this section, we present non-linear actions in three spacetime dimensions that, when linearised around a constant curvature background, reduce to a sum of free actions for massless fields with various spin. The simplest example is the Einstein–Hilbert action in three dimensions: its linearised equations of motion do not admit wave solutions, but they take the same form as in a spacetime of dimension four or higher, where local degrees of freedom are present. When the spin is greater than two, one can reverse the logic: linear equations with an analogous structure as those describing massless spin- s particles in $D \geq 4$ can be defined in $D = 3$ too. They do not admit wave solutions as for $s = 2$, but we consider any non-linear completion thereof that preserves the same amount of gauge symmetries as in the free theory as a higher-spin gauge theory. Remarkably, if one ignores matter couplings, in three dimensions higher-spin gauge theories have essentially the same structure as Einstein’s gravity. For this reason, we first briefly recall the main features of the Einstein–Hilbert action in $D = 2 + 1$ and then we show how to include higher spins.

2.1 Three-dimensional gravity as a Chern–Simons theory

To generalise the construction of three-dimensional gravity to higher spins, it is convenient to work in a first-order formalism.³ In terms of the dreibein $e^a = e_\mu^a dx^\mu$ and the spin connection $\omega^{ab} = \omega_\mu^{ab} dx^\mu$, the Einstein–Hilbert action in $D = 2 + 1$ dimensions reads

$$S_{\text{EH}} = \frac{1}{16\pi G} \int \epsilon_{abc} \left(e^a \wedge R^{bc} + \frac{1}{3\ell^2} e^a \wedge e^b \wedge e^c \right), \quad (2.1)$$

where $R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c^b$ is the Riemann curvature, ℓ is related to the cosmological constant by $\Lambda = -\ell^{-2}$ and G is Newton’s constant, which has dimension of length in three dimensions. The equation of motion for the dreibein imposes that the curvature must be constant on shell, thus confirming the absence of local degrees of freedom.

Here and in the following, a, b, \dots denote Lorentz frame indices, while μ, ν, \dots denote base-manifold indices. The latter will often be omitted. We use the following conventions for the Minkowski metric and the Levi–Civita tensor:

$$\eta_{ab} = (-, +, +) \quad \text{and} \quad \epsilon^{012} = 1. \quad (2.2)$$

As usual, η_{ab} is used to raise and lower Lorentz frame indices.

In three dimensions, one can dualise the spin connection and define

$$\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{bc}, \quad R^a = d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge \omega_c. \quad (2.3)$$

The action (2.1) can then be rewritten as

$$S_{\text{EH}} = \frac{1}{8\pi G} \int \left(e_a \wedge R^a + \frac{1}{6\ell^2} \epsilon_{abc} e^a \wedge e^b \wedge e^c \right), \quad (2.4)$$

and its equations of motions read

$$\mathcal{T}^a := de^a + \epsilon^{abc} \omega_b \wedge e_c = 0, \quad (2.5a)$$

$$\mathcal{R}^a := R^a + \frac{1}{2\ell^2} \epsilon^{abc} e_b \wedge e_c = 0. \quad (2.5b)$$

The torsion constraint (2.5a) allows one to rewrite the dualised spin connection in terms of the dreibein, while (2.5b) states that the curvature is constant.

Exercise 1.* Show that the torsion constraint $\mathcal{T}^a = 0$ implies

$$\omega_\mu^a = \epsilon^{abc} e_b^\nu (\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) - \frac{1}{2} \epsilon^{bcd} (e_b^\nu e_c^\rho \partial_\nu e_{\rho d}) e_\mu^a, \quad (2.6)$$

³ An extensive introduction to three-dimensional gravity, covering both its first and second order formulations, is given in [23].

where e^μ_a denotes the inverse dreibein, satisfying $e^\mu_a e_\mu^b = \delta_a^b$ and $e^\mu_a e_\nu^a = \delta^\mu_\nu$.

As in spacetimes of dimension $D \geq 4$, the Einstein–Hilbert action is invariant under diffeomorphisms and local Lorentz transformations. When $D = 3$, the action is also invariant under local translations, so that its symmetries are given by

$$\delta e^a = \nabla \xi^a + \epsilon^{abc} e_b \Lambda_c, \quad \delta \omega^a = \nabla \Lambda^a + \frac{1}{\ell^2} \epsilon^{abc} e_b \xi_c, \quad (2.7)$$

where we introduced the Lorentz-covariant derivative ∇ , acting on tangent-space vectors as

$$\nabla f^a = \mathbf{d}f^a + \epsilon^{abc} \omega_b f_c. \quad (2.8)$$

Exercise 2.* Show that a local translation with parameter $\xi^a = e_\mu^a v^\mu$ is equivalent on shell to the diffeomorphism generated by the vector field v^μ ,

$$\delta e_\mu^a = v^\nu \partial_\nu e_\mu^a + e_\nu^a \partial_\mu v^\nu, \quad (2.9)$$

up to a local Lorentz transformation with parameter $\Lambda^a = \omega_\mu^a v^\mu$. This implies that the invariance under diffeomorphisms does not constitute an additional symmetry of the action.

One can further simplify the form (2.4) of the action by introducing the fields $E = \frac{1}{G} e^a P_a$ and $\Omega = \omega^a L_a$, where P_a and L_a are the generators of the isometry algebra of the vacuum:

$$[P_a, P_b] = \left(\frac{G}{\ell}\right)^2 \epsilon_{abc} L^c, \quad [L_a, P_b] = \epsilon_{abc} P^c, \quad [L_a, L_b] = \epsilon_{abc} L^c. \quad (2.10)$$

For $\ell \rightarrow \infty$ one gets the three-dimensional Poincaré algebra $\mathfrak{iso}(1,2)$, for $\ell^2 < 0$ one gets the de Sitter algebra $\mathfrak{so}(1,3)$, and for $\ell^2 > 0$ one gets the anti-de Sitter algebra $\mathfrak{so}(2,2)$. The customary presentation of the vacuum isometry algebras, applying to spacetimes of any dimension, is recovered by dualising the Lorentz generators as $L_a = \frac{1}{2} \epsilon_{abc} M^{bc}$. One can then rewrite the Einstein–Hilbert action (2.4) as

$$S_{\text{EH}} = \frac{1}{8\pi} \int \text{Tr} \left(E \wedge R + \frac{1}{3} E \wedge E \wedge E \right), \quad (2.11)$$

where $R = \mathbf{d}\Omega + \Omega \wedge \Omega = R^a L_a$ and where we introduced the bilinear form⁴

$$\text{Tr}(P_a P_b) = 0, \quad \text{Tr}(P_a L_b) = \eta_{ab}, \quad \text{Tr}(L_a L_b) = 0. \quad (2.12)$$

Up to boundary terms, the action (2.11) is equivalent to a Chern–Simons action for the field $\mathcal{A} = E + \Omega$ with gauge algebra $\mathfrak{iso}(1,2)$, $\mathfrak{so}(1,3)$ or $\mathfrak{so}(2,2)$, depending on

⁴ For the Lie algebras $\mathfrak{iso}(1,2)$ and $\mathfrak{so}(2,2)$, which are not simple, there is also another independent invariant bilinear form that, however, does not lead to the Einstein–Hilbert action [29].

the value of ℓ [29]:⁵

$$S_{\text{EH}} = \frac{1}{16\pi} \int \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right). \quad (2.13)$$

All solutions of the Chern–Simons field equations,

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0, \quad (2.14)$$

are flat connections, so that one can see once again that three-dimensional gravity has no local degrees of freedom.

For a negative cosmological constant, one can also profit from the isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. The two copies of $\mathfrak{sl}(2, \mathbb{R})$ are generated by

$$J_a^\pm = \frac{1}{2} \left(L_a \pm \frac{\ell}{G} P_a \right), \quad (2.15)$$

that, indeed, satisfy

$$[J_a^\pm, J_b^\pm] = \epsilon_{ab}{}^c J_c^\pm, \quad [J_a^\pm, J_b^\mp] = 0. \quad (2.16)$$

The bilinear form (2.12) then induces

$$\text{Tr}(J_a^\pm J_b^\pm) = \pm \frac{1}{2} \frac{\ell}{G} \eta_{ab}, \quad \text{Tr}(J_a^\pm J_b^\mp) = 0. \quad (2.17)$$

Splitting the $\mathfrak{so}(2,2)$ connection as $\mathcal{A} = A^a J_a^+ + \tilde{A}^a J_a^-$ gives

$$A^a = \omega^a + \frac{1}{\ell} e^a, \quad \tilde{A}^a = \omega^a - \frac{1}{\ell} e^a, \quad (2.18)$$

and substituting this splitting of \mathcal{A} in the action (2.13) one can rewrite it as the difference of two Chern–Simons actions [28, 29]. The latter form of the three-dimensional Einstein–Hilbert action with negative cosmological constant is the most common one in the literature:

$$S_{\text{EH}} = S_{\text{CS}}[A] - S_{\text{CS}}[\tilde{A}], \quad (2.19)$$

with

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.20)$$

We introduced here the fields $A = A^a J_a$ and $\tilde{A} = \tilde{A}^a J_a$, taking values in a single $\mathfrak{sl}(2, \mathbb{R})$ algebra with bilinear form

⁵ In our conventions, the field \mathcal{A} is dimensionless, and this explains the overall dimensionless parameter. Newton’s constant has been absorbed in the definition of E .

$$\mathrm{tr}(J_a J_b) = \frac{1}{2} \eta_{ab}. \quad (2.21)$$

The opposite sign in the scalar products (2.17) of the generators J_a^\pm is taken into account by the relative sign between the two Chern–Simons actions in (2.19). Furthermore, we absorbed the dimensionless factor $\frac{\ell}{G}$ in the definition of the level k :

$$k = \frac{\ell}{4G}. \quad (2.22)$$

Let us stress again that the Einstein–Hilbert action can only be rewritten as the difference of two $\mathfrak{sl}(2, \mathbb{R})$ Chern–Simons actions when the cosmological constant is negative, i.e. for $\ell^2 > 0$. Nevertheless, it can always be rewritten as a Chern–Simons action with a $\mathfrak{iso}(1, 2)$, $\mathfrak{so}(1, 3)$ or $\mathfrak{so}(2, 2)$ gauge algebra, since (2.13) holds for arbitrary values of ℓ^2 .

Yet another useful rewriting of the Einstein–Hilbert action results from considering the combinations

$$e = \frac{\ell}{2} (A - \tilde{A}), \quad \omega = \frac{1}{2} (A + \tilde{A}), \quad (2.23)$$

where all fields are again assumed to take values in the single $\mathfrak{sl}(2, \mathbb{R})$ algebra with Killing metric (2.21).

Exercise 3.* Show that one can rewrite the action (2.19) as

$$S_{EH} = \frac{1}{4\pi G} \int \mathrm{tr} \left(e \wedge R + \frac{1}{3\ell^2} e \wedge e \wedge e \right), \quad (2.24)$$

and argue that this form of the action actually holds for arbitrary values of the cosmological constant.

Exercise 4. Check that the infinitesimal gauge transformations of the Chern–Simons fields,

$$\delta A = d\lambda + [A, \lambda], \quad \delta \tilde{A} = d\tilde{\lambda} + [\tilde{A}, \tilde{\lambda}], \quad (2.25)$$

can be rewritten as

$$\delta e = d\xi + [\omega, \xi] + [e, \Lambda], \quad \delta \omega = d\Lambda + [\omega, \Lambda] + \frac{1}{\ell^2} [e, \xi], \quad (2.26)$$

with

$$\xi = \frac{\ell}{2} (\lambda - \tilde{\lambda}), \quad \Lambda = \frac{1}{2} (\lambda + \tilde{\lambda}), \quad (2.27)$$

and check that the transformations (2.26) correspond to the symmetries (2.7) of the Einstein–Hilbert action.

2.2 Including higher-spin fields

To include higher-spin gauge fields, we begin by presenting their free equations of motion in two approaches, first generalising the “metric”, second-order formulation of Einstein’s gravity, and then generalising the “frame”, first-order formulation that we reviewed in the previous section. We then show that Chern–Simons actions give interacting theories that decompose into a sum of free higher-spin frame-like actions upon linearisation.

2.2.1 Metric-like formulation

A massless spin- s particle in Minkowski space can be described by a symmetric tensor field $\phi_{\mu_1 \dots \mu_s}$ fulfilling the Fronsdal equation [48]

$$\square \phi_{\mu_1 \dots \mu_s} - s \partial^\sigma \partial_{(\mu_1} \phi_{\mu_2 \dots \mu_s) \sigma} + \frac{s(s-1)}{2} \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3 \dots \mu_s)}^\sigma \sigma = 0. \quad (2.28)$$

Here and below, indices enclosed between parentheses are symmetrised, and dividing by the number of terms in the sum is understood (weight-one convention). For instance, $A_{(\mu} B_{\nu)} := \frac{1}{2} (A_\mu B_\nu + A_\nu B_\mu)$. This equation is left invariant by gauge transformations

$$\delta \phi_{\mu_1 \dots \mu_s} = s \partial_{(\mu_1} \epsilon_{\mu_2 \dots \mu_s)} \quad (2.29)$$

with a traceless gauge parameter,

$$\epsilon_{\mu_1 \dots \mu_{s-3}}^\sigma \sigma = 0. \quad (2.30)$$

For $s = 1$ and $s = 2$, Fronsdal’s equations reduce, respectively, to Maxwell’s equations and to linearised Einstein’s equations.

The equations of motion (2.28) can be defined on a Minkowski space of any dimension D . Together with the gauge symmetry (2.29), they imply that, out of the $\binom{D+s-1}{s}$ independent components of the field $\phi_{\mu_1 \dots \mu_s}$, only the $\binom{D+s-4}{s} \frac{D+2s-4}{D+s-4}$ degrees of freedom of a massless spin- s particle propagate on shell (see, e.g., [7]). When $D = 3$, Fronsdal’s equations can be defined as well but, consistently with the previous counting, they do not admit wave solutions like the spin-two Fierz–Pauli equations.

In the following, we shall be mainly interested in massless higher-spin fields propagating on an AdS background with metric $\bar{g}_{\mu\nu}$. Their equations of motion can be obtained by replacing partial derivatives with Levi–Civita covariant derivatives $\bar{\nabla}$ for the AdS metric in (2.28) and by adding suitable mass terms to restore the gauge symmetry

$$\delta \phi_{\mu_1 \dots \mu_s} = s \bar{\nabla}_{(\mu_1} \epsilon_{\mu_2 \dots \mu_s)} \quad \text{with} \quad \bar{g}^{\rho\sigma} \epsilon_{\mu_1 \dots \mu_{s-3} \rho \sigma} = 0. \quad (2.31)$$

The additional mass terms allow one to compensate the contributions to the gauge variation produced by the commutator of covariant derivatives,

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu]v_\rho = \frac{1}{\ell^2} (\bar{g}_{\nu\rho}v_\mu - \bar{g}_{\mu\rho}v_\nu). \quad (2.32)$$

The resulting equations of motion for a massless spin- s field in AdS [49] are then given by

$$(\square - m_s^2)\phi_{\mu_1 \dots \mu_s} - s \bar{\nabla}^\sigma \bar{\nabla}_{(\mu_1} \phi_{\mu_2 \dots \mu_s)\sigma} + \frac{s(s-1)}{2} \bar{\nabla}_{(\mu_1} \bar{\nabla}_{\mu_2} \phi_{\mu_3 \dots \mu_s)}{}^\sigma{}_\sigma = 0, \quad (2.33)$$

with the mass coefficient⁶

$$m_s^2 = \frac{2(s-1)(D+s-3)}{\ell^2}. \quad (2.34)$$

In AdS₃ one thus obtains $m_s^2 = 2s(s-1)\ell^{-2}$.

Exercise 5. Check the invariance of the equations (2.33) under the gauge transformations (2.31).

2.2.2 Frame-like formulation

In three dimensions, the equations of motion (2.33) can be reformulated in a first-order form by introducing the higher-spin vielbeins and spin connections

$$e^{a_1 \dots a_{s-1}} = e_\mu^{a_1 \dots a_{s-1}} dx^\mu, \quad \omega^{a_1 \dots a_{s-1}} = \omega_\mu^{a_1 \dots a_{s-1}} dx^\mu, \quad (2.35)$$

which are symmetric and traceless in their Lorentz indices a_i and admit the gauge symmetries

$$\delta e^{a_1 \dots a_{s-1}} = \nabla \xi^{a_1 \dots a_{s-1}} + (s-1) \bar{e}_b \epsilon^{bc(a_1} \Lambda^{a_2 \dots a_{s-1})}{}_c, \quad (2.36a)$$

$$\delta \omega^{a_1 \dots a_{s-1}} = \nabla \Lambda^{a_1 \dots a_{s-1}} + \frac{s-1}{\ell^2} \bar{e}_b \epsilon^{bc(a_1} \xi^{a_2 \dots a_{s-1})}{}_c, \quad (2.36b)$$

where \bar{e} and $\bar{\omega}$ denote the dreibein and spin connection of the AdS background, and the gauge parameters $\xi^{a_1 \dots a_{s-1}}$ and $\Lambda^{a_1 \dots a_{s-1}}$ are symmetric and traceless like the fields. Moreover, ∇ denotes the Lorentz covariant derivative

$$\nabla f^{a_1 \dots a_n} = df^{a_1 \dots a_n} + n \epsilon^{bc(a_1} \bar{\omega}_b \wedge f^{a_2 \dots a_n)}{}_c, \quad (2.37)$$

which satisfies

$$\nabla^2 f^{a_1 \dots a_n} = \frac{n}{\ell^2} \bar{e}^b \wedge \bar{e}^{(a_1} \wedge f^{a_2 \dots a_n)}{}_b. \quad (2.38)$$

For $s=2$ one recovers the dreibein and the dualised spin connection introduced in (2.3), together with the linearisation of the transformations (2.7). When $D > 3$,

⁶ In AdS the order of derivatives does matter: often in the literature the derivatives in the second term of (2.33) are exchanged and this gives a different mass term involving also the trace of the field. Notice also that, for $s=2$, (2.33) agrees with the linearisation of the vacuum Einstein equations $R_{\mu\nu} - \frac{2\Lambda}{D-2} g_{\mu\nu} = 0$.

one must actually introduce spin connections of the form $\omega^{b,a_1\cdots a_{s-1}}$ satisfying $\omega^{(b,a_1\cdots a_{s-1})} = 0$ (see, e.g., [12, 7] and appendix A for more details). When $D = 3$, the latter irreducibility condition allows one to dualise them to recover the fields introduced in (2.35).

For any value of the spin, one can then consider the action

$$S = \frac{1}{8\pi G} \int \left(e_{a_1\cdots a_{s-1}} \wedge \nabla \omega^{a_1\cdots a_{s-1}} + \frac{s-1}{2} \epsilon_{bcd} \bar{e}^b \wedge \omega^c{}_{a_1\cdots a_{s-1}} \wedge \omega^{da_1\cdots a_{s-1}} + \frac{s-1}{2\ell^2} \epsilon_{bcd} \bar{e}^b \wedge e^c{}_{a_1\cdots a_{s-1}} \wedge e^{da_1\cdots a_{s-1}} \right), \quad (2.39)$$

which is left invariant by the gauge transformations (2.36) and, for $s = 2$, corresponds to the linearisation of the Einstein–Hilbert action (2.4). It also corresponds to the three-dimensional version of the linearised action for a massless spin- s particle in arbitrary dimension that we discuss in section A. When $D = 3$, one could also consider other gauge invariant quadratic actions in the fields (2.35), while for generic values of D the action is uniquely fixed.

One can also introduce the following curvatures for the fields e and ω :

$$\mathcal{F}^{a_1\cdots a_{s-1}} = \nabla e^{a_1\cdots a_{s-1}} + (s-1) \epsilon^{bc(a_1} \bar{e}_b \wedge \omega^{a_2\cdots a_{s-1})}{}_c, \quad (2.40a)$$

$$\mathcal{R}^{a_1\cdots a_{s-1}} = \nabla \omega^{a_1\cdots a_{s-1}} + \frac{s-1}{\ell^2} \epsilon^{bc(a_1} \bar{e}_b \wedge e^{a_2\cdots a_{s-1})}{}_c. \quad (2.40b)$$

These two-forms are invariant under the transformations (2.36) and satisfy the Bianchi identities

$$\nabla \mathcal{F}^{a_1\cdots a_{s-1}} + (s-1) \epsilon^{bc(a_1} \bar{e}_b \wedge \mathcal{R}^{a_2\cdots a_{s-1})}{}_c = 0, \quad (2.41a)$$

$$\nabla \mathcal{R}^{a_1\cdots a_{s-1}} + \frac{s-1}{\ell^2} \epsilon^{bc(a_1} \bar{e}_b \wedge \mathcal{F}^{a_2\cdots a_{s-1})}{}_c = 0. \quad (2.41b)$$

In analogy with the spin two example, the equations of motion of the action (2.39) are the zero-curvature conditions

$$\mathcal{F}^{a_1\cdots a_{s-1}} = 0, \quad (2.42a)$$

$$\mathcal{R}^{a_1\cdots a_{s-1}} = 0. \quad (2.42b)$$

This is again a peculiarity of the three-dimensional setup: when $D > 3$, in analogy with gravity, the equations of motion only imply the vanishing of a certain projection of the analogue of the curvature (2.42b) – see again section A for more details.

Exercise 6. Prove the gauge invariance of the action (2.39) under the transformations (2.36) and check that its equations of motion are those in (2.42).

Equations (2.42) are equivalent to the Fronsdal equation (2.33). This can be shown by following the same steps as in exercise 1, i.e., by first expressing the generalised spin connection in terms of the higher-spin vielbein and its first derivative using the torsion constraint (2.42a), and then by substituting the resulting

$\omega^{a_1 \dots a_{s-1}} = \omega^{a_1 \dots a_{s-1}}(e, \nabla e)$ in (2.42b) (see [34] for more details or section A for the analogous, and somehow simpler, solution (A.16) of the torsion constraint in any D). The Fronsdal equation can eventually be recovered by rewriting (2.42b) in terms of

$$\phi_{\mu_1 \dots \mu_s} = \bar{e}^{a_1}{}_{(\mu_1} \dots \bar{e}^{a_{s-1}}{}_{\mu_{s-1}} e_{\mu_s)}{}_{a_1 \dots a_{s-1}}. \quad (2.43)$$

2.2.3 Chern–Simons formulation

As for gravity, one can combine the spin connections $\omega^{a_1 \dots a_{s-1}}$ and the vielbeins $e^{a_1 \dots a_{s-1}}$ into the linear combinations

$$A^{a_1 \dots a_{s-1}} = \left(\omega + \frac{1}{\ell} e \right)^{a_1 \dots a_{s-1}}, \quad \tilde{A}^{a_1 \dots a_{s-1}} = \left(\omega - \frac{1}{\ell} e \right)^{a_1 \dots a_{s-1}}. \quad (2.44)$$

We now wish to introduce Chern–Simons fields A and \tilde{A} including the gravitational connections discussed in section 2.1, together with the fields above contracted with the generators $J_{a_1 \dots a_{s-1}}$ of a higher-spin Lie algebra that we still need to identify.

These generators, besides being symmetric under permutations of their indices, cannot be all independent as they must satisfy the constraints $\eta^{ab} J_{abc_1 \dots c_{s-3}} = 0$ induced by the contraction with traceless fields. Moreover, we can fix the commutation relations between J_a and $J_{b_1 \dots b_{s-1}}$ by imposing that the linearisation of the Chern–Simons equations of motion reproduces the free equations of motion for higher-spin fields on the AdS background. The vanishing of the curvatures for A and \tilde{A} , when expressed in terms of e and ω , imposes

$$d\omega + \omega \wedge \omega + \frac{1}{\ell^2} e \wedge e = 0, \quad de + e \wedge \omega + \omega \wedge e = 0 \quad (2.45)$$

(see also exercise 3, where the precise form of the gauge algebra did not play any role). Expanding the fields in terms of the generators t_A of the Lie algebra satisfying $[t_A, t_B] = f_{AB}{}^C t_C$, we obtain

$$d\omega^A + \frac{1}{2} f_{BC}{}^A \omega^B \wedge \omega^C + \frac{1}{2\ell^2} f_{BC}{}^A e^B \wedge e^C = 0, \quad de^A + f_{BC}{}^A e^B \wedge e^C = 0. \quad (2.46)$$

Linearising around a background $\bar{e} = \bar{e}^a J_a$, $\bar{\omega} = \bar{\omega}^a J_a$ one obtains

$$d\omega^A + f_{bC}{}^A \bar{\omega}^b \wedge \omega^C + \frac{1}{\ell^2} f_{bC}{}^A \bar{e}^b \wedge e^C = 0, \quad de^A + f_{bC}{}^A \bar{\omega}^b \wedge e^C + f_{bC}{}^A \bar{e}^b \wedge e^C = 0, \quad (2.47)$$

where capital indices can be traded for the multi-indices $(a_1 \dots a_{s-1})$. Comparing to (2.42) we can read off the Lie bracket

$$[J_a, J_{b_1 \dots b_{s-1}}] = (s-1) \epsilon^d{}_{a(b_1} J_{b_2 \dots b_{s-1})d}. \quad (2.48)$$

The zero-curvature equations of motion (2.45) may be obtained starting from several actions, as noticed in the gravity case in [29]. On the other hand, we also wish to recover upon linearisation the action (2.39), that directly generalises the Einstein–Hilbert action and corresponds to the $D = 3$ instance of the free higher-spin action (A.1). To this end, one can start from an action of the form (2.24) that, as shown in exercise 3, can also be written as the difference of two Chern–Simons actions when the cosmological constant is negative (the rewriting does not rely on the form of the gauge algebra). A direct comparison with the linearised action (2.39) then fixes the invariant bilinear form in (2.24) to

$$\mathrm{tr} (J_a J_b) = \frac{1}{2} \eta_{ab}, \quad \mathrm{tr} (J_a J_{b_1 \dots b_{s-1}}) = 0, \quad (2.49a)$$

$$\mathrm{tr} (J_{a_1 \dots a_{s-1}} J_{b_1 \dots b_{s-1}}) = \frac{1}{2} \eta_{a_1(b_1} \eta_{a_2|b_2} \dots \eta_{b_{s-1})a_{s-1}} + \dots, \quad (2.49b)$$

where in the last line we omitted the terms needed to separately enforce a traceless projection in the indices a_i and b_i (when writing the action the projection is already automatically taken into account by the contraction with the traceless fields).⁷

Exercise 7. * Check (2.48) and (2.49) starting, respectively, from (2.47), (2.42) and (2.24), (2.39).

A Chern–Simons theory with gauge algebra $\mathfrak{g} \oplus \mathfrak{g}$ and action (2.19) thus reproduces Fronsdal’s equations in AdS_3 upon linearisation, provided that \mathfrak{g} admits the non-degenerate invariant bilinear form (2.49) and its generators can be collected in traceless symmetric tensors $J_{a_1 \dots a_{s-1}}$ satisfying the Lie bracket (2.48). At this point, one should classify all algebras fulfilling the previous conditions to classify all possible interacting higher-spin gauge theories.⁸ In principle, this task can be achieved by solving the Jacobi identities with the initial data (2.48); still, we stress that any non-compact Lie algebra with a non-degenerate bilinear form fits into this scheme upon selecting a distinguished $\mathfrak{sl}(2, \mathbb{R})$ subalgebra to be interpreted as the gravitational one, possibly modulo the flip in the overall sign of (2.49b) for some values of s . We shall comment more on this general setup at the end of this section, while we now illustrate the key features of three-dimensional higher-spin algebras in a prototypical example.

To begin with, we observe that a symmetric and traceless product of $\mathfrak{sl}(2, \mathbb{R})$ generators satisfies the commutator (2.48), so that we can set

⁷ Notice that we fixed conventionally the overall sign in (2.49b) independently of the value of s . While keeping the same relative sign between the kinetic terms of different fields is a reasonable requirement, we stress that this is not mandatory in the current context in which there are no propagating degrees of freedom and, therefore, no risks to introduce ghosts. At the level of the gauge algebra, introducing relative signs in (2.49b) would correspond to selecting different real forms of the same complex Lie algebra.

⁸ A priori this classification would only cover interacting theories that can be cast in the Chern–Simons form, but one can prove that the latter exhaust the space of all possible interactions in the absence of matter couplings [50, 51].

$$J_{a_1 \cdots a_{s-1}} \sim J_{\{a_1 \cdots a_{s-1}\}}, \quad (2.50)$$

where braces denote a symmetrisation and a traceless projection.

Defining (2.50) requires introducing an associative product, out of which one can realise the Lie bracket as a commutator. The simplest way of doing so is to consider a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$ and use the matrix product. For concreteness, let us first focus on the three-dimensional representation. We can then consider the combinations

$$J_a, \quad J_{\{a} J_b\}} = J_{(a} J_b) - \frac{1}{3} \eta_{ab} C_2, \quad (2.51)$$

where

$$C_2 = \eta^{ab} J_a J_b \quad (2.52)$$

is the $\mathfrak{sl}(2, \mathbb{R})$ Casimir operator. As you are invited to check in the following exercise, these eight matrices are all traceless and linearly independent: as a result, they span all traceless 3×3 -matrices and generate $\mathfrak{sl}(3, \mathbb{R})$. The resulting Chern–Simons theory contains fields of spin two and three corresponding, respectively, to the generators J_a and $J_{\{a} J_b\}}$.

Exercise 8. Use the following three-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$,

$$J_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (2.53)$$

to show that the matrices in (2.51) are linearly independent and traceless.

Starting from a N -dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$ leads to an analogous construction: symmetrised and traceless products of $s - 1 \leq N$ generators are independent and form the fundamental representation of $\mathfrak{sl}(N, \mathbb{R})$ [52]. As a result, a Chern–Simons theory with gauge algebra $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ describes gauge fields of spin $s = 2, 3, \dots, N$. Each of them corresponds, respectively, to the components proportional to

$$J_{a_1 \cdots a_{s-1}} = \sqrt{\frac{(2s-1)!}{6(s-1)!^2} \prod_{i=2}^{s-1} \frac{2}{N^2 - i^2}} J_{\{a_1 \cdots a_{s-1}\}}, \quad 2 \leq s \leq N, \quad (2.54)$$

where we fixed the normalisation such that it agrees with the normalisation of the bilinear invariant form introduced in (2.49).

This construction can be generalised to include infinite-dimensional representations of $\mathfrak{sl}(2, \mathbb{R})$ so as to obtain higher-spin theories with an unbounded spectrum of fields. To this end, let us recall that in a N -dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$ the quadratic Casimir (2.52) takes the value

$$C_2 = \frac{1}{4} (N^2 - 1) \mathbb{I}. \quad (2.55)$$

The previous construction can thus be recovered by defining the combinations (2.54) in the universal enveloping algebra⁹ of $\mathfrak{sl}(2, \mathbb{R})$ and then imposing (2.55) by factoring out the corresponding relation. One can then generalise the procedure to include other values of C_2 so as to define the associative algebras

$$B[\lambda] = \frac{\mathcal{U}(\mathfrak{sl}(2, \mathbb{R}))}{\langle C_2 - \frac{1}{4}(\lambda^2 - 1)\mathbb{I} \rangle}. \quad (2.56)$$

We can turn this associative algebra into a Lie algebra with Lie bracket given by the commutator. As a Lie algebra, $B[\lambda]$ decomposes as

$$B[\lambda] = \mathbb{R} \oplus \mathfrak{hs}[\lambda], \quad (2.57)$$

since the identity \mathbb{I} commutes with all generators. For all values of λ , the Lie algebra $\mathfrak{hs}[\lambda]$ is a higher-spin algebra spanned by generators $J_a, J_{\{a}J_{b\}}, \dots$, because any antisymmetric combination can be expressed in terms of the Lie bracket, and any trace part can be expressed in terms of the Casimir operator.¹⁰ A Chern–Simons theory with gauge algebra $\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ thus describes an interacting higher-spin theory for gauge fields of spin $s = 2, 3, 4, \dots$, while different values of λ give inequivalent theories since the gauge algebras are not isomorphic (see, e.g., [27, 57]).

To build a Chern–Simons action, one also needs to define an invariant, non-degenerate bilinear form on $\mathfrak{hs}[\lambda]$. This can be done introducing a trace in $B[\lambda]$ defined by

$$\mathrm{tr} \mathbb{I} = \mathrm{const}, \quad \mathrm{tr} J_{\{a_1} \cdots J_{a_{s-1}\}} = 0. \quad (2.58)$$

The corresponding invariant bilinear form on $\mathfrak{hs}[\lambda]$ is non-degenerate for non-integer λ [59]. If $\lambda = N > 1$ is an integer, the bilinear form degenerates, and there is a non-trivial ideal

$$I_N = \{V \in \mathfrak{hs}[N] \mid \text{for all } W \in \mathfrak{hs}[N] : \mathrm{tr}(VW) = 0\}. \quad (2.59)$$

Factoring out this ideal leads to $\mathfrak{sl}(N, \mathbb{R})$,

$$\frac{\mathfrak{hs}[\lambda = N]}{I_N} \cong \mathfrak{sl}(N, \mathbb{R}), \quad (2.60)$$

and thus reproduces the previous construction in terms of products of matrices of finite-dimensional representations of the $\mathfrak{sl}(2, \mathbb{R})$ algebra.

⁹ For any Lie algebra \mathfrak{g} , its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is built as follows. One first introduces the space of linear combinations of all possible products of generators, defined by concatenation, and then quotients out the two-sided ideal generated by $J_a J_b - J_b J_a - f_{ab}{}^c J_c$ to ensure that the Lie bracket is realised as the commutator. One can show that $\mathcal{U}(\mathfrak{g})$ is spanned by symmetrised products of the generators since any antisymmetrised combination can be replaced by the commutator. For more details, see, e.g., [53, chapter 14].

¹⁰ This infinite-dimensional algebra has been introduced from different perspectives in [54, 27, 55–58]. In section 5.1.2, we shall also discuss in detail an alternative construction of $\mathfrak{hs}[\lambda]$ in terms of oscillator variables.

We now introduce a convenient basis for $\mathfrak{hs}[\lambda]$ that we shall use extensively later. To begin with, we define the linear combinations

$$L_{\pm 1} = J_0 \pm J_1, \quad L_0 = J_2 \quad (2.61)$$

that form a basis for the $\mathfrak{sl}(2, \mathbb{R})$ algebra with commutation relations

$$[L_m, L_n] = (m - n)L_{m+n}. \quad (2.62)$$

The invariant bilinear form (2.21) in this basis takes the form

$$\mathrm{tr} L_0 L_0 = \frac{1}{2}, \quad \mathrm{tr} L_1 L_{-1} = -1, \quad \mathrm{tr} L_0 L_{\pm 1} = 0. \quad (2.63)$$

For any fixed s the generators $J_{\{a_1 \cdots a_{s-1}\}}$ form a $(2s - 1)$ -dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$ with respect to the adjoint action. This directly follows from standard consideration on the tensoring of $\mathfrak{sl}(2, \mathbb{R})$ representations, and can also be made more manifest by using as a basis the generators W_m^s (with $-s + 1 \leq m \leq s - 1$) defined as follows. One first introduces the generator

$$W_{s-1}^s = (L_1)^{s-1} \quad (2.64)$$

and then defines recursively further $2s - 2$ generators using

$$W_{m-1}^s = -\frac{1}{m + s - 1} [L_{-1}, W_m^s] \quad (2.65)$$

until one gets $W_{1-s}^s = (L_{-1})^{s-1}$. These generators are in one-to-one correspondence with the independent components of the symmetric and traceless tensor $J_{a_1 \cdots a_{s-1}}$ of rank $s - 1$: their number is $2s - 1$ and they are all independent because they are eigenvectors of L_0 with different eigenvalues (see the following exercise).

Exercise 9.* Check that (2.64) and (2.65) imply

$$[L_m, W_n^s] = ((s - 1)m - n)W_{m+n}^s. \quad (2.66)$$

The definition we gave thus uniquely fixes the commutation relations of the basis elements with the $\mathfrak{sl}(2, \mathbb{R})$ generators. In particular, the ‘‘mode number’’ n corresponds to minus the eigenvalue of L_0 under the adjoint action. One can then check that (2.66) implies

$$[L_0, [W_m^s, W_n^s]] = -(m + n)[W_m^s, W_n^s] \quad (2.67)$$

and therefore that the mode number is additive under the Lie bracket.

In general, all commutation relations of $\mathfrak{hs}[\lambda]$ can be cast in the form¹¹

¹¹ The structure of the commutators shows that one can rescale $W_m^{2n+1} \rightarrow iW_m^{2n+1}$ to obtain another real form of $\mathfrak{hs}[\lambda]$, which allows for unitary representations for $0 < \lambda < 1$ [60]. For $\lambda = N$, it corresponds to the algebra $\mathfrak{su}(\frac{N-1}{2}, \frac{N+1}{2})$ for N odd and $\mathfrak{su}(\frac{N}{2}, \frac{N}{2})$ for N even. One can also see

$$[W_m^{s_1}, W_n^{s_2}] = \sum_{k=0}^{\min(s_1, s_2) - 2} c_\lambda[m, n] W_{m+n}^{s_1 + s_2 - 2k - 2}. \quad (2.68)$$

We refer to, e.g., [55] for more details about this statement, while here we propose to analyse its simplest manifestation in the following exercises.

Exercise 10.* Prove that the commutator $[W_m^3, W_n^3]$ must be a combination of W_{m+n}^4 and L_{m+n} , and then check that the m, n -dependence of the structure constants is completely fixed by (2.66) via the Jacobi identities. That is, prove that

$$[W_m^3, W_n^3] = \alpha[\lambda](m-n)W_{m+n}^4 + \frac{\beta[\lambda]}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n}. \quad (2.69)$$

Exercise 11.* Determine $\alpha[\lambda]$ and $\beta[\lambda]$ by computing the commutator (2.69) using concrete expressions for W_m^3 and W_n^4 obtained starting from (2.64) and the recursion relation (2.65). Notice that the result,

$$\alpha[\lambda] = 2, \quad \beta[\lambda] = \frac{4 - \lambda^2}{5}, \quad (2.70)$$

can be obtained using only a selected number of m and n components; one also has to use that

$$C_2 = L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) = \frac{1}{4}(\lambda^2 - 1)\mathbb{I}. \quad (2.71)$$

General expressions for the structure constants of $\mathfrak{hs}[\lambda]$ were postulated in [55] and later proven in [66]. The invariant bilinear form can be defined via the trace in (2.58) [59]. If we normalize the trace such that it coincides with the standard form (2.21) on the $\mathfrak{sl}(2)$ subalgebra, it is explicitly given by (see for example [63])

$$\mathrm{tr}(W_m^s W_n^t) = (-1)^m \frac{6(s-1)!^2 (s+m-1)!(s-m-1)!}{(2s-1)!(2s-2)!} \prod_{i=2}^{s-1} (\lambda^2 - i^2) \delta^{s,t} \delta_{m+n,0}. \quad (2.72)$$

From this expression one can directly read off that the bilinear form is degenerate for any integer $\lambda = N > 1$.

Taking symmetrised products of $\mathfrak{sl}(2)$ generators is a natural way to build higher-spin algebras in three dimensions and mimics the construction of higher-spin algebras in any spacetime dimensions [67–70].¹² On the other hand, we can start with any Lie algebra \mathfrak{g} admitting a non-degenerate bilinear form and that contains a $\mathfrak{sl}(2, \mathbb{R})$ subalgebra. The decomposition of \mathfrak{g} into irreducible representations of $\mathfrak{sl}(2, \mathbb{R})$ with respect to the adjoint action then determines the field content:

from (2.68) that it is possible to consistently truncate the algebra to only contain even-spin generators, which for integer λ results in a real form of $\mathfrak{so}(2N)$. We refer, e.g., to [61–65] for more details and for applications of these observations.

¹² In a spacetime of dimension D , higher-spin algebras are defined as quotients of the universal enveloping algebra of $\mathfrak{so}(2, D-1)$ and are used to organise the non-linear deformations of the free field equations, see, e.g., [12]. See also [71] for an explicit comparison between the previous construction and the one that applies in generic spacetime dimensions.

the $2s - 1$ generators belonging to each irreducible representation can indeed be grouped into a tensor $J_{a_1 \dots a_{s-1}}$ satisfying (2.48) or, equivalently, into a tensor W_m^s satisfying (2.66). Let us stress that different embeddings of $\mathfrak{sl}(2, \mathbb{R})$ into \mathfrak{g} lead to different spectra (see, e.g., [63, 72]). For example, under the regular embedding of $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{sl}(3, \mathbb{R})$ as 2×2 -matrix blocks,¹³ we have the decomposition

$$\mathfrak{sl}(3, \mathbb{R}) \simeq \mathbf{3} + 2 \cdot \mathbf{2} + \mathbf{1}, \quad (2.73)$$

where we labelled $\mathfrak{sl}(2)$ representations by their dimension in boldface. We can interpret the corresponding Chern–Simons theory as a gauge theory containing gravity coupled to two bosonic spin- $\frac{3}{2}$ fields and one spin-1 field.

In the following, we shall anyway focus on higher-spin theories with $\mathfrak{hs}[\lambda]$ or $\mathfrak{sl}(N, \mathbb{R})$ gauge algebras, often resorting to their simplest $N = 3$ instance to introduce new concepts. For this reason, we close this section with an exercise aiming at fixing all conventions for the $\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(3, \mathbb{R})$ Chern–Simons theory describing the coupling of a spin-three field to gravity.

Exercise 12. Starting from the three-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$ given in (2.53), check that the explicit matrix representations of the generators L_m and W_m^3 is

$$\begin{aligned} L_1 &= \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & L_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & L_{-1} &= \sqrt{2} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \\ W_2^3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, & W_1^3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, & W_0^3 &= \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & (2.74) \\ W_{-2}^3 &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & W_{-1}^3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Using this representation, verify that the normalised trace in (2.72) is related to the matrix trace $\text{tr}_{3 \times 3}$ by a factor $\frac{1}{4}$,

$$\text{tr}(L_m L_n) = \frac{1}{4} \text{tr}_{3 \times 3}(L_m L_n) = (-1)^m \frac{(1+m)!(1-m)!}{2} \delta_{m+n,0}, \quad (2.75a)$$

$$\text{tr}(W_m^3 W_n^3) = \frac{1}{4} \text{tr}_{3 \times 3}(W_m^3 W_n^3) = (-1)^m \frac{(2+m)!(2-m)!}{4!} \delta_{m+n,0}. \quad (2.75b)$$

By explicit computation, check the commutation relations of $\mathfrak{sl}(3, \mathbb{R})$ in this basis:

¹³ For $N = 3$, the construction that we discussed before, in which $\mathfrak{sl}(2, \mathbb{R})$ is embedded into $\mathfrak{sl}(3, \mathbb{R})$ as the 3×3 matrices (2.53), corresponds to another embedding called the principal one.

$$[L_m, L_n] = (m - n)L_{m+n}, \quad (2.76a)$$

$$[L_m, W_n^3] = (2m - n)W_{m+n}^3, \quad (2.76b)$$

$$[W_m^3, W_n^3] = -\frac{1}{12}(m - n)(2m^2 + 2n^2 - mn - 8)L_{m+n}. \quad (2.76c)$$

3 Asymptotic symmetries

In this section, we study the asymptotic symmetries of three-dimensional higher-spin gauge theories on a spacetime which is asymptotically AdS. These are transformations of the fields that – although they take the same form as a gauge transformation – correspond to global symmetries of the system rather than parameterising a redundancy in its description. They are identified among the gauge transformations that preserve the boundary conditions on the fields (that one has to impose due to the non-compact nature of spacetime) while acting on the boundary data. As we shall discuss, transformations of this sort contain a subset that is canonically generated by global charges defined on the boundary of spacetime and that has to be identified with global symmetries.

We work in the Chern–Simons formulation that we introduced in the last section, first explaining the basic concepts using boundary conditions that lead to an affine Lie algebra of asymptotic symmetries. We then include additional constraints that characterise asymptotically AdS field configurations. These conditions lead to a reduction of the affine Lie algebra to a classical \mathscr{W} -algebra known as Drinfeld–Sokolov reduction. In the case of gravity, this reproduces the Virasoro algebra first identified by Brown and Henneaux [73], whereas for higher-spin gravity we get asymptotic non-linear \mathscr{W} -algebras as first observed in [33, 34].

We follow the approach of [74, 34], while distinguishing the role of the time and radial coordinates, as later suggested in [75, 43]. We assume here that the reader is familiar with Dirac’s treatment of Hamiltonian systems with constraints, while recalling the basic facts we need in section C.

3.1 Chern–Simons theory with boundary

We wish to describe gravity and higher-spin gravity on three-dimensional geometries which are asymptotically anti-de Sitter. The conformal completion of AdS₃ is diffeomorphic to a solid cylinder $\Sigma = \mathbb{R} \times D_2$ with boundary $\partial\Sigma = \mathbb{R} \times S^1$: for this reason, in the following we shall consider Chern–Simons theories defined on this manifold.¹⁴ We also choose coordinates t, θ, ρ , where t is the time coordinate along \mathbb{R} , θ is the 2π -periodic angular coordinate on the disk D_2 , and ρ is the radial coordi-

¹⁴ A full characterisation of the space of solutions might require to also consider different topologies, but asymptotically the manifolds should take the form of a solid cylinder. For this reason,

nate on D_2 , for which we find it convenient to choose the range from $\rho = 0$ (centre of the disk) to $\rho = \infty$ (boundary of the disk).

To begin with, we consider a generic Chern–Simons action,

$$S_{\text{CS}} = \frac{k}{4\pi} \int_{\Sigma} \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (3.1)$$

where A is a Lie-algebra-valued field, for a Lie algebra \mathfrak{g} admitting an invariant non-degenerate bilinear form here denoted by tr . This action is already in Hamiltonian form, with conjugate fields A_θ and A_ρ , and a Lagrange multiplier A_t :

$$S_{\text{CS}} = \frac{k}{4\pi} \left(\int_{\Sigma} dt d\rho d\theta \text{tr} (A_\theta \dot{A}_\rho - A_\rho \dot{A}_\theta + 2A_t F_{\rho\theta}) - \int_{\partial\Sigma} dt d\theta \text{tr} (A_t A_\theta) \right), \quad (3.2)$$

where \dot{A}_ρ , \dot{A}_θ denote the t -derivative of A_ρ and A_θ , respectively. The bulk Hamiltonian vanishes as it should be the case for any action that is invariant under diffeomorphisms while, as anticipated, A_t only enters algebraically and enforces the constraint

$$F_{\rho\theta} := \partial_\rho A_\theta - \partial_\theta A_\rho + [A_\rho, A_\theta] = 0. \quad (3.3)$$

The variation of the action reads

$$\begin{aligned} \delta S_{\text{CS}} = & \frac{k}{2\pi} \int_{\Sigma} dt d\rho d\theta \text{tr} \left((\dot{A}_\rho - \partial_\rho A_t + [A_t, A_\rho]) \delta A_\theta - (\dot{A}_\theta - \partial_\theta A_t + [A_t, A_\theta]) \delta A_\rho \right. \\ & \left. + F_{\rho\theta} \delta A_t \right) - \frac{k}{4\pi} \int_{\partial\Sigma} dt d\theta \text{tr} (A_\theta \delta A_t - A_t \delta A_\theta). \end{aligned} \quad (3.4)$$

The bulk part leads to the equations of motion for A_ρ and A_θ and to the constraint $F_{\rho\theta} = 0$. At this stage, we do not want to put boundary conditions on the field A_θ . To cancel the boundary contribution that contains δA_θ , we then add the boundary term

$$S_{\text{bdy}} = -\frac{k}{4\pi} \int_{\partial\Sigma} dt d\theta \text{tr} (A_t A_\theta) \quad (3.5)$$

to the action. The total boundary contribution to the variation of the action now only consists of a term that contains δA_t . We can set the latter to zero by fixing A_t at the boundary, i.e. by imposing $\delta A_t|_{\partial\Sigma} = 0$. Whereas A_t plays the role of a Lagrange multiplier in the bulk, its boundary value can thus be viewed as a parameter of the theory.

Since A is a Lie-algebra-valued field, we can decompose it with respect to a basis of generators t_A of \mathfrak{g} as $A = A^A t_A$. The equal-time Poisson brackets between the fields A_ρ^A and A_θ^B are given by¹⁵

focussing on Chern–Simons theories on a solid cylinder suffices to discuss asymptotic symmetries.

¹⁵ We restrict here to a fixed time slice and suppress the time dependence of the fields; it will later be restored when we study the time evolution.

$$\{A_\rho^A(\rho, \theta), A_\theta^B(\rho', \theta')\} = \frac{2\pi}{k} \gamma^{AB} \delta(\rho - \rho') \delta(\theta - \theta'), \quad (3.6)$$

where γ^{AB} is the inverse of the matrix $\gamma_{AB} = \text{tr}(t_A t_B)$. For phase-space functionals \mathcal{F} and \mathcal{H} with well-defined functional derivatives with respect to A_ρ and A_θ , the Poisson bracket is

$$\{\mathcal{F}, \mathcal{H}\} = \frac{2\pi}{k} \int_{D_2} d\rho d\theta \text{tr} \left(\frac{\delta \mathcal{F}}{\delta A_\rho(\rho, \theta)} \frac{\delta \mathcal{H}}{\delta A_\theta(\rho, \theta)} - \frac{\delta \mathcal{F}}{\delta A_\theta(\rho, \theta)} \frac{\delta \mathcal{H}}{\delta A_\rho(\rho, \theta)} \right). \quad (3.7)$$

Exercise 13. Show that (3.6) implies (3.7).

$F_{\rho\theta}$ is a single constraint, and it is compatible with time evolution. As such, it can only be a first-class constraint, and we can build from it phase-space functionals $G(\Lambda)$ which generate the gauge transformations $A \rightarrow A + d\Lambda + [A, \Lambda]$ with the help of the Poisson brackets. Let us discuss this in more detail because it becomes subtle in the presence of a boundary. Indeed, in this case one has to write the generators of gauge transformations as [76]

$$G(\Lambda) = \frac{k}{2\pi} \int_{D_2} d\rho d\theta \text{tr} (\Lambda F_{\rho\theta}) + Q(\Lambda), \quad (3.8)$$

where we introduced a boundary term to ensure that G has a well-defined functional derivative with respect to A_ρ and A_θ , i.e. to ensure that the variation of $G(\Lambda)$,

$$\begin{aligned} \delta G &= \frac{k}{2\pi} \int_{D_2} d\rho d\theta \text{tr} \left((\partial_\theta \Lambda + [A_\theta, \Lambda]) \delta A_\rho - (\partial_\rho \Lambda + [A_\rho, \Lambda]) \delta A_\theta \right) \\ &\quad + \frac{k}{2\pi} \int_{S^1} d\theta \left(\text{tr} (\Lambda \delta A_\theta) + \delta Q(\Lambda) \right), \end{aligned} \quad (3.9)$$

does not contain any boundary terms. This condition fixes the functional variation of $Q(\Lambda)$ to be

$$\delta Q(\Lambda) = -\frac{k}{2\pi} \int_{S^1} d\theta \text{tr} (\Lambda \delta A_\theta). \quad (3.10)$$

In the current setup, in which we do not impose any constraint on A_θ , the parameter Λ is just an arbitrary Lie-algebra-valued function, so that the variation (3.10) can be straightforwardly integrated to get¹⁶

$$Q(\Lambda) = -\frac{k}{2\pi} \int_{S^1} d\theta \text{tr} (\Lambda A_\theta). \quad (3.11)$$

Notice, however, that the expression (3.10) for the variation of the charge also applies to more general contexts that we shall encounter later, in which the gauge parameter may depend on the fields. This is so because any possible contribution

¹⁶ One could also add an arbitrary function on the boundary, provided that it does not depend on the boundary value of the dynamical field A_θ . We omitted it in agreement with the most common expression for $Q(\Lambda)$ in the literature, but this issue will be relevant when we shall discuss the time dependence of the charge in (3.30) below.

in $\delta\Lambda$ in the variation of G would anyway be proportional to the constraint, as it is manifest from (3.8). On the other hand, for field-dependent gauge parameters – as, e.g., those corresponding to diffeomorphisms (see exercise 2) – the integration of $\delta Q(\Lambda)$ may lead to different results compared to (3.11) or require additional boundary conditions.

Exercise 14. Verify the variation (3.9) and show that it implies

$$\delta_\Lambda A_\rho = \{G(\Lambda), A_\rho\} = \partial_\rho \Lambda + [A_\rho, \Lambda], \quad (3.12a)$$

$$\delta_\Lambda A_\theta = \{G(\Lambda), A_\theta\} = \partial_\theta \Lambda + [A_\theta, \Lambda]. \quad (3.12b)$$

The equations of motion for A_ρ and A_θ , which follow from the variation of the action $S = S_{\text{CS}} + S_{\text{bdy}}$, show that the time evolution is a gauge transformation with A_t having the interpretation of a gauge parameter per unit time,

$$\dot{A}_\rho = \{G(A_t), A_\rho\}, \quad \dot{A}_\theta = \{G(A_t), A_\theta\}. \quad (3.13)$$

This comes without surprise because the Hamiltonian vanishes in the action (3.2) and the total bulk Hamiltonian of the system is a pure constraint, $H_{\text{tot}} \propto G(A_t)$. On the other hand, when one applies a possibly time-dependent gauge transformation with parameter $\Lambda(t, \rho, \theta)$, one also changes the precise time evolution of A_θ and A_ρ along the gauge orbit. To be consistent with the time evolution determined by (3.13), the gauge transformation has to be accompanied by a variation of A_t :

$$\delta A_t = \partial_t \Lambda + [A_t, \Lambda]. \quad (3.14)$$

The gauge transformations of the fields and of the Lagrange multiplier combine into

$$\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda], \quad (3.15)$$

which is the usual covariant infinitesimal gauge symmetry of the Chern–Simons action (3.1).

We have discussed how to recover in the Hamiltonian formalism the gauge symmetry (3.15), but the boundary term (3.11) in the generator of gauge transformations leads to several subtleties in their interpretation. This already becomes apparent in the Poisson bracket of two gauge generators,

$$\{G(\Lambda), G(\Gamma)\} = G([\Lambda, \Gamma]) + \frac{k}{2\pi} \int_{S^1} d\theta \operatorname{tr} (\Lambda \partial_\theta \Gamma), \quad (3.16)$$

that develops a central extension which depends on the boundary values of Λ and Γ .

Exercise 15.* Prove (3.16).

The most important consequence of a non-trivial $Q(\Lambda)$ is that the corresponding transformation is actually not a gauge symmetry, but a global symmetry canonically generated by the boundary charge $Q(\Lambda)$. In other words, a variation (3.15) associated to a non-trivial boundary charge sends solutions of the equations of motion into

physically inequivalent solutions even if it takes the same form as a gauge transformation. Symmetries of this kind are known as asymptotic symmetries.

To analyse this statement in detail, it is convenient to add a gauge-fixing condition on top of the first-class constraint (3.3). As we shall see shortly, the additional constraint we are going to impose completely fixes the proper gauge symmetry, so that no redundancy in the description of the space of solutions is left. The gauge-fixing, however, still allows for residual transformations of the form (3.15), to be interpreted as global (or asymptotic) symmetries.

To this end, we choose a specific form for A_ρ which, close to the boundary, we take as

$$A_\rho(\rho) = b^{-1}(\rho)\partial_\rho b(\rho) \quad (3.17)$$

for a chosen group-valued function $b(\rho)$. Close to the boundary one can always reach this gauge off-shell: to bring a gauge field A' into the form (3.17), one has to perform a gauge transformation satisfying

$$U^{-1}A'_\rho U + U^{-1}\partial_\rho U = b^{-1}\partial_\rho b. \quad (3.18)$$

Writing $U = U'b$, this condition holds if

$$\partial_\rho U' = -A'_\rho U', \quad (3.19)$$

which can be solved by a path-ordered exponential,

$$U = \mathcal{P}e^{-\int_{\rho_0}^{\rho} A'_r dr} U_0 b. \quad (3.20)$$

U_0 is a constant of integration that must be chosen so as to preserve the boundary condition $\delta A_t|_{\partial\Sigma} = 0$, and one can always do this on the solid cylinder.¹⁷

On the constraint surface, where we implement $F_{\rho\theta} = 0$, the ρ -dependence of A_θ is also fixed,

$$\partial_\rho A_\theta + [A_\rho, A_\theta] = 0 \quad \implies \quad \partial_\rho (b A_\theta b^{-1}) = 0. \quad (3.21)$$

A_θ is then of the form

$$A_\theta(t, \rho, \theta) = b^{-1}(\rho) a_\theta(t, \theta) b(\rho) \quad (3.22)$$

with some ρ -independent term $a_\theta(t, \theta)$, where we restored the time dependence to stress that this condition is respected along time evolution.

The residual transformations of the form (3.15) that leave A_ρ invariant satisfy

$$\partial_\rho \Lambda + [A_\rho, \Lambda] = 0, \quad (3.23)$$

hence Λ has to be of the form

¹⁷ If other boundaries are present, it might not be possible to select a U_0 compatible with all boundary conditions. In other words, in some cases the gauge (3.17) might not be reachable, or it may be necessary to select specific group-valued functions in order to reconstruct globally certain classes of solutions. We refer, e.g., to [77, 78] for more details.

$$\Lambda(t, \rho, \theta) = b^{-1}(\rho) \lambda(t, \theta) b(\rho). \quad (3.24)$$

Similarly, the Lagrange multiplier A_t needs to satisfy

$$A_t(t, \rho, \theta) = b^{-1}(\rho) a_t(t, \theta) b(\rho), \quad (3.25)$$

such that the gauge choice for A_ρ is preserved along time evolution. The function a_t parameterises the freedom to choose the boundary value of A_t .

Now we want to argue that the residual transformations generated by gauge parameters of the form (3.24) cease to be gauge symmetries, and they parameterise global symmetries instead. One way to see this is by looking at the evolution in time. If we have a proper gauge symmetry, we can choose an arbitrary time dependence for the gauge parameter because, on each time slice, configurations related by gauge transformations should be completely equivalent. The transformation of a_t induced by a Λ of the form (3.24) is however

$$a_t(t, \theta) \rightarrow a_t(t, \theta) + \partial_t \lambda(t, \theta) + [a_t(t, \theta), \lambda(t, \theta)]. \quad (3.26)$$

As a_t is a fixed parameter of the theory in the current setup, it should be invariant under this transformation. This fixes the time dependence of λ , which therefore cannot correspond to a proper gauge transformation.

Moreover, adding the constraint (3.17) to the first-class constraint (3.3) one obtains a set of second-class constraints, as it should be the case for any complete gauge-fixing. Indeed, the Poisson bracket of the constraints can be computed as

$$\{G(\mathcal{E}), A_\rho\} = \partial_\rho \mathcal{E} + [A_\rho, \mathcal{E}] =: D_\rho \mathcal{E}, \quad (3.27)$$

where we introduced a smearing function \mathcal{E} for the constraint $F_{\rho\theta} = 0$, while we omitted the term $b^{-1} \partial_\rho b$ in the constraint (3.17) since it does not depend on phase-space variables. Notice that here $G(\mathcal{E})$ has to depend only on the first-class constraint (3.3), so that the smearing function \mathcal{E} must vanish at the boundary. The unique solution of the first-order differential equation $D_\rho \mathcal{E} = 0$ with this boundary condition is $\mathcal{E} = 0$ everywhere, thus showing that the bracket (3.27) is not degenerate.

Conversely, non-trivial transformations preserving the gauge-fixing (3.17) have a non-vanishing boundary charge (3.11), so that they are not generated by a first-class constraint:

$$Q(\Lambda) = -\frac{k}{2\pi} \int_{S^1} d\theta \operatorname{tr} (b^{-1} \lambda b b^{-1} a_\theta b) = -\frac{k}{2\pi} \int_{S^1} d\theta \operatorname{tr} (\lambda a_\theta) =: Q(\lambda). \quad (3.28)$$

This is yet another way of concluding that they cannot be proper gauge transformations (see, e.g., [79, 76, 80] for an ampler discussion of this point in the Hamiltonian formalism).

Notice that the charge given in (3.28) is conserved on-shell if one chooses the function a_t to be constant:

$$\frac{d}{dt}Q(\lambda) = -\frac{k}{2\pi} \int_{S^1} d\theta \operatorname{tr} (\lambda \partial_\theta a_t), \quad (3.29)$$

where we stress that one has to impose the equations of motion (3.13) to get this result. For generic values of a_t the charge is not conserved, but its time variation only depends on the latter and not on the boundary value of the dynamical field a_θ (see also [81]). As such, since a_t is just a parameter of the theory, one can still define a quantity that is conserved on shell along time evolution:¹⁸

$$\hat{Q}(\lambda) = Q(\lambda) + \frac{k}{2\pi} \int_{t_0}^t dt' \int_{S^1} d\theta \operatorname{tr} (\lambda \partial_\theta a_{t'}). \quad (3.30)$$

Since the constraints (3.3) and (3.17) form a set of second-class constraints, one can define a Poisson structure on the reduced phase space, here parameterised by the functions $a_\theta(t, \theta)$, via the Dirac bracket. The latter can be efficiently computed by realising that the Dirac bracket of the charges $Q(\Lambda)$ with any phase-space functional corresponds to the Poisson bracket. Indeed, denoting collectively the two constraints (3.3) and (3.17) by ϕ_i , the Dirac bracket $\{\cdot, \cdot\}_*$ is defined as (see also (C.8))

$$\{Q(\Lambda), \mathcal{F}\}_* = \{G(\Lambda), \mathcal{F}\}_* = \{G(\Lambda), \mathcal{F}\} - \{G(\Lambda), \phi_i\} C^{ij} \{\phi_j, \mathcal{F}\}, \quad (3.31)$$

where C^{ij} is the inverse of the matrix $C_{ij} = \{\phi_i, \phi_j\}$. On the other hand, the constraints are invariant under the residual symmetries generated by gauge parameters of the form (3.24),

$$\{G(\Lambda), \phi_i\} = \delta_\Lambda \phi_i \approx 0, \quad (3.32)$$

where we used the symbol \approx to stress that the identity holds on the constraint surface. Eq. (3.24) was indeed obtained by demanding the invariance of the constraint (3.17), while the constraint $F_{\rho\theta} = 0$ is also invariant on the constraint surface since

$$\delta_\Lambda F_{\mu\nu} = [F_{\mu\nu}, \Lambda]. \quad (3.33)$$

As a result, the boundary charges canonically generate the global symmetries induced by non-trivial $\lambda(t, \theta)$, that is

$$\delta_\lambda \mathcal{F} = \{G(\Lambda), \mathcal{F}\} \approx \{Q(\lambda), \mathcal{F}\}_*, \quad (3.34)$$

and the algebra of boundary charges on the constrained phase space is the same as that of the generators $G(\Lambda)$ (given in (3.16)),

$$\{Q(\lambda_1), Q(\lambda_2)\}_* = Q([\lambda_1, \lambda_2]) + \frac{k}{2\pi} \int_{S^1} d\theta \operatorname{tr} (\lambda_1 \partial_\theta \lambda_2). \quad (3.35)$$

From this result we can infer the Poisson structure on the reduced phase space parameterised by a_θ , and a similar strategy can be employed also when some con-

¹⁸ Boundary conditions leading to truly non-conserved surface charges in three dimensions have been considered, e.g., in [82–85].

straints on a_θ are imposed as part of the boundary conditions of the theory (as in the next section).

In the current example, it is convenient to expand $a_\theta(t, \theta)$ in Fourier modes and in a basis t_A of generators of \mathfrak{g} ,

$$a_\theta(t, \theta) = \frac{1}{k} \sum_{p \in \mathbb{Z}} a_p^A(t) e^{-ip\theta} t_A. \quad (3.36)$$

The modes a_p^A can be expressed in terms of the boundary charges as

$$a_p^A = -\gamma^{AB} Q(t_B e^{ip\theta}), \quad (3.37)$$

where we recall that γ^{AB} is the inverse of the Killing metric $\gamma_{AB} = \text{tr}(t_A t_B)$. We can then determine the brackets between the modes from (3.35):

$$\begin{aligned} \{a_p^A, a_q^B\}_* &= Q([t^A, t^B] e^{i(p+q)\theta}) + \frac{k}{2\pi} \int d\theta \text{tr}(t^A t^B) i q e^{i(p+q)\theta} \\ &= -f^{AB}{}_C a_{p+q}^C + i q k \gamma^{AB} \delta_{p+q, 0}. \end{aligned} \quad (3.38)$$

This gives the central extension of the loop algebra of \mathfrak{g} (which one calls the affine Lie algebra $\hat{\mathfrak{g}}$, see also (4.64)), or more precisely, the modes a_p^A provide a representation (via the Dirac bracket) of the central extension of the loop algebra at a fixed level k . To summarise, on the reduced phase space of a Chern–Simons theory with gauge algebra \mathfrak{g} and with fixed boundary values of A_t we have found a realisation of the affine Lie algebra $\hat{\mathfrak{g}}_k$.

3.2 Asymptotically AdS solutions: the Drinfeld–Sokolov reduction

In the previous section, we discussed how some of the residual gauge transformations compatible with the boundary conditions might become asymptotic symmetries. To illustrate this point, we used boundary conditions in which we completely fixed A_ρ while leaving the boundary value of A_θ arbitrary. These boundary conditions are useful to discuss the subtleties that are brought by the presence of a boundary, but they are not appropriate to describe asymptotically AdS₃ spaces, at least according to the definition given in the seminal paper by Brown and Henneaux [73].¹⁹ In the following, we recall how asymptotically AdS₃ spaces can be described in the Chern–Simons formulation of gravity, and we then generalise their definition to the higher-spin gauge theories that we introduced in section 2.2.3.

To begin with, we recall that a global parameterisation of AdS₃ is given by

$$ds^2 = -\cosh^2 \rho dt^2 + \ell^2 (d\rho^2 + \sinh^2 \rho d\theta^2), \quad (3.39)$$

¹⁹ See, e.g., [86, 81–85] and references therein for generalisations of the Brown–Henneaux boundary conditions.

where the radial coordinate ρ is taken to be dimensionless, but it still takes the same range as in the previous subsection. Rewriting the hyperbolic functions in terms of exponentials and performing for later convenience the shift $\rho \rightarrow \rho + \log(2)$ we obtain

$$ds^2 = \ell^2 d\rho^2 + \left(e^{2\rho} + \frac{1}{16} e^{-2\rho} \right) (-dt^2 + \ell^2 d\theta^2) - \frac{1}{2} (dt^2 + \ell^2 d\theta^2). \quad (3.40)$$

In the Chern–Simons formulation, this metric can be reproduced starting from the $\mathfrak{sl}(2)$ -valued connections

$$A^{\text{AdS}} = L_0 d\rho + (e^\rho L_1 + \frac{1}{4} e^{-\rho} L_{-1}) \left(\frac{dt}{\ell} + d\theta \right), \quad (3.41a)$$

$$\tilde{A}^{\text{AdS}} = -L_0 d\rho - (\frac{1}{4} e^{-\rho} L_1 + e^\rho L_{-1}) \left(\frac{dt}{\ell} - d\theta \right), \quad (3.41b)$$

that are related to the vielbein and the spin connection as in (2.23). We used here the basis (2.62) for $\mathfrak{sl}(2)$, that we recall for the reader's convenience:

$$[L_m, L_n] = (m - n) L_{m+n}. \quad (3.42)$$

Exercise 16. Check eqs. (3.41) by first verifying that they satisfy the zero-curvature condition. Then verify that the corresponding vielbein reproduces the metric (3.40) with the conventions

$$ds^2 = \eta_{ab} e^a e^b = 2 \text{tr}(e^2), \quad (3.43)$$

where the vielbein is defined as in (2.23) while the trace is defined as in (2.63) (see also [34, 87] for a more general discussion on how to rewrite metric-like fields as traces over Lie-algebra-valued fields).

The connections A^{AdS} and \tilde{A}^{AdS} satisfy the gauge-fixing condition (3.17) that we imposed in the previous section with, respectively, the group-valued elements

$$b(\rho) = e^{\rho L_0}, \quad \tilde{b}(\rho) = b^{-1}(\rho). \quad (3.44)$$

These connections thus fit in our previous discussion with the following expressions for the canonical variables:

$$A_\rho = b^{-1}(\rho) \partial_\rho b(\rho), \quad \tilde{A}_\rho = b(\rho) \partial_\rho b^{-1}(\rho), \quad (3.45a)$$

$$A_\theta = b^{-1}(\rho) a(t, \theta) b(\rho), \quad \tilde{A}_\theta = b(\rho) \tilde{a}(t, \theta) b^{-1}(\rho), \quad (3.45b)$$

with

$$a^{\text{AdS}} = L_1 + \frac{1}{4} L_{-1}, \quad \tilde{a}^{\text{AdS}} = \frac{1}{4} L_1 + L_{-1}. \quad (3.46)$$

The asymptotically AdS_3 spacetimes of [73] can be recovered by imposing the natural conditions that the corresponding connections take the form (3.45) and approach those of global anti-de Sitter space close to the boundary, that is

$$A_\theta - A_\theta^{\text{AdS}} = \mathcal{O}(1) \quad \text{and} \quad \tilde{A}_\theta - \tilde{A}_\theta^{\text{AdS}} = \mathcal{O}(1) \quad \text{for } \rho \rightarrow \infty \quad (3.47)$$

at any time t . The conditions (3.44), (3.45) and (3.47) can be extended verbatim to higher-spin theories in the Chern–Simons formulation [34] and this will be our definition of asymptotically AdS₃ field configurations.²⁰

Since the ρ -dependence is completely encoded in the group-valued $b(\rho)$ defined in (3.44), the constraints (3.47) translate into constraints on the form of $a(t, \theta)$ and $\tilde{a}(t, \theta)$. For simplicity, we now restrict the discussion to $\mathfrak{sl}(N)$ or $\mathfrak{hs}[\lambda]$ valued connections. These can be conveniently expanded in the basis W_m^s that we introduced in (2.65), whose generators satisfy

$$[L_m, W_n^s] = ((s-1)m - n)W_{m+n}^s. \quad (3.48)$$

In this basis, a generic $\mathfrak{sl}(N)$ -valued connection takes the form

$$a(t, \theta) = \sum_{m=-1}^1 \ell^m(t, \theta)L_m + \sum_{l=2}^{N-1} \sum_{m=-l}^l w_{l+1}^m(t, \theta)W_m^{l+1}, \quad (3.49)$$

while the AdS conditions (3.47) constrain a and \tilde{a} to be of the form

$$a(t, \theta) = L_1 + \sum_{m=-1}^0 \ell^m(t, \theta)L_m + \sum_{l=2}^{N-1} \sum_{m=-l}^0 w_{l+1}^m(t, \theta)W_m^{l+1}, \quad (3.50a)$$

$$\tilde{a}(t, \theta) = L_{-1} + \sum_{m=0}^1 \tilde{\ell}^m(t, \theta)L_m + \sum_{l=2}^{N-1} \sum_{m=0}^l \tilde{w}_{l+1}^m(t, \theta)W_m^{l+1}. \quad (3.50b)$$

The second expansion can be recovered by sending $m \rightarrow -m$ in the first, so that in the following we shall focus only on the connection $a(t, \theta)$. The corresponding results for $\tilde{a}(t, \theta)$ can be easily recovered. In the $\mathfrak{hs}[\lambda]$ case, the connections take the same form, but one has to consider an unbounded sum over l .

Exercise 17.* Check that the AdS conditions (3.47) imply the expansion (3.50).

We gave a definition of the space of asymptotically AdS₃ configurations that fits within the discussion of section 3.1, but where A_θ is bounded to satisfy the constraint (3.47) on any time slice. The boundary term (3.5) continues to guarantee a well-posed action principle provided that the Lagrange multiplier A_t is fixed at the boundary, but the latter is not any more arbitrary. Preserving the form (3.47) of A_θ under time evolution imposes some constraints on the Lagrange multiplier A_t that we discuss in specific examples in the following. For what concerns the asymptotic symmetries, the additional constraints lead to a reduction of the affine Lie algebra $\hat{\mathfrak{g}}_k$ that we identified in the previous section, which in mathematics is known as the Drinfeld–Sokolov reduction (see, e.g., [91–93] and references therein).

²⁰ The Brown–Henneaux boundary conditions were first translated in the Chern–Simons formulation in slightly different terms in [88]. This reformulation later allowed to characterise asymptotically AdS₃ configurations in supergravity [89, 90] and higher-spin theories [33, 34].

We discuss it in detail in the following, starting from the illuminating examples of gravity and of the $\mathfrak{sl}(3) \oplus \mathfrak{sl}(3)$ Chern–Simons theory describing the coupling of a spin-three field to gravity. We then conclude with a discussion of the $\mathfrak{sl}(N)$ case and some comments on the $\mathfrak{hs}[\lambda]$ case.

3.3 The $\widehat{\mathfrak{sl}(2)}$ example: asymptotic symmetries of gravity

We now derive the asymptotic symmetries of gravity with the Brown–Henneaux boundary conditions (3.47) by focussing on the connection A_μ . As already mentioned, the contribution of the connection \tilde{A}_μ can be recovered following similar steps. The following analysis amounts to implement the Drinfeld–Sokolov reduction of the $\widehat{\mathfrak{sl}(2)}_k$ affine algebra (3.38) to the Virasoro algebra.

In this example, on any fixed time slice, the generic expansion (3.49) of the ρ -independent part of $A_\theta = b^{-1}(\rho)a(\theta)b(\rho)$ reads

$$a(\theta) = \ell^1(\theta)L_1 + \ell^0(\theta)L_0 + \ell^{-1}(\theta)L_{-1}. \quad (3.51)$$

Before imposing the Drinfeld–Sokolov condition (3.50a), the phase space is parameterised by this $a(\theta)$. The corresponding modes, defined by the expansion

$$\ell^m(\theta) = \frac{1}{k} \sum_{p \in \mathbb{Z}} \ell_p^m e^{-ip\theta}, \quad (3.52)$$

satisfy the affine Lie algebra (3.38), that in this case reads

$$\{\ell_p^1, \ell_q^0\} = 2\ell_{p+q}^1, \quad \{\ell_p^1, \ell_q^{-1}\} = \ell_{p+q}^0 + ipk \delta_{p+q,0}, \quad (3.53a)$$

$$\{\ell_p^0, \ell_q^{-1}\} = 2\ell_{p+q}^{-1}, \quad \{\ell_p^0, \ell_q^0\} = -2ipk \delta_{p+q,0}, \quad (3.53b)$$

where here and in the following we omit the $*$ subscript denoting the Dirac bracket. Indeed, the origin of this phase space as a constrained surface in an ampler phase space will be immaterial in the following.

The Drinfeld–Sokolov condition (3.50) implies

$$\ell^1(\theta) - 1 \approx 0 \quad \iff \quad \ell_p^1 - k \delta_{p,0} \approx 0, \quad (3.54)$$

where we used the symbol \approx to stress that we are imposing a constraint on the phase space parameterised by the modes (3.52) with the Poisson structure (3.53). This is a first-class constraint because

$$\{\ell_p^1, \ell_q^1\} = 0, \quad (3.55)$$

so that it generates a gauge symmetry on the phase space. From the viewpoint of the connection $a(\theta)$, this corresponds to all transformations generated by parameters of the form (3.24) that preserve the constraint (3.54) [91].

A convenient way to proceed is to impose an additional gauge-fixing condition to work with a reduced phase space on which one can compute the Dirac bracket. For instance, the bracket of the mode $\ell_p^1 - k\delta_{p,0}$ of the constraint with ℓ_q^0 reads

$$\{\ell_p^1, \ell_q^0\} = 2\ell_{p+q}^1 \approx 2k\delta_{p+q,0}, \quad (3.56)$$

thus implying that ℓ_{-p}^0 can be set to any value by using the gauge transformation generated by the mode ℓ_p^1 of the constraint, that is $\delta\ell_{-p}^0 = 2k\epsilon_{-p}$, with a constant ϵ_{-p} . A possible complete gauge-fixing is therefore²¹

$$\ell^0 = 0. \quad (3.57)$$

This gauge choice corresponds to the stronger condition $A_\theta - A_\theta^{\text{AdS}} \xrightarrow{\rho \rightarrow \infty} 0$, and the once more reduced phase space is parameterised only by $\mathcal{L}(\theta) := \ell^{-1}(\theta)$.

We now want to understand the structure of the Dirac bracket on this reduced space. We recall that given a family $\{\phi_i\}$ of second-class constraints, the Dirac bracket of two phase-space functionals f and g reads (see (C.8))

$$\{f, g\}_* = \{f, g\} - \{f, \phi_i\} C^{ij} \{\phi_j, g\}, \quad (3.58)$$

where C^{ij} is the inverse of the matrix defined by $C_{ij} = \{\phi_i, \phi_j\}$. In this case, it is convenient to label the constraints using the mode labels, introducing $\phi_{(1,p)} = \ell_p^1 - k\delta_{p,0}$ and $\phi_{(0,p)} = \ell_p^0$. Then

$$C_{(\cdot,p)(\cdot,q)} \approx 2k\delta_{p+q,0} \begin{pmatrix} 0 & 1 \\ -1 & -ip \end{pmatrix} \quad (3.59)$$

and

$$C^{(\cdot,p)(\cdot,q)} \approx \frac{1}{2k}\delta_{p+q,0} \begin{pmatrix} ip & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.60)$$

The Dirac bracket for the modes $\mathcal{L}_p = \ell_p^{-1}$ is then obtained as

$$\{\mathcal{L}_p, \mathcal{L}_q\}_* \approx i(p-q)\mathcal{L}_{p+q} - i\frac{k}{2}p^3\delta_{p+q,0}. \quad (3.61)$$

Exercise 18. Confirm the result for the Dirac bracket (3.61) using (3.53) and (3.60).

By defining $\hat{\mathcal{L}}_p = -\mathcal{L}_p + \frac{k}{4}\delta_{p,0}$ we find the standard form of the Virasoro algebra,

$$i\{\hat{\mathcal{L}}_p, \hat{\mathcal{L}}_q\}_* = (p-q)\hat{\mathcal{L}}_{p+q} + \frac{c}{12}(p^3 - p)\delta_{p+q,0}, \quad (3.62)$$

with central charge $c = 6k$.

²¹ In the $\mathfrak{sl}(2)$ case, a complete gauge-fixing can be achieved only by fixing ℓ^0 . On the other hand, it is also possible to analyse the Drinfeld–Sokolov reduction by performing the partial gauge-fixing $\ell^{-1} = 0$. This leads to a simpler presentation of the reduced algebra, although one still has to handle the residual gauge symmetry; see, e.g., [94].

We can perform a similar analysis for the modes of the field \tilde{A} leading to another copy of the Virasoro algebra. This reproduces the classical result by Brown and Henneaux [73]: the asymptotic symmetry algebra of three-dimensional gravity with a negative cosmological constant is given by two copies of the Virasoro algebra with central charge $c = \frac{3\ell}{2G}$ (see (2.22) for the relation between k , ℓ and G).

Before we move on to higher-spin theories, we now rederive this result following a different approach that is easier to generalise to more involved gauge connections. Along the way, we also exhibit the constraints on the Lagrange multiplier A_t induced by the additional constraints that we are imposing on A_θ . The starting point is similar: we still use the gauge transformations generated by the Drinfeld–Sokolov constraint to fix the ρ -independent part of A_θ as

$$a(t, \theta) = L_1 + \mathcal{L}(t, \theta)L_{-1}, \quad (3.63)$$

where we reinstated the time dependence to stress that we are imposing this condition all along time evolution. As we have already discussed, these transformations are generated by a first-class constraint and, therefore, they have to be interpreted as proper gauge transformations rather than asymptotic symmetries. In other words, fixing the form of a as in (3.63) is not affecting the global symmetries of the system. The same conclusion can also be reached by checking explicitly that the transformation setting $\ell^0 = 0$ does not have an associated boundary charge. Generic infinitesimal gauge transformations preserving the radial gauge-fixing (3.17) indeed act on a as

$$\delta_\lambda a(t, \theta) = \partial_\theta \lambda(t, \theta) + [a(t, \theta), \lambda(t, \theta)], \quad (3.64)$$

where we recall that λ is the ρ -independent part of the gauge parameter, see (3.24). Choosing

$$\lambda = \epsilon L_{-1} \implies \delta_\lambda a = 2\epsilon L_0 + (\partial_\theta \epsilon + 2\ell^0 \epsilon) L_{-1}, \quad (3.65)$$

one obtains a variation that can be used to set a generic connection (3.51) in the form (3.63) while giving a vanishing $\delta Q(\lambda)$ when substituted in (3.10). The constants ϵ_p that we encounter below (3.56) can be interpreted as the Fourier modes of the gauge parameter $\epsilon(\theta)$.

We can now identify the residual gauge transformations preserving the gauge-fixed form of the connection (3.63). Setting

$$\lambda(t, \theta) = \epsilon(t, \theta)L_1 + \epsilon^0(t, \theta)L_0 + \epsilon^{-1}(t, \theta)L_{-1} \quad (3.66)$$

and requiring δa to only have a L_{-1} component fixes ϵ^0 and ϵ^{-1} in terms of ϵ as

$$\epsilon^0 = -\epsilon', \quad (3.67a)$$

$$\epsilon^{-1} = \frac{1}{2}\epsilon'' + \epsilon\mathcal{L}, \quad (3.67b)$$

where $f' := \partial_\theta f$. The residual gauge transformations then act on the L_{-1} component of the connection as follows:

$$\delta\mathcal{L} = \epsilon\mathcal{L}' + 2\epsilon'\mathcal{L} + \frac{1}{2}\epsilon'''. \quad (3.68)$$

Exercise 19. Check eqs. (3.67) and (3.68).

Eqs. (3.67) characterise the asymptotic symmetries, and they also fix the form of the Lagrange multiplier A_t . As we discussed around (3.13), time evolution corresponds indeed to a gauge transformation generated by A_t . Preserving the form (3.64) of A_θ for any t therefore implies

$$A_t = b^{-1}(\rho) \left[\mu(t, \theta) \left(L_1 + \mathcal{L}(t, \theta)L_{-1} \right) - \mu'(t, \theta)L_0 + \frac{1}{2}\mu''(t, \theta)L_{-1} \right] b(\rho), \quad (3.69)$$

where $\mu(t, \theta)$ is an arbitrary boundary function, often dubbed chemical potential because of the way it enters the variation of the entropy of black hole solutions [75, 43]. The previous expression can be obtained by replacing (3.67) in the general form (3.25) taken by the Lagrange multiplier as a result of the radial gauge-fixing. Via (3.13) this result also implies the following time evolution for the function \mathcal{L} parameterising the connection:

$$\dot{\mathcal{L}}(t, \theta) = \mu(t, \theta)\mathcal{L}'(t, \theta) + 2\mu'(t, \theta)\mathcal{L} + \frac{1}{2}\mu'''(t, \theta). \quad (3.70)$$

Although $\mu(t, \theta)$ can be an arbitrary function, it is customary to set it to one. This choice implies $A_t - A_\theta = 0$ and $\mathcal{L} = \mathcal{L}'$, so that \mathcal{L} becomes a chiral function. This is the option chosen, e.g., in [76, 74] that was later generalised to higher spins in [33, 34]. While this convenient choice does not affect the analysis of asymptotic symmetries, it still imposes a restriction on the space of allowed solutions. This is harmless in gravity, while its analogue in higher spin theories does not allow one to access black holes solutions [43] (see also [41, 95]).

We now go back to the residual transformations of the form (3.64) that preserve the Drinfeld–Sokolov boundary conditions on any time slice. Since δa only has a component along L_{-1} , the variation (3.10) of the charge is

$$\delta Q(\lambda) = \frac{k}{2\pi} \int d\theta \epsilon \delta \mathcal{L} \quad (3.71)$$

thanks to $\text{tr}(L_1 L_{-1}) = -1$ (see (2.63)). Even if the $\mathfrak{sl}(2)$ -valued parameter λ depends on the function $\mathcal{L}(\theta)$, the variation of the charge is insensitive to this and can be integrated to

$$Q(\epsilon) = \frac{k}{2\pi} \int d\theta \epsilon(\theta) \mathcal{L}(\theta) \quad (3.72)$$

on each time slice. Being associated to a non-vanishing boundary charge, the residual transformations generated by a $\mathfrak{sl}(2)$ -valued parameter satisfying (3.66) and (3.67) are therefore asymptotic symmetries.

The additional constraint we imposed compared to section 3.1 is, by construction, invariant under these residual symmetries. As a result, the considerations that led to (3.34) are still valid, and the boundary charge canonically generates the asymptotic

symmetries:

$$\delta\mathcal{L} = \{Q(\epsilon), \mathcal{L}\}_*. \quad (3.73)$$

In analogy with the derivation of the $\widehat{\mathfrak{sl}}(2)_k$ algebra in (3.38), we can use this result to fix the Poisson structure of the functions parameterising the phase space, obtaining

$$\{\mathcal{L}(\theta), \mathcal{L}(\theta')\}_* = -\frac{2\pi}{k} \left(\delta(\theta - \theta') \mathcal{L}'(\theta) + 2\delta'(\theta - \theta') \mathcal{L}(\theta) + \frac{1}{2} \delta'''(\theta - \theta') \right), \quad (3.74)$$

where $\delta'(\theta - \theta') = \partial_\theta \delta(\theta - \theta')$. One can then recover the Virasoro algebra in the form (3.61) by expanding \mathcal{L} as

$$\mathcal{L}(\theta) = \frac{1}{k} \sum_{p \in \mathbb{Z}} \mathcal{L}_p e^{-ip\theta}. \quad (3.75)$$

Exercise 20.* Check (3.74) by substituting (3.68) in (3.73) and introducing an integral over θ also on the left-hand side.

Alternatively, the same result can be obtained by evaluating the variation (3.73) for $\epsilon(\theta) = \epsilon_{(0)} e^{im\theta}$ with a fixed $m \in \mathbb{Z}$ and a constant $\epsilon_{(0)}$, and at the same time expanding \mathcal{L} again as in (3.75).

We then find

$$\delta\mathcal{L}_n = i\epsilon_{(0)}(m-n)\mathcal{L}_{m+n} - \frac{i\epsilon_{(0)}k}{2}m^3\delta_{m,-n}. \quad (3.76)$$

On the other hand, for the chosen ϵ the boundary charge takes the form

$$Q(\epsilon) = \epsilon_{(0)} \mathcal{L}_m. \quad (3.77)$$

By substituting (3.76) and (3.77) in (3.73) we obtain again (3.61).

3.4 The $\mathfrak{sl}(3)$ case

We now move on to a Chern–Simons theory with a $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ gauge algebra, describing the coupling of a spin-three gauge field to gravity. As in the previous example, we focus on a single copy of the Chern–Simons connection and we consider the basis (2.76) for the eight-dimensional $\mathfrak{sl}(3, \mathbb{R})$ algebra where, for simplicity, we rename as $W_m := W_m^3$ the five generators that complement the gravitational $\mathfrak{sl}(2, \mathbb{R})$ subalgebra (spanned by the generators L_m as in the previous section). The invariant bilinear form is given in (2.75).

As before, we expand the ρ -independent part of A_θ as

$$a(\theta) = \sum_{m=-1}^1 \ell^m(\theta) L_m + \sum_{m=-2}^2 w^m(\theta) W_m. \quad (3.78)$$

The Drinfeld–Sokolov constraint (3.50) takes the form

$$\ell^1(\theta) - 1 = 0, \quad w^1(\theta) = 0, \quad w^2(\theta) = 0. \quad (3.79)$$

In terms of the modes of ℓ^m and w^m defined as in (3.36), the constraints read

$$\ell_p^1 - k\delta_{p,0} = 0 \quad w_p^1 = 0 \quad w_p^2 = 0. \quad (3.80)$$

As in the $\mathfrak{sl}(2)$ case, these are first-class constraints: the Poisson brackets between them are zero except for the bracket $\{w_p^1, \ell_q^1\}$ which is proportional to w_{p+q}^2 and thus vanishes on the constraint surface. To proceed, one can impose further gauge-fixing constraints, so as to obtain a set of second-class constraints allowing one to compute the Dirac bracket on the reduced phase space as we did to get (3.61). The main difference is that in this case, one can impose different gauge-fixings, leading to different bases for the asymptotic symmetry algebra. We refer to [34] for a computation of the Dirac bracket along the lines of (3.59) with the additional gauge-fixing $\ell^0 = w^0 = 0$. Here we focus on the approach deriving the Poisson bracket on the reduced phase space via the canonical realisation of the residual gauge symmetries, along the lines of (3.73). Even when employing this strategy, one has to first completely fix the proper gauge symmetry. In the following, we review successively two widely used complete gauge-fixings: the *highest-weight gauge* and the *single-row gauge* (also known as *u-gauge*).²²

3.4.1 The highest-weight gauge

When a satisfies the Drinfeld–Sokolov constraint (3.79), we can write it as

$$a(\theta) = L_1 + \sum_{m=-1}^0 \ell^m(\theta) L_m + \sum_{m=-2}^0 w^m(\theta) W_m =: L_1 + u(\theta), \quad (3.81)$$

where u is a linear combination of $\mathfrak{sl}(3)$ basis elements with non-positive mode numbers. A transformation of the form (3.64) with $\lambda(\theta) = \epsilon(\theta)L_{-1}$ leads to

$$\delta_\lambda a = \partial_\theta \epsilon L_{-1} + 2\epsilon L_0 + \epsilon[u, L_{-1}]. \quad (3.82)$$

As $[u, L_{-1}]$ is a combination of $\mathfrak{sl}(3)$ basis elements with strictly negative mode numbers, there is only one term in L_0 , and such transformations can be used to set the L_0 -component of a to 0, $\ell^0 = 0$. Similarly, we can achieve $w^0 = 0$ by transformations with $\lambda = \epsilon W_{-1}$ because of $[L_1, W_{-1}] = 3W_0$. Furthermore, by transformations with $\lambda = \epsilon W_{-2}$ we can set $w^{-1} = 0$. Notice that in each step the involved infinitesimal gauge transformations can be integrated to finite ones because the com-

²² In the $\mathfrak{sl}(3)$ case these two options exhaust all possible complete gauge-fixings, while for gauge algebras of bigger rank other options are possible, see, e.g., [91]. As for gravity, one can also characterise the asymptotic symmetries without fixing completely the proper gauge symmetry, see, e.g., [94].

ponents along, respectively, L_0 , W_0 and W_{-1} of the gauge variations do not depend on the fields. Moreover, these transformations are associated to vanishing boundary charges for the same mechanism that we discussed for $\mathfrak{sl}(2)$ in (3.65). We finally arrive at the *highest-weight gauge* in which we constrain all components of a except the “highest modes” $\mathcal{L} := \ell^{-1}$ and $\mathcal{W} := w^{-2}$, so that a has the form

$$a(\theta) = L_1 + \mathcal{L}(\theta)L_{-1} + \mathcal{W}(\theta)W_{-2}. \quad (3.83)$$

Similarly to the second approach in the last subsection, we can infer the Dirac brackets on the constraint surface from the transformations that leave the constrained form (3.83) invariant. In order to do this, we start with a general gauge parameter

$$\lambda(\theta) = \epsilon(\theta)L_1 + \sum_{m=-1}^0 \epsilon^m(\theta)L_m + \chi(\theta)W_2 + \sum_{m=-2}^1 \chi^m(\theta)W_m. \quad (3.84)$$

The condition that the transformation leaves the form (3.83) untouched means that $\delta_\lambda a$ only has components corresponding to the generators L_{-1} and W_{-2} , the coefficients in front of all other generators vanish. This leads to a system of differential equations that determines $\epsilon^m(\theta)$ and $\chi^m(\theta)$ in terms of the functions $\epsilon(\theta)$ and $\chi(\theta)$ and their derivatives. The transformation of a then takes the form

$$\delta\mathcal{L} = \epsilon\mathcal{L}' + 2\epsilon'\mathcal{L} + \frac{1}{2}\epsilon''' - 2\chi\mathcal{W}' - 3\chi'\mathcal{W}, \quad (3.85a)$$

$$\begin{aligned} \delta\mathcal{W} = & \epsilon\mathcal{W}' + 3\epsilon'\mathcal{W} + \frac{1}{12}(2\chi\mathcal{L}''' + 9\chi'\mathcal{L}'' + 15\chi''\mathcal{L}' + 10\chi'''\mathcal{L}) \\ & + \frac{1}{24}\chi^{(5)} + \frac{8}{3}(\chi\mathcal{L}\mathcal{L}' + \chi'\mathcal{L}^2), \end{aligned} \quad (3.85b)$$

where $\chi^{(5)}$ denotes the fifth derivative of χ .

Exercise 21.* Derive the expressions for $\epsilon^m(\theta)$, $\chi^m(\theta)$ in terms of $\epsilon(\theta)$, $\chi(\theta)$ and verify (3.85b).

Now we can follow the same procedure that led to the Dirac brackets (3.61) in the last section. To obtain the Dirac bracket involving the modes of \mathcal{L} we start with a transformation with parameters $\epsilon(\theta) = \epsilon_{(0)}e^{im\theta}$ and $\chi = 0$. From (3.85a) and (3.85b) we obtain the transformation of the modes \mathcal{L}_n and \mathcal{W}_n (where the expansion into Fourier modes is done similarly to (3.75)):

$$\delta\mathcal{L}_n = i\epsilon_{(0)}(m-n)\mathcal{L}_{m+n} - \frac{i\epsilon_{(0)}k}{2}m^3\delta_{m,-n}, \quad (3.86a)$$

$$\delta\mathcal{W}_n = i\epsilon_{(0)}(2m-n)\mathcal{W}_{m+n}. \quad (3.86b)$$

The transformation of \mathcal{L}_n is identical to the $\mathfrak{sl}(2)$ case (see (3.76)).

Exercise 22.* Check (3.86b).

Whereas the transformation of \mathcal{L} leads to the same Dirac brackets (3.61) as in the $\mathfrak{sl}(2)$ example, we can read off the Dirac brackets of \mathcal{L}_m and \mathcal{W}_n from (3.86b):

$$\{\mathcal{L}_m, \mathcal{W}_n\}_* = i(2m-n)\mathcal{W}_{m+n}. \quad (3.87)$$

Similarly, setting $\epsilon(\theta) = 0$ and $\chi(\theta) = \chi_{(0)} e^{im\theta}$ we obtain the transformations

$$\delta\mathcal{L}_n = i\chi_{(0)}(2n-m)\mathcal{W}_{m+n}, \quad (3.88a)$$

$$\begin{aligned} \delta\mathcal{W}_n = i\chi_{(0)} \frac{1}{12} & \left(\frac{k}{2} m^5 \delta_{m+n,0} - (m-n)(2m^2 + 2n^2 - mn) \mathcal{L}_{m+n} \right. \\ & \left. + \frac{16}{k} (m-n) \sum_{q \in \mathbb{Z}} \mathcal{L}_{m+n+q} \mathcal{L}_{-q} \right). \end{aligned} \quad (3.88b)$$

Exercise 23.* Verify (3.88b).

From here we can read off the Dirac brackets of \mathcal{W}_m with \mathcal{L}_n (consistent with (3.87)) and the Dirac brackets of \mathcal{W}_m and \mathcal{W}_n . As before, we define $\hat{\mathcal{L}}_m = -\mathcal{L}_m + \frac{k}{4} \delta_{m,0}$, and we introduce the quadratic field

$$\Lambda_p = \sum_{q \in \mathbb{Z}} \hat{\mathcal{L}}_{p+q} \hat{\mathcal{L}}_{-q}. \quad (3.89)$$

We finally obtain the classical \mathcal{W}_3 -algebra in standard form:

$$i\{\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n\}_* = (m-n)\hat{\mathcal{L}}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad (3.90a)$$

$$i\{\hat{\mathcal{L}}_m, \mathcal{W}_n\}_* = (2m-n)\mathcal{W}_{m+n}, \quad (3.90b)$$

$$\begin{aligned} i\{\mathcal{W}_m, \mathcal{W}_n\}_* = \frac{1}{12} & \left((m-n)(2m^2 + 2n^2 - mn - 8)\hat{\mathcal{L}}_{m+n} + \frac{96}{c}(m-n)\Lambda_{m+n} \right. \\ & \left. + \frac{c}{12}m(m^2 - 1)(m^2 - 4)\delta_{m+n,0} \right). \end{aligned} \quad (3.90c)$$

Exercise 24.* Derive (3.90c) from (3.88b).

Notice that this non-linear algebra includes a Virasoro subalgebra. Furthermore, in the $c \rightarrow \infty$ limit the non-linear terms vanish, and if one focuses on the modes $m, n \in -2, \dots, 2$ in (3.90c) the term proportional to the central charge does not contribute. Therefore, in this limit one identifies a $\mathfrak{sl}(3)$ subalgebra, cf. (2.76), which is called the wedge algebra of \mathcal{W}_3 .

3.4.2 The single-row gauge

The Drinfeld–Sokolov condition restricts a to the form $a(\theta) = L_1 + u(\theta)$ as in (3.81) where u is a combination of generators with non-positive mode numbers. By inspecting the explicit matrix realisation of $\mathfrak{sl}(3)$ in (2.74), we observe that u is a general traceless upper triangular matrix. As in the previous discussion of the highest-weight gauge, we can achieve that $\ell^0 = 0$ and $w^0 = 0$, such that u is strictly upper triangular. Concretely, it has the form

$$u = \begin{pmatrix} 0 - \frac{1}{\sqrt{2}}(2\ell^{-1} + w^{-1}) & 2w^{-2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\sqrt{2}}(2\ell^{-1} - w^{-1}) \begin{pmatrix} \\ \\ 0 \end{pmatrix}. \quad (3.91)$$

The remaining gauge freedom can be used to fix the coefficient w^{-1} at will, and we can choose it as $w^{-1} = 2\ell^{-1}$ such that the only non-vanishing entries of u are in the first row (*single-row gauge*),

$$u = \begin{pmatrix} 0 & u_2 & u_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.92)$$

Now we determine the most general transformation parameter λ , such that the transformation $\delta_\lambda a$ only has components in the first row. In matrix components, we have

$$\delta_\lambda a_{ij} = \partial \lambda_{ij} + \sqrt{2} \lambda_{i-1j} - \sqrt{2} \lambda_{ij+1} - u_j \lambda_{i1} + \delta_{i1} (u_2 \lambda_{2j} + u_3 \lambda_{3j}), \quad (3.93)$$

where from now on we shall denote with ∂ a derivative with respect to θ . We then require that all entries of δa , except the two last entries in the first row, vanish. One can solve the corresponding system of equations and express the most general allowed matrix λ in terms of the coefficients λ_{21} and λ_{31} in the first column, which are unconstrained. The corresponding transformations of u_2 and u_3 are then given as

$$\begin{aligned} \delta u_2 = & -\frac{1}{2\sqrt{2}} \lambda_{31}^{(4)} + \frac{1}{2} \partial^2 (u_2 \lambda_{31}) + \frac{3}{\sqrt{2}} u_3 \lambda'_{31} + \sqrt{2} u'_3 \lambda_{31} \\ & - \lambda_{21}^{(3)} + \sqrt{2} u_2 \lambda'_{21} + \frac{1}{\sqrt{2}} u'_2 \lambda_{21}, \end{aligned} \quad (3.94a)$$

$$\begin{aligned} \delta u_3 = & \frac{1}{6} \lambda_{31}^{(5)} - \frac{\partial^3 (u_2 \lambda_{31})}{3\sqrt{2}} - \frac{1}{3\sqrt{2}} u_2 \lambda_{31}^{(3)} + \frac{1}{3} u_2 \partial (u_2 \lambda_{31}) - u'_3 \lambda'_{31} - \frac{1}{2} \lambda_{31} u''_3 \\ & + \frac{\lambda_{21}^{(4)}}{2\sqrt{2}} + \frac{u'_3 \lambda_{21}}{\sqrt{2}} + \frac{3u_3 \lambda'_{21}}{\sqrt{2}} - \frac{1}{2} u_2 \lambda''_{21}. \end{aligned} \quad (3.94b)$$

Exercise 25.* Work out the details that lead to (3.94a) and (3.94b).

From here one can read off the Dirac brackets for u_2 and u_3 . We obtain the same algebraic structure as in the highest-weight gauge if we identify

$$\mathcal{L}(\theta) = -\frac{1}{2\sqrt{2}} u_2(\theta), \quad (3.95a)$$

$$\mathcal{W}(\theta) = \frac{1}{2} \left(u_3(\theta) + \frac{1}{2\sqrt{2}} u'_2(\theta) \right). \quad (3.95b)$$

Expressed in terms of modes, one recovers the Dirac brackets (3.90).

$$\partial^k f = \sum_{i=0}^{\infty} \binom{k}{i} f^{(i)} \partial^{k-i}. \quad (3.100)$$

For a positive integer k the sum truncates, and we get the usual rule. For negative k , we formally consider combinations of differential operators with arbitrarily negative powers. We also introduce the operator

$$L = \partial^N + u_2 \partial^{N-2} + \cdots + u_N. \quad (3.101)$$

The vanishing of $\delta_\lambda a_{ij}$ given in (3.98) for $i \geq 2$ can be expressed in terms of differential operators as

$$0 = \sum_{j=1}^N \delta_\lambda a_{ij} \partial^{N-j} = \partial \lambda_i - \lambda_{i-1} - \lambda_{i1} L. \quad (3.102)$$

Exercise 26.* Show (3.102).

Composing (3.102) with ∂^{-N+i-1} from the left and summing over i we find

$$\begin{aligned} 0 &= \sum_{i=2}^N (\partial^{-N+i} \lambda_i - \partial^{-N+i-1} \lambda_{i-1}) - (\lambda^1 - \partial^{-N} \lambda_{11}) L \\ &= \lambda_N - \partial^{-N+1} \lambda_1 - (\lambda^1 - \partial^{-N} \lambda_{11}) L. \end{aligned} \quad (3.103)$$

For a pseudo-differential operator $X = \sum_{i=-\infty}^M x_i \partial^i$, we denote by $(X)_+ = \sum_{i=0}^M x_i \partial^i$ its truncation to the part containing non-negative powers of ∂ . Similarly, $(X)_- = X - (X)_+$ denotes the projection to the part containing only negative powers of ∂ . From truncating (3.103) in this way, we obtain

$$\lambda_N = (\lambda^1 L)_+, \quad (3.104)$$

which expresses the last row of λ in terms of its first column. We can obtain the other rows by using (3.102) as a recursion relation,

$$\lambda_{i-1} = \partial \lambda_i - \lambda_{i1} L. \quad (3.105)$$

The solution is

$$\lambda_{N-i} = \partial^i (\lambda^1 L)_+ - (\partial^i \lambda^1)_+ L. \quad (3.106)$$

Exercise 27.* Check that (3.106) solves the recursion relation (3.105).

Similarly to (3.102), the transformation of the u_j can be summarised as

$$\delta_\lambda L = \sum_j \delta_\lambda a_{1j} \partial^{N-j} = \partial \lambda_1 - \lambda_{11} L + \sum_{k=2}^N u_k \lambda_k. \quad (3.107)$$

With the help of the expression for λ_i in terms of λ^1 , one can show that

$$\delta_\lambda L = L(\lambda^1 L)_+ - (L\lambda^1)_+ L. \quad (3.108)$$

Exercise 28.* Show (3.108).

We have not yet implemented the condition $\delta_\lambda a_{11} = 0$. Therefore, the results above are also valid if we consider $\mathfrak{gl}(N)$ instead of $\mathfrak{sl}(N)$ and allow for a non-zero entry u_1 in the top-left corner of the matrix u . The Poisson brackets (often referred to as second Gelfand–Dickey Poisson structure) that we would then obtain from the transformation (3.108) lead to a \mathscr{W} -algebra with fields u_1, \dots, u_N .

To complete the reduction for $\mathfrak{g} = \mathfrak{sl}(N)$, it remains to require $\delta_\lambda a_{11} = 0$ or $u_1 = 0$. This translates into the condition that the coefficient of the highest possible power ∂^{N-1} in $\delta_\lambda L$ vanishes. Using $(X)_+ = X - (X)_-$ in (3.108), we get

$$\begin{aligned} \delta_\lambda L &= -L(\lambda^1 L)_- + (L\lambda^1)_- L \\ &= -(\partial^N + \dots)((\text{res } \lambda^1 L)\partial^{-1} + \dots) + ((\text{res } L\lambda^1)\partial^{-1} + \dots)(\partial^N + \dots) \\ &= \text{res}[L, \lambda^1]\partial^{N-1} + \dots \end{aligned} \quad (3.109)$$

Here, we introduced the notation

$$\text{res}\left(\sum_i x_i \partial^i\right) = x_{-1} \quad (3.110)$$

for the coefficient of ∂^{-1} of a pseudo-differential operator. When we enforce $\delta_\lambda a_{11} = 0$, we obtain

$$\begin{aligned} 0 &= \text{res}[L, \lambda^1] \\ &= \text{res}[\partial^N, \partial^{-N}\lambda_{11}] + \text{res}[L, \hat{\lambda}^1] \\ &= N\lambda'_{11} + \text{res}[L, \hat{\lambda}^1], \end{aligned} \quad (3.111)$$

which allows expressing λ'_{11} in terms of the remaining coefficients of the first column collected in

$$\hat{\lambda}^1 = \sum_{i=2}^N \partial^{-N+i-1}\lambda_{i1}. \quad (3.112)$$

On the other hand, we can rewrite the transformation (3.108) of L as

$$\begin{aligned} \delta_\lambda L &= L(\partial^{-N}\lambda_{11}L)_+ - (L\partial^{-N}\lambda_{11})_+ L + L(\hat{\lambda}^1 L)_+ - (L\hat{\lambda}^1)_+ L \\ &= [L, \lambda_{11}] + L(\hat{\lambda}^1 L)_+ - (L\hat{\lambda}^1)_+ L, \end{aligned} \quad (3.113)$$

where we separated the contribution of λ_{11} from the rest. Using

$$[\partial^k, f] = \sum_{j=0}^{k-1} \partial^j f' \partial^{k-1-j}, \quad (3.114)$$

we find (defining $u_0 = 1$)

$$\begin{aligned} [L, \lambda_{11}] &= \sum_{i=0}^N u_i [\partial^{N-i}, \lambda_{11}] \\ &= \sum_{i=0}^N u_i \sum_{j=0}^{N-i-1} \partial^j \lambda'_{11} \partial^{N-i-1-j} \\ &= \sum_{k=0}^{N-1} \sum_{i=0}^k u_i \partial^{k-i} \lambda'_{11} \partial^{N-1-k} \\ &= \sum_{k=0}^{N-1} (L \partial^{k-N})_+ \lambda'_{11} \partial^{N-1-k}. \end{aligned} \quad (3.115)$$

In total, we obtain

$$\delta_\lambda L = \sum_{k=0}^{N-1} (L \partial^{k-N})_+ \lambda'_{11} \partial^{N-1-k} + L(\hat{\lambda}^1 L)_+ - (L \hat{\lambda}^1)_+ L, \quad (3.116)$$

where

$$\lambda'_{11} = -\frac{1}{N} \text{res} [L, \hat{\lambda}^1]. \quad (3.117)$$

The transformation (3.116) encodes the Dirac brackets. Because L depends linearly on the fields, it is obvious from the formula above that in this basis the Dirac bracket is at most quadratic in the fields u_i .

An analogous analysis can be performed for the higher-spin algebra $\mathfrak{hs}[\lambda]$. One obtains similar expressions for the transformations, where the exponent N is formally replaced by the parameter λ and the sums become untruncated (formal) series (thus generalising to pseudo-differential operators involving not necessarily integer powers of ∂). The Drinfeld–Sokolov reduction for this case (without employing $u_1 = 0$) has been performed in [58]. Roughly, one can argue that the coefficients in the Dirac brackets $\{u_i, u_j\}_*$ are polynomials in the parameter λ , or equivalently that the coefficients in $\delta_\lambda u_i$ for a transformation involving only finitely many non-zero λ_{j1} ($j \neq 1$) are polynomials in λ . Replacing N by λ in the exponents appearing in (3.108) yields an expression that precisely has this property and by construction coincides with it for positive integer $\lambda = N$. The resulting \mathscr{W} -algebra contains infinitely many fields including u_1 , it is called the $\mathscr{W}_{1+\infty}[\lambda]$ -algebra. Its further reduction by implementing $u_1 = 0$ leads to the $\mathscr{W}_\infty[\lambda]$ -algebra [96].

For computational purposes, it is useful to realise the algebras \mathscr{W}_N and $\mathscr{W}_\infty[\lambda]$ by a free-field construction. This is achieved by the Miura transformation [97],

$$(\partial + v_1) \cdots (\partial + v_N) = \partial^N + \sum_{s=2}^N u_s(\theta) \partial^{N-s}. \quad (3.118)$$

Expanding the left-hand side, one can read off expressions for u_s in terms of the fields v_i . The Poisson bracket for the u_s is then induced from the simple (free field) Poisson brackets

$$\{v_k(\theta), v_l(\theta')\} = \frac{2\pi N(N^2 - 1)}{6k} \left(\delta_{kl} - \frac{1}{N} \right) \delta'(\theta - \theta'). \quad (3.119)$$

The possibility to express the fields u_s in terms of free fields allows one to construct a quantum version of the classical \mathscr{W}_N -algebra by quantising the free fields. This will be discussed in section 4.4.

The fields v_k can be understood as the elements of a diagonal matrix D for a gauge choice $a = L_1 + D$ in the Drinfeld–Sokolov reduction. This *diagonal gauge* however is not a complete gauge-fixing. The algebra \mathscr{W}_N is the subalgebra of the Poisson algebra generated by the v_k that is invariant under the residual gauge symmetries [94] (see also [91, 98, 99]).

The fact that the Poisson brackets for the fields u_j induced by the Miura transformation coincide with the Poisson brackets obtained by the Drinfeld–Sokolov reduction in the single-row gauge is known as Kupershmidt–Wilson theorem in the context of pseudo-differential operators [97]. This is explained in section B.

The Miura transformation provides expressions for the fields u_j in terms of polynomials of v_k and their derivatives. The coefficients in the Poisson brackets then depend rationally on N which follows from the explicit formula (B.5) for the Poisson bracket and the condition (B.16). Replacing N in these expressions by λ , one obtains the classical $\mathscr{W}_\infty[\lambda]$ based on a free-field construction. Similarly to what we discussed after (3.90), the wedge algebra of $\mathscr{W}_\infty[\lambda]$ is the algebra $\mathfrak{hs}[\lambda]$.

4 Quantum \mathscr{W} -algebras and minimal-model holography

4.1 Motivation

In the previous section, we have seen the appearance of classical \mathscr{W} -algebras as asymptotic symmetries of higher-spin gauge theories on AdS_3 backgrounds. These are classical Poisson algebras, and the obvious question arises whether they would also appear in a quantised theory. Even without having the quantisation of higher-spin gauge theories under control, we can ask whether quantum versions of the classical \mathscr{W} -algebras exist.

The simplest case is the Virasoro algebra that we found in the last section (see (3.62)),

$$i\{\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n\} = (m - n)\hat{\mathcal{L}}_{m+n} + \delta_{m,-n} \frac{c}{12} m(m^2 - 1). \quad (4.1)$$

In this case we can just replace $\hat{\mathcal{L}}_m$ by quantum operators L_m and replace $i\{\cdot, \cdot\}$ by the commutator $[\cdot, \cdot]$, so that the L_m satisfy the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \frac{c}{12} m(m^2 - 1). \quad (4.2)$$

Such a quantum Virasoro algebra (or better two copies – left-moving and right-moving – of it) appears in two-dimensional conformal quantum field theories. The appearance of this algebra is expected in the spirit of a holographic correspondence between a gravitational theory on asymptotically AdS_{d+1} spacetimes and a conformal field theory on its conformal d -dimensional boundary, for $d = 2$.²³

Finding a quantum version of the Virasoro algebra was simple because the corresponding Poisson algebra is linear. This is far less trivial for non-linear \mathscr{W} -algebras. When we replace the classical modes by quantum operators, we face two problems: on the one hand, there is some freedom in translating the classical Poisson brackets to quantum commutators because we have to choose an ordering in the non-linear terms. On the other hand, after having defined the commutators using some ordering prescription, the classical Jacobi identity does not guarantee that the quantum Jacobi identity is satisfied by the commutators. As an example, we look at the \mathscr{W}_3 -algebra. Recall from the previous section the Poisson bracket of two spin-3 modes, which reads (see (3.90c))

$$i\{\mathscr{W}_m, \mathscr{W}_n\} = \frac{1}{12} \left((m-n)(2m^2 + 2n^2 - mn - 8)\hat{\mathcal{L}}_{m+n} + \frac{96}{c}(m-n)\Lambda_{m+n} + \frac{c}{12}m(m^2 - 1)(m^2 - 4)\delta_{m,-n} \right), \quad (4.3)$$

with $\Lambda_p = \sum_{q \in \mathbb{Z}} \hat{\mathcal{L}}_{p+q} \hat{\mathcal{L}}_{-q}$.

Exercise 29.* Show that

$$i\{\hat{\mathcal{L}}_m, \Lambda_n\} = (3m-n)\Lambda_{m+n} + \frac{c}{6}m(m^2 - 1)\hat{\mathcal{L}}_{m+n}. \quad (4.4)$$

When we replace $\hat{\mathcal{L}}_m$ and \mathscr{W}_n by quantum operators L_m and W_n , respectively, we first have to decide what we mean by Λ : which ordering prescription do we use? Notice that there is from the start a problem to make sense of the infinite series: the naive expressions $\sum_{p \in \mathbb{Z}} L_{m+p} L_{-p}$ and $\sum_{p \in \mathbb{Z}} L_{-p} L_{m+p}$ differ by $\sum_{p \in \mathbb{Z}} (m+2p)L_m$ (for $m \neq 0$), which is ill-defined. We shall require that the operators Λ_m are well-defined on states that are annihilated by modes L_p for large enough mode number p . Then Λ has to be of the form (we do not distinguish the classical Λ and the quantum counterpart in notation, and hope that the meaning will be clear from the context)

$$\Lambda_m = \sum_{p \in \mathbb{Z}} : L_{m+p} L_{-p} : + f(m)L_m, \quad (4.5)$$

where the colons denote mode normal ordering, i.e. operators L_q are ordered such that the highest mode numbers appear on the right. Requiring the Jacobi identity to hold will then, on the one hand, fix $f(m)$ (which will be determined in exercise 34), and, on the other hand, it leads to a shift in the structure constants: the coefficient

²³ For a general introduction into the holographic AdS/CFT correspondence, see, e.g., [100–102].

$\frac{96}{c}$ in front of Λ in the Poisson bracket (4.3) of two $\mathscr{W}^{(3)}$ -modes is replaced in the quantum commutator by $\frac{96}{c+22/5}$. Working out the restrictions imposed by the Jacobi identity directly is straightforward but tedious. A more elegant way of understanding how Λ has to be defined, and how the shift by $22/5$ arises, is by using the language of two-dimensional conformal field theories, which we shall now introduce.

4.2 \mathscr{W} -algebras and operator product expansion

A conformal field theory is a quantum field theory which is invariant under conformal transformations. In two dimensions, in the Euclidean formulation, we can consider a field theory on the compactified complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and the group of global conformal transformations mapping $\overline{\mathbb{C}}$ to itself is $PSL(2, \mathbb{C})$. There is a special set of fields, the quasi-primary fields, which transform covariantly under such conformal transformations $z \mapsto w = f(z)$,

$$\phi_{h,\bar{h}}(z,\bar{z}) \mapsto \tilde{\phi}_{h,\bar{h}}(w,\bar{w}) = (f'(z))^{-h} (\overline{f'(z)})^{-\bar{h}} \phi_{h,\bar{h}}(z,\bar{z}). \quad (4.6)$$

They are characterised by the conformal weights h, \bar{h} (real and non-negative in a unitary theory), or equivalently by the scaling dimension $\Delta = h + \bar{h}$ and the spin $s = h - \bar{h}$. All other fields in the theory are obtained from quasi-primary fields by taking derivatives.

In a unitary two-dimensional scale-invariant theory, conservation of a current implies that its z - and its \bar{z} -component are separately conserved. Therefore, the currents can be split into a set of holomorphic currents $W_{s_i}(z)$ (of spin s_i and scaling dimension $\Delta_i = s_i$), and a set of antiholomorphic currents $\overline{W}_{s_i}(\bar{z})$ (of spin $-s_i$ and scaling dimension $\Delta_i = s_i$). Let us focus on the holomorphic currents. The set of such currents is closed under operator product expansion (OPE): if we take the operator product of two holomorphic fields, only holomorphic fields appear in the short-distance expansion. These fields then define an algebraic structure that is called (quantum) \mathscr{W} -algebra. For a basic introduction to conformal field theories, see, e.g., [103–105], while we refer to [99, 92, 106, 107] for an ampler introduction to quantum \mathscr{W} -algebras.

There are different ways to think about fields in a quantum field theory, one way is to think of them as objects that can be plugged into correlation functions resulting in some function (or distribution) depending on the insertion points. In many cases it is useful to have an operator realisation, where we can view the fields as operator-valued functions (or distributions) which act on some Hilbert space, and the correlation functions can be understood as expectation values of products of these field operators. In two-dimensional Euclidean conformal field theories, there is a particularly advantageous operator realisation called radial quantisation. Here we think of the radial direction as representing (Euclidean) time, and scaling corresponds to time translation. This quantisation picture results from the ordinary quantisation picture on the Euclidean cylinder $\mathbb{R} \times S^1$ by a conformal transformation mapping

the infinite Euclidean past on the cylinder to the origin of the plane. Correlation functions are then given by expectation values of radially ordered products of field operators.

These field operators can be expanded in modes,

$$W^{(s)}(z) = \sum_{n \in \mathbb{Z}} W_n^{(s)} z^{-n-s}, \quad (4.7)$$

with operators $W_n^{(s)}$ resulting from contour integrals of $W^{(s)}(z)$,

$$W_n^{(s)} = \frac{1}{2\pi i} \oint dz z^{n+s-1} W^{(s)}(z). \quad (4.8)$$

These mode operators $W_n^{(s)}$ satisfy commutation relations determined by the OPEs of the fields (see below). A \mathcal{W} -algebra can be equivalently described in terms of commutators of the modes, or in terms of OPEs.

The commutation relation between mode operators $W_m^{(s)}$ and $W_n^{(t)}$ can be expressed in terms of the fields as

$$\left[W_m^{(s)}, W_n^{(t)} \right] = \left[\frac{1}{2\pi i} \oint dz z^{m+s-1} W^{(s)}(z), \frac{1}{2\pi i} \oint dw w^{n+t-1} W^{(t)}(w) \right]. \quad (4.9)$$

Introducing the radial ordering symbol \mathcal{R} for operators,²⁴ we can rewrite the expression above as

$$\left[W_m^{(s)}, W_n^{(t)} \right] = \frac{1}{(2\pi i)^2} \left\{ \oint_{|z|>|w|} \oint_{|z|<|w|} - \oint_{|z|<|w|} \oint_{|z|>|w|} \right\} dz dw z^{m+s-1} w^{n+t-1} \mathcal{R} \left(W^{(s)}(z) W^{(t)}(w) \right). \quad (4.10)$$

Assuming analyticity of the radially ordered operator product, we can deform the contour of the z -integral and find

$$\left[W_m^{(s)}, W_n^{(t)} \right] = \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^{m+s-1} w^{n+t-1} \mathcal{R} \left(W^{(s)}(z) W^{(t)}(w) \right). \quad (4.11)$$

The commutator is therefore determined by the poles in the OPE of $W^{(s)}$ and $W^{(t)}$.

Let us look at an example. The OPE of a (normalised) spin-1 current J with itself reads (we shall omit the radial ordering symbol \mathcal{R} from now on),

$$J(z)J(w) = \frac{1}{(z-w)^2} + \text{regular}. \quad (4.12)$$

For the modes J_m in the mode decomposition $J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$ we find the commutator

²⁴ Analogously to the time ordering operator it reorders the operators according to the absolute value of the complex coordinate of the insertion point; smaller values are ordered to the right.

$$\begin{aligned}
[J_m, J_n] &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^m w^n \frac{1}{(z-w)^2} \\
&= \frac{1}{2\pi i} \oint_0 dw w^n m w^{m-1} \\
&= m \delta_{m,-n}.
\end{aligned} \tag{4.13}$$

Exercise 30.* Starting from the OPE of the energy-momentum tensor with itself,

$$T(z)T(w) = \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) + \text{regular}, \tag{4.14}$$

show that its modes L_m in the mode decomposition $T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}$ satisfy the Virasoro algebra (see (4.2)).

Another useful concept in conformal field theories is the operator-state correspondence, which states that there is a one-to-one map between local fields and states in the Hilbert space \mathcal{H} (in radial quantisation), on which the operator-valued fields act. The idea is that in radial quantisation, a state at infinite Euclidean past (corresponding to taking $z \rightarrow 0$ on the plane) is specified by the insertion of a local operator at $z = 0$.

In particular, we can look at the subspace of states which are annihilated by the zero-mode \bar{L}_0 of the antiholomorphic component $\bar{T}(\bar{z})$ of the energy-momentum tensor,

$$\mathcal{H}^{\text{hol}} = \{v \in \mathcal{H}, \bar{L}_0 v = 0\}. \tag{4.15}$$

The vectors in \mathcal{H}^{hol} are in one-to-one correspondence with the holomorphic currents. In \mathcal{H}^{hol} there is one vector Ω that is called the vacuum. It is annihilated by all modes $W_m^{(s)}$ with $m > -s$,

$$W_m^{(s)} \Omega = 0 \quad \text{for } m > -s. \tag{4.16}$$

We denote the holomorphic field corresponding to a vector $\phi \in \mathcal{H}^{\text{hol}}$ by $V(\phi; z)$. It has the property that

$$\lim_{z \rightarrow 0} V(\phi; z) \Omega = \phi, \tag{4.17}$$

so the insertion of the local field $V(\phi; z)$ at $z = 0$ generates the state ϕ .

Exercise 31.* Show that $W^{(s)}(z) = V(W_{-s}^{(s)} \Omega; z)$.

The operator-state correspondence allows us to rewrite the OPE of two fields. Consider the action of a field $V(\phi; z-w)$ on a state ψ (where we assume that both are eigenstates of L_0 with eigenvalues h_ϕ and h_ψ , respectively). It can be expanded in powers of $z-w$,

$$V(\phi; z-w) \psi = \sum_{n \geq 0} (z-w)^{-h_\phi - h_\psi + n} \chi_n, \tag{4.18}$$

where the vectors χ_n are eigenvectors of L_0 satisfying $L_0\chi_n = n\chi_n$. This decomposition on the space of states has a corresponding decomposition on the space of fields, which is just the OPE,

$$V(\phi; z)V(\psi; w) = V(V(\phi; z-w)\psi; w) = \sum_{n \geq 0} (z-w)^{-h_\phi-h_\psi+n} V(\chi_n; w). \quad (4.19)$$

We have seen above that the OPE determines the commutation relations of the modes. With the help of the last equation, we can use the commutation relations to determine OPEs. Let us look at the example of the OPE of the energy-momentum tensor $T(z) = V(L_{-2}\mathcal{Q}; z)$ with itself. When we apply the relation (4.19), we obtain

$$\begin{aligned} T(z)T(w) &= V(T(z-w)L_{-2}\mathcal{Q}; w) \\ &= \sum_{n \geq 0} (z-w)^{n-4} V(L_{-n+2}L_{-2}\mathcal{Q}; w) \\ &= \frac{1}{(z-w)^4} \frac{c}{2} + \frac{2}{(z-w)^2} T(w) + \frac{1}{(z-w)} \partial_w T(w) \\ &\quad + V(L_{-2}L_{-2}\mathcal{Q}; w) + \dots \end{aligned} \quad (4.20)$$

by using the commutation relations of the modes L_m . In the last step we used that $L_{-1}L_{-2}\mathcal{Q} = L_{-3}\mathcal{Q}$ is the state corresponding to ∂T which can be checked straightforwardly.

The regular term in the OPE is called the normal ordered product of the fields whose OPE is considered. In this case we find that the normal ordered product of T with itself – which we denote by (TT) – is given by

$$(TT)(w) = V(L_{-2}L_{-2}\mathcal{Q}; w). \quad (4.21)$$

Exercise 32.* Show that the normal ordered product $(W^{(s)}W^{(t)})(z)$ of two holomorphic currents of spin s and t , respectively, is given by $V(W_{-s}^{(s)}W_{-t}^{(t)}\mathcal{Q}; z)$.

We have seen how we can determine the states corresponding to the fields that appear in the OPE. Similar techniques can be used to obtain the fields themselves. By a straightforward computation,²⁵ we find that

$$(TT)(z) = \sum_{n \in \mathbb{Z}} \left(\sum_{p \geq 2} L_{-p}L_{n+p} + \sum_{p \leq 1} L_{n+p}L_{-p} \right) z^{-n-4}. \quad (4.22)$$

Notice that the normal ordering that we defined using the OPE does not coincide with the normal ordering of modes by their mode numbers.

As we said in the beginning, all fields can be generated from the set of quasi-primary fields by taking derivatives. Quasi-primary fields are specified by their behaviour under conformal transformations. This translates into a specific structure of

²⁵ See, e.g., [105, section 2.7] or [103, section 6.5].

their OPE with the energy-momentum tensor, namely that there is no third-order pole. For a quasi-primary field ϕ of weight h , the OPE with T then reads

$$T(z)\phi(w) = \text{higher-order poles} + \frac{0}{(z-w)^3} + \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w} + \text{regular}. \quad (4.23)$$

The states corresponding to quasi-primary fields are also called quasi-primary, they are characterised by the condition

$$\phi \text{ quasi-primary} \iff L_1\phi(0)\Omega = 0. \quad (4.24)$$

The energy-momentum tensor is quasi-primary. The normal-ordered product (TT) is not, but

$$\mathcal{T} = (TT) - \frac{3}{10}\partial^2 T \quad (4.25)$$

is, as we shall see now. The state corresponding to \mathcal{T} is

$$\mathcal{T}(0)\Omega = \left(L_{-2}L_{-2} - \frac{3}{5}L_{-4} \right)\Omega. \quad (4.26)$$

Applying L_1 leads to

$$L_1\mathcal{T}(0)\Omega = L_1\left(L_{-2}L_{-2} - \frac{3}{5}L_{-4} \right)\Omega = \left(3L_{-3} - \frac{3}{5}5L_{-3} \right)\Omega = 0, \quad (4.27)$$

so indeed \mathcal{T} is quasi-primary.

Exercise 33.* Let $W^{(s)}(z) = \sum_{n \in \mathbb{Z}} W_n^{(s)} z^{-n-s}$ be a holomorphic current of spin s . Show that $W^{(s)}$ is quasi-primary if and only if

$$[L_m, W_n^{(s)}] = ((s-1)m - n)W_{m+n}^{(s)} \quad \text{for } m \in \{-1, 0, 1\}. \quad (4.28)$$

Fields $W^{(s)}$ whose modes satisfy (4.28) for all m are called *primary fields*. We now have all the tools at our disposal to study the structure of the quantum \mathscr{W} -algebras in more detail.

4.3 The quantum \mathscr{W}_3 -algebra

Let us consider the quantum \mathscr{W}_3 -algebra generated by modes L_m and $W_n = W_n^{(3)}$. This is a prototypical example of a nonlinear \mathscr{W} -algebra, and the first one that was considered [108]. As we discussed before, we follow here the idea to take the classical \mathscr{W}_3 -algebra, and replace the classical modes by operators and the Poisson bracket by the commutator. This works fine as long as the Poisson bracket relations are linear, the first obstacle is met when we consider the Poisson bracket between two spin-3 modes, which contains a term quadratic in the L_m 's.

When going from the Poisson algebra to the quantum algebra, we make the following ansatz:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \quad (4.29a)$$

$$[L_m, W_n] = (2m-n)W_{m+n}, \quad (4.29b)$$

$$[W_m, W_n] = \frac{1}{12} \left((m-n)(2m^2+2n^2-mn-8)L_{m+n} + \frac{96}{c}\alpha(m-n)\Lambda_{m+n} + \frac{c}{12}m(m^2-1)(m^2-4)\delta_{m,-n} \right). \quad (4.29c)$$

This ansatz is motivated by the following considerations:

- The bracket for two L_m 's is not modified, so we find the Virasoro algebra as a subalgebra of the \mathcal{W}_3 -algebra. The Jacobi identity for three L_m 's is automatically satisfied.
- The linear commutator of L_m with the modes W_n is also unchanged; in this way also the Jacobi identity involving two L_m 's and one W_n is directly satisfied. The modes W_n can be seen to obey the expected commutation relations with the Virasoro modes L_m corresponding to modes of a primary field of spin 3.
- In the ansatz for the bracket of two W -modes we assume that the linear and central parts are unchanged,²⁶ and we put all modifications into a possible redefinition of the quadratic piece Λ . For convenience, we also put a free coefficient α in this term.

We have to check that the proposed commutation relations are consistent, i.e. that they satisfy the Jacobi identity. As already discussed, the Jacobi identity involving three L_m 's, and the identity involving two L_m 's and one W_n are automatically satisfied. Let us then consider the Jacobi identity for one L_m and two W -modes. Using the fixed commutation relation between L_m and W_n we arrive at the condition

$$[L_m, [W_p, W_q]] = (2m-q)[W_p, W_{q+m}] + (2m-p)[W_{p+m}, W_q]. \quad (4.30)$$

Both sides of the equation contain central, linear and quadratic terms. From the Poisson bracket

$$i\{\mathcal{L}_m, \Lambda_n\} = (3m-n)\Lambda_{m+n} + \frac{c}{6}m(m^2-1)\mathcal{L}_{m+n} \quad (4.31)$$

(see (4.4)) we see that for $m \in \{-1, 0, 1\}$ there is no mixing between central, linear and quadratic terms in the classical bracket. Because we did not modify the linear and central terms when replacing the Poisson brackets by the commutators (4.29), in the quantum algebra these terms will automatically cancel in (4.30), and we find that

$$[L_m, \Lambda_n] = (3m-n)\Lambda_{m+n} \quad \text{for } m \in \{-1, 0, 1\}. \quad (4.32)$$

²⁶ There is no general reason why these terms should be unmodified, but in the case at hand it turns out to be the right guess.

We conclude that Λ_n are the modes of a quasi-primary field Λ of spin 4 (see (4.28)). Up to normalisation there is only one such field,²⁷ namely $\mathcal{T} = (TT) - \frac{3}{10}\partial^2 T$ which we introduced in the last subsection (see (4.25)). So we set $\Lambda = \mathcal{T}$.

Exercise 34.* Work out the modes Λ_m in terms of the Virasoro modes L_n , and determine the function $f(m)$ in (4.5). Note that it might be useful to treat the two cases of m being even or odd separately.

The OPE of T and Λ is then

$$T(z)\Lambda(w) = \sum_{n \leq 4} (z-w)^{-n-2} V(L_n(L_{-2}L_{-2} - \frac{3}{5}L_{-4})\Omega; w), \quad (4.33)$$

and a quick computation shows that

$$T(z)\Lambda(w) = \frac{1}{(z-w)^4} \left(c + \frac{22}{5} \right) T(w) + \frac{4}{(z-w)^2} \Lambda(w) + \frac{1}{z-w} \partial \Lambda(w) + \text{regular}. \quad (4.34)$$

Exercise 35. Verify this.

The OPE determines the commutation relation of the modes, and by a straightforward computation we find

$$[L_m, \Lambda_n] = (3m-n)\Lambda_{m+n} + \frac{c + \frac{22}{5}}{6} m(m^2-1)L_{m+n}. \quad (4.35)$$

We observe that the quantum commutator differs from the corresponding Poisson bracket (see (4.31)) by a shift $c \rightarrow c + \frac{22}{5}$.

Let us go back to the Jacobi identity (4.30) of one L_m and two W_n . For $|m| > 1$, the different terms start to mix. The terms in Λ automatically cancel because the Λ -term on the right-hand side of (4.35) is the same as for the Poisson bracket. Similarly, the central terms cancel. On the left-hand side of (4.30), the commutator of L_m with Λ_{p+q} introduces a linear term which is – compared to the Poisson bracket – modified by a factor $\alpha \frac{c + \frac{22}{5}}{c}$. All other linear terms are unmodified. Because the Jacobi identity holds for the Poisson bracket, the factor has to be 1, and therefore

$$\alpha = \frac{c}{c + \frac{22}{5}}. \quad (4.36)$$

We conclude that with this choice of α and the identification of Λ with \mathcal{T} , the Jacobi identity is satisfied for three L_m 's, two L_m 's and one W_n , and one L_m and two W_n . It remains to check the Jacobi identity of three W_m 's. As we have already fixed all ambiguities in our ansatz for the commutation relations, it is a highly non-trivial requirement that also this last Jacobi identity holds. Checking the Jacobi identity is

²⁷ The space of states at spin 4 is spanned by $L_{-4}\Omega$, $L_{-2}L_{-2}\Omega$ and $W_{-4}\Omega$. It is straightforward to check that there is only a one-dimensional subspace of quasi-primary states.

tedious. A more elegant way is to verify the associativity of the operator algebra by showing crossing symmetry of the four-point functions, which holds in the case of the \mathscr{W}_3 -algebra [108].

4.4 \mathscr{W}_N and the quantum Miura transform

It is in general a highly non-trivial question whether a given classical \mathscr{W} -algebra possesses a quantum deformation. The classical \mathscr{W}_N -algebras that we obtain from $\mathfrak{sl}(N)$ via a Drinfeld–Sokolov reduction of the corresponding affine Lie algebra (as we discussed in section 3) can all be turned into quantum \mathscr{W}_N -algebras. How can one see this? Writing all Poisson brackets, deforming them to commutators and checking Jacobi identity does not appear to be a practicable approach. Various alternative approaches to this problem are discussed, e.g., in [92]. Here, we leverage upon the realisation of the classical \mathscr{W}_N -algebra in terms of free fields by the classical Miura transformation that we discussed in section 3.5 (see (3.118)). There is indeed an analogous construction, the quantum Miura transformation, by which the quantum \mathscr{W}_N -algebra is realised in terms of free fields.

We start with $N - 1$ spin-1 currents K_a , $a = 1, \dots, N - 1$, with the standard OPE

$$K_a(z)K_b(w) = \frac{\delta_{ab}}{(z-w)^2} + \text{regular}. \quad (4.37)$$

We combine them into a vector $K = (K_1, \dots, K_{N-1})$. Let ϵ_k ($k = 1, \dots, N$) be N vectors in \mathbb{R}^{N-1} satisfying

$$\epsilon_i \cdot \epsilon_j = \delta_{ij} - \frac{1}{N} \quad (4.38)$$

with respect to the canonical scalar product (they can be thought of as the weight vectors of the vector representation of $\mathfrak{sl}(N)$). It follows that $\sum_i \epsilon_i = 0$. With the help of these vectors we can form N spin-1 currents

$$J_i(z) = \epsilon_i \cdot K(z) \quad (4.39)$$

as linear combinations of the components of K . They satisfy the OPE²⁸

$$J_j(z)J_k(w) = \frac{\delta_{jk} - \frac{1}{N}}{(z-w)^2} + \text{regular}. \quad (4.40)$$

Similarly to the classical construction in (3.118), we now implicitly define fields U_s of spin s by the quantum Miura transform [99],

²⁸ This OPE should be compared to the Poisson bracket in (3.119), the currents J_k being the quantum analogues of the classical fields v_k .

$$\left((\alpha_0 \partial - J_1) \cdots (\alpha_0 \partial - J_N) \right) = (\alpha_0 \partial)^N - \sum_{s=2}^N U_s (\alpha_0 \partial)^{N-s}. \quad (4.41)$$

Here, the product on the left-hand side is the normal ordered product, and α_0 is a parameter. By expanding the left-hand side in powers of the derivative, we find explicit expressions for the fields U_s in terms of derivatives and normal ordered products of the J_i . Notice that there is no term of order ∂^{N-1} because $\sum J_i = 0$.

The fields U_s defined in this way are the quantum analogues of the fields u_s that we used in section 3.5. They form a closed operator algebra. This is proven explicitly in [99]. Inside the free field algebra generated by the J_i , the quantum \mathscr{W}_N -algebra is realised as a subalgebra of fields which commute with a certain set of ‘‘screening charges’’. In this way it can be understood in a larger framework of a ‘‘quantum Drinfeld–Sokolov reduction’’ (see, e.g., [92]).

Consider as an example the field U_2 which arises from the coefficient of $(\alpha_0 \partial)^{N-2}$. There are two contributions on the left-hand side, one coming from products of two J_i and one from a derivative hitting a J_i ,

$$U_2(z) = - \sum_{i < j} (J_i J_j)(z) + \alpha_0 \sum_j (j-1) \partial J_j(z). \quad (4.42)$$

One can then check that it satisfies the standard OPE of an energy-momentum tensor,

$$U_2(z)U_2(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}U_2(w) + \frac{1}{z-w}\partial U_2(w) + \text{regular} \quad (4.43)$$

with central charge

$$c = (N-1)(1 - N(N+1)\alpha_0^2). \quad (4.44)$$

We write $T(z) = U_2(z)$. For $N = 2$ the construction stops here, and the algebra generated by the modes of T is the Virasoro algebra.

In the following we sketch some results on the \mathscr{W}_N -algebras obtained by this construction without presenting details of the computation.

Consider the case $N > 2$. The field at spin 3, $U_3(z)$, turns out to not be quasi-primary. Its quasi-primary projection is

$$W^{(3)} := U_3 - \alpha_0 \frac{N-2}{2} \partial T. \quad (4.45)$$

One can then check that $W^{(3)}$ not only is quasi-primary, but primary. For $N = 3$, there are no further fields, and we have reproduced the quantum \mathscr{W}_3 -algebra. For $N > 3$ the construction continues. Again, the field U_4 is not quasi-primary, and we denote its quasi-primary projection as

$$\tilde{U}_4 = U_4 - \alpha_0 \frac{N-3}{2} \partial U_3 + \alpha_0^2 \frac{(N-2)(N-3)}{10} \partial^2 T. \quad (4.46)$$

This field is not primary, but there is (up to normalisation) exactly one primary field of spin 4 which is given by

$$W^{(4)} = \tilde{U}_4 + \frac{(N-2)(N-3)}{2N} \frac{5 - N(5N+7)\alpha_0^2}{17 + 5N - 5N(N^2-1)\alpha_0^2} \Lambda. \quad (4.47)$$

For any $N > 3$, the identification of the primary fields $W^{(3)}$ and $W^{(4)}$ is unique up to normalisation. In our convention the normalisation is given by

$$W^{(s)}(z)W^{(s)}(w) = \frac{n_s c}{(z-w)^{2s}} + \text{subleading poles} \quad (4.48)$$

with²⁹

$$n_3 = \frac{(N-2)(4 - N(N+2)\alpha_0^2)}{6N}, \quad (4.49a)$$

$$n_4 = n_3 \frac{3(N+1)(N-3)(1 - N(N-1)\alpha_0^2)(9 - N(N+3)\alpha_0^2)}{2N(17 + 5N - 5N(N^2-1)\alpha_0^2)}. \quad (4.49b)$$

To better understand the structure of the \mathscr{W} -algebra, we consider the OPE of $W^{(3)}$ and $W^{(4)}$. Its leading term is

$$W^{(3)}(z)W^{(4)}(w) = \frac{c_{34}^3}{(z-w)^4} W^{(3)}(w) + \text{subleading poles} \quad (4.50)$$

with coefficient (checked until $N = 10$ using again `OPEDefs` [109])

$$c_{34}^3 = \frac{6(N+1)(N-3)(1 - N(N-1)\alpha_0^2)(9 - N(N+3)\alpha_0^2)}{N(17 + 5N - 5N(N^2-1)\alpha_0^2)} = 4 \frac{n_4}{n_3}. \quad (4.51)$$

A normalisation independent quantity that characterises the \mathscr{W}_N -algebra is the ratio

$$C(N, c) = \frac{(c_{34}^3)^2}{n_4} = \frac{144(c+2)(N-3)(2(N-1)(4N+3) + c(N+3))}{22 + 5c(N-2)((N-1)(3N+2) + c(N+2))}. \quad (4.52)$$

As one can see, one can obtain very explicit results on the structure constants of the \mathscr{W}_N -algebra using the free field construction. In the next section we discuss how it can be used to also study $\mathscr{W}_\infty[\lambda]$.

²⁹ The formula for n_3 has been checked explicitly, the expression for n_4 is an extrapolation from concrete computations at low N and checked until $N = 10$ with the help of the Mathematica package `OPEDefs` [109].

4.5 The $\mathscr{W}_\infty[\lambda]$ quantum algebra and its triality

Until now, we have looked at the quantum version of the \mathscr{W}_N -algebra. Eventually we shall be interested in the quantum version of $\mathscr{W}_\infty[\lambda]$, where we have higher-spin currents $W^{(s)}$ for every spin $s \geq 2$ (with $W^{(2)} = T$). It can be constructed by observing that the structure constants (like (4.49a), (4.49b) or (4.51)) that appear in the OPEs of the \mathscr{W}_N -algebra are rational functions of N . These rational functions can be continued to real parameters, and we can simply replace N by λ . In this way one automatically obtains a closed operator algebra that can be seen as quantum version of the classical $\mathscr{W}_\infty[\lambda]$ algebra.³⁰

It has been observed in [110], further elaborated upon in [111] and then proven in [112] that for fixed central charge c there only exists a one-parameter family of quantum \mathscr{W} -algebras that is generated by one higher-spin current $W^{(s)}$ for each spin $s \geq 2$. It is uniquely characterised by the normalisation independent ratio $C = (c_{34}^3)^2/n_4$ that we computed in the last section for the \mathscr{W}_N -algebras (see (4.52)). When we replace N by λ , we find the characteristic parameter for the quantum $\mathscr{W}_\infty[\lambda]$ -algebra:

$$C(\lambda, c) = \frac{144(c+2)(\lambda-3)(2(\lambda-1)(4\lambda+3)+c(\lambda+3))}{22+5c(\lambda-2)((\lambda-1)(3\lambda+2)+c(\lambda+2))}. \quad (4.53)$$

\mathscr{W}_∞ -algebras with the same parameter C (and the same central charge) are isomorphic. As the above expression for C involves a cubic rational function in λ , one observes that there are generically three different values of λ that lead to the same value of C (where the central charge c is fixed),

$$C(\lambda_1, c) = C(\lambda_2, c) = C(\lambda_3, c). \quad (4.54)$$

One can show that for given C and c the solutions λ_i satisfy [111, section 2.2.3]

$$0 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad (4.55a)$$

$$c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1), \quad (4.55b)$$

$$C = \frac{144(c+2)(\lambda_1-3)(\lambda_2-3)(\lambda_3-3)}{22+5c(\lambda_1-2)(\lambda_2-2)(\lambda_3-2)}. \quad (4.55c)$$

This three-fold symmetry of $\mathscr{W}_\infty[\lambda]$ is called triality [110].

It is interesting to analyse the fate of triality in the classical limit. For $c \rightarrow \infty$, one obtains

$$C_{\text{classical}}(\lambda) = \lim_{c \rightarrow \infty} C(\lambda, c) = \frac{144}{5} \frac{\lambda^2 - 9}{\lambda^2 - 4}. \quad (4.56)$$

³⁰ This can be compared with the classical construction that we discussed in the previous chapter: the Poisson brackets obtained from the classical Miura transform with N continued to λ results in the same classical $\mathscr{W}_\infty[\lambda]$ algebra as the Drinfeld–Sokolov reduction based on $\mathfrak{hs}[\lambda]$.

One can check that this coincides with the direct computation of $C_{\text{classical}}$ using the explicit expressions for the Poisson brackets that can be found in [63, Appendix C]. The classical remnant of triality is the symmetry $\lambda \mapsto -\lambda$, which is a symmetry of $\mathfrak{hs}[\lambda]$.

The above derivation of triality is not the original one, and it is instructive to sketch the arguments in [110] that led to its discovery because one gets a better idea of the general structure of the algebra (see also [64, 65]). For a moment, we forget the results from the quantum Miura transformation, and more generally ask what is the family of quantum \mathscr{W} -algebras corresponding to the classical \mathscr{W}_∞ -algebras.

When we consider a \mathscr{W} -algebra generated by T and primary fields $W^{(s)}$ ($s = 3, 4, \dots$), the commutator of the modes of the spin-3 current will get an extra term compared to the bracket (4.29c) in the \mathscr{W}_3 -algebra,

$$[W_m^{(3)}, W_n^{(3)}] = 2(m-n)W_{m+n}^{(4)} + \dots, \quad (4.57)$$

involving the spin-4 current. This is a linear term that we directly get from the Poisson brackets of the classical \mathscr{W} -algebra. With this commutator, the normalisation of $W^{(3)}$ and $W^{(4)}$ are fixed (if also the term involving the Virasoro modes is fixed to its standard normalisation). If one then looks at $[W^{(3)}, W^{(4)}]$, there is a piece proportional to $W^{(3)}$, and its coefficient \tilde{C} is a true characteristic of the quantum algebra,

$$[W_n^{(3)}, W_n^{(4)}] = \dots W_{m+n}^{(5)} + \dots \Lambda_{m+n}^{(5)} + \dots \theta_{m+n}^{(6)} - \tilde{C}(n^3 - 5m^3 - 3mn^2 + 5m^2n - 9n + 17m)W_{m+n}^{(3)}. \quad (4.58)$$

From computing Jacobi identities for higher fields, one finds that all higher structure constants are determined by this coefficient \tilde{C} as was observed in [110] and proven in [112]. This statement does not rely on the fact that one started from a known classical algebra, but it is valid for any algebra that one builds out of higher-spin currents $W^{(s)}$, one for each spin $s \geq 2$. The coefficient \tilde{C} is related to the coefficient $C = (c_{34}^3)^2/n_4$ by

$$C = \frac{2016}{5}\tilde{C}. \quad (4.59)$$

We know the λ -dependence of the coefficient \tilde{C} for the classical Poisson algebra (which follows from (4.56), see also [110]),

$$\tilde{C}_{\text{classical}}(\lambda) = \frac{1}{14} \frac{\lambda^2 - 9}{\lambda^2 - 4}. \quad (4.60)$$

What is the corresponding relation in the quantum algebra? The quantum Miura transformation gives one answer, but in general the question is not well-defined. By definition, it is clear what we mean by \tilde{C} in the quantum algebra, but it is a priori not clear what we mean by λ in the quantum algebra. We have learned that we can characterise our quantum \mathscr{W}_∞ algebra by the central charge c and the coefficient \tilde{C} , so where does λ come into play? We expect that corresponding to the classical family of $\mathscr{W}_\infty[\lambda]$, we have a family of quantum $\mathscr{W}_\infty[\lambda]$ -algebras which are characterised

by the coefficient $\tilde{C}(\lambda, c)$ in dependence of λ and the central charge c . We require that the classical algebra arises from the quantum algebra in the limit $c \rightarrow \infty$, so we demand that

$$\lim_{c \rightarrow \infty} \tilde{C}(\lambda, c) = \tilde{C}_{\text{classical}}(\lambda) = \frac{1}{14} \frac{\lambda^2 - 9}{\lambda^2 - 4}. \quad (4.61)$$

This alone does of course not fix $\tilde{C}(\lambda, c)$. On the other hand, we know that if $\lambda = N \in \mathbb{N}$, the classical algebra can be truncated to an algebra generated by $N - 1$ currents. We can demand that this is also true for the quantum family $\mathscr{W}_\infty[\lambda]$, namely that they reduce to the quantum \mathscr{W}_N -algebra for $\lambda = N$. Although the structure constants of the quantum \mathscr{W}_N -algebra are not completely known, one can fix $\tilde{C}(N, c)$ indirectly by requiring that one finds known representations of the quantum \mathscr{W}_N -algebra [110]. This then fixes $\tilde{C}(N, c)$, but it still does not fix the function $\tilde{C}(\lambda, c)$ completely. On the other hand, one finds a unique solution if one requires $C(\lambda, c)$ to be a rational function of λ , which is then given by

$$\tilde{C}(\lambda, c) = \frac{1}{14} \frac{c+2}{c + \frac{22}{5}} \frac{(\lambda-3)(2(\lambda-1)(4\lambda+3) + c(\lambda+3))}{(\lambda-2)((\lambda-1)(3\lambda+2) + c(\lambda+2))}. \quad (4.62)$$

One should be aware that the outlined procedure contains some arbitrariness, and the solution for $\tilde{C}(\lambda, c)$ given in (4.62) depends on the choices we made. For instance, we could have required instead that we recover \mathscr{W}_N for $\lambda = -N$ ($\mathfrak{hs}[\lambda]$ has the symmetry $\lambda \mapsto -\lambda$), and we would have obtained a different solution (indeed, $\tilde{C}(\lambda, c)$ in (4.62) is not invariant under $\lambda \mapsto -\lambda$).

The solution in (4.62) coincides with the result (4.53) obtained from the quantum Miura transformation (taking into account the relation (4.59) between C and \tilde{C}). This should not come as a surprise because the quantum Miura transformation by construction yields the \mathscr{W}_N -algebra for $\lambda = N$ and the coefficients that appear are rational functions of λ .

We focused here on the structure of the $\mathscr{W}_\infty[\lambda]$ algebra, but we wish to stress that its representation theory has been studied, e.g., in [111] and this led to identify an interesting link with the affine $\widehat{\mathfrak{gl}}_1$ Yangian [113–115].

4.6 Conformal field theories with \mathscr{W} -symmetry

We now look for candidates for a CFT dual of higher-spin gauge theories in three dimensions. There are several families of CFTs known with \mathscr{W}_N - or \mathscr{W}_∞ -symmetry:

- \mathscr{W}_N -minimal models [116] labelled by a positive integer k (the level): for $0 < c < N - 1$ there are only finitely many unitary irreducible representations of \mathscr{W}_N for a given c . In fact within this range there is only a discrete set of values for c , for which there exist unitary representations at all:

$$\begin{aligned}
c_{N,k} &= (N-1) \left(1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right) \\
&= 2k \left(1 - \frac{(k+1)(k + \frac{3N+1}{2})}{(k+N)(k+N+1)} \right). \tag{4.63}
\end{aligned}$$

For each such value, one can build a CFT whose spectrum is given by the corresponding unitary representations – these are the \mathscr{W}_N -minimal models.

- conformal $\mathfrak{sl}(N)$ Toda theories [117]: this is a continuous family of CFTs with central charges $c \geq (N-1)(1+4N(N+1))$ with a continuous spectrum of primary fields.
- m complex free bosons: the singlet sector of the chiral algebra gives rise to a $\mathscr{W}_\infty[\lambda=1]$ -algebra with $c=2m$ [118, 119].
- m complex free fermions: their symmetry algebra contains a $\mathscr{W}_\infty[\lambda=0]$ -algebra with $c=m-1$ as a subalgebra [119]. It can be obtained by a $U(1)$ coset construction from the free fermion theory.

Let us focus on the \mathscr{W}_N -minimal models. They can be realised by a procedure known as coset construction [120]. One starts with the affine algebra $\widehat{\mathfrak{sl}(N)}$ with generators J_m^a ($m \in \mathbb{Z}$, $a=1, \dots, \dim \mathfrak{sl}(N)$) and a central element \hat{k} , obeying the commutation relations

$$[J_m^a, J_n^b] = f^{ab}_c J_{m+n}^c + \hat{k} \gamma^{ab} m \delta_{m,-n}. \tag{4.64}$$

Here, f^{ab}_c are the structure constants of $\mathfrak{sl}(N)$. We now build the algebra $U_k(\mathfrak{sl}(N))$ which is obtained from the universal enveloping algebra of the affine $\widehat{\mathfrak{sl}(N)}$ by dividing out the ideal generated by $\hat{k} - k \cdot \mathbf{1}$ for some integer level k ,

$$U_k(\mathfrak{sl}(N)) = \mathscr{U}(\widehat{\mathfrak{sl}(N)}) / (\hat{k} - k \cdot \mathbf{1}). \tag{4.65}$$

When we consider the tensor product of such algebras at level k and level 1, it contains the diagonally embedded algebra $U_{k+1}(\mathfrak{sl}(N))$,

$$U_{k+1}(\mathfrak{sl}(N)) \subset U_k(\mathfrak{sl}(N)) \otimes U_1(\mathfrak{sl}(N)). \tag{4.66}$$

We then define the coset algebra to be the commutant of $U_{k+1}(\mathfrak{sl}(N))$ inside $U_k(\mathfrak{sl}(N)) \otimes U_1(\mathfrak{sl}(N))$, i.e. the algebra of all elements that commute with all elements of $U_{k+1}(\mathfrak{sl}(N))$. The coset algebra is often denoted by (the slightly misleading quotient notation)

$$\frac{\mathfrak{sl}(N)_k \times \mathfrak{sl}(N)_1}{\mathfrak{sl}(N)_{k+1}} := \{a \in U_k(\mathfrak{sl}(N)) \otimes U_1(\mathfrak{sl}(N)) : \forall b \in U_{k+1}(\mathfrak{sl}(N)) \ ab = ba\}. \tag{4.67}$$

This realises the \mathscr{W}_N -algebra at central charge $c_{N,k}$. For example, the spin-2 current (the energy-momentum tensor) T is constructed as

$$T = T^{\mathfrak{sl}(N),k} + T^{\mathfrak{sl}(N),1} - T^{\mathfrak{sl}(N),k+1}, \tag{4.68}$$

where

$$T^{\mathfrak{sl}(N),k} = \frac{1}{2(k+N)} \kappa_{ab}(J^a J^b) \quad (4.69)$$

is the Sugawara energy-momentum tensor constructed in $U_k(\mathfrak{sl}(N))$.

Exercise 36. Denote by $J_m^{(k),a}$ and $J_m^{(1),a}$ the generators of $U_k(\mathfrak{sl}(N))$ and $U_1(\mathfrak{sl}(N))$, respectively. The generators of the embedded $U_{k+1}(\mathfrak{sl}(N))$ are then given by

$$J_m^{(k+1),a} = J_m^{(k),a} \otimes \mathbf{1} + \mathbf{1} \otimes J_m^{(1),a}. \quad (4.70)$$

The modes $L_m^{(k)}$ of the Sugawara energy-momentum tensor $T^{\mathfrak{sl}(N),k}$ satisfy

$$[L_m^{(k)}, J_n^{(k),a}] = -n J_{m+n}^{(k),a}. \quad (4.71)$$

The analogous relation holds for level 1, and also for the modes $L_m^{(k+1)}$ built out of the embedded generators $J_n^{(k+1),a}$.

Show that the modes

$$L_m = L_m^{(k)} \otimes \mathbf{1} + \mathbf{1} \otimes L_m^{(1)} - L_m^{(k+1)} \quad (4.72)$$

commute with all generators $J_n^{(k+1),a}$ and therefore are part of the coset algebra.

The power of the coset construction lies in the possibility to construct representations of \mathcal{W} -algebras out of representations of affine Lie algebras. Let $\mathcal{H}_R^{N,k}$ and $\mathcal{H}_{R'}^{N,1}$ be the representation spaces of the unitary irreducible representations R and R' of $U_k(\mathfrak{sl}(N))$ and $U_1(\mathfrak{sl}(N))$, respectively. We then consider the decomposition of the tensor product with respect to the subalgebra $U_{k+1}(\mathfrak{sl}(N))$ and obtain

$$\mathcal{H}_R^{N,k} \otimes \mathcal{H}_{R'}^{N,1} = \bigoplus_S \mathcal{H}_{(R,R';S)} \otimes \mathcal{H}_S^{N,k+1}. \quad (4.73)$$

The spaces $\mathcal{H}_{(R,R';S)}$ then form a representation space for the coset algebra. In this way, one obtains unitary representations of the $\mathcal{W}_{N,k}$ algebra. In the cases that are of interest here, they are irreducible, but this is not true in general.

The unitary dominant highest-weight representations of $U_k(\mathfrak{sl}(N))$ are labelled by Young diagrams R with maximally $N-1$ rows and maximally k columns. For given representations R and R' , not all representations S can occur in the decomposition above. The selection rule can be formulated as the requirement that the total number of boxes in R and R' equals the number of boxes in S up to multiples of N . So if we denote the number of boxes of a Young diagram R by $B(R)$, we demand that

$$B(R) + B(R') = B(S) \pmod{N}. \quad (4.74)$$

For level 1, there are only N different Young diagrams allowed (the empty diagram or diagrams consisting of only one column with length 1 to $N-1$),

$$0, \square, \begin{array}{|c|} \hline \square \\ \hline \end{array}, \dots, \left. \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} (N-1). \quad (4.75)$$

By the above selection rule, for given R and S , there is a unique representation R' such that S occurs in the decomposition of the product of R and R' . The coset representations that appear in the decomposition can therefore be labelled by pairs (R, S) . This labelling is, however, not unique: there can be different pairs (R_1, S_1) and (R_2, S_2) that label the same representation

$$\mathcal{H}_{(R_1, S_1)} \cong \mathcal{H}_{(R_2, S_2)}. \quad (4.76)$$

To get all pairs that label the same representation as (R, S) , we can perform the following algorithm: we add a row of length k on the top of the Young diagram R , and a row of length $k+1$ on top of S . Then we reduce the new diagrams by killing all columns of length N , and we arrive at a pair (\hat{R}, \hat{S}) that labels the same coset representation (see figure 4.6). The orbit of pairs under this operation has length N , so there will be N pairs which label the same coset representation. The conformal

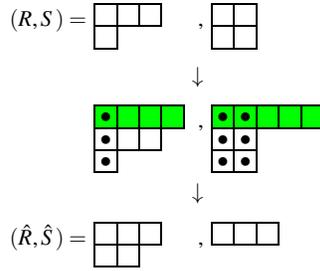


Fig. 1 An example for $N = 3$ and $k = 4$: starting from Young diagrams R and S we add a row of $k = 4$ and $k + 1 = 5$ boxes (in green), respectively. Then we delete the columns of length $N = 3$ (marked with a bullet).

weight $h_{R,S}$ of the ground states of the representation $\mathcal{H}_{(R,S)}$ is given in terms of the row lengths r_i and s_i of the Young diagrams R and S , respectively. Namely, introduce $t_i := (k + N + 1)r_i - (k + N)s_i$, then

$$h_{(R,S)} = \frac{1}{2(k+N)(k+N+1)} \left(\sum_{i=1}^{N-1} t_i(t_i + 2N - 2i) + \sum_{i,j=1}^{N-1} t_i(t_j + 2N - 2j) \right). \quad (4.77)$$

The \mathcal{W}_N -minimal models are CFTs whose Hilbert space is built precisely from those coset representations

$$\mathcal{H} = \bigoplus_{(R,S)} \mathcal{H}_{(R,S)} \otimes \mathcal{H}_{(R,S)}, \quad (4.78)$$

where the prime on the sum means that we only sum over inequivalent pairs. Notice that we have two copies of the \mathscr{W} -algebra present (one in the holomorphic and one in the antiholomorphic sector), therefore we find tensor products of representations in the total Hilbert space.

After this brief excursion into minimal models, we have the prerequisites to discuss the proposal for a CFT dual of higher-spin gauge theories in three dimensions. For more details on minimal models see [121, 116, 32], or more generally on coset models see, e.g., [103].

4.7 Minimal-model holography

We now have all ingredients to look for CFTs that are dual to higher-spin gauge theories on AdS_3 . We do not have a quantum description of the higher-spin theories, so we only have control over the theories for small gravitational coupling, $G/\ell \ll 1$. For the dual CFT this means to consider large central charges,

$$c = \frac{3\ell}{2G} \gg 1. \quad (4.79)$$

When we want to propose a duality, we need to have a family of CFTs that has a large c limit.

Consider the \mathscr{W}_N -minimal models. Their central charge is bounded by $\min(N-1, 2k)$, so to allow for an arbitrarily high central charge, we have to consider a limit where both N and k go to infinity. On the other hand, it seems that in this way the \mathscr{W} -algebra structure never stabilises, and it is unclear a priori how one could obtain a $\mathscr{W}_\infty[\lambda]$ -algebra with a finite value of λ . Here, triality comes to the rescue. For central charge $c = c_{N,k}$, and $\lambda_1 = N$, the other two values of λ leading to the same \mathscr{W} -algebra structure are

$$\lambda_2 = \frac{N}{N+k} \quad , \quad \lambda_3 = -\frac{N}{N+k+1}. \quad (4.80)$$

Therefore if we keep the ratio

$$\Lambda = \frac{N}{N+k} \quad (4.81)$$

fixed while sending N and k to infinity, the \mathscr{W} -algebra structure approaches (in the appropriate sense) $\mathscr{W}_\infty[\Lambda]$. This is very encouraging because it means that we can indeed find a family of CFTs that have a large c limit in which the symmetry algebra is given by a \mathscr{W}_∞ -algebra [31]. This limit of \mathscr{W}_N -minimal models has been dubbed 't Hooft limit, in analogy with the limit of Yang-Mills theories typically entering the boundary side of the AdS/CFT correspondence.

Besides the algebra, we can also analyse the spectrum. The simplest primary state in the \mathscr{W}_N -minimal models (except for the vacuum) is the state where

$$(R, S) = (\square, 0). \quad (4.82)$$

Here, 0 stands for the empty Young diagram. By considering (4.77) we see that the corresponding conformal weight is

$$h_{(\square, 0)} = \frac{N-1}{2N} \left(1 + \frac{N+1}{N+k} \right) \xrightarrow{N, k \rightarrow \infty} \frac{1}{2}(1 + \Lambda). \quad (4.83)$$

As the primary fields in the minimal models have identical conformal weights in the left- and right-moving sector, we find an operator with scaling dimension

$$\Delta = h + \bar{h} = 1 + \Lambda \quad (4.84)$$

in the spectrum.

According to the AdS/CFT correspondence (see, e.g., [100]) this operator should correspond to a scalar bulk field with a definite scaling behaviour at the conformal boundary of AdS, which in turn restricts its mass to be

$$M^2 = \frac{\Delta(\Delta-2)}{\ell^2} = \frac{\Lambda^2-1}{\ell^2}. \quad (4.85)$$

Hence, in addition to the higher-spin gauge fields, which are responsible for the \mathcal{W} -symmetry of the CFT, we also expect a massive scalar field in the bulk with a specific mass.

Indeed it turns out that the $\mathfrak{hs}[\lambda]$ higher-spin gauge theories can be coupled to a massive scalar for $M^2 = \frac{\Lambda^2-1}{\ell^2}$. To describe those theories we have to go beyond the Chern–Simons formulation – this will be the subject of the following section.

Before doing so, let us consider further states in the CFT spectrum. Another natural example to consider is the state

$$(R, S) = (\bar{\square}, 0), \quad (4.86)$$

where $\bar{\square}$ denotes the Young diagram conjugate to \square ,

$$\bar{\square} = \left. \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} (N-1). \quad (4.87)$$

Whereas \square corresponds to the fundamental representation of $\mathfrak{sl}(N)$, $\bar{\square}$ corresponds to the antifundamental representation.³¹ The state $(\bar{\square}, 0)$ has the same conformal weight as $(\square, 0)$, so we expect to find two scalars of the same mass (and with the same scaling behaviour at the boundary) in the bulk theory (or one complex scalar). Indeed, if one works out the contribution of all representations of the form $(R, 0)$ to the CFT partition function, it precisely equals the contribution of one complex

³¹ In general, the Young diagram of the representation conjugate to a representation with Young diagram R (with row lengths r_i) has row lengths $\bar{r}_i = r_1 - r_{N-i+1}$.

scalar to the one-loop partition function on thermal AdS [122]. This again is very encouraging!

Similarly, we can also consider the representation

$$(R, S) = (0, \square). \quad (4.88)$$

Here the conformal weight is

$$h_{(0, \square)} = \frac{N-1}{2N} \left(1 - \frac{N+1}{N+k+1} \right) \xrightarrow{N, k \rightarrow \infty} \frac{1}{2} (1 - \Lambda). \quad (4.89)$$

On the bulk side, this again looks like the contribution of a scalar of the same mass $M^2 = \frac{\Lambda^2 - 1}{\ell^2}$ but with a different scaling behaviour at the boundary (corresponding to a different quantisation). However, as we discuss below, this contribution should not be interpreted as a scalar.

On the other hand, we can for example look at the representation

$$(R, S) = (\square, \square). \quad (4.90)$$

Here we find the conformal weight

$$h_{(\square, \square)} = \frac{N^2 - 1}{2N(N+k)(N+k+1)} \xrightarrow{N, k \rightarrow \infty} \frac{\Lambda^2}{2N}, \quad (4.91)$$

which goes to zero in the limit. We find a similar behaviour for representations (R, R) , so it seems that we find infinitely many states whose conformal weight approaches zero. From the bulk side, this would correspond to a huge number of ultra-light fields, which are hard to interpret within the AdS/CFT correspondence.

So in spite of the encouraging matches that we observed, the spectrum is not easy to interpret in the holographic setting. One could think of different explanations:

- The difficulties with the interpretation indicate that there might not be a consistent quantisation of the classical higher-spin gauge theory, and that instead they could be resolved by adding to the spectrum additional perturbative fields. One proposal going in this direction involves a higher-spin theory on $\text{AdS}_3 \times \text{S}^1$ [123, 124]. Another option might be that the only solution is to embed the higher-spin states in string theory in a tensionless limit (see the discussion in section 6).
- The 't Hooft limit has to be modified. Indeed, it seems that the unwanted part of the spectrum decouples in the limit [31]. Taking limits of CFTs can have subtle effects on the spectrum [125], and one could try to define a limit in which the unwanted states are not present.
- The 't Hooft limit is not truly a semi-classical limit. Although c goes to infinity, the spectrum changes when we let N and k go to infinity, so it is a priori not clear whether this is really the appropriate limit.

In view of the last option, one might look for other ways of taking a semi-classical limit of the minimal models. Indeed, there is a proposal [126, 110, 127] which is

more a semi-classical continuation than a semi-classical limit: we consider the \mathcal{W}_N -minimal models, and in all quantities that we can compute, the level k is analytically continued such that it approaches $k \rightarrow -N - 1$. In this limit the central charge behaves as

$$c_{N,k} \xrightarrow{k \rightarrow -N-1} \frac{N(N^2 - 1)}{N + k + 1}, \quad (4.92)$$

so it goes to infinity. The conformal weights of some of the states behave as

- $(\square, 0)$:

$$h_{(\square, 0)} = \frac{N-1}{2N} \left(1 + \frac{N+1}{N+k} \right) \rightarrow \frac{1}{2}(1-N) \quad (4.93)$$

corresponding to a scalar of mass $M^2 = \frac{N^2-1}{\ell^2}$.

- $(0, \square)$:

$$h_{(0, \square)} = \frac{N-1}{2N} \left(1 - \frac{N+1}{N+k+1} \right) \rightarrow -\frac{c}{2N^2}. \quad (4.94)$$

The conformal weight grows with the central charge, so one would expect those states to correspond to non-perturbative states on the bulk side.

- (\square, \square) :

$$h_{(\square, \square)} = \frac{N^2 - 1}{2N(N+k)(N+k+1)} \rightarrow -\frac{c}{2N^2}, \quad (4.95)$$

so again those correspond to non-perturbative states.

In this semi-classical limit (or continuation) there is no apparent problem: we find the contributions of one complex scalar (corresponding to $(\square, 0)$ and $(\bar{\square}, 0)$) and non-perturbative contributions that should correspond to classical solutions of the higher-spin gauge theory. The problem of finding infinitely many light states has disappeared.

There is indeed a class of classical solutions of the higher-spin gauge theories, called conical defect solutions, that explains the full spectrum [126]. These solutions can be labelled by a Young diagram S , and in the holographic picture they correspond to the states $(0, S)$. This can be confirmed by comparing the higher-spin charges carried by the solutions to the higher-spin charges of the CFT states [126, 128]. In this interpretation, the states (R, S) correspond to excitations of the scalar on the classical solutions [127].

The picture that emerges in this semi-classical limit nicely matches the spectra on the two sides. On the other hand, it is not obvious how to make rigorous sense of the continuation as a limit of CFTs. This is furthermore complicated by the fact that negative conformal weights appear, as it is manifest in (4.94), so that the corresponding CFTs cannot be unitary. This non-unitary but more controllable version of AdS/CFT has been further studied in [94] (see also [129, 130]), while the very different behaviour of the $(R, 0)$ and $(0, S)$ states in this limit has been used to argue that they should be treated differently in the 't Hooft limit too. In particular, this led to refine the original conjecture of [31] postulating that also in the 't Hooft limit the

$(R, 0)$ states should correspond to a perturbative scalar, while the $(0, S)$ states should correspond to non-perturbative bulk states [110, 127].

There are a number of further checks that have been performed on the proposed dualities. In particular, one can compare correlators on the two sides of the correspondence. For example, the 3-point correlators $\langle O\bar{O}J^{(s)} \rangle$ for the scalar primary O (corresponding to $(\square, 0)$), its complex conjugate \bar{O} and a higher-spin current $J^{(s)}$ match on the two sides [131]. Another check shows that the 3-point functions of perturbative $(R, 0)$ states factorise in the limit in a way that is consistent with the interpretation that they correspond to multiparticle states built from the elementary scalar field O [132]. Further checks and details of the proposed duality are discussed in the review [32].

The higher-spin $\text{AdS}_3/\text{CFT}_2$ correspondence has been generalised in various ways, for example to even spin minimal-model holography (where only even spins appear in the spectrum) [61, 62], supersymmetric minimal-model holography [133] or matrix-extended minimal-model holography [134, 135]. The latter also appears to describe a subsector of a correspondence of string theory on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ to a symmetric orbifold theory on the boundary [136], as we shall briefly discuss in section 6.

5 Coupling to matter

In this section, we discuss the coupling of a propagating scalar field to higher-spin gauge fields. To this end, we start in section 5.1 by reformulating the equations of motion of a free scalar field on AdS_3 in a so-called *unfolded* form, that allows to couple it to a higher-spin background. We then present the oscillator formulation of $\mathfrak{hs}[\lambda]$, that provides a useful tool to handle the unfolded equations of motion. The latter is also used in the approach by Prokushkin–Vasiliev [47] towards a non-linear theory of a complex scalar coupled to higher-spin fields that we introduce in section 5.2. Our presentation partly follows the one in [137].

5.1 Free scalar field in the unfolded formulation

The goal is to describe a scalar field that is coupled to gravity and to higher-spin fields. For this purpose, we need a description of the scalar that is compatible with the first-order frame-like formulation of higher-spin gravity that has been discussed in the previous sections. A first-order action describing the coupling of a scalar to gravity can be obtained by introducing a single auxiliary field; see, e.g., [138, chapter I.5]. On the other hand, it is unclear how this formulation could be modified so as to allow one to couple higher-spin fields too. An alternative first-order formula-

tion that appears better adapted to discuss such coupling is provided by the unfolded formulation, which we introduce below.³²

5.1.1 Matter fields

To motivate the unfolded re-formulation of the Klein–Gordon equation, we consider for simplicity a flat background with a constant vielbein $\bar{h}^a = \delta_\mu^a dx^\mu$. We can expand the differential of a scalar field in this basis of one-forms,

$$d\Phi = \bar{h}_a C^a, \quad (5.1)$$

with coefficient functions $C^a = \bar{h}^{\mu a} \partial_\mu \phi$ that encode the derivatives of the scalar field. Taking another differential we obtain

$$0 = d^2\Phi = -\bar{h}_a \wedge dC^a. \quad (5.2)$$

The solution of this equation can be parameterised by a symmetric tensor \tilde{C}^{ab} ,

$$dC^a = \bar{h}_b \tilde{C}^{ab}. \quad (5.3)$$

The coefficients are given by second derivatives of the scalar field,

$$\tilde{C}^{ab} = \bar{h}^{\mu a} \bar{h}^{\nu b} \partial_\mu \partial_\nu \Phi. \quad (5.4)$$

Imposing the Klein–Gordon equation $\square\Phi = M^2\Phi$ constrains the trace part, and we can write

$$dC^a = \bar{h}_b C^{ab} + \frac{1}{3} M^2 \bar{h}^a \Phi \quad (5.5)$$

with an undetermined symmetric traceless part C^{ab} . The structure continues: one can apply a differential to (5.5) which leads to a condition on dC^{ab} , and its solution is parameterised by a symmetric traceless tensor C^{abc} , and so on. At the end, one arrives at a system of first-order equations

$$dC^{a_1 \dots a_n} = \bar{h}_b C^{a_1 \dots a_n b} + \frac{n}{3} M^2 \bar{h}^{(a_1} C^{a_2 \dots a_n)}. \quad (5.6)$$

Exercise 37.* Show the consistency of (5.6) by checking that the differential d annihilates the right-hand side.

In this way, one has encoded the Klein–Gordon equation in a first-order system of *unfolded form*. In general, equations of motion are said to be in unfolded form if they possess the structure

$$d\Phi^i = F^i(\Phi), \quad (5.7)$$

³² Actions describing the coupling of higher-spin Chern–Simons theories to (topological) matter have also been introduced by using a 2-form and a 0-form to model the matter sector [139, 140].

where Φ^i denotes various fields of possibly different form degrees (see, e.g., [141, 11, 12, 7] for reviews). Consistency ($d^2 = 0$) requires

$$F^j \wedge \frac{\partial F^i}{\partial \Phi^j} = 0. \quad (5.8)$$

Note that in particular the equations of motion for the gauge fields (2.45) are in unfolded form. Obtaining a formulation of the scalar on the same footing appears to be a good strategy if we want to couple the scalar to a gauge field background.

What we have presented for a flat background can also be repeated for a scalar on AdS_3 with a background vielbein \bar{e}^a and spin connection $\bar{\omega}^a$. Following the same strategy as above and replacing the differential d by the Lorentz covariant differential ∇ that was introduced in (2.37), we arrive at a first order system of the form

$$\nabla C^{a_1 \dots a_n} = \bar{e}_b C^{a_1 \dots a_n b} + \left(\frac{n}{3} M^2 - \frac{n(n-1)}{\ell^2} \right) \bar{e}^{(a_1} C^{a_2 \dots a_n)}. \quad (5.9)$$

The difference to (5.6) comes from the fact that the square of the covariant differential ∇ is not zero, but given by the AdS curvature (see (2.38)).

Exercise 38.* Check consistency of (5.9) by applying ∇ on both sides and using (2.38).

Eventually we want to couple the scalar to higher-spin fields. As we presented in section 2.2.3, higher-spin fields can be encoded in Chern–Simons connections A and \bar{A} with values in a Lie algebra with generators $J_{a_1 \dots a_{s-1}}$ which are symmetric and traceless in the indices a_n . An unfolded scalar consists of a system of symmetric traceless tensors, and they can be contracted with these generators to obtain an algebra valued object that could be used to describe a coupling to higher-spin fields. Concretely, we need a Lie algebra where all values of the spin occur once. A natural framework is the algebra $\mathfrak{hs}[\lambda]$, that we introduced in section 2.2.3 considering traceless symmetrised products of the $\mathfrak{sl}(2, \mathbb{R})$ generators J_a in the universal enveloping algebra. The scalar field itself has no indices and should be contracted with the identity, therefore we have to consider the associative algebra behind $\mathfrak{hs}[\lambda]$, that includes the unit and that we denoted as $B[\lambda]$ in (2.56). To realise two copies of this algebra (which is needed to realise the full higher-spin algebra) we introduce a variable ϕ which squares to 1 and commutes with the generators J_a ,

$$\phi^2 = 1, \quad \phi J_a = J_a \phi. \quad (5.10)$$

In this way one obtains the algebra $B[\lambda] \oplus B[\lambda]$, where projections on the first and second component are given by

$$\Pi_{\pm} = \frac{1}{2}(1 \pm \phi). \quad (5.11)$$

The $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Lie subalgebra is spanned by $J_a^{\pm} = \Pi_{\pm} J_a$, and the generators of the AdS_3 isometry algebra are then realised as

$$L_a = J_a, \quad P_a = \frac{G}{\ell} \phi J_a. \quad (5.12)$$

The scalar field and its undetermined derivatives can be combined into an algebra valued 0-form (we use a calligraphic letter for the components $\mathcal{C}^{a_1 \dots a_n}$ as they can differ from the coefficients $C^{a_1 \dots a_n}$ in normalisation)

$$\mathcal{C}(x) = \sum_{n=0}^{\infty} \mathcal{C}^{a_1 \dots a_n}(\phi|x) J_{\{a_1 \dots a_n\}}, \quad (5.13)$$

where the component without indices that comes with the unit element is the scalar field itself, $\mathcal{C}^{\emptyset} = \Phi$, and we recall that braces denote a symmetrisation and a traceless projection as, e.g., in (2.51). The Lorentz covariant derivative that we introduced in (2.37) can be generalised to any differential form F , and can be formulated in terms of algebra valued objects as

$$\nabla F = dF + \bar{\omega} \wedge F - (-1)^{|F|} F \wedge \bar{\omega}, \quad (5.14)$$

where $\bar{\omega} = \bar{\omega}^b L_b$ is the AdS background spin connection. Using the commutator (2.48), one can check that the action of the Lorentz covariant derivative on the zero-form defined above is given by

$$\nabla \mathcal{C} = \sum_{n=0}^{\infty} \underbrace{(d\mathcal{C}^{a_1 \dots a_n} + n \epsilon^{bc(a_1} \bar{\omega}_b \mathcal{C}_c^{a_2 \dots a_n})}_{\nabla \mathcal{C}^{a_1 \dots a_n}} J_{\{a_1 \dots a_n\}}. \quad (5.15)$$

To encode the unfolded Klein–Gordon equation (5.9) in an equation involving objects valued in this algebra, a natural ansatz where the right-hand side is linear in \mathcal{C} and in the vielbein³³ $\bar{e} = \bar{e}^a J_a \phi$ is

$$\nabla \mathcal{C} = \frac{\alpha_1}{\ell} \bar{e} \mathcal{C} + \frac{\alpha_2}{\ell} \mathcal{C} \bar{e}, \quad (5.16)$$

with some constants α_1, α_2 . This ansatz needs to satisfy consistency when applying the covariant differential ∇ again. Acting on the left-hand side of (5.16) leads to

$$\begin{aligned} \nabla^2 \mathcal{C} &= \sum_{n=0}^{\infty} \nabla^2 \mathcal{C}^{a_1 \dots a_n} J_{\{a_1 \dots a_n\}} \\ &= \sum_{n=0}^{\infty} \frac{n}{\ell^2} \bar{e}^b \wedge \bar{e}^{(a_1} \mathcal{C}^{a_2 \dots a_n)}_{b} J_{\{a_1 \dots a_n\}} \\ &= \frac{1}{\ell^2} [\mathcal{C}, \bar{e} \wedge \bar{e}]. \end{aligned} \quad (5.17)$$

³³ Notice that the Lie-algebra valued vielbein and spin-connection we use in this section take values in the whole AdS₃ isometry algebra, so that they are closer to those we introduced in (2.11) rather than to those we used in (2.24) and in section 2.2.3. The same conventions will be used for their higher-spin generalisations, that will thus take values in the $\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ algebra.

Exercise 39. Check (5.17).

Acting with ∇ on the right-hand side of (5.16) gives

$$\begin{aligned}\nabla\left(\frac{\alpha_1}{\ell}\bar{e}\mathcal{C}+\frac{\alpha_2}{\ell}\mathcal{C}\bar{e}\right) &= -\frac{\alpha_1}{\ell}\bar{e}\wedge\nabla\mathcal{C}+\frac{\alpha_2}{\ell}\nabla\mathcal{C}\wedge\bar{e} \\ &= -\frac{\alpha_1^2}{\ell^2}\bar{e}\wedge\bar{e}\mathcal{C}+\frac{\alpha_2^2}{\ell^2}\mathcal{C}\bar{e}\wedge\bar{e}.\end{aligned}\quad (5.18)$$

Comparing (5.17) and (5.18), we conclude that

$$\alpha_1^2 = \alpha_2^2 = 1. \quad (5.19)$$

Hence, the right-hand side of (5.16) is given by the commutator or by the anticommutator. The commutator would lead to a system of uncoupled equations for the components $\mathcal{C}^{a_1\dots a_n}$ because the commutator of \bar{e} with $J_{\{a_1\dots a_{s-1}\}}$ produces again a generator that corresponds to the same spin s . Such an equation does not describe a physical scalar field, but describes the so-called twisted sector of the Prokushkin–Vasiliev model [47] as we shall discuss in section 5.1.4. For the anticommutator, we are left with the possibilities $\alpha_1 = \alpha_2 = \pm 1$. As we are in principle free to swap the sign of the vielbein, we can set $\alpha_1 = \alpha_2 = -1$, and we arrive at the equation

$$\nabla\mathcal{C} = -\frac{1}{\ell}\{\bar{e}, \mathcal{C}\}. \quad (5.20)$$

The anticommutator of J_a with another basis element is given by

$$\begin{aligned}\{J_a, J_{\{b_1\dots b_n\}}\} &= 2J_{\{aJ_{b_1}\dots J_{b_n}\}} \\ &+ \frac{2n}{2n+1}\left(C_2 - \frac{n^2-1}{4}\right)\left(\eta_{a(b_1}J_{\{b_2\dots b_n\}} - \frac{n-1}{2n-1}\eta_{(b_1b_2}J_{\{b_3\dots b_n\}}J_a\right).\end{aligned}\quad (5.21)$$

With this information, we can now extract the components of (5.20). The identity component gives

$$\nabla\Phi = -\frac{2}{3\ell}C_2\phi\mathcal{C}^a\bar{e}_a, \quad (5.22)$$

leading to

$$\mathcal{C}^a = -\frac{3\ell}{2C_2}\phi\bar{e}^{\mu a}\partial_\mu\Phi. \quad (5.23)$$

The J_a -component of (5.20) is given by

$$\nabla_\mu\mathcal{C}^a = -\frac{2}{\ell}\bar{e}_\mu{}^a\Phi\phi - \frac{4}{5\ell}\left(C_2 - \frac{3}{4}\right)\bar{e}_{\mu c}\mathcal{C}^{ca}\phi. \quad (5.24)$$

When one contracts this with the inverse background vielbein $\bar{e}^\mu{}_a$, one obtains

$$\bar{e}^\mu{}_a\nabla_\mu\mathcal{C}^a = -\frac{6}{\ell}\phi\Phi. \quad (5.25)$$

Inserting the expression (5.23) for \mathcal{C}^a , we finally arrive at a Klein–Gordon equation

$$\square\Phi = \bar{g}^{\mu\nu}\bar{\nabla}_\mu\partial_\nu\Phi = \underbrace{\frac{4}{\ell^2}C_2}_{M^2}\Phi, \quad (5.26)$$

where $\bar{\nabla}$ denotes the Levi–Civita covariant derivative of the AdS background. The mass squared is given by

$$M^2 = \frac{4}{\ell^2}C_2 = \frac{\lambda^2 - 1}{\ell^2}. \quad (5.27)$$

The other components of (5.20) do not lead to any further conditions, and (5.20) provides an unfolded formulation of the scalar on AdS as before. Because Φ can depend on ϕ , we get two scalar fields $\Pi_\pm\Phi$ of the same mass. The mass depends on the parameter λ of the algebra $\mathfrak{hs}[\lambda]$. Only for this value of the mass it is possible to write the unfolded equation for the scalar in a way that is compatible with the algebra structure and to couple the scalar also to a non-trivial higher-spin background. Let us also stress that the value for the mass we obtained in (5.27) precisely fits the prediction we made in (4.85) only looking at the spectrum of minimal models in the 't Hooft limit. This had to be expected since the value of the mass is fixed by symmetry considerations both on the CFT side and in the previous bulk computation, but this is a first encouraging evidence of the consistency of the holographic conjecture that we reviewed in section 4.7.

The generalisation to an arbitrary higher-spin background is straightforward. We replace the vielbein and the spin connection by their higher-spin generalisations e and ω and obtain

$$d\mathcal{C} + [\omega, \mathcal{C}] = -\frac{1}{\ell}\{e, \mathcal{C}\}. \quad (5.28)$$

This is consistent as long as ω and e satisfy the equations of motion (2.45) (see exercise 40 below). Because of the ϕ -dependence, we can split this in two equations. Each component of \mathcal{C} we split as

$$\mathcal{C}^{a_1\dots a_n}(\phi|x) = C^{a_1\dots a_n}(x)\Pi_+ + \tilde{C}^{a_1\dots a_n}(x)\Pi_-, \quad (5.29)$$

and we obtain two $B[\lambda]$ -valued fields C and \tilde{C} , such that $\mathcal{C} = C\Pi_+ + \tilde{C}\Pi_-$. As in section 2.2.3, we introduce gauge fields

$$A = \sum_{n=1}^{\infty} A^{a_1\dots a_n} J_{\{a_1\dots a_n\}} = \sum_{n=1}^{\infty} \left(\omega^{a_1\dots a_n} + \frac{1}{\ell} e^{a_1\dots a_n} \right) J_{\{a_1\dots a_n\}}, \quad (5.30a)$$

$$\tilde{A} = \sum_{n=1}^{\infty} \tilde{A}^{a_1\dots a_n} J_{\{a_1\dots a_n\}} = \sum_{n=1}^{\infty} \left(\omega^{a_1\dots a_n} - \frac{1}{\ell} e^{a_1\dots a_n} \right) J_{\{a_1\dots a_n\}}, \quad (5.30b)$$

which take values in $\mathfrak{hs}[\lambda] \subset B[\lambda]$. Higher-spin vielbeins and spin connections taking values in the whole higher-spin algebra can then be defined as $\omega = \frac{1}{2}(A + \tilde{A})$ and $e = \phi \frac{\ell}{2}(A - \tilde{A})$ (see also footnote 33). Finally, we obtain from (5.28) two equations (each being an equation in $B[\lambda]$)

$$dC = -AC + C\tilde{A}, \quad (5.31a)$$

$$d\tilde{C} = -\tilde{A}\tilde{C} + \tilde{C}A. \quad (5.31b)$$

For backgrounds that include non-trivial higher-spin gauge fields these equations lead to two equivalent generalisations of the Klein–Gordon equation involving higher derivatives, see [131].

Exercise 40.* Check the consistency of (5.31) for flat connections A and \tilde{A} .

5.1.2 Oscillator realisation of $\mathfrak{hs}[\lambda]$

In this section, we introduce an oscillator realisation of the higher-spin algebra $\mathfrak{hs}[\lambda]$. This is a useful tool to handle infinite-dimensional algebras (see, e.g., [59, 70, 66] and references therein for some applications of oscillator techniques in this and related contexts), and we shall rely on it in section 5.2 to set up a full non-linear interacting theory of a scalar coupled to higher-spin gauge fields along the lines of [47].

We begin by introducing an abstract algebra with generating objects y_α with $\alpha = 0, 1$ and a multiplication denoted by \star , and demand the commutation relations

$$[y_\alpha, y_\beta]_\star = 2i\epsilon_{\alpha\beta}. \quad (5.32)$$

Indices can be lowered and raised by using the antisymmetric epsilon tensor $\epsilon_{\alpha\beta}$ and its inverse $\epsilon^{\alpha\beta}$ with $\epsilon_{01} = \epsilon^{01} = 1$,

$$y^\alpha := \epsilon^{\alpha\beta}y_\beta, \quad y_\alpha = y^\beta\epsilon_{\beta\alpha}, \quad (5.33)$$

where we have used $\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta^\alpha_\beta$.

These oscillators allow us to realise $\mathfrak{sl}(2, \mathbb{R})$, which is isomorphic to $\mathfrak{sp}(2, \mathbb{R})$. Using the commutator (5.32), one can indeed show that the generators

$$L_{\alpha\beta} = -\frac{i}{2}y_{(\alpha}\star y_{\beta)} \quad (5.34)$$

satisfy the $\mathfrak{sp}(2, \mathbb{R})$ commutation relations

$$[L_{\alpha\beta}, L_{\alpha'\beta'}]_\star = \epsilon_{\alpha\alpha'}L_{\beta\beta'} + \epsilon_{\beta\beta'}L_{\alpha\alpha'} + \epsilon_{\alpha\beta'}L_{\beta\alpha'} + \epsilon_{\beta\alpha'}L_{\alpha\beta}. \quad (5.35)$$

Exercise 41.* Using the commutation relations (5.35), verify the isomorphism $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{sp}(2, \mathbb{R})$ by checking that the generators $L_{-1} = \frac{1}{2}L_{11}$, $L_{+1} = \frac{1}{2}L_{00}$ and $L_0 = \frac{1}{2}L_{10} = \frac{1}{2}L_{01}$ obey the $\mathfrak{sl}(2, \mathbb{R})$ commutation relations $[L_m, L_n] = (m-n)L_{m+n}$.

This specific realisation of $\mathfrak{sl}(2, \mathbb{R})$ can be used to obtain a realisation of a specific higher-spin algebra. Indeed, symmetrised polynomials in the y -oscillators of even degree form an associative algebra under star multiplication. This associative algebra is isomorphic to $B[\lambda]$ for $\lambda = \frac{1}{2}$, as can be seen by comparing with the quadratic Casimir of the oscillator realisation of $\mathfrak{sp}(2, \mathbb{R})$,

$$C_2 := -\frac{1}{8} L^{\alpha\beta} \star L_{\alpha\beta} = -\frac{1}{16} \{L^{\alpha\beta}, L_{\alpha\beta}\} \star = -\frac{3}{16}, \quad (5.36)$$

which indeed corresponds to $\lambda = \frac{1}{2}$ in (2.56).

Exercise 42. Verify (5.36) using (5.32).

The higher-spin algebra is obtained from the associative algebra $B[\lambda]$ by

$$B[\lambda] = \mathbb{C} \oplus \mathfrak{hs}[\lambda], \quad (5.37)$$

where, with respect to (2.57), we stressed that we can also complexify the algebra. Therefore, even monomials in the oscillators of degree at least 2 form a basis of a Lie algebra (with respect to the star commutator) that is isomorphic to $\mathfrak{hs}[\lambda]$ for $\lambda = \frac{1}{2}$.

The star product that we introduced abstractly can also be realised as a Moyal–Weyl product. To this end, we can think of the oscillators as ordinary commuting variables,

$$y^\alpha y^\beta = y^\beta y^\alpha, \quad (5.38)$$

where we distinguish this product from the previous star product by omitting the product symbol.

For functions f, g of the oscillators y_α , one then defines the star product by

$$(f \star g)(y) = \frac{1}{(2\pi)^2} \int d^2 u d^2 v f(y+u) g(y+v) \exp(ivu), \quad (5.39)$$

where we used the notation $v u = -u v := v^\alpha u_\alpha$. Let us stress that the contraction of two identical commuting oscillators vanishes,

$$y y = u u = v v = 0. \quad (5.40)$$

Furthermore, we shall use the notation

$$\partial_\alpha^y = \frac{\partial}{\partial y^\alpha}, \quad \partial_y^\alpha := \epsilon^{\alpha\beta} \partial_\beta^y, \quad (5.41)$$

from which we deduce the following relations

$$\partial_\alpha^y y^\beta = -\partial_y^\beta y_\alpha = \delta_\alpha^\beta, \quad (5.42a)$$

$$\partial_y^\alpha y^\beta = \epsilon^{\alpha\beta}, \quad (5.42b)$$

$$\partial_\alpha^y y_\beta = \epsilon_{\alpha\beta}. \quad (5.42c)$$

As a word of warning we remark that relation (5.42a) implies that $\partial_y^\alpha = -\frac{\partial}{\partial y_\alpha}$.

A differential version of the star product (as long as the functions are analytic) can also be obtained from its definition (5.39):

$$(f \star g)(y) = f(y) e^{-i \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_y} g(y). \quad (5.43)$$

Exercise 43.* Use Taylor's theorem, $f(y+u) = e^{u\partial^y} f(y)$ and the identity $\delta^{(2)}(u) = \frac{1}{(2\pi)^2} \int d^2v e^{ivu}$ to obtain the differential version of the star product.

If one of the factors is just an oscillator variable, we obtain the following expressions for the star product (which follow straightforwardly from (5.43)):

$$y_\alpha \star f(y) = (y_\alpha + i\partial_\alpha^y) f(y), \quad (5.44a)$$

$$f(y) \star y_\alpha = (y_\alpha - i\partial_\alpha^y) f(y). \quad (5.44b)$$

From these we obtain the relations

$$[y_\alpha, f(y)]_\star = 2i\partial_\alpha^y f(y), \quad (5.45a)$$

$$\frac{1}{2} \{y_\alpha, f(y)\}_\star = y_\alpha f(y). \quad (5.45b)$$

We observe from (5.45b) that the symmetrised star-product of y_α -oscillators is simply the ordinary product, $y_{(\alpha} \star y_{\beta)} = y_\alpha y_\beta$. On the other hand, when we insert $f(y) = y_\beta$ into (5.45a), we recover the star commutation relation (5.32) of the oscillators.

Exercise 44.* Check equations (5.44) and (5.45).

Exercise 45.* Using (5.44) and (5.45) check the identities

$$[L_{\alpha\beta}, f(y)]_\star = 2y_{(\alpha} \partial_{\beta)}^y f(y), \quad (5.46a)$$

$$\{L_{\alpha\beta}, f(y)\}_\star = -i \left(y_\alpha y_\beta - \partial_\alpha^y \partial_\beta^y \right) f(y). \quad (5.46b)$$

Use the latter identity to verify once again the result (5.36) for the eigenvalue of the Casimir operator.

With the construction above, we could realise $B[\lambda]$ for the specific value of $\lambda = \frac{1}{2}$. To get an oscillator realisation for other values of λ , we introduce *deformed oscillators* \hat{y}_α with $\alpha = 0, 1$ obeying the commutation relations

$$[\hat{y}_\alpha, \hat{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 + \nu k), \quad (5.47)$$

where $\nu \in \mathbb{R}$ is some real parameter, while the symbol k denotes an object called *outer Kleinian* obeying

$$k \hat{y}_\alpha = -\hat{y}_\alpha k, \quad k^2 = 1. \quad (5.48)$$

As we shall see shortly, the parameter ν is related to the parameter λ that appears in $B[\lambda]$. Note that for $\nu = 0$, eq. (5.47) reduces to the undeformed commutation relations (5.32) discussed previously. One can then show that

$$L_{\alpha\beta} = -\frac{i}{2} \hat{y}_{(\alpha} \star \hat{y}_{\beta)} \quad (5.49)$$

fulfils the $\mathfrak{sp}(2, \mathbb{R})$ commutation relations (5.35). This can be seen by observing that

$$\begin{aligned} [L_{\alpha\beta}, \hat{y}_\gamma]_\star &= -\frac{i}{2} \hat{y}_{(\alpha} \star [\hat{y}_{\beta)}, \hat{y}_\gamma]_\star + \frac{i}{2} [\hat{y}_\gamma, \hat{y}_{(\beta)}]_\star \star \hat{y}_{\alpha)} \\ &= -\epsilon_{\gamma(\beta} \hat{y}_{\alpha)} (1 + \nu k) - \epsilon_{\gamma(\beta} \hat{y}_{\alpha)} (1 - \nu k) \\ &= -2\epsilon_{\gamma(\alpha} \hat{y}_{\beta)}. \end{aligned} \quad (5.50)$$

Note that we have used $\hat{y}_\alpha k = -k \hat{y}_\alpha$ to obtain the second line which is crucial for the final expression to be ν independent. By using (5.50) twice, it immediately follows that $L_{\alpha\beta}$ obeys the $\mathfrak{sp}(2, \mathbb{R})$ commutation relations (5.35). For the quadratic Casimir, we obtain

$$C_2 = -\frac{1}{8} L^{\alpha\beta} \star L_{\alpha\beta} = -\frac{1}{16} (3 + 2\nu k - \nu^2). \quad (5.51)$$

Exercise 46.* Verify this statement by using the commutation relations (5.47) to show that

$$\hat{y}^\alpha \star \hat{y}_\alpha = -2i(1 + \nu k), \quad L_{\alpha\beta} = -\frac{i}{2} \hat{y}_\alpha \star \hat{y}_\beta - \frac{1}{2} \epsilon_{\alpha\beta} (1 + \nu k), \quad (5.52)$$

and use these relations to calculate the value of C_2 .

The quadratic Casimir (5.51) is not a pure number, but contains the outer Kleinian k . By using the projectors

$$P_\pm = \frac{1}{2} (1 \pm k) \quad (5.53)$$

to restrict to a definite parity of $k = \pm 1$, we can then realise the associative higher-spin algebra $B[\lambda]$ (see (2.56)) for arbitrary values of λ with the identification

$$\lambda = \frac{1}{2} (\nu \pm 1) \quad \text{for } k = \mp 1. \quad (5.54)$$

A basis for this realisation is given by even and symmetric monomials in the deformed oscillators \hat{y}_α projected on a sector of definite k -parity, i.e.

$$L_{\alpha_1 \dots \alpha_{2n}} := P_\pm \hat{y}_{(\alpha_1} \star \dots \star \hat{y}_{\alpha_{2n})}. \quad (5.55)$$

By turning the associative algebra into a Lie algebra and using the decomposition (5.37), a realisation for $\mathfrak{hs}[\lambda]$ for arbitrary λ is thus obtained.

5.1.3 Higher-spin and matter fields using oscillators

We now wish to rewrite the unfolded equations for the higher-spin and matter fields introduced in section 5.1.1 using the oscillator realisation of $\mathfrak{hs}[\lambda]$, in order to facilitate the introduction of interactions in the following section. We start by realising the AdS_3 background in this setup. The AdS_3 isometry algebra is $\mathfrak{so}(2, 2)$. This algebra is isomorphic to two copies of $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{sp}(2, \mathbb{R})$, that we realise as before

by introducing an additional variable ϕ with the properties

$$\phi^2 = 1, \quad \hat{y}_\alpha \phi = \phi \hat{y}_\alpha, \quad \phi k = k \phi. \quad (5.56)$$

As in section 5.1.1, the two commuting copies of $\mathfrak{sp}(2, \mathbb{R})$ (and thus the AdS_3 isometry algebra $\mathfrak{so}(2, 2)$) are obtained with the help of the projectors $\Pi_\pm = \frac{1}{2}(1 \pm \phi)$ by introducing the generators $\Pi_+ L_{\alpha\beta}$ and $\Pi_- L_{\alpha\beta}$.

Using this realisation of $\mathfrak{so}(2, 2)$, we can describe the AdS_3 background by defining the connections

$$A^{AdS} = -\frac{1}{2} \left(\bar{\omega} + \frac{1}{\ell} \bar{e} \right)^{\alpha\beta} L_{\alpha\beta}, \quad (5.57a)$$

$$\tilde{A}^{AdS} = -\frac{1}{2} \left(\bar{\omega} - \frac{1}{\ell} \bar{e} \right)^{\alpha\beta} L_{\alpha\beta}, \quad (5.57b)$$

where \bar{e} and $\bar{\omega}$ describe the AdS_3 vielbein and spin connection, respectively. We normalise the vielbein \bar{e} such that

$$\bar{e}_{\mu\alpha\beta} \bar{e}_\nu^{\alpha\beta} = -\frac{1}{2} g_{\mu\nu}, \quad (5.58a)$$

$$\bar{e}^\mu_{\alpha_1\alpha_2} \bar{e}_\mu^{\beta_1\beta_2} = -\frac{1}{2} \delta^{\beta_1}_{\alpha_1} \delta^{\beta_2}_{\alpha_2}. \quad (5.58b)$$

Here, first letters of the Greek alphabet always describe spinorial indices, $\alpha, \beta, \dots \in \{1, 2\}$, whereas letters μ, ν, \dots denote spacetime indices. Note that these identities allow us to convert the spinorial indices of a completely symmetric tensor $f^{\alpha_1 \dots \alpha_{2s}}$ to spacetime indices:

$$f^{\alpha_1 \dots \alpha_{2s}} \bar{e}^{\mu_1}_{\alpha_1\alpha_2} \dots \bar{e}^{\mu_s}_{\alpha_{2s-1}\alpha_{2s}} = \left(-\frac{1}{2}\right)^s f^{\mu_1 \dots \mu_s}. \quad (5.59)$$

The tensor $f^{\mu_1 \dots \mu_s}$ is completely symmetric and traceless.

Exercise 47.* Prove that $f^{\mu_1 \dots \mu_s}$ is traceless by using (5.58b).

The AdS_3 connections A^{AdS} and \tilde{A}^{AdS} satisfy the zero-curvature condition (2.14). As in section 2.1, it is convenient to combine A^{AdS} and \tilde{A}^{AdS} into a single connection

$$\mathcal{A}^{AdS} = \Pi_+ A^{AdS} + \Pi_- \tilde{A}^{AdS} = -\frac{1}{2} \bar{\omega}^{\alpha\beta} L_{\alpha\beta} - \frac{1}{2G} \bar{e}^{\alpha\beta} P_{\alpha\beta} = \bar{\omega} + \frac{1}{\ell} \bar{e}, \quad (5.60)$$

where we defined $P_{\alpha\beta} := \phi \frac{G}{\ell} L_{\alpha\beta}$ consistently with (5.12). \mathcal{A}^{AdS} is also a solution of the Chern–Simons equation of motion (2.14),

$$d_{\mathcal{A}^{AdS}} + \mathcal{A}^{AdS} \wedge \star \mathcal{A}^{AdS} = 0. \quad (5.61)$$

As in section 2.2.3, we describe higher-spin fields by considering connections A and \tilde{A} to be valued in $\mathfrak{hs}(\lambda)$, which now we realise using the oscillators,

$$\mathcal{A} = \Pi_+ A + \Pi_- \tilde{A} = \omega + \frac{1}{\ell} e. \quad (5.62)$$

Here ω and e are even polynomials in the \hat{y}_α -oscillators, i.e.

$$\omega = \frac{1}{2} \sum_{s=0}^{\infty} \frac{i}{(2s)!} \omega_{\alpha_1 \dots \alpha_{2s}}(x) P_{\pm} \hat{y}^{(\alpha_1} \star \dots \star \hat{y}^{\alpha_{2s})}, \quad (5.63a)$$

$$e = \frac{1}{2} \sum_{s=0}^{\infty} \frac{i}{(2s)!} e_{\alpha_1 \dots \alpha_{2s}}(x) \frac{\phi}{\ell} P_{\pm} \hat{y}^{(\alpha_1} \star \dots \star \hat{y}^{\alpha_{2s})}. \quad (5.63b)$$

In complete analogy to our discussion in section 2.2.2, the components $e_{\alpha_1 \dots \alpha_{2s}}(x)$ and $\omega_{\alpha_1 \dots \alpha_{2s}}(x)$ are the higher-spin vielbeins and spin connections, respectively. They correspond to tensors which are fully symmetric and traceless.

On an AdS_3 background $\mathcal{A}^{AdS} = \bar{\omega} + \frac{1}{\ell} \bar{e}$, the zero-curvature conditions (2.42) translate to the equation of motion

$$D^{AdS} \mathcal{A} := d\mathcal{A} + \mathcal{A}^{AdS} \wedge \star \mathcal{A} + \mathcal{A} \wedge \star \mathcal{A}^{AdS} = 0. \quad (5.64)$$

Here, D^{AdS} denotes the AdS_3 covariant derivative that includes the background spin connection and the vielbein. On a differential form F of degree $|F|$, it acts as

$$D^{AdS} F := dF + \mathcal{A}^{AdS} \wedge \star F - (-1)^{|F|} F \wedge \star \mathcal{A}^{AdS}, \quad (5.65)$$

and it is related to the Lorentz covariant differential as

$$D^{AdS} F = \nabla F + \frac{1}{\ell} \bar{e} \wedge \star F - \frac{1}{\ell} (-1)^{|F|} F \wedge \star \bar{e}. \quad (5.66)$$

The equation of motion (5.64) is invariant under the gauge transformation

$$\delta \mathcal{A} = D^{AdS} \xi, \quad (5.67)$$

with some arbitrary zero-form ξ generically depending on \hat{y}^α , ϕ and spacetime coordinates x^μ . The gauge invariance follows immediately from the fact that the covariant derivative D^{AdS} is nilpotent

$$D^{AdS} D^{AdS} F = 0. \quad (5.68)$$

Exercise 48.* Check the nilpotency (5.68).

After having discussed the higher-spin fields, we now reformulate the unfolded scalar in terms of the oscillators. It is described by a zero-form

$$\mathcal{C}(y, \phi | x) = \sum_{s=0}^{\infty} \frac{1}{s!} \mathcal{C}_{\alpha_1 \dots \alpha_s}(\phi, k | x) \hat{y}^{(\alpha_1} \star \dots \star \hat{y}^{\alpha_s)}, \quad (5.69)$$

where we restrict the sum to even s .

Consistently with (5.15), the action of the Lorentz covariant derivative on this zero-form is given by

$$\nabla \mathcal{C} = \sum_{s=0}^{\infty} \frac{1}{s!} \underbrace{(d\mathcal{C}_{\alpha_1 \dots \alpha_s} + s \bar{\omega}^{\beta}_{\alpha_1} \mathcal{C}_{\beta \alpha_2 \dots \alpha_s})}_{\nabla \mathcal{C}_{\alpha_1 \dots \alpha_s}} \hat{y}^{(\alpha_1} \star \dots \star \hat{y}^{\alpha_s)}, \quad (5.70)$$

where we defined the Lorentz covariant derivative acting on the fully symmetric components $\mathcal{C}_{\alpha_1 \dots \alpha_s}$. The Klein–Gordon equation on AdS_3 in unfolded form (5.20) in the oscillator formalism then reads

$$\nabla \mathcal{C} = -\{\bar{e}, \mathcal{C}\}_{\star}. \quad (5.71)$$

We reformulated the whole free theory in the oscillator language. In the remaining part of this subsection, we shall revisit the proof that (5.71) is equivalent to the Klein–Gordon equation by using oscillator techniques, so as to familiarise with them. Our path towards interactions restarts in section 5.1.4, where another class of fields that enter the interacting theory is discussed.

To analyse the content of eq. (5.71), it is convenient to first express the anticommutator of $L_{\alpha_1 \alpha_2}$ with an oscillator that is given by

$$\begin{aligned} \{L_{\alpha_1 \alpha_2}, \hat{y}_{(\beta_1} \star \dots \star \hat{y}_{\beta_s)}\}_{\star} &= -i \hat{y}_{(\alpha_1} \star y_{\alpha_2} \star \dots \star \hat{y}_{\beta_s)} \\ &\quad - i s \frac{\nu^2 - 2k\nu - (s^2 - 1)}{s + 1} \epsilon_{\alpha_1 (\beta_1} \epsilon_{|\alpha_2| \beta_2} \hat{y}_{\beta_3} \star \dots \star \hat{y}_{\beta_s)} \end{aligned} \quad (5.72)$$

for even s .

Exercise 49.* Verify this statement by first using the commutation relation (5.47) to show that

$$\begin{aligned} \hat{y}_{\alpha} \star \hat{y}_{(\beta_1} \star \dots \star \hat{y}_{\beta_s)} &= \hat{y}_{(\alpha} \star \hat{y}_{\beta_1} \star \dots \star \hat{y}_{\beta_s)} \\ &\quad + i \left(s + \frac{\frac{1}{2}(1 - (-1)^s) + s}{s + 1} \nu k \right) \epsilon_{\alpha (\beta_1} \hat{y}_{\beta_2} \star \dots \star \hat{y}_{\beta_s)} \end{aligned} \quad (5.73)$$

and analogously

$$\begin{aligned} \hat{y}_{(\beta_1} \star \dots \star \hat{y}_{\beta_s)} \star \hat{y}_{\alpha} &= \hat{y}_{(\alpha} \star \hat{y}_{\beta_1} \star \dots \star \hat{y}_{\beta_s)} \\ &\quad - i \left(s + \frac{\frac{1}{2}(1 - (-1)^s) + (-1)^{s+1} s}{s + 1} \nu k \right) \epsilon_{\alpha (\beta_1} \hat{y}_{\beta_2} \star \dots \star \hat{y}_{\beta_s)}. \end{aligned} \quad (5.74)$$

By using both results twice deduce (5.72). Recall that (5.72) holds for even s only.

We now evaluate (5.71) by replacing $\bar{e} = -\frac{\phi}{2\bar{r}} \bar{e}^{\alpha\beta} L_{\alpha\beta}$ and using (5.72). In components, (5.71) is then given by

$$\nabla \mathcal{C}_{\alpha_1 \dots \alpha_s} = \frac{\phi}{2i\bar{r}} \left(s(s-1) \bar{e}_{(\alpha_1 \alpha_2} \mathcal{C}_{\alpha_3 \dots \alpha_s)} - \left(1 - \frac{\nu(\nu-2k)}{(s+1)(s+3)} \right) \bar{e}^{\beta_1 \beta_2} \mathcal{C}_{\beta_1 \beta_2 \alpha_1 \dots \alpha_s} \right). \quad (5.75)$$

To analyse this equation further, we contract its spacetime index with a vielbein $\bar{e}_{\gamma_1\gamma_2}^\mu$ which yields

$$\begin{aligned} \bar{e}_{\gamma_1\gamma_2}^\mu \nabla_\mu \mathcal{C}_{\alpha_1 \dots \alpha_s} &= \frac{\phi}{2i\ell} \left(s(s-1) \underbrace{\bar{e}_{\gamma_1\gamma_2}^\mu \bar{e}_{\mu(\alpha_1\alpha_2)}}_{\sim \epsilon_{\gamma_1(\alpha_1} \epsilon_{\alpha_2)\gamma_2}} \mathcal{C}_{\alpha_3 \dots \alpha_s} \right. \\ &\quad \left. - \frac{4}{3} \ell^2 \underbrace{\frac{3}{4\ell^2} \left(1 - \frac{\nu(\nu-2k)}{(s+1)(s+3)} \right)}_{=: M_s^2} \underbrace{\bar{e}_{\gamma_1\gamma_2}^\mu \bar{e}_{\mu}^{\beta_1\beta_2}}_{=-\frac{1}{2} \delta_{\gamma_1}^{(\beta_1} \delta_{\gamma_2}^{\beta_2)}} \mathcal{C}_{\beta_1\beta_2\alpha_1 \dots \alpha_s} \right). \end{aligned} \quad (5.76)$$

To analyse this equation further, we first consider its symmetrisation over all γ_i and α_i indices. The first summand on the right hand side does not contribute, and we obtain

$$\mathcal{C}_{\alpha_1 \dots \alpha_s} = \frac{3i}{M_{s-2}^2} \frac{\phi}{\ell} \bar{e}^{\mu(\alpha_1\alpha_2} \nabla_\mu \mathcal{C}_{\alpha_3 \dots \alpha_s)}, \quad (5.77)$$

where we also used $\phi^2 = 1$. From this expression we see that by recursion, all higher tensor components $\mathcal{C}_{\alpha_1 \dots \alpha_s}$ can be expressed as derivatives of the lowest component $\Phi = \mathcal{C}_\emptyset = \mathcal{C}|_{y=0}$. To read off the remaining information contained in (5.76) we now consider its projection to the part that is antisymmetric under $\gamma_1 \leftrightarrow \alpha_1$ and $\gamma_2 \leftrightarrow \alpha_2$. To this end, we contract it with $\epsilon^{\gamma_1\alpha_1} \epsilon^{\gamma_2\alpha_2}$, which annihilates the second term on the right-hand side. With the help of the identity (5.58a) we obtain

$$\epsilon^{\gamma_1\alpha_1} \epsilon^{\gamma_2\alpha_2} \bar{e}_{\gamma_1\gamma_2}^\mu \nabla_\mu \mathcal{C}_{\alpha_1 \dots \alpha_s} = -\frac{3}{4i} \frac{\phi}{\ell} s(s-1) \mathcal{C}_{\alpha_3 \dots \alpha_s}. \quad (5.78)$$

Evaluating this equation for $s = 2$ implies

$$-\frac{3}{2i} \frac{\phi}{\ell} \Phi = \bar{e}_\mu^{\alpha\beta} \nabla^\mu \mathcal{C}_{\alpha\beta} = \frac{3i}{M_0^2} \frac{\phi}{\ell} \bar{e}_\mu^{\alpha\beta} \nabla^\mu \bar{e}_{\alpha\beta}^\nu \nabla_\nu \Phi, \quad (5.79)$$

where we have used (5.77) in the last step. Using $\nabla \bar{e} = 0$, which follows from the ϕ -component of the AdS₃ flatness condition (5.61), we obtain the Klein–Gordon equation

$$g^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \Phi = M^2 \Phi, \quad (5.80)$$

with

$$M^2 = M_0^2 = -\frac{3}{4\ell^2} \left(1 - \frac{\nu(\nu-2k)}{3} \right) = \frac{4}{\ell^2} C_2, \quad (5.81)$$

which reproduces our previous result (5.26). We have thus seen how to reformulate free higher-spin and matter fields in terms of the oscillators.

5.1.4 Twisted fields

Before we move to the nonlinear equations, let us compare the equations of motion for the higher-spin gauge fields encoded in \mathcal{A} and the scalar field encoded in \mathcal{C} on an AdS background. The equation of motion $D^{AdS} \mathcal{A} = 0$ for the one-form \mathcal{A} can be written as

$$\begin{aligned} D^{AdS} \mathcal{A} &= \nabla \mathcal{A} + \bar{e} \wedge \star \mathcal{A} + \mathcal{A} \wedge \star \bar{e} \\ &= \nabla \mathcal{A} + [\bar{e}_\mu, \mathcal{A}_\nu]_\star dx^\mu \wedge dx^\nu. \end{aligned} \quad (5.82)$$

We observe that – as we are used to in gauge theories – the Lie bracket of the gauge algebra occurs; we say that the one-form \mathcal{A} transforms in the adjoint representation of the higher-spin algebra. On the other hand, the equation of motion (5.71) for \mathcal{C} involves the anticommutator; for this reason the zero-form \mathcal{C} is said to transform in the *twisted adjoint representation*. Recall that to obtain the Klein–Gordon equation, the occurrence of the anticommutator in (5.71) is essential as it relates different components of the zero-form \mathcal{C} contrary to the commutator.³⁴

The equation of motion for \mathcal{C} cannot directly be written with the help of the covariant derivative D^{AdS} because $D^{AdS} \mathcal{C}$ involves the commutator rather than the anticommutator. There is an elegant way to again bring it into the form $D^{AdS}(\cdot) = 0$. To this end, let us introduce an additional variable ψ obeying

$$\psi^2 = 1, \quad \psi \phi = -\phi \psi, \quad \psi \hat{y}_\alpha = \hat{y}_\alpha \psi. \quad (5.83)$$

We can then rewrite the equation of motion (5.71) as

$$D^{AdS}(\mathcal{C} \psi) = \nabla(\mathcal{C} \psi) - \frac{1}{2} \bar{e}^{\alpha\beta} [\phi L_{\alpha\beta}, \mathcal{C} \psi]_\star = (\nabla \mathcal{C} + \{\bar{e}, \mathcal{C}\}_\star) \psi = 0, \quad (5.84)$$

where we have used the fact that the Lorentz covariant derivative ∇ is independent of ϕ and $[\phi f(\hat{y}), g(\hat{y}, \phi)]_\star = \{\phi f(\hat{y}), g(\hat{y}, \phi)\}_\star \psi$, which follows immediately from $\psi \phi = -\phi \psi$.

After having introduced ψ , at the formal level it is natural to also include “twisted fields” in the description and to consider the combinations

$$\mathbf{A}(\hat{y}, \phi, \psi|x) = \mathcal{A}(\hat{y}, \phi|x) + \tilde{\mathcal{A}}(\hat{y}, \phi|x) \psi, \quad (5.85a)$$

$$\mathbf{C}(\hat{y}, \phi, \psi|x) = \mathcal{C}(\hat{y}, \phi|x) \psi + \tilde{\mathcal{C}}(\hat{y}, \phi|x), \quad (5.85b)$$

where the fields are functions of ψ . Note that the twisted field $\tilde{\mathcal{A}}$ for the one-form is given by the ψ -dependent part whereas the zero-form has the opposite decomposition, and we shall see in the next section that this extension of the field content has been used by Prokushkin and Vasiliev to introduce interactions.

The twisted zero-form $\tilde{\mathcal{C}}$ obeys the equation of motion

³⁴ This can be seen by comparing the star product identities for the anticommutator (5.72) with the analogous expression (5.50) for the commutator.

$$D^{AdS} \tilde{\mathcal{C}} = 0, \quad (5.86)$$

corresponding to the adjoint representation. This equation of motion does not mix different components of $\tilde{\mathcal{C}}$ due to the fact that it only contains commutators of $L_{\alpha\beta}$.³⁵ Therefore, it is not equivalent to a Klein–Gordon equation. Note also that by (5.67) this is precisely the form of a covariantly constant gauge parameter ξ . In a slight abuse of terminology, the components of the twisted zero-form $\tilde{\mathcal{C}}$ are therefore sometimes referred to as Killing tensors.

On the other hand, the twisted one-form $\tilde{\mathcal{A}}$ is defined to satisfy the equation

$$D^{AdS}(\tilde{\mathcal{A}}\psi) = 0, \quad (5.87)$$

and it corresponds to the twisted adjoint representation. Note that this relation does not decompose into independent equations for each component of $\tilde{\mathcal{A}}$ and therefore it does not describe a multiplet of higher-spin fields.

The equations of motion for twisted and untwisted fields can be summarised as

$$D^{AdS} \mathbf{A} = 0, \quad (5.88a)$$

$$D^{AdS} \mathbf{C} = 0, \quad (5.88b)$$

which are invariant under

$$\delta \mathbf{A} = D^{AdS} \xi, \quad (5.89a)$$

$$\delta \mathbf{C} = 0, \quad (5.89b)$$

where ξ is an arbitrary zero-form which can now also depend on ψ in addition to ϕ , \hat{y}^α and x^m . The non-linear equations of motion that we shall discuss in the following section will reproduce (5.88) upon linearisation.

5.2 Non-Linear Theory

In this section, we introduce Vasiliev’s approach to a non-linear theory of higher-spin gauge fields coupled to matter. It is based on unfolded equations for master fields which encode the equations of motion of the physical fields. Their linearisation leads to the free equations of motion for higher-spin and scalar fields discussed in section 5.1.3, together with the free equations of motion for the twisted fields that we introduced in section 5.1.4. For the remainder of this section, we set the AdS radius to one ($\ell = 1$) to simplify notation.

³⁵ It corresponds to the other solution of the consistency condition for the ansatz (5.16) of an unfolded equation for the scalar.

5.2.1 Master fields and Vasiliev Equations

To formulate Prokushkin–Vasiliev theory [47, 142], we first introduce another type of commuting oscillators z_α in addition to the y_α and define a star product for functions of both these oscillators as

$$(f \star g)(y, z) = \frac{1}{(2\pi)^2} \int d^2u d^2v f(y+u, z+u) g(y+v, z-v) \exp(iv^\alpha u_\alpha). \quad (5.90)$$

This definition reduces to the previous star product (5.39) for functions that only depend on y . This looks like a restriction to the undeformed oscillators, but, as we shall see, the deformed algebra arises upon the expansion around a non-trivial background value for the master field describing the scalar field. The new variables z_α satisfy

$$[z_\alpha, z_\beta]_\star = -2i \epsilon_{\alpha\beta}, \quad [z_\alpha, y_\beta]_\star = 0, \quad (5.91)$$

and – similarly to (5.45a) – we have

$$[y_\alpha, f(y, z)]_\star = 2i \partial_\alpha^y f(y, z), \quad (5.92a)$$

$$[z_\alpha, f(y, z)]_\star = -2i \partial_\alpha^z f(y, z). \quad (5.92b)$$

We now introduce master fields \mathcal{W} , \mathcal{B} and \mathcal{S}_α , which depend on y_α - and z_α -oscillators, as well as on the variables ϕ , ψ and the idempotent outer Kleinian k that will allow us to also realise the undeformed algebra later. k anticommutes with the oscillators y_α and z_α , but commutes with ϕ and ψ ,

$$k y_\alpha = -y_\alpha k, \quad k z_\alpha = -z_\alpha k, \quad k \phi = \phi k, \quad k \psi = \psi k, \quad k^2 = 1. \quad (5.93)$$

The z -independent part of \mathcal{W} and \mathcal{B} are interpreted as gauge field \mathbf{A} and scalar field \mathbf{C} ,

$$\mathcal{W}(y, z, \phi, \psi, k|x) \Big|_{z=0} = -\mathbf{A}(y, \phi, \psi, k|x), \quad (5.94a)$$

$$\mathcal{B}(y, z, \phi, \psi, k|x) \Big|_{z=0} = \mathbf{C}(y, \phi, \psi, k|x). \quad (5.94b)$$

The z -dependent part describes yet another auxiliary sector besides that of twisted fields already included in \mathbf{A} and \mathbf{C} . To formulate the nonlinear equations for the master fields, we need to introduce another idempotent variable ρ , that anticommutes with k and commutes with the rest,

$$\rho^2 = 1, \quad \rho k = -k \rho, \quad \rho \phi = \phi \rho, \quad \rho \psi = \psi \rho, \quad \rho y_\alpha = y_\alpha \rho, \quad \rho z_\alpha = z_\alpha \rho. \quad (5.95)$$

The Prokushkin–Vasiliev equations then read³⁶

³⁶ We have chosen to make the ρ -dependence manifest to have ρ -independent master fields.

$$d\mathcal{W} = \mathcal{W} \wedge \star \mathcal{W}, \quad (5.96a)$$

$$d\mathcal{B} = [\mathcal{W}, \mathcal{B}]_\star, \quad (5.96b)$$

$$\rho d\mathcal{S}_\alpha = [\mathcal{W}, \rho\mathcal{S}_\alpha]_\star, \quad (5.96c)$$

$$0 = [\mathcal{B}, \rho\mathcal{S}_\alpha]_\star, \quad (5.96d)$$

$$[\rho\mathcal{S}_\alpha, \rho\mathcal{S}_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 + \mathcal{B} \star k\kappa), \quad (5.96e)$$

where we defined the (inner) Kleinian $\kappa := e^{iyz}$. It satisfies

$$\kappa \star \kappa = 1, \quad \kappa \star f(y, z) \star \kappa = f(-y, -z). \quad (5.97)$$

The equations (5.96) are invariant with respect to the gauge transformations

$$\delta\mathcal{W} = d\xi - [\mathcal{W}, \xi]_\star, \quad (5.98a)$$

$$\delta\mathcal{B} = [\xi, \mathcal{B}]_\star, \quad (5.98b)$$

$$\rho\delta\mathcal{S}_\alpha = [\xi, \rho\mathcal{S}_\alpha]_\star, \quad (5.98c)$$

where the gauge parameter $\xi = \xi(y, z, \phi, \psi, k|x)$ depends on the same variables as the master fields. We call the first two Prokushkin–Vasiliev equations (5.96a)–(5.96b) *dynamical equations* because they are similar to the unfolded equations for the gauge and scalar fields, now generalised to the corresponding master fields.³⁷ The other three equations (5.96c)–(5.96e) will be referred to as *non-dynamical equations*.

We are interested here in bosonic higher-spin and matter fields, and this requires to apply a bosonic projection on the master fields,³⁸

$$\kappa \star \mathcal{W} \star \kappa = \mathcal{W}, \quad \kappa \star \mathcal{B} \star \kappa = \mathcal{B}, \quad \kappa \star \mathcal{S}_\alpha \star \kappa = -\mathcal{S}_\alpha. \quad (5.99)$$

As one can see from (5.97), this constrains \mathcal{W} and \mathcal{B} to even functions in y_α , z_α , and \mathcal{S}_α to an odd function. As in [47], we introduce a hermitian conjugation for the variables as

$$(y_\alpha)^\dagger = y_\alpha, \quad (z_\alpha)^\dagger = -z_\alpha, \quad \phi^\dagger = \phi, \quad \psi^\dagger = \psi, \quad k^\dagger = k, \quad \rho^\dagger = \rho, \quad (5.100)$$

and require the fields to satisfy the (anti-)hermiticity conditions

$$\mathcal{W}^\dagger = -\mathcal{W}, \quad \mathcal{S}_\alpha^\dagger = -\mathcal{S}_\alpha, \quad \mathcal{B}^\dagger = \mathcal{B}. \quad (5.101)$$

It is straightforward to verify that equations (5.96) are compatible with imposing these conditions. The gauge transformations (5.98) that are consistent with the (anti-)hermiticity conditions have gauge parameters that satisfy

$$\kappa \star \xi \star \kappa = \xi, \quad (\xi)^\dagger = -\xi. \quad (5.102)$$

³⁷ The first equation has the form of a zero-curvature condition, but the dependence of the master fields on the oscillators z_α leads to a non-trivial interaction between higher-spin and scalar fields.

³⁸ Without the projection, the master fields also encode fermionic fields, and the theory describes a higher-spin generalisation of $\mathcal{N} = 2$ supergravity [47].

As the master field \mathcal{B} contains the unfolded scalar as $\mathbf{C} = \mathcal{C} + \mathcal{C}\psi$, the condition $\mathcal{B}^\dagger = \mathcal{B}$ implies

$$(\Phi_\pm)^\dagger = \Phi_\mp, \quad (5.103)$$

where $\Phi_\pm = \Pi_\pm \mathcal{C}(y=0)$. The theory therefore contains one complex scalar field. On the other hand, the condition $\mathcal{W}^\dagger = -\mathcal{W}$ leads to a reality condition for the higher-spin fields. A further truncation of the theory to describe a single real scalar field has been also introduced in [47, 143].

5.2.2 Higher-spin background solutions

In this section we consider solutions of the Vasiliev equations (5.96) with a vanishing scalar, $\mathcal{C} = 0$. We make the ansatz of a constant value for the twisted scalar field by setting

$$\mathcal{B}^{(0)} = \nu \quad (5.104)$$

with a constant parameter ν (which will turn out to be the parameter ν of the deformed oscillators). This directly solves two of Vasiliev's equations, namely (5.96b) and (5.96d). The remaining equations are

$$d\mathcal{W} = \mathcal{W} \wedge \star \mathcal{W}, \quad (5.105a)$$

$$\rho d\mathcal{S}_\alpha = [\mathcal{W}, \rho \mathcal{S}_\alpha]_\star, \quad (5.105b)$$

$$[\rho \mathcal{S}_\alpha, \rho \mathcal{S}_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 + \nu k \varkappa). \quad (5.105c)$$

For $\nu = 0$, the last equation is solved by $\mathcal{S}_\alpha = z_\alpha$ (this follows directly from (5.91)). A solution for $\nu \neq 0$ is given by

$$\mathcal{S}_\alpha = \hat{z}_\alpha := z_\alpha + \nu w_\alpha k, \quad (5.106)$$

where

$$w_\alpha = (z_\alpha + y_\alpha) \int_0^1 dt t e^{it y z}. \quad (5.107)$$

Exercise 50. Check that (5.106) solves (5.105c).

One can also realise the deformed oscillators \hat{y}_α in a similar way by setting

$$\hat{y}_\alpha = y_\alpha + \nu w_\alpha \star k \varkappa. \quad (5.108)$$

Then

$$[\rho \hat{z}_\alpha, \rho \hat{z}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 + \nu k \varkappa), \quad (5.109a)$$

$$[\hat{y}_\alpha, \hat{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 + \nu k), \quad (5.109b)$$

$$[\hat{y}_\alpha, \rho \hat{z}_\beta]_\star = 0. \quad (5.109c)$$

For the background value (5.106) for \mathcal{S}_α , the equation (5.105b) becomes

$$[\mathcal{W}, \rho \hat{z}_\alpha]_\star = 0. \quad (5.110)$$

To solve this equation, we choose \mathcal{W} to be a combination of (star) products of \hat{y}_α , ϕ , and k . The remaining equation (5.105a) is then solved by any $\mathcal{W} = -\mathcal{A}(\hat{y}_\alpha, \phi, k)$ which solves the flatness condition

$$d\mathcal{A} + \mathcal{A} \wedge \star \mathcal{A} = 0, \quad (5.111)$$

which is precisely the equation of motion for a Chern–Simons field \mathcal{A} . When we project to a definite k -parity, for example by requiring $P_+\mathcal{A} = \mathcal{A}$, we can identify \mathcal{A} as a higher-spin gauge field for the algebra $\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ with $\lambda = \frac{1}{2}(\nu - 1)$. We conclude that every solution of Chern–Simons theory for this algebra gives rise to a solution of Vasiliev equations with constant (twisted) scalar $\mathcal{B} = \nu$.

5.2.3 Linear Perturbations

We shall now consider linear perturbations around the higher-spin background of the last section. To this end, we expand

$$\mathcal{S}_\alpha = \hat{z}_\alpha + \epsilon \mathcal{S}_\alpha^{(1)} + \epsilon^2 \mathcal{S}_\alpha^{(2)} + \dots, \quad (5.112a)$$

$$\mathcal{W} = -\mathcal{A} + \epsilon \mathcal{W}^{(1)} + \epsilon^2 \mathcal{W}^{(2)} + \dots, \quad (5.112b)$$

$$\mathcal{B} = \nu + \epsilon \mathcal{B}^{(1)} + \epsilon^2 \mathcal{B}^{(2)} + \dots, \quad (5.112c)$$

with a formal expansion parameter ϵ . Inserting this into the Prokushkin–Vasiliev equations (5.96), we find at linear order in ϵ :

$$0 = d\mathcal{W}^{(1)} + \mathcal{A} \wedge \star \mathcal{W}^{(1)} + \mathcal{W}^{(1)} \wedge \star \mathcal{A}, \quad (5.113a)$$

$$0 = d\mathcal{B}^{(1)} + [\mathcal{A}, \mathcal{B}^{(1)}]_\star, \quad (5.113b)$$

$$[\mathcal{W}^{(1)}, \rho \hat{z}_\alpha]_\star = \rho d\mathcal{S}_\alpha^{(1)} + [\mathcal{A}, \rho \mathcal{S}_\alpha^{(1)}]_\star, \quad (5.113c)$$

$$[\mathcal{B}^{(1)}, \rho \hat{z}_\alpha]_\star = 0, \quad (5.113d)$$

$$[\rho \hat{z}_\alpha, \rho \mathcal{S}_\alpha^{(1)\alpha}]_\star = -2i \mathcal{B}^{(1)} \star k \varkappa. \quad (5.113e)$$

As discussed in [47], the general solution of the equation (5.113d) for the scalar master field is any function of \hat{y}_α , ϕ , ψ , k , and the spacetime coordinates,

$$\mathcal{B}^{(1)} = \mathbf{C}^{(1)}(\hat{y}, \phi, \psi, k|x). \quad (5.114)$$

From (5.113b) we then recover the free equation of motion (5.88b) for the unfolded scalar $\mathbf{C}^{(1)}$ in a higher-spin background,

$$D\mathbf{C}^{(1)} = 0. \quad (5.115)$$

5.2.4 Subtleties of the twisted sector

As we have shown above, the linearised Vasiliev equations reproduce the expected equations for the unfolded scalar. In the analysis of the remaining equations at the linear level, we shall find that the twisted part of the gauge field is sourced by the scalar field. A priori this would be problematic since one would wish to decouple the unphysical twisted sector. It was shown in [144], however, that one can remove this source term by a field redefinition as we review in the following.

To simplify the discussion, from now on we set $\nu = 0$. The variable k then is not necessary any more: we can redefine $\mathcal{B} \rightarrow \mathcal{B}k$ and take all fields otherwise to be independent of k . Furthermore, the variable ρ drops out of the equations, and we have $\hat{z}_\alpha = z_\alpha$. For bosonic fields (satisfying (5.102)), the linear equations then simplify to

$$D\mathcal{W}^{(1)} = 0, \quad (5.116a)$$

$$D\mathcal{B}^{(1)} = 0, \quad (5.116b)$$

$$2i\partial_\alpha^z \mathcal{W}^{(1)} = D\mathcal{S}_\alpha^{(1)}, \quad (5.116c)$$

$$\partial_\alpha^z \mathcal{B}^{(1)} = 0, \quad (5.116d)$$

$$\partial_\alpha^z \mathcal{S}^{(1)\alpha} = \mathcal{B}^{(1)} \star \varkappa, \quad (5.116e)$$

where we also used (5.92b) to replace the commutator with z_α by a derivative.

The non-dynamical equations (5.116c)-(5.116e) are first-order differential equations with respect to z , and they can be solved using the following general results³⁹

$$\partial_\alpha^z f^\alpha(y, z) = g(y, z) \quad \Longrightarrow \quad f_\alpha(y, z) = \partial_\alpha^z \epsilon(y, z) + z_\alpha \Gamma_1 \langle g(y, z) \rangle, \quad (5.117a)$$

$$\partial_\alpha^z f(y, z) = g_\alpha(y, z) \quad \Longrightarrow \quad f(y, z) = \epsilon(y) + z^\alpha \Gamma_0 \langle g_\alpha(y, z) \rangle. \quad (5.117b)$$

Here, $\Gamma_n \langle \bullet \rangle$ stands for *homotopy integrals* defined as

$$\Gamma_n \langle f \rangle (z) := \int_0^1 dt t^n f(tz), \quad (5.118)$$

and $\epsilon(y, z)$ and $\epsilon(y)$ are arbitrary functions of the indicated arguments.

Therefore, the solutions for the non-dynamical equations are given by (where we suppress the dependence on ϕ, ψ and x)

³⁹ The second equation (5.117b) only has a solution if the following compatibility condition is satisfied,

$$\partial_z^\alpha g_\alpha(y, z) = 0,$$

which holds in the case of (5.116c) to which we are going to apply this solution.

$$\mathcal{B}^{(1)} = \mathbf{C}^{(1)}(y), \quad (5.119a)$$

$$\mathcal{S}_\alpha^{(1)} = \partial_\alpha^z \epsilon^{(1)}(y, z) + z_\alpha \Gamma_1 \langle \mathbf{C}^{(1)} \star \varkappa \rangle, \quad (5.119b)$$

$$\mathcal{W}^{(1)} = -\mathbf{A}^{(1)}(y) - \frac{i}{2} z^\alpha \Gamma_0 \langle D \mathcal{S}_\alpha^{(1)} \rangle. \quad (5.119c)$$

Note that these equations fully determine the z_α -dependence of the master fields. This statement also holds at higher orders in perturbation theory.

We now impose the following gauge condition on the master field \mathcal{S}_α ,

$$z^\alpha \mathcal{S}_\alpha = 0, \quad (5.120)$$

which is usually referred to as *Schwinger–Fock gauge*.⁴⁰ The Schwinger–Fock gauge implies that the homogeneous solution $\partial_\alpha^z \epsilon^{(1)}(y, z)$ in (5.119b) vanishes. This is because $z^\alpha \partial_\alpha^z$ is the z_α -number operator which implies that in the Schwinger–Fock gauge $\epsilon^{(1)}(y, z) = \epsilon^{(1)}(y)$ and therefore

$$\partial_\alpha^z \epsilon^{(1)}(y, z) = 0. \quad (5.121)$$

To determine the residual gauge transformations, we also expand the gauge parameter starting from linear order,

$$\xi = \epsilon \xi^{(1)} + \dots \quad (5.122)$$

The first order transformations are then

$$\delta \mathcal{W}^{(1)} = D \xi^{(1)}, \quad (5.123a)$$

$$\delta \mathcal{B}^{(1)} = 0, \quad (5.123b)$$

$$\delta \mathcal{S}_\alpha^{(1)} = 2i \partial_\alpha^z \xi^{(1)}, \quad (5.123c)$$

where we used the restrictions (5.102) on ξ from the bosonic projection. When we compare with the gauge-fixing condition (5.121) we find

$$z^\alpha \delta \mathcal{S}_\alpha^{(1)} = 2i z^\alpha \partial_\alpha^z \xi^{(1)}(y, z) \stackrel{!}{=} 0 \quad \Longrightarrow \quad \xi^{(1)}(y, z) = \xi^{(1)}(y). \quad (5.124)$$

It follows that the residual gauge freedom preserving the Schwinger–Fock gauge is given at first order by z -independent gauge parameters $\xi^{(1)}(y)$.

Plugging the solutions (5.119) of the non-dynamical equations in the dynamical equations (5.116a) and (5.116b), we obtain in Schwinger–Fock gauge

$$D \left(\mathcal{E}^{(1)}(y) \psi + \tilde{\mathcal{E}}^{(1)}(y) \right) = 0, \quad (5.125a)$$

$$D \left(\mathcal{A}^{(1)}(y) + \tilde{\mathcal{A}}^{(1)}(y) \psi \right) = -\frac{i}{2} D \left(z^\alpha \Gamma_0 \left\langle D \left(z_\alpha \Gamma_1 \langle \mathbf{C}^{(1)} \star \varkappa \rangle \right) \right\rangle \right). \quad (5.125b)$$

⁴⁰ In some parts of the literature, the Schwinger–Fock gauge is also known as the Vasiliev gauge.

Notice that the fields still depend on ϕ and the spacetime variable x . The left-hand side of (5.125b) is z -independent, and therefore also its right-hand side has to share this property. This implies that we can evaluate it for $z = 0$ as all z -dependent terms have to cancel out anyway. Note however that the star product (5.90) does not commute with setting the z_α -oscillators to zero. Therefore, we need to first evaluate all star products and only afterwards set all z -dependent factors to zero.

To simplify the computations, we now restrict the background to AdS, $\mathcal{A} = \mathcal{A}^{AdS}$. After some algebra, one obtains

$$\begin{aligned} D^{AdS} \left(z^\alpha \Gamma_0 \left\langle D^{AdS} \left(z_\alpha \Gamma_1 \langle \mathbf{C}^{(1)} \star \mathcal{Z} \rangle \right) \right\rangle \right) \Big|_{z=0} \\ = \frac{1}{8} E^{\alpha\beta} (y_\alpha + i\partial_\alpha^u)(y_\beta + i\partial_\beta^u) \mathcal{C}^{(1)}(u) \psi \Big|_{u=0}, \end{aligned} \quad (5.126)$$

where we have defined

$$E^{\alpha\beta} = \bar{e}^\alpha_\gamma \wedge \bar{e}^{\beta\gamma}. \quad (5.127)$$

It was shown by Vasiliev [144] that one can remove the source term (5.126) by a field redefinition of the twisted one-form $\mathcal{A}^{(1)} \rightarrow \mathcal{A}^{(1)} + M$ with

$$M = -\frac{i}{8} \phi \bar{e}^{\alpha\beta} \int_0^1 dt (t^2 - 1) (y_\alpha + it^{-1} \partial_\alpha^y)(y_\beta + it^{-1} \partial_\beta^y) \mathcal{C}^{(1)}(ty). \quad (5.128)$$

After performing this field redefinition, we obtain the equations of motion

$$D^{AdS} \left(\mathcal{C}^{(1)}(y) \psi + \tilde{\mathcal{C}}^{(1)}(y) \right) = 0, \quad (5.129a)$$

$$D^{AdS} \left(\mathcal{A}^{(1)}(y) + \tilde{\mathcal{A}}^{(1)}(y) \psi \right) = 0, \quad (5.129b)$$

which indeed coincide with the free unfolded equations (5.88) introduced earlier. Therefore, the Prokushkin–Vasiliev equations provide us with a non-linear theory of higher-spin gauge fields coupled to a complex scalar field (as well as additional twisted fields). Note that we can choose vanishing solutions for the twisted fields, i.e. $\tilde{\mathcal{C}}^{(1)} = \tilde{\mathcal{A}}^{(1)} = 0$, as their equations of motion do not contain any source terms (after we have performed the field redefinition (5.128)).

Exercise 51.* Check that (5.126) holds. Consider the case $\tilde{\mathcal{C}}^{(1)} = 0$ and neglect all terms involving the background spin connection for simplicity.

Exercise 52.* Show that the field redefinition (5.128) indeed removes the source term (5.126).

5.2.5 Comments on higher orders

In the last section, we have seen how to perturbatively extract equations of motion from Vasiliev equations (5.96). At first order, we have verified in this way that the linearised equations of motion describe a free scalar field and free higher-spin fields

(together with a twisted sector that can however be decoupled at this order). In principle, one can now go on and compute the nonlinear contributions to the equations of motion. Field redefinitions as the one we discussed for the twisted sector at linear order become necessary at higher orders also in the untwisted sector.

This becomes first apparent at second order when one analyses the backreaction of the matter fields to the higher-spin fields. The naive computation results in a non-local backreaction, and a non-local field redefinition is needed to bring the backreaction to the expected local form [145]. On the other hand, one cannot allow for arbitrary non-local field redefinitions because one could otherwise remove all interactions [47]. The necessity of a particular non-local field redefinition can be interpreted by saying that the identification of the physical field inside the master fields has to be modified (for example [146] by using a shifted version of the homotopy integrals (5.118)). It is an open question what the correct prescription is at all orders. These issues might be related to the non-localities that appear in four dimensions, for a discussion of this topic see, e.g., [147–151].

Let us also stress that these issues are relevant to determine the status of the Prokushkin–Vasiliev model as a candidate bulk dual within minimal model holography. Indeed, it is unclear what the twisted sector could correspond to on the CFT side. As we mentioned, accepting certain non-local field redefinitions, the twisted sector can be decoupled up to second order, but it is not known if a similar scenario applies also to all orders. To conclude, we mention that, while Vasiliev’s equations in three and four dimensions have been originally defined using oscillator techniques that are peculiar to these dimensions, interacting equations of motion have been later defined in a language that applies to arbitrary spacetime dimensions [10]. A priori, these equations of motion do not involve a twisted sector, so that their three-dimensional instance could provide an alternative proposal for a coupling of matter to higher spin fields.

6 Summary and further developments

These lecture notes focus on three-dimensional models describing the interactions of relativistic massless higher-spin fields and their couplings to scalar matter, mainly on an anti-de Sitter background, as well as on their holographic description in terms of \mathscr{W}_N -minimal models. To this end, an introduction to quantum \mathscr{W} -symmetries and to their realisation in conformal field theory is also provided. The selected material has been chosen so as to provide the bases to appreciate the key ideas underlying minimal-model holography [31], which we review in section 4.7.

In section 2, we first review the Chern–Simons formulation of three-dimensional gravity, and we then show how it can be naturally extended to describe higher spins by enlarging the gauge algebra. We focus on extensions involving $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ gauge algebras, which describe fields of spin $2, 3, \dots, N$ on AdS_3 , and on their $N \rightarrow \infty$ limit, involving two copies of the infinite-dimensional $\mathfrak{hs}[\lambda]$ algebra, whose construction we review in section 2.2.3.

In section 3, we identify the asymptotic symmetries of the previous field theories, starting from the example of gravity and then moving to the higher-spin case, showing that they are given by non-linear \mathscr{W} -algebras. We study asymptotic symmetries using Hamiltonian techniques, and we stress the link with the mathematical literature on the Drinfeld–Sokolov reduction. In particular, in section 3.5 we compute the asymptotic symmetries of $\mathfrak{sl}(N, \mathbb{R})$ Chern–Simons theories in a basis in which their algebra admits at most quadratic non-linearities and in which explicit expressions for the structure constants can be derived. We achieve this goal by encoding the relevant conditions into suitable pseudodifferential operators, following the standard treatment in the mathematical literature. We then briefly discuss how these techniques can be adapted to deal with the infinite-dimensional $\mathfrak{hs}[\lambda]$ case.

In section 4, we deal with the quantum version of the \mathscr{W} -algebras introduced in the previous section, discussing the normal-ordering subtleties brought by the non-linearities in the commutators. We first present the example of the \mathscr{W}_3 -algebra, and we then discuss generic \mathscr{W}_N -algebras using their realisation in terms of the Miura transform, which is the quantum counterpart of the classical construction of section 3.5. This allows us to eventually consider the $N \rightarrow \infty$ limit, resulting in the $\mathscr{W}_\infty[\lambda]$ family of infinite-dimensional non-linear algebras, labelled by the parameter λ and by the central charge. We discuss the emergence of a remarkable triality symmetry relating different values of these parameters, and we introduce conformal field theories admitting the previous symmetry algebras. We begin by defining \mathscr{W}_N -minimal models, whose global symmetries are given by \mathscr{W}_N -algebras, and we then define their large- N limit that enters minimal-model holography, highlighting the crucial role played by triality. We argue that the resulting CFT should admit a holographic description in terms of a higher-spin theory coupled to scalar matter, and we discuss the criticalities of this proposal.

The Chern–Simons theories that we consider in section 2 only describe massless fields of spin $s \geq 2$. Coupling them to matter introduces additional difficulties that we discuss in section 5. We then present the Prokushkin–Vasiliev model [47], that provides an explicit example of such a coupling. It does so using an approach, dubbed unfolded formalism, in which the matter equations of motion are reformulated in a first order form and using an infinite number of auxiliary fields, encoding the derivatives of the Klein–Gordon field. We first review the peculiarities of this formulation of the dynamics for a free Klein–Gordon field on AdS_3 , and then we show how to describe its propagation on a higher-spin background. We conclude by reviewing how one can include scalar backreaction along the lines of [47]. The latter reference relies on an oscillator realisation of the $\mathfrak{hs}[\lambda]$ algebras (introduced in section 2.2.3) that we also detail along the way.

Most of the material reviewed here already found various applications that go well beyond the study of minimal-model holography. For this reason, we close these lecture notes with a quick overview of selected further developments related to higher spins in three dimensions.

Higher-spin black holes

Shortly after the observation that higher-spin gauge theories in three spacetime dimensions have extended asymptotic symmetries [33,34], higher-spin generalisations of the BTZ black hole [21,22] have been constructed [41]. Since higher-spin gauge transformations can map a given metric into another one with a different causal structure (see, e.g., [95,152]), the proposed definition of higher-spin black holes relies on their thermal properties rather than on the presence of an event horizon. In practice, these solutions are first defined in Euclidean spacetime, assuming the same torus topology as that of a BTZ black hole and demanding regularity of the Chern–Simons connection in its interior, and then continued to Lorentzian signature; see, e.g., [44–46] for a review. This somehow indirect way of defining higher-spin black holes is well adapted to look for holographic counterparts of these classical solutions [153–155], but introduces various subtleties. For instance, the actual dependence of the black-hole entropy on higher-spin charges have been debated in the literature [156–159,42], before leading to a finer characterisation of these solutions [43,160]. Possible generalisations of the usual metric-based definition of a Lorentzian causal structure have also been discussed in [161,77].

Semiclassical methods to compute observables in CFTs with \mathscr{W} -symmetry

Independently of the details of a precise holographic duality, minimal-model holography suggests the option to compute several, possibly non-local, observables in conformal field theories with extended \mathscr{W} -symmetries by means of semiclassical computations in higher-spin Chern–Simons theories. This option has been explored, e.g., in [162–169] often relying on the following logic: once a holographic prescription to compute a CFT observable by means of a semiclassical computation in gravity is known, in three dimensions one can first reformulate it in a Chern–Simons language and then naturally extend it to higher-spin theories.

Higher spins and strings

As mentioned in the introduction, it is expected that string theory in the tensionless limit develops higher-spin symmetries; see, e.g., [6,7] for a review. This has been made precise for strings on $\text{AdS}_3 \times S^3 \times T^4$, where it was shown that the tensionless limit has a CFT dual given by a symmetric orbifold of T^4 , which indeed shows a large symmetry algebra of global higher-spin charges that should signal the presence of higher-spin gauge fields in the bulk [136,170,171]. The latter non-linear algebra, whose wedge algebra has been dubbed as higher-spin square, contains a $\mathcal{N} = 4$ generalisation of the algebra $\mathscr{W}_\infty[\lambda]$ that we discussed in these notes, but it is much bigger. A higher-spin theory displaying this symmetry was built from scratch in [172].

Three-dimensional higher-spin theories in flat space

In the very first paper on the subject [26], it was already noticed that the Chern–Simons formulation of higher-spin gauge theories does not require a cosmological constant. In the last years, three-dimensional gravity proved to be an interesting setup where to test various ideas on flat space holography and this naturally led to explore higher-spin gauge theories on three-dimensional Minkowski space; see, e.g., [173] for a review. Their asymptotic symmetries have been characterised in [174–176] and the representation theory of the resulting algebras has been studied in [177, 178]. Coupling these models to matter is instead much subtler than in AdS; see, e.g., [176]. The main reason that allows to easily construct higher-spin theories in three-dimensional Minkowski space is that the higher-spin algebras discussed in section 2.2.3 admit contractions that can be neatly interpreted as an appropriate gauge algebra for a Chern–Simons theory in Minkowski space. These steps are much subtler in four dimensions due to the presence of additional generators; on the other hand, the structure of three-dimensional flat-space higher-spin algebras recently suggested a path to introduce similar contractions in four dimensions too [71].

Metric-like formulation

In these notes we have reviewed Chern–Simons higher-spin gauge theories, which are based on a first-order frame-like formulation of the dynamics. In principle, such theories can also be formulated as second-order theories in terms of metric-like Fronsdal fields, adding interaction vertices to the free theories reviewed in section 2.2.1. In [158, 179, 87] it was shown how one can perturbatively obtain such field theories from the Chern–Simons formulation in a weak field expansion, but similarly to gravity they involve interaction vertices of arbitrarily high order in the fields (see also [180, 139] for an alternative proposal to rewrite Chern–Simons actions in terms of metric-like fields involving additional auxiliary fields). Alternatively, one can classify perturbatively all possible interactions between higher-spin gauge fields directly within the metric-like formulation [181, 182]. This approach confirms – in agreement with the Chern–Simons formulation – that cubic vertices determine all interactions of higher-spin gauge fields, in the sense that no further independent coupling constants can enter the interacting action [183, 51]. Although this perturbative approach is less efficient in describing massless fields compared to the Chern–Simons formulation, let us stress that it provides an alternative and conceptually straightforward setup where to analyse and classify possible matter couplings, with potential applications in the quest for alternative holographic models with respect to the Prokushkin–Vasiliev model.

Other higher-spin theories in three dimensions

We focused on massless bosonic fields and their couplings to scalar matter, but other classes of higher-spin theories have been studied too in three dimensions. Massless fermions can be described employing Chern–Simons theories with gauge superalgebras, in the same spirit of the Chern–Simons formulation of three-dimensional supergravity; see, e.g., [28,90,184–187]. Supersymmetric extensions of the Prokushkin–Vasiliev bosonic model reviewed here have also been introduced [47], as well as supersymmetric extensions of minimal-model holography [188–190]. On constant-curvature backgrounds one can also introduce non-propagating three-dimensional generalisations of partially-massless theories, see e.g. [191,51], while on a Minkowski background one can consider similar theories with exotic indecomposable spectra [192]. All these classes of non-propagating fields admit interactions that can always be rewritten in a Chern–Simons form [51]. Massive higher-spin fields, instead, do propagate local degrees of freedom in three dimensions too [18]; this does not allow for a compact description in terms of Chern–Simons actions, but their interaction vertices have been studied perturbatively, e.g., in [193, 194]. Topologically-massive higher-spin theories have been also considered, e.g., in [195–201] and massive higher-spin supermultiplets have been considered, e.g., in [202–204]. Fractional and continuous spin representations and their field theory realisations have been explored too [205–207].

Non-relativistic three-dimensional higher-spin theories

We already mentioned that the higher-spin algebras of section 2.2.3 allow for contractions that can be used to define higher-spin theories in Minkowski space via the Chern–Simons formulation. Other contractions are also possible, and they have been shown to lead to Chern–Simons theories that can be interpreted as non-relativistic field theories [208–214], with potential applications in the study of two-dimensional condensed matter systems. Higher-rank tensorial fields in $2 + 1$ dimensions have also been employed in effective theories describing various features of the fractional quantum Hall effect; see, e.g., [215–217].

Acknowledgements We thank G. Lucena Gómez and P. Kessel for collaboration on this project at an initial stage. We also thank M. Henneaux, P. Kessel, O. Krüger, K. Mkrtchyan, S. Pfenninger, T. Prochazka, J. Raeymaekers and S. Theisen for collaboration on the topics reviewed in these lecture notes as well as G. Barnich, N. Boulanger, A. Delfante, J. Fischer, D. Francia, M. Gaberdiel, H. González, R. Lomartire, T. Nutma, S. Pekar, M. Riegler, B. Oblak, A. Sagnotti, E. Schnabel, E. Skvortsov, and M. Taronna for discussions. AC is a research associate of the Fund for Scientific Research – FNRS. His work was partially supported by FNRS through the grants No. F.4503.20, T.0022.19 and T.0047.24. We acknowledge the Galileo Galilei Institute for Theoretical Physics in Florence for the opportunity to present a part of the material collected in these lecture notes.

A Frame-like formulation in arbitrary dimension

In the main text, we have presented the frame-like formulation of higher-spin fields in three spacetime dimensions with the action (2.39). In this appendix, we contrast it with the frame-like formulation in generic spacetime dimensions [218, 219] (see also [2] for a review).

In arbitrary spacetime dimensions, a frame-like free action can be obtained by introducing a vielbein-like field $e^{a_1 \cdots a_{s-1}}$ and a spin-connection-like field $\omega^{b, a_1 \cdots a_{s-1}}$. Both fields are traceless in all the Lorentz indices, fully symmetric in a_i and the spin connection obeys the irreducibility condition $\omega^{(a_1, a_2 \cdots a_s)} = 0$, which will play a key role in the proof of most of the identities discussed in the following. When $D = 3$, this auxiliary field can be dualised as $\omega^{a(s-1)} \sim \omega^{c, b(a_1 \cdots a_{s-2} \epsilon^{a_{s-1})}_{bc}}$ to recover the field we introduced in (2.35).

The frame-like action is given by⁴¹

$$S = \frac{1}{16\pi G} \int K_{abc} \wedge \left(e^a_{q_{s-2}} \wedge \nabla \omega^{b, cq_{s-2}} - \frac{1}{2(D-2)(s-1)} \bar{e}^a \wedge \omega^{b, q_{s-1}} \wedge \omega^{c, q_{s-1}} - \frac{1}{2(D-2)} \bar{e}^a \wedge \omega_{d, q_{s-2}}{}^b \wedge \omega^{d, cq_{s-2}} + \frac{s(D+s-4)}{2(D-2)\ell^2} \bar{e}^a \wedge e^b_{q_{s-2}} \wedge e^{cq_{s-2}} \right), \quad (\text{A.1})$$

where

$$K_{abc} := \epsilon_{p_1 \cdots p_{D-3} abc} \bar{e}^{p_1} \wedge \cdots \wedge \bar{e}^{p_{D-3}}, \quad (\text{A.2})$$

and where an index with a subscript denotes a group of symmetrised indices. For instance, $e^{a_{s-1}} := e^{a_1 \cdots a_{s-1}}$. Moreover, repeated covariant or contravariant indices denote a symmetrisation. For instance, $A^a B^a := \frac{1}{2} (A^{a_1} B^{a_2} + A^{a_2} B^{a_1})$. This action is invariant under the gauge transformations

⁴¹ For some specific values of D , as for instance $D = 3$, additional terms may be written. Notice also that here we presented the free action as, e.g., in [220]. An equivalent presentation of this action, used in [218, 2], can be obtained by rewriting the terms quadratic in the spin connection thanks to the identity

$$\begin{aligned} 0 &= (D+1) \bar{e}^{[d} \wedge \omega_{d, a[q_{s-2}]} \wedge \omega^{b, c}_{q_{s-2}} \wedge \underbrace{\bar{e}^{p_1} \wedge \cdots \wedge \bar{e}^{p_{D-3}}}_{K_{abc}} \epsilon_{p_1 \cdots p_{D-3} abc} \\ &= (D-2) \bar{e}^d \wedge \omega_{d, aq_{s-2}} \wedge \omega^{b, c}_{q_{s-2}} \wedge K_{abc} \\ &\quad + \bar{e}^a \wedge \left(\omega_{d, bq_{s-2}} \wedge \omega^{d, c}_{q_{s-2}} + \frac{1}{s-1} \omega^{b, q_{s-1}} \wedge \omega^{c, q_{s-1}} \right) \wedge K_{abc}, \end{aligned}$$

where the first line vanishes because of the antisymmetrisation over $D+1$ indices and the two terms in the third line are those that enter (A.1).

$$\delta e^{a_{s-1}} = \nabla \xi^{a_{s-1}} + \bar{e}_b \Lambda^{b, a_{s-1}}, \quad (\text{A.3a})$$

$$\begin{aligned} \delta \omega^{b, a_{s-1}} = & \nabla \Lambda^{b, a_{s-1}} + \bar{e}_c \theta^{bc, a_{s-1}} + \frac{(s-1)(D+s-4)}{(D-2)\ell^2} (\bar{e}^b \xi^{a_{s-1}} - \bar{e}^a \xi^{ba_{s-2}}) \\ & - \frac{(s-1)(s-2)}{(D-2)\ell^2} (\eta^{ab} \bar{e}_c \xi^{ca_{s-2}} - \eta^{aa} \bar{e}_c \xi^{ba_{s-3}}), \end{aligned} \quad (\text{A.3b})$$

where we fixed conventionally the relative factor in the first variation. The terms in η^{ab} , here and in the following gauge variations, just implement a traceless projection, in agreement with the properties of the fields that are varied.

For $s = 2$, the action (A.1) reduces to the linearisation of the Einstein-Hilbert action. For arbitrary values of s , the previous action contains all possible terms that are quadratic in the higher-spin fields, while the relative coefficients are fixed by demanding the presence of a gauge symmetry of the form $\delta e^{a_{s-1}} = \nabla \xi^{a_{s-1}} + \dots$ and $\delta \omega^{b, a_{s-1}} = \nabla \Lambda^{b, a_{s-1}} + \dots$. These gauge transformations involve parameters with the same tensorial structure as the fields and generalise the symmetry under local translations and local Lorentz transformations of linearised gravity. This requirement fixes uniquely the coefficients in the action, but notice that the result displays an additional symmetry generated by a parameter $\theta^{bc, a_1 \dots a_{s-1}}$ which is fully traceless and satisfies the irreducibility condition $\theta^{b(a_1, a_2 \dots a_s)} = 0$.

In the following, we shall show that the equations of motion implied by the action (A.1) are equivalent to the Fronsdal equations on AdS of (2.33). To this end, it will be convenient to rewrite them in terms of (linearised) higher-spin curvatures, that are gauge-invariant two-forms linear in the fields. They read

$$\mathcal{F}^{a_{s-1}} = \nabla e^{a_{s-1}} + \bar{e}_b \wedge \omega^{b, a_{s-1}}, \quad (\text{A.4a})$$

$$\begin{aligned} \mathcal{R}^{b, a_{s-1}} = & \nabla \Lambda^{b, a_{s-1}} + \frac{(s-1)(D+s-4)}{(D-2)\ell^2} (\bar{e}^b \wedge e^{a_{s-1}} - \bar{e}^a \wedge e^{ba_{s-2}}) \\ & - \frac{(s-1)(s-2)}{(D-2)\ell^2} (\eta^{ab} \bar{e}_c \wedge e^{ca_{s-2}} - \eta^{aa} \bar{e}_c \wedge e^{bca_{s-3}}) + \bar{e}_c \wedge \mathcal{Q}^{bc, a_{s-1}}, \end{aligned} \quad (\text{A.4b})$$

where in the following we shall often denote $\mathcal{F}^{a_{s-1}}$ as the torsion two-form. Notice that asking for gauge invariance with respect to $\Lambda^{b, a_1 \dots a_{s-1}}$ requires to introduce a further connection that transforms as

$$\begin{aligned} \delta \Omega^{bb, a_{s-1}} = & \nabla \theta^{bb, a_{s-1}} - \frac{2(s-2)(D+s-3)}{D\ell^2} \left(\bar{e}^b \Lambda^{b, a_{s-1}} - \frac{s-1}{s-2} \bar{e}^a \Lambda^{b, ba_{s-2}} \right) \\ & + \frac{2(s-1)(s-2)}{D(D-2)\ell^2} \left(\frac{D+s-3}{s-1} \eta^{bb} \bar{e}_c \Lambda^{c, a_{s-1}} - \frac{D+2s-6}{s-2} \eta^{ab} \bar{e}_c \Lambda^{c, ba_{s-2}} \right. \\ & \left. + \frac{(s-3)(D-2)}{s-2} \eta^{ab} \bar{e}_c \Lambda^{b, ca_{s-2}} - (D-2) \eta^{aa} \bar{e}_c \Lambda^{b, bca_{s-3}} + \eta^{aa} \bar{e}_c \Lambda^{c, bba_{s-3}} \right) + \dots \end{aligned} \quad (\text{A.5})$$

The procedure iterates: building a gauge-invariant curvature for $\Omega^{b_2, a_{s-1}}$ would require to introduce yet another connection that would naturally transform as $\delta\Omega^{b_3, a_{s-1}} = \nabla\theta^{b_3, a_{s-1}} + \dots$ and the new gauge parameter will appear in the terms that we omitted in (A.5). Proceeding in this way one would introduce the additional auxiliary fields $\Omega^{b_r, a_{s-1}}$ with $2 \leq r \leq s-1$ that enter Vasiliev's equations in four and higher dimensions (see, e.g., [221, 219]), while this step will not be relevant in the ensuing discussion.

We can now move to study the equations of motion. The variation of the action with respect to the spin connection gives

$$\delta_\omega S = \frac{1}{16\pi G} \int K_{abc} \wedge (\mathcal{T}^a_{q_{s-2}} \wedge \delta\omega^{b, cq_{s-2}} + \mathcal{K}^{abc}), \quad (\text{A.6})$$

where the three-form \mathcal{K}^{abc} reads

$$\begin{aligned} \mathcal{K}^{abc} &= \bar{e}_f \wedge \omega^{f, a}_{q_{s-2}} \wedge \delta\omega^{b, cq_{s-2}} \\ &+ \frac{1}{D-2} \left(\frac{1}{(s-1)} \bar{e}^a \wedge \omega^{b, q_{s-1}} \wedge \delta\omega^{c, q_{s-1}} + \bar{e}^a \wedge \omega_{d, q_{s-2}}{}^b \wedge \delta\omega^{d, cq_{s-2}} \right). \end{aligned} \quad (\text{A.7})$$

Its contribution to the variation vanishes thanks to

$$K_{abc} \wedge \mathcal{K}^{abc} = -3!(D-3)! \delta_{abc}^{mnr} \mathcal{K}_{mnr; abc} \bar{e} d^D x = 0, \quad (\text{A.8})$$

where \bar{e} denotes the determinant of the background vielbein, $\delta_{abc}^{mnr} = \delta^m_{[a} \delta^n_b \delta^r_{c]}$, and we introduced the form components as $\mathcal{K}^{abc} = \bar{e}^m \wedge \bar{e}^n \wedge \bar{e}^r \mathcal{K}_{mnr; abc}$. The last identity can be verified by substituting the explicit expression (A.7).

The equation of motion for the spin connection therefore reads

$$K_{abq} \wedge \mathcal{T}_{q_{s-2}}{}^a = 0 \quad (\text{A.9})$$

since this expression already has the same irreducibility properties under permutations of its indices as the spin connection. Taking the wedge product with another background vielbein and introducing the components of the two-form \mathcal{T} as in (A.8) leads to

$$\mathcal{T}_{aq; bq_{s-2}} = 0. \quad (\text{A.10})$$

In spite of the symmetrisation over the indices q , this expression suffices to conclude that the full torsion tensor vanishes, since its components can be rewritten as

$$\mathcal{T}_{ab; q_{s-1}} = \sum_{k=1}^{s-1} \mathcal{T}_{a(b; q_1 \dots \widehat{q_k} \dots q_{s-1}) q_k} - (s-2) \mathcal{T}_{aq; bq_{s-2}} = 0. \quad (\text{A.11})$$

The variation of the action with respect to the vielbein reads

$$\begin{aligned}
\delta_e \mathcal{S} &= \frac{1}{16\pi G} \int K_{abq} \wedge \left(\nabla \omega^{a,b}_{q_{s-2}} + \frac{s(D+s-4)}{(D-2)\ell^2} \bar{e}^a \wedge e^b_{q_{s-2}} \right) \wedge \delta e^{q_{s-1}} \\
&= \frac{1}{16\pi G} \int K_{abq} \wedge \left(\mathcal{R}^{a,b}_{q_{s-2}} - \bar{e}_c \wedge \Omega^{ac,b}_{q_{s-2}} \right) \wedge \delta e^{q_{s-1}}.
\end{aligned} \tag{A.12}$$

When we rewrite this in components, we obtain

$$\delta_e \mathcal{S} = \frac{1}{16\pi G} \int d^D x \bar{e} \delta_{abq}^{mnr} \left(\mathcal{R}_{mn};^{a,b}_{q_{s-2}} - \delta_{[m}^c \Omega_{n];c}^{a,b}_{q_{s-2}} \right) \delta e_{r; q_{s-1}}. \tag{A.13}$$

Notice however that the functional derivative with respect to the hook component⁴² of $e^{r; q_{s-1}}$ vanishes due to the invariance under local Lorentz transformations of the action, which implies that the latter actually does not depend on this component. Therefore, it is enough to consider the variation of the action under variations $\delta e^{q; q_{s-1}}$. The symmetrisation eliminates the terms in Ω , so that one can rewrite the equations of motion in terms of \mathcal{R} even if Ω does not enter the action (A.1). This results in the equations of motion

$$\eta_{qq} \mathcal{R}_{ab};^{a,b}_{q_{s-2}} - \mathcal{R}_{aq};^a_{q_{s-1}} + \mathcal{R}_{aq; q}^a_{q_{s-2}} = 0. \tag{A.14}$$

One can also check explicitly that Ω drops out of this particular projection of the curvature tensor, which displays the same symmetries under permutation of its indices as the field we varied. Taking a trace of (A.14) and substituting back, we finally arrive at the equations of motion

$$\mathcal{T}_{mn; q_{s-1}} = 0, \tag{A.15a}$$

$$\mathcal{R}_{aq};^a_{q_{s-1}} = 0. \tag{A.15b}$$

The torsion equation (A.15a) allows one to express the following part of the spin connection in terms of the vielbein:

$$\omega_{[m; n]; q_{s-1}} = \bar{\nabla}_{[m} e_{n]; q_{s-1}}. \tag{A.16}$$

This is actually the only part of the spin connection entering the equation of motion (A.15b),⁴³ that can be rewritten as

⁴² By hook component we mean the projection of the vielbein on its irreducible component satisfying $e^{q; q_{s-1}} = 0$.

⁴³ This is the case because $\omega_{[m; n]; q_{s-1}}$ is actually the only part of the spin connection entering the action (A.1). Indeed, $\omega_{m; n; q_{s-1}}$ can be decomposed into its symmetric and antisymmetric components in the indices m and n , and the former can be further decomposed as

$$\omega_{m; m; q_{s-1}} = 2\omega_{[q; m]; m q_{s-2}} + \frac{s-2}{s-1} \left(\omega_{m; m; q_{s-1}} + \frac{s-1}{2} \omega_{q; q; q_{s-3} m m} \right).$$

The first term can be expressed in terms of the vielbein using again (A.16), while the combination between parentheses corresponds to the $\{s-1, 2\}$ Young projection of $\omega_{m; n; q_{s-1}}$. As such, it can be gauged away using the Stueckelberg symmetry with parameter θ in (A.3b), thus implying that this combination actually does not even enter the action (A.1).

$$\begin{aligned}
0 = \mathcal{R}_{aq; q_{s-1}}^a &= \nabla^a \omega_{[q;a], q_{s-1}} + (s-1) \nabla_q \omega_{[a;q], q_{s-2}}^a \\
&+ \frac{s-1}{2\ell^2} \left((D+s-4) \phi_{q_s} + \frac{s}{2} \eta_{qq} \phi_{q_{s-2}}^a \right), \tag{A.17}
\end{aligned}$$

where we also introduced the symmetric component of the vielbein,

$$\phi_{q_s} = e_{q; q_{s-1}}. \tag{A.18}$$

Substituting the solution (A.16) to the torsion constraint and using the commutator

$$\nabla_{[\mu} \nabla_{\nu]} v^a = \frac{1}{\ell^2} \bar{e}_{[\mu}{}^b \bar{e}_{\nu]}{}^a v_b, \tag{A.19}$$

one eventually recovers Fronsdal's equation (2.33) for the field ϕ_{q_s} .

B More on the classical Miura transformation

We start by analysing the Drinfeld–Sokolov reduction of $\mathfrak{gl}(N)$ in the single-row gauge (where u_1 can be different from zero). The transformations are generated by the charges

$$\begin{aligned}
Q(\lambda) &= -\frac{k}{2\pi} \int d\theta \operatorname{tr} \lambda u \\
&= -\frac{k}{2\pi} \int d\theta \frac{6}{N(N^2-1)} \operatorname{tr}_{N \times N} \lambda u \\
&= -\frac{k}{2\pi} \frac{6}{N(N^2-1)} \int d\theta (\lambda_{11} u_1 + \dots + \lambda_{N1} u_N) \\
&= -\frac{k}{2\pi} \frac{6}{N(N^2-1)} \int d\theta \operatorname{res}(L\lambda^1). \tag{B.1}
\end{aligned}$$

Here, we expressed the normalised trace tr (that satisfies $\operatorname{tr} L_0^2 = \frac{1}{2}$) in terms of the matrix trace, and in the last step we wrote the integrand in terms of the residue (see (3.110)) of the product of the differential operators L (containing the fields u_j , see (3.101)) and λ^1 (containing the elements of the first column of the matrix λ , see (3.112)).

We can infer the Poisson bracket by requiring that the charges generate the transformation (3.108) of the fields u_j (encoded in L),

$$\delta_\lambda L = \{Q(\lambda), L\} \tag{B.2a}$$

$$\implies L(\lambda^1 L)_+ - (L\lambda^1)_+ L = -\frac{k}{2\pi} \frac{6}{N(N^2-1)} \left\{ \int d\theta' \operatorname{res}(L\lambda^1), L \right\}. \tag{B.2b}$$

From here we can derive the Poisson bracket of linear functionals of the field u_j :

$$\begin{aligned}
& \left\{ \int d\theta' \operatorname{res} L(\theta') \lambda^1(\theta'), \int d\theta \operatorname{res} L(\theta) \mu^1(\theta) \right\} \\
&= \int d\theta \operatorname{res} \left(\left\{ \int d\theta' \operatorname{res} L(\theta') \lambda^1(\theta'), L(\theta) \right\} \mu^1(\theta) \right) \\
&= -\frac{2\pi N(N^2-1)}{6k} \int d\theta \operatorname{res} (L(\lambda^1 L)_+ \mu^1 - (L\lambda^1)_+ L\mu^1)(\theta) \\
&= \frac{2\pi N(N^2-1)}{6k} \int d\theta \operatorname{res} ((L\lambda^1)_+ L\mu^1 - (\lambda^1 L)_+ \mu^1 L)(\theta). \quad (\text{B.3})
\end{aligned}$$

In the last step we used that the integral over a residue is cyclic,

$$\int d\theta \operatorname{res} XY = \int d\theta \operatorname{res} YX. \quad (\text{B.4})$$

The Poisson bracket can be straightforwardly generalised to arbitrary functionals F and G of the fields,

$$\{F, G\} = \frac{2\pi N(N^2-1)}{6k} \int d\theta \operatorname{res} \left(\left(L \frac{\delta F}{\delta L} \right)_+ L \frac{\delta G}{\delta L} - \left(\frac{\delta F}{\delta L} L \right)_+ \frac{\delta G}{\delta L} \right). \quad (\text{B.5})$$

Here, the functional derivative by the u 's is encoded in the pseudo-differential operator setting as

$$\frac{\delta F}{\delta L} = \sum_{k=1}^N \partial^{-N+k-1} \frac{\delta F}{\delta u_k}. \quad (\text{B.6})$$

Then for example

$$\frac{\delta}{\delta L} \int d\theta \operatorname{res} L\lambda^1 = \sum_{k=1}^N \partial^{-N+k-1} \frac{\delta}{\delta u_k} \sum_{i=1}^N u_i \lambda_{i1} = \lambda^1. \quad (\text{B.7})$$

We now use this formulation to show that the Poisson bracket is induced by the Poisson bracket of free fields via the Miura transformation. To this end we employ the following theorem stating that for a factorised differential operator $L = L_1 L_2$ we have

$$\begin{aligned}
& \int d\theta \operatorname{res} \left(\left(L \frac{\delta F}{\delta L} \right)_+ L \frac{\delta G}{\delta L} - \left(\frac{\delta F}{\delta L} L \right)_+ \frac{\delta G}{\delta L} \right) \\
&= \sum_{i=1}^2 \int d\theta \operatorname{res} \left(\left(L_i \frac{\delta F}{\delta L_i} \right)_+ L_i \frac{\delta G}{\delta L_i} - \left(\frac{\delta F}{\delta L_i} L_i \right)_+ \frac{\delta G}{\delta L_i} \right). \quad (\text{B.8})
\end{aligned}$$

For a proof we follow [97]. We denote the order of L_i by N_i , then $N = N_1 + N_2$ is the order of L . First, we note that

$$\delta F = \int d\theta \operatorname{res} \frac{\delta F}{\delta L} \delta L = \int d\theta \operatorname{res} \left(L_2 \frac{\delta F}{\delta L} \delta L_1 + \frac{\delta F}{\delta L} L_1 \delta L_2 \right), \quad (\text{B.9})$$

and hence

$$\frac{\delta F}{\delta L_1} = \left(L_2 \frac{\delta F}{\delta L} \right)_+ + \dots \quad \text{and} \quad \frac{\delta F}{\delta L_2} = \left(\frac{\delta F}{\delta L} L_1 \right)_+ + \dots, \quad (\text{B.10})$$

where ... refers to terms of the form $\partial^{-k} a_k$ with $k > N_i$ which are undetermined by (B.9), but which are irrelevant for the following computations. Then

$$\begin{aligned} & \int d\theta \operatorname{res} \left(\left(L_1 \frac{\delta F}{\delta L_1} \right)_+ L_1 \frac{\delta G}{\delta L_1} - \left(\frac{\delta F}{\delta L_1} L_1 \right)_+ \frac{\delta G}{\delta L_1} L_1 \right) \\ &= - \int d\theta \operatorname{res} \left(\left(L_1 L_2 \frac{\delta F}{\delta L} \right)_- L_1 \left(L_2 \frac{\delta G}{\delta L} \right)_- - \left(L_2 \frac{\delta F}{\delta L} L_1 \right)_- \left(L_2 \frac{\delta G}{\delta L} \right)_- L_1 \right) \\ &= - \int d\theta \operatorname{res} \left(\left(L_1 L_2 \frac{\delta F}{\delta L} \right)_- L_1 L_2 \frac{\delta G}{\delta L} - \left(L_2 \frac{\delta F}{\delta L} L_1 \right)_- L_2 \frac{\delta G}{\delta L} L_1 \right. \\ &\quad \left. - \left(L_1 L_2 \frac{\delta F}{\delta L} \right)_- L_1 \left(L_2 \frac{\delta G}{\delta L} \right)_+ + \left(L_2 \frac{\delta F}{\delta L} L_1 \right)_- \left(L_2 \frac{\delta G}{\delta L} \right)_+ L_1 \right) \\ &= - \int d\theta \operatorname{res} \left(\left(L_1 L_2 \frac{\delta F}{\delta L} \right)_- L_1 L_2 \frac{\delta G}{\delta L} - \left(L_2 \frac{\delta F}{\delta L} L_1 \right)_- L_2 \frac{\delta G}{\delta L} L_1 \right), \quad (\text{B.11}) \end{aligned}$$

where in the last step one can omit in the second row the subscript $-$ such that the terms cancel. A similar computation gives

$$\begin{aligned} & \int d\theta \operatorname{res} \left(\left(L_2 \frac{\delta F}{\delta L_2} \right)_+ L_2 \frac{\delta G}{\delta L_2} - \left(\frac{\delta F}{\delta L_2} L_2 \right)_+ \frac{\delta G}{\delta L_2} L_2 \right) \\ &= - \int d\theta \operatorname{res} \left(\left(L_2 \frac{\delta F}{\delta L} L_1 \right)_- L_2 \frac{\delta G}{\delta L} L_1 - \left(\frac{\delta F}{\delta L} L_1 L_2 \right)_- \frac{\delta G}{\delta L} L_1 L_2 \right). \quad (\text{B.12}) \end{aligned}$$

Using these results to evaluate the right-hand side of (B.8) we straightforwardly obtain the left-hand side.

Using the above theorem iteratively on the Miura transformation $L = L_1 \dots L_N$ with $L_i = \partial + v_i$ (see (3.118)) we see that the Poisson bracket for the fields u_k is induced by a Poisson bracket⁴⁴ for the fields v_i which in terms of modes reads

⁴⁴ This is the content of the Kupershmidt–Wilson theorem [97].

$$\begin{aligned}
\{v_{i,m}, v_{i,n}\} &= \left\{ \frac{1}{2\pi} \int d\theta v_i(\theta) e^{im\theta}, \frac{1}{2\pi} \int d\theta v_i(\theta) e^{in\theta} \right\} \\
&= \frac{N(N^2-1)}{12\pi k} \int d\theta \operatorname{res} \left((L_i \partial^{-1} e^{im\theta})_+ L_i \partial^{-1} e^{in\theta} \right. \\
&\quad \left. - (\partial^{-1} e^{im\theta} L_i)_+ \partial^{-1} e^{in\theta} L_i \right) \\
&= \frac{N(N^2-1)}{12\pi k} \int d\theta (in) e^{i(m+n)\theta} \\
&= -\frac{N(N^2-1)}{6k} im \delta_{m,-n}. \tag{B.13}
\end{aligned}$$

These are the Poisson brackets of a free field.

We now discuss the Drinfeld–Sokolov reduction based on $\mathfrak{sl}(N)$. When we constrain the variations of L such that $u_1 = 0$, there is some ambiguity in the definition of the functional derivative. Demanding that

$$\delta F = \int d\theta \operatorname{res} \frac{\delta F}{\delta L} \delta L \tag{B.14}$$

for the constrained variations, we see that we have the freedom to redefine $\frac{\delta F}{\delta L} \rightarrow \frac{\delta F}{\delta L} + \partial^{-N} a(\theta)$ for an arbitrary function a . We know that the Poisson bracket is given by

$$-\frac{k}{2\pi} \frac{6}{N(N^2-1)} \left\{ \int d\theta' \operatorname{res} (L \lambda^1), L \right\} = L(\lambda^1 L)_+ - (L \lambda^1)_+ L \tag{B.15}$$

where λ^1 satisfies $\operatorname{res} [L, \lambda^1] = 0$ (see (3.111)). This can be generalised to a Poisson bracket of arbitrary functionals F and G as in (B.5), but where we now use the ambiguity of the functional derivative to require

$$\operatorname{res} \left[L, \frac{\delta F}{\delta L} \right] = \operatorname{res} \left[L, \frac{\delta G}{\delta L} \right] = 0. \tag{B.16}$$

The Poisson bracket can be reproduced by free fields. For a factorised operator $L = L_1 L_2$, the computation above leading to the result (B.8) carries over as long as we still use the relation (B.10) for the functional derivative with respect to L_1 and L_2 in terms of $\frac{\delta F}{\delta L}$. We then have

$$\sum_i \operatorname{res} \left[L_i, \frac{\delta F}{\delta L_i} \right] = \operatorname{res} \left[L_1, \left(L_2 \frac{\delta F}{\delta L} \right)_- \right] + \operatorname{res} \left[L_2, \left(\frac{\delta F}{\delta L} L_1 \right)_- \right], \tag{B.17}$$

where the subscript $-$ can be omitted. Writing out the commutators, two terms cancel, and we arrive at

$$\sum_i \operatorname{res} \left[L_i, \frac{\delta F}{\delta L_i} \right] = \operatorname{res} \left[L, \frac{\delta F}{\delta L} \right]. \tag{B.18}$$

Fixing the ambiguity of the functional derivative by requiring (B.16) is then equivalent to

$$\sum_i \text{res} \left[L_i, \frac{\delta F}{\delta L_i} \right] = 0. \quad (\text{B.19})$$

For $L = (\partial + v_1) \dots (\partial + v_N)$, the requirement on the functional derivative becomes

$$\sum_{i=1}^N \frac{\delta F}{\delta v_i} = 0. \quad (\text{B.20})$$

When we apply this to the modes of the fields v_i , we find

$$\frac{\delta}{\delta v_k(\theta)} \int d\theta' v_i(\theta') e^{im\theta'} = \left(\delta_{ik} - \frac{1}{N} \right) e^{im\theta}. \quad (\text{B.21})$$

The Poisson brackets then become

$$\{v_{i,m}, v_{j,n}\} = -\frac{N(N^2-1)}{6k} \left(\delta_{ij} - \frac{1}{N} \right) im \delta_{m,-n}, \quad (\text{B.22})$$

which is the mode version of the result stated in (3.119).

C Constrained Hamiltonian systems

In this appendix, we briefly summarise some important concepts that are useful to discuss Hamiltonian systems with constraints. For simplicity, we restrict the discussion to finite-dimensional systems, but similar considerations apply to field theories in Hamiltonian form. See [222–224] for ampler introductions to this material.

We consider a dynamical system with Hamiltonian H on a phase space with canonical coordinates $q^1, \dots, q^n, p_1, \dots, p_n$ and the Poisson bracket

$$\{q^i, p_j\} = \delta^i_j. \quad (\text{C.1})$$

We then assume that the system is confined to a constraint surface M defined by the vanishing of some phase space functions $\phi_i(q, p)$ ($i = 1, \dots, N$), which we denote as constraints. We demand a certain regularity for these functions such that locally around a point in M , we can take the functions ϕ_i as the first N coordinates of a local coordinate system on phase space. An equality for phase space functions that only holds on the constraint surface M is denoted by the symbol \approx , so in particular we have

$$\phi_i \approx 0. \quad (\text{C.2})$$

We further assume that the constraints are consistent with the time evolution, i.e., we demand that $\dot{\phi}_i \approx 0$. Notice that on the constraint surface, the Hamiltonian can

be modified by adding functions that vanish on the constraint surface,⁴⁵

$$H_T = H + \sum u^i \phi_i, \quad (\text{C.3})$$

for some functions u^i . Consistency of the constraints with time evolution is then formulated as

$$\dot{\phi}_i = \{\phi_i, H_T\} \approx \{\phi_i, H\} + \sum_j u^j \{\phi_i, \phi_j\} \approx 0, \quad (\text{C.4})$$

so that adding the terms in u^i to the total Hamiltonian can avoid the introduction of further constraints. When the matrix

$$C_{ij} = \{\phi_i, \phi_j\} \quad (\text{C.5})$$

is not degenerate on the constraint surface ($\det(C_{ij}) \not\approx 0$), the Lagrange multipliers u^i are determined by consistency. Otherwise, there is some freedom to redefine them, which signals the appearance of a gauge symmetry.

A constraint ϕ that has vanishing Poisson brackets with all constraints on the constraint surface,

$$\{\phi, \phi_i\} \approx 0 \quad \text{for all } i, \quad (\text{C.6})$$

is called a *first-class constraint*, otherwise it is called *second-class*. If a first-class constraint ϕ is present, then, as we emphasised above, the matrix C_{ij} is degenerate and the total Hamiltonian H_T is not uniquely defined, leaving the ambiguity of adding a term $u\phi$ which is not fixed by (C.4). An infinitesimal time evolution of a phase space function then has the ambiguity

$$\delta f = u \delta t \{f, \phi\}. \quad (\text{C.7})$$

The different results of time evolution have to be considered as equivalent, and we should interpret (C.7) as a gauge transformation, with u interpreted as the gauge parameter per unit of time. A priori, this freedom can be fixed by introducing additional constraints to be interpreted as gauge-fixing conditions. A full gauge fixing is then leading to an extended set of constraints that are all of second class.

If only second-class constraints are present (i.e. if the matrix C_{ij} is not degenerate on the constraint surface), there are no gauge symmetries, and we can take M as a constrained phase space with a new Poisson bracket (called the *Dirac bracket*) defined by

$$\{f, g\}_* = \{f, g\} - \sum_{ij} \{f, \phi_i\} C^{ij} \{\phi_j, g\}, \quad (\text{C.8})$$

where C^{ij} denotes the coefficients of the inverse of the matrix (C_{ij}) . The Dirac bracket satisfies the Jacobi identity and

⁴⁵ If H is derived from a Lagrangian theory in which some of the relations between coordinates and momenta are not invertible, then it is necessary to add the corresponding constraints to H_T ; see, e.g., [223].

$$\{f, \phi_i\}_* \approx 0, \quad \dot{f} \approx \{f, H\}_*. \quad (\text{C.9})$$

Therefore, the dynamical system can be treated as a system without constraints defined only on the constraint surface.

D Solutions to selected exercises

Solution 1. Considering the relation $\epsilon^{abc} \mathcal{T}_c = 0$ and writing explicitly the spacetime indices one obtains

$$\omega_{[\mu}{}^a e_{\nu]}{}^b - \omega_{[\mu}{}^b e_{\nu]}{}^a = \epsilon^{abc} \partial_{[\mu} e_{\nu]}{}^c. \quad (\text{D.1})$$

Contracting with the inverse vielbein $e^{\nu}{}_b$ one then gets

$$\omega_{\mu}{}^a + \omega_b{}^b e_{\mu}{}^a = 2\epsilon^{abc} e^{\nu}{}_b \partial_{[\mu} e_{\nu]}{}^c. \quad (\text{D.2})$$

The trace of the spin connection $\omega_b{}^b := e^{\nu}{}_b \omega_{\nu}{}^b$ can be computed by contracting (D.2) with $e^{\mu}{}_a$. Substituting the result in the same equation one obtains (2.6).

Solution 2. Plugging $\xi^a = e_{\mu}{}^a v^{\mu}$ and $\Lambda^a = \omega_{\mu}{}^a v^{\mu}$ into the variation $\delta e_{\mu}{}^a$ (see (2.7)), we obtain after some algebra

$$\delta e_{\mu}{}^a = v^{\nu} \partial_{\nu} e_{\mu}{}^a + e_{\nu}{}^a \partial_{\mu} v^{\nu} + 2v^{\nu} (\partial_{[\mu} e_{\nu]}{}^a + \epsilon^a{}_{bc} \omega_{[\mu}{}^b e_{\nu]}{}^c). \quad (\text{D.3})$$

The last term in parentheses contains the components of the torsion \mathcal{T}^a (see (2.5a)) and vanishes on-shell.

Solution 3. Plugging the definitions $A = (\omega^a + \frac{1}{\ell} e^a) J_a$ and $\tilde{A} = (\omega^a - \frac{1}{\ell} e^a) J_a$ in (2.19), and using partial integration to show that $\int \text{tr}(de \wedge \omega) = \int \text{tr}(e \wedge d\omega)$, we obtain

$$\frac{\ell}{16\pi G} \int \text{tr} \left(\frac{4}{\ell} e \wedge d\omega + \frac{4}{\ell} e \wedge \omega \wedge \omega + \frac{4}{3\ell^3} e \wedge e \wedge e \right), \quad (\text{D.4})$$

where we have also used the cyclicity of the trace, that is $\text{tr}(e \wedge \omega \wedge \omega) = \text{tr}(\omega \wedge e \wedge \omega) = \text{tr}(\omega \wedge \omega \wedge e)$ and similarly for other terms. Recalling $R = d\omega + \omega \wedge \omega$, we see that this is precisely (2.24).

Using the definition of the bilinear form (2.21), one can then rewrite the action (2.24) in terms of e^a and ω^a obtaining the Einstein–Hilbert action (2.4). The latter expression is valid for arbitrary values of the cosmological constant, so that the same is true for (2.24).

Solution 7. The zero-curvature condition (2.42b) can be rewritten (using the definition (2.40b) of $\mathcal{R}^{a_1 \dots a_{s-1}}$) as

$$d\omega^{a_1 \dots a_{s-1}} + (s-1) \epsilon_{bc_1}{}^{(a_1} \delta_{c_2}^{a_2} \dots \delta_{c_{s-1}}^{a_{s-1})} \left(\bar{\omega}^b \wedge \omega^{c_1 \dots c_{s-1}} + \frac{1}{\ell^2} \bar{e}^b \wedge e^{c_1 \dots c_{s-1}} \right) = 0. \quad (\text{D.5})$$

When we compare this to (2.47), we can directly read off the commutation relations (2.48). The analogous inspection of (2.42a) leads to the same result.

To determine the trace, one can insert $e = \bar{e}_a J^a + e_{a_1 \dots a_{s-1}} J^{a_1 \dots a_{s-1}}$ and $\omega = \bar{\omega}_a J^a + \omega_{a_1 \dots a_{s-1}} J^{a_1 \dots a_{s-1}}$ in the action (2.24) and expand to quadratic order in the higher-spin fields. Comparison to the higher-spin action (2.39) then leads to the results (2.49) for the trace tr .

Solution 9. The relation (2.66) for $m = -1$,

$$[L_{-1}, W_n^s] = -(s-1+n)W_{n-1}^s, \quad (\text{D.6})$$

directly follows from the definition (2.65). For $m = 0$, one can prove inductively that

$$[L_0, W_n^s] = -n W_n^s. \quad (\text{D.7})$$

For this purpose one starts from

$$[L_0, W_{s-1}^s] = [L_0, L_1^{s-1}] = -(s-1)L_1^{s-1} = -(s-1)W_{s-1}^s, \quad (\text{D.8})$$

and in the induction step one uses Jacobi identity:

$$\begin{aligned} [L_0, W_m^s] &= -\frac{1}{m+s} [L_0, [L_{-1}, W_{m+1}^s]] \\ &= -\frac{1}{m+s} ([L_{-1}, [L_0, W_{m+1}^s]] + [[L_0, L_{-1}], W_{m+1}^s]) \\ &= -\frac{1}{m+s} (-(m+1)[L_{-1}, W_{m+1}^s] + [L_{-1}, W_{m+1}^s]) \\ &= -m W_m^s. \end{aligned} \quad (\text{D.9})$$

The relation (2.66) for $m = 1$,

$$[L_1, W_n^s] = (s-1-n)W_{n+1}^s, \quad (\text{D.10})$$

can be proven analogously. It trivially holds for $n = s-1$, and for $n = s-2$ one obtains

$$[L_1, W_{s-2}^s] = -\frac{1}{2s-2} [L_1, [L_{-1}, L_1^{s-1}]] = -\frac{1}{2s-2} [[L_1, L_{-1}], L_1^{s-1}] = W_{s-1}^s. \quad (\text{D.11})$$

Similarly one can prove the relation for lower n by induction.

Again by employing Jacobi identity one concludes that

$$[L_0, [W_m^s, W_n^t]] = [[L_0, W_m^s], W_n^t] + [W_m^s, [L_0, W_n^t]] = -(m+n)[W_m^s, W_n^t], \quad (\text{D.12})$$

hence $[W_m^s, W_n^t]$ has to be a linear combination of terms with mode number $m+n$.

Solution 10. Thanks to the results of exercise 9, the commutator with the highest mode number, $[W_2^3, W_1^3]$, can only be proportional to W_3^4 ,

$$[W_2^3, W_1^3] = \alpha W_3^4. \quad (\text{D.13})$$

Indeed, it cannot be proportional to any W_3^s with $s > 4$ since the Jacobi identities imply that it is annihilated by the adjoint action of L_1 and this property is satisfied only by W_3^4 .

In this particular example, this first result already suffices to rule out the presence of W_{m+n}^3 in the expansion of the commutator in a basis of the $\mathfrak{hs}[\lambda]$ algebra. This is so because there are at most 10 independent commutators $[W_m^3, W_n^3]$ and we must already have 7 independent W_{m+n}^4 generators in the expansion. The remaining 3 independent generators can thus only be proportional to the 3 independent components of L_{m+n} .

The same outcome could be obtained by considering that $[W_2^3, W_0^3]$ is the only commutator with “total mode number” 2. As a result, it must be proportional to W_2^4 , thus excluding a contribution in W_2^3 . At the next step, on the other hand, one encounters two commutators with mode number 1, i.e. $[W_2^3, W_{-1}^3]$ and $[W_1^3, W_0^3]$, so that they can both be proportional to a linear combination of W_1^4 and L_1 . Notice that this argument can be generalised to fix the structure of commutators involving generators with the same “total angular momentum” quantum number as follows:

$$[W_m^s, W_n^s] = \sum_{k=1}^{s-1} c_k[m, n] W_{m+n}^{2(s-k)}. \quad (\text{D.14})$$

We conclude that for $s = 3$ we have the Lie bracket

$$[W_m^3, W_n^3] = c_0[m, n] W_{m+n}^4 + c_1[m, n] L_{m+n}. \quad (\text{D.15})$$

Both sides of the equation have a definite behaviour under the adjoint action with L_p , and this completely fixes the dependence on m and n of each $c_i[m, n]$ up to a constant factor. Indeed, acting on the commutator (D.13) with L_{-1} leads (upon using the Jacobi identity on the left-hand side) to

$$(-3)[W_2^3, W_0^3] = \alpha(-6)W_2^4 \implies [W_2^3, W_0^3] = 2\alpha W_2^4, \quad (\text{D.16})$$

which determines the coefficient $c_0[2, 0]$ in terms of $c_0[2, 1] = \alpha$. Acting once more with L_{-1} we obtain

$$(-4)[W_1^3, W_0^3] + (-2)[W_2^3, W_{-1}^3] = 2\alpha(-5)W_1^4. \quad (\text{D.17})$$

Additionally we can look for a linear combination of the terms on the right-hand side that vanishes when we apply L_1 and therefore has to be proportional to L_1 ,

$$(-3)[W_1^3, W_0^3] + [W_2^3, W_{-1}^3] = \frac{5}{2}\beta L_1, \quad (\text{D.18})$$

where the factor $\frac{5}{2}$ has been chosen to get the final result in a convenient form. The previous two equations determine $[W_1^3, W_0^3]$ and $[W_2^3, W_{-1}^3]$, and one finds

$$[W_1^3, W_0^3] = \alpha W_1^4 - \frac{1}{2}\beta L_1, \quad [W_2^3, W_{-1}^3] = 3\alpha W_1^4 + \beta L_1. \quad (\text{D.19})$$

Analogously, all other commutators $[W_m^3, W_n^3]$ are determined in this way leaving only two constants α and β ,

$$[W_1^3, W_{-1}^3] = 2\alpha W_0^4 - \frac{\beta}{2} L_0, \quad (\text{D.20a})$$

$$[W_2^3, W_{-2}^3] = 4\alpha W_0^4 + 4\beta L_0, \quad (\text{D.20b})$$

$$[W_1^3, W_{-2}^3] = 3\alpha W_{-1}^4 + \beta L_{-1}, \quad (\text{D.20c})$$

$$[W_0^3, W_{-1}^3] = \alpha W_{-1}^4 - \frac{\beta}{2} L_{-1}, \quad (\text{D.20d})$$

$$[W_0^3, W_{-2}^3] = 2\alpha W_{-2}^4, \quad (\text{D.20e})$$

$$[W_{-1}^3, W_{-2}^3] = \alpha W_{-3}^4. \quad (\text{D.20f})$$

One can then explicitly check that the formula (2.69) precisely summarises these commutators.

Solution 11. Starting from $W_2^3 = (L_1)^2$ one can use the recursion relation (2.65) to obtain

$$W_1^3 = -\frac{1}{4}[L_{-1}, W_2^3] = \frac{1}{2}(L_0 L_1 + L_1 L_0), \quad (\text{D.21a})$$

$$W_0^3 = -\frac{1}{3}[L_{-1}, W_1^3] = \frac{1}{6}(L_{-1} L_1 + L_1 L_{-1} + 4(L_0)^2), \quad (\text{D.21b})$$

$$W_{-1}^3 = -\frac{1}{2}[L_{-1}, W_0^3] = \frac{1}{2}(L_{-1} L_0 + L_0 L_{-1}), \quad (\text{D.21c})$$

$$W_{-2}^3 = -[L_{-1}, W_{-1}^3] = (L_{-1})^2. \quad (\text{D.21d})$$

In a similar way we can determine the first few basis elements W_m^4 starting from $W_3^4 = (L_1)^3$:

$$W_2^4 = -\frac{1}{6}[L_{-1}, W_3^4] = L_{(0} L_1 L_1), \quad (\text{D.22a})$$

$$W_1^4 = -\frac{1}{5}[L_{-1}, W_2^4] = \frac{1}{5}L_{(-1} L_1 L_1) + \frac{4}{5}L_{(0} L_0 L_1). \quad (\text{D.22b})$$

Notice that

$$L_{(-1} L_1 L_1) = \frac{1}{2}(L_{-1} L_1^2 + L_1^2 L_{-1}) - \frac{1}{3}L_1, \quad (\text{D.23a})$$

$$\begin{aligned} L_{(0} L_0 L_1) &= \frac{1}{2}(L_0^2 L_1 + L_1 L_0^2) - \frac{1}{6}L_1 \\ &= C_2 L_1 + \frac{1}{2}(L_{-1} L_1^2 + L_1^2 L_{-1}) - \frac{2}{3}L_1, \end{aligned} \quad (\text{D.23b})$$

which can be checked by elementary computations. Using the explicit value for the Casimir we can write W_1^4 as

$$W_1^4 = \frac{1}{2}(L_{-1}L_1^2 + L_1^2L_{-1}) + \frac{\lambda^2 - 4}{5}L_1. \quad (\text{D.24})$$

From the explicit expressions for W_m^3 we compute

$$[W_2^3, W_1^3] = \frac{1}{2}[L_1^2, L_0L_1 + L_1L_0] = 2L_1^3 = 2W_3^4, \quad (\text{D.25})$$

and by comparison to the general form (2.69) of the commutator we conclude that $\alpha[\lambda] = 2$. A straightforward computation shows that

$$\begin{aligned} [W_2^3, W_{-1}^3] &= \frac{1}{2}[L_1^2, L_{-1}L_0 + L_0L_{-1}] \\ &= 3(L_1^2L_{-1} + L_{-1}L_1^2) + (\lambda^2 - 4)L_1. \end{aligned} \quad (\text{D.26})$$

On the other hand, using $\alpha[\lambda] = 2$ in the general commutator (2.69) we obtain

$$[W_2^3, W_{-1}^3] = 6W_1^4 + \beta[\lambda]L_1, \quad (\text{D.27})$$

and by comparison with the previous expression we can read off

$$\beta[\lambda] = \frac{4 - \lambda^2}{5}. \quad (\text{D.28})$$

Solution 15. From the definition (3.7) for the Poisson bracket we obtain

$$\begin{aligned} \{G(\Lambda), G(\Gamma)\} &= \frac{2\pi}{k} \int_{D_2} d\rho d\theta \operatorname{tr} \left(\frac{\delta G(\Lambda)}{\delta A_\rho} \frac{\delta G(\Gamma)}{\delta A_\theta} - (\Lambda \leftrightarrow \Gamma) \right) \\ &= \frac{k}{2\pi} \int_{D_2} d\rho d\theta \operatorname{tr} \left((\partial_\theta \Lambda + [A_\theta, \Lambda])(-\partial_\rho \Gamma - [A_\rho, \Gamma]) - (\Lambda \leftrightarrow \Gamma) \right), \end{aligned} \quad (\text{D.29})$$

where we used the variation (3.9) of G . Rearranging the terms and using partial integration in θ , we find

$$\begin{aligned} \{G(\Lambda), G(\Gamma)\} &= \frac{k}{2\pi} \int_{D_2} d\rho d\theta \operatorname{tr} \left(\partial_\rho (\Lambda \partial_\theta \Gamma) - \partial_\rho (A_\theta [\Lambda, \Gamma]) + [\Lambda, \Gamma] F_{\rho\theta} \right) \\ &= G([\Lambda, \Gamma]) + \frac{k}{2\pi} \int_{S^1} d\theta \operatorname{tr} (\Lambda \partial_\theta \Gamma), \end{aligned} \quad (\text{D.30})$$

reproducing (3.16).

Solution 17. From (3.45b) we read off the relation between a_θ and A_θ , which then determines the ρ -dependence. Considering the expansion of a_θ in terms of the basis elements L_m and W_m^s , we have to understand the ρ -dependence of $b^{-1}(\rho)W_m^s b(\rho)$.

From the commutation relations (3.48) we see that

$$L_0 W_m^s = W_m^s (L_0 - m) \quad (\text{D.31})$$

which means that whenever we move a mode W_m^s through L_0 from the right, L_0 is shifted by $-m$. Hence

$$e^{-\rho L_0} W_m^s e^{\rho L_0} = e^{m\rho} W_m^s. \quad (\text{D.32})$$

Any mode W_m^s with a positive mode number m then corresponds to a term that diverges for $\rho \rightarrow \infty$. Imposing the AdS condition (3.47) then leads to the expansion (3.50a) of a_θ . A similar argument leads to the expansion (3.50b) of \tilde{a}_θ .

Solution 20. We can rewrite (3.68) as

$$\begin{aligned} \delta \mathcal{L}(\theta') &= \int d\theta \delta(\theta - \theta') \left(\epsilon(\theta) \mathcal{L}'(\theta) + 2\epsilon'(\theta) \mathcal{L}(\theta) + \frac{1}{2} \epsilon'''(\theta) \right) \\ &= \int d\theta \epsilon(\theta) \left(-\delta(\theta - \theta') \mathcal{L}'(\theta) - 2\delta'(\theta - \theta') \mathcal{L}(\theta) - \frac{1}{2} \delta'''(\theta - \theta') \right). \end{aligned} \quad (\text{D.33})$$

When we compare this with (3.73) where we insert the charge (3.72),

$$\delta \mathcal{L}(\theta') = \frac{k}{2\pi} \int d\theta \epsilon(\theta) \{ \mathcal{L}(\theta), \mathcal{L}(\theta') \}, \quad (\text{D.34})$$

we read off the Poisson bracket (3.74).

Solution 21. We start from the highest-weight form (3.83) of a and a general transformation parameter λ given in (3.84). Using the commutation relations (2.76) of $\mathfrak{sl}(3, \mathbb{R})$, one can compute

$$\begin{aligned} \delta_\lambda a &= \lambda' + [a, \lambda] \\ &= (\epsilon' + \epsilon^0) L_1 \\ &\quad + ((\epsilon^0)' + 2\epsilon^{-1} - 2\mathcal{L}\epsilon + 4\mathcal{W}\chi) L_0 \\ &\quad + ((\epsilon^{-1})' - \mathcal{L}\epsilon^0 + \mathcal{W}\chi^1) L_{-1} \\ &\quad + (\chi' + \chi^1) W_2 \\ &\quad + ((\chi^1)' + 2\chi^0 - 4\mathcal{L}\chi) W_1 \\ &\quad + ((\chi^0)' + 3\chi^{-1} - 3\mathcal{L}\chi^1) W_0 \\ &\quad + ((\chi^{-1})' + 4\chi^{-2} - 2\mathcal{L}\chi^0 - 4\mathcal{W}\epsilon) W_{-1} \\ &\quad + ((\chi^{-2})' - 2\mathcal{L}\chi^{-1} - 2\mathcal{W}\epsilon^0) W_{-2}. \end{aligned} \quad (\text{D.35})$$

Requiring the vanishing of all coefficients except for those of L_{-1} and W_{-2} , one obtains a system of differential equations that allows us to express all ϵ^m and χ^n in terms of (derivatives of) ϵ and χ :

$$\epsilon^0 = -\epsilon', \quad (\text{D.36a})$$

$$\epsilon^{-1} = \frac{1}{2}\epsilon'' + \mathcal{L}\epsilon - 2\mathcal{W}\chi, \quad (\text{D.36b})$$

$$\chi^1 = -\chi', \quad (\text{D.36c})$$

$$\chi^0 = \frac{1}{2}\chi'' + 2\mathcal{L}\chi, \quad (\text{D.36d})$$

$$\chi^{-1} = -\frac{1}{6}\chi^{(3)} - \frac{5}{3}\mathcal{L}\chi' - \frac{2}{3}\mathcal{L}'\chi, \quad (\text{D.36e})$$

$$\chi^{-2} = \frac{1}{24}\chi^{(4)} + \frac{2}{3}\mathcal{L}\chi'' + \frac{7}{12}\mathcal{L}'\chi' + \frac{1}{6}\mathcal{L}''\chi + \mathcal{L}^2\chi + \mathcal{W}\epsilon. \quad (\text{D.36f})$$

Inserting these expressions into (D.35) one can read off the transformation (3.85b) of \mathcal{W} .

Solution 22. The transformation of \mathcal{W} is given in (3.85b). Similarly to (3.75) we expand

$$\mathcal{W}(\theta) = \frac{1}{k} \sum_{n \in \mathbb{Z}} \mathcal{W}_n e^{-in\theta}. \quad (\text{D.37})$$

With $\chi = 0$ and $\epsilon = \epsilon_{(0)} e^{im\theta}$ we obtain

$$\frac{1}{k} \sum_n \delta \mathcal{W}_n e^{-in\theta} = \epsilon_{(0)} \sum_n (-in) e^{-i(n-m)\theta} \mathcal{W}_n + 3\epsilon_{(0)} \sum_n (im) e^{-i(n-m)\theta} \mathcal{W}_n. \quad (\text{D.38})$$

From here we can deduce the transformation $\delta \mathcal{W}_n$ as in (3.86b).

Solution 23. The transformation of \mathcal{W} with $\epsilon = 0$ and $\chi = \chi_{(0)} e^{im\theta}$ can be obtained from (3.85b). Expanding \mathcal{W} and \mathcal{L} in Fourier modes as in (D.37) and (3.75), we find

$$\begin{aligned} \sum_n \delta \mathcal{W}_n e^{-in\theta} &= \frac{k}{24} (im)^5 \chi_{(0)} e^{im\theta} + \frac{5}{6} (im)^3 \chi_{(0)} \sum_n \mathcal{L}_n e^{i(m-n)\theta} \\ &+ \frac{15}{12} (im)^2 \chi_{(0)} \sum_n (-in) \mathcal{L}_n e^{i(m-n)\theta} \\ &+ \frac{9}{12} (im) \chi_{(0)} \sum_n (-in)^2 \mathcal{L}_n e^{i(m-n)\theta} \\ &+ \frac{1}{6} \chi_{(0)} \sum_n (-in)^3 \mathcal{L}_n e^{i(m-n)\theta} \\ &+ \frac{1}{k} \frac{8}{3} (im) \chi_{(0)} \sum_{n,p} \mathcal{L}_n \mathcal{L}_{-p} e^{i(m-n+p)\theta} \\ &+ \frac{1}{k} \frac{8}{3} \chi_{(0)} \sum_{n,p} \mathcal{L}_n \mathcal{L}_{-p} e^{i(m-n+p)\theta}. \end{aligned} \quad (\text{D.39})$$

By changing the summation variable as $n \mapsto n + m$ in the first four lines and $n \mapsto n + m + p$ in the last two lines we can deduce the transformation of \mathcal{W}_n ,

$$\begin{aligned}
\delta\mathcal{W}_n &= i\frac{k}{24}m^5\chi_{(0)}\delta_{m+n,0} \\
&\quad + \frac{i}{12}\chi_{(0)}(-10m^3 + 15m^2(n+m) - 9m(n+m)^2 + 2(n+m)^3)\mathcal{L}_{m+n} \\
&\quad + \frac{i}{k}\frac{8}{3}\sum_p(m+p)\mathcal{L}_{n+m+p}\mathcal{L}_{-p}. \tag{D.40}
\end{aligned}$$

The sum in the last term can be rewritten as

$$\begin{aligned}
\sum_p(m+p)\mathcal{L}_{n+m+p}\mathcal{L}_{-p} &= \frac{1}{2}\left(\sum_p(m+p)\mathcal{L}_{n+m+p}\mathcal{L}_{-p} + \sum_p(-n-p)\mathcal{L}_{n+m+p}\mathcal{L}_{-p}\right) \\
&= (m-n)\sum_p\mathcal{L}_{n+m+p}\mathcal{L}_{-p}, \tag{D.41}
\end{aligned}$$

where in the first step in one of the sums we have changed the summation label $p \mapsto -m-n-p$. After rearranging some terms we arrive at the result for $\delta\mathcal{W}_n$ given in (3.88b).

Solution 24. The transformation of a is generated by the charge $Q(\lambda)$ through the Dirac bracket,

$$\delta_\lambda a = \{Q(\lambda), a\}. \tag{D.42}$$

Generalising the discussion that led to (3.72), the charge $Q(\lambda)$ for a gauge parameter λ of the form (3.84) is given as

$$Q(\lambda) = \frac{k}{2\pi} \int d\theta (\epsilon(\theta)\mathcal{L}(\theta) - \chi(\theta)\mathcal{W}(\theta)) \tag{D.43}$$

(note that $\text{tr}(W_2W_{-2}) = 1$ (see (2.75)) compared to $\text{tr}(L_1L_{-1}) = -1$). If $\epsilon = 0$ and $\chi = \chi_{(0)}e^{im\theta}$, we find

$$\begin{aligned}
Q(\lambda) &= -\frac{k}{2\pi} \int d\theta \chi_{(0)} e^{im\theta} \frac{1}{k} \sum_n \mathcal{W}_n e^{-in\theta} \\
&= -\chi_{(0)}\mathcal{W}_m. \tag{D.44}
\end{aligned}$$

We then get

$$\begin{aligned}
i\{\mathcal{W}_m, \mathcal{W}_n\} &= -\frac{i}{\chi_{(0)}}\delta\mathcal{W}_n \\
&= \frac{1}{12}\left(\frac{k}{2}m^5\delta_{m+n,0} - (m-n)(2m^2 + 2n^2 - mn)\mathcal{L}_{m+n}\right. \\
&\quad \left. + \frac{16}{k}(m-n)\sum_{q\in\mathbb{Z}}\mathcal{L}_{m+n+q}\mathcal{L}_{-q}\right), \tag{D.45}
\end{aligned}$$

where we used the result (3.88b) for $\delta\mathcal{W}_n$. Finally, we want to express this in terms of the modes $\hat{\mathcal{L}}_m = -\mathcal{L}_m + \frac{k}{4}\delta_{m,0}$ and the quadratic field Λ given in (3.89),

$$\begin{aligned}
\Lambda_p &= \sum_q \mathcal{L}_{p+q} \mathcal{L}_{-q} \\
&= \sum_q \mathcal{L}_{p+q} \mathcal{L}_{-q} + \frac{k}{2} \mathcal{L}_p - \left(\frac{k}{4}\right)^2 \delta_{p,0}. \tag{D.46}
\end{aligned}$$

Using this result in (D.45) and replacing $k = \frac{\epsilon}{6}$, we obtain the Dirac bracket given in (3.90c).

Solution 25. The transformation of a with a general matrix-valued parameter λ takes the form

$$\delta_{\lambda} a = \begin{pmatrix} \lambda'_{11} - \sqrt{2}\lambda_{12} + u_{11} & \lambda'_{12} - \sqrt{2}\lambda_{13} - u_2\lambda_{11} + u_{12} & \lambda'_{13} - u_3\lambda_{11} + u_{13} \\ \lambda'_{21} + \sqrt{2}(\lambda_{11} - \lambda_{22}) & \lambda'_{22} + \sqrt{2}(\lambda_{12} - \lambda_{23}) - u_2\lambda_{21} & \lambda'_{23} + \sqrt{2}\lambda_{13} - u_3\lambda_{21} \\ \lambda'_{31} + \sqrt{2}(\lambda_{21} - \lambda_{32}) & \lambda'_{32} + \sqrt{2}(\lambda_{22} - \lambda_{33}) - u_2\lambda_{31} & \lambda'_{33} + \sqrt{2}\lambda_{23} - u_3\lambda_{31} \end{pmatrix}, \tag{D.47}$$

where

$$u_{1j} = u_2\lambda_{2j} + u_3\lambda_{3j}. \tag{D.48}$$

Requiring that the second and third row vanish, we can express the coefficients λ_{ij} in terms of the entries λ_{ji} of the first column as we discuss now. First notice that the 3-1-entry and the 2-1-entry of (D.47) allows us to conclude

$$\lambda_{32} = \frac{1}{\sqrt{2}}\lambda'_{31} + \lambda_{21}, \tag{D.49a}$$

$$\lambda_{22} = \frac{1}{\sqrt{2}}\lambda'_{21} + \lambda_{11}. \tag{D.49b}$$

Now we can use the 3-2-entry to determine

$$\lambda_{33} = \frac{1}{2}\lambda''_{31} + \sqrt{2}\lambda'_{21} + \lambda_{11} - \frac{1}{\sqrt{2}}u_2\lambda_{31}. \tag{D.50}$$

Using the entries 3-3, 2-3 and 2-2 of (D.47) we arrive at

$$\lambda_{23} = -\frac{1}{2\sqrt{2}}\lambda_{31}^{(3)} - \lambda''_{21} - \frac{1}{\sqrt{2}}\lambda'_{11} + \frac{1}{2}\partial(u_2\lambda_{31}) + \frac{1}{\sqrt{2}}u_3\lambda_{31}, \tag{D.51a}$$

$$\lambda_{13} = \frac{1}{4}\lambda_{31}^{(4)} + \frac{1}{\sqrt{2}}\lambda_{21}^{(3)} + \frac{1}{2}\lambda''_{11} - \frac{1}{2\sqrt{2}}\partial^2(u_2\lambda_{31}) - \frac{1}{2}\partial(u_3\lambda_{31}) + \frac{1}{\sqrt{2}}u_3\lambda_{21}, \tag{D.51b}$$

$$\lambda_{12} = -\frac{1}{2\sqrt{2}}\lambda_{31}^{(3)} - \frac{3}{2}\lambda''_{21} - \sqrt{2}\lambda'_{11} + \frac{1}{2}\partial(u_2\lambda_{31}) + \frac{1}{\sqrt{2}}(u_2\lambda_{21} + u_3\lambda_{31}). \tag{D.51c}$$

It remains to implement the condition that the 1-1-entry of (D.47) vanishes. Using the expression (D.51c) of λ_{12} , we get a condition on λ_{11} ,

$$\lambda'_{11} = -\frac{1}{6}\lambda_{31}^{(3)} - \frac{1}{\sqrt{2}}\lambda''_{21} + \frac{1}{3\sqrt{2}}\partial(u_2\lambda_{31}). \tag{D.52}$$

We can then read off the transformation of u_2 and u_3 from the 1-2- and 1-3-entry of (D.47), and we arrive at (3.94a) and (3.94b).

Solution 26. From the expression (3.98) for $\delta_\lambda a_{ij}$ we find

$$\sum_{j=1}^N \delta_\lambda a_{ij} \partial^{N-j} = \sum_{j=1}^N (\lambda'_{ij} - \lambda_{i-1j} + \lambda_{ij+1} - u_j \lambda_{i1}) \partial^{N-j}. \quad (\text{D.53})$$

We rewrite

$$\lambda'_{ij} \partial^{N-j} = \partial \circ (\lambda_{ij} \partial^{N-j}) - \lambda_{ij} \partial^{N-j+1}, \quad (\text{D.54})$$

and obtain (using the definition (3.99a) of λ_i)

$$\begin{aligned} \sum_{j=1}^N \delta_\lambda a_{ij} \partial^{N-j} &= \partial \circ \lambda_i - \lambda_{i-1} - \sum_{j=1}^N (\lambda_{ij} \partial^{N-j+1} - \lambda_{ij+1} \partial^{N-j} + \lambda_{i1} u_j \partial^{N-j}) \\ &= \partial \circ \lambda_i - \lambda_{i-1} - \lambda_{i1} \partial^N - \lambda_{i1} \sum_{j=1}^N u_j \partial^{N-j}, \end{aligned} \quad (\text{D.55})$$

from which the result (3.102) follows.

Solution 27. We insert the proposed solution (3.106) into the right-hand side of the recursion relation (3.105) (setting i to $N-i$) and obtain

$$\begin{aligned} \partial \lambda_{N-i} - \lambda_{N-i1} L &= \partial^{i+1} (\lambda^1 L)_+ - \partial (\partial^i \lambda^1)_+ L - \lambda_{N-i1} L \\ &= \partial^{i+1} (\lambda^1 L)_+ - \partial \sum_{j=N+1-i}^N \partial^{-N-1+i+j} \lambda_{j1} L - \lambda_{N-i1} L \\ &= \partial^{i+1} (\lambda^1 L)_+ - \sum_{j=N-i}^N \partial^{-N+i+j} \lambda_{j1} L \\ &= \partial^{i+1} (\lambda^1 L)_+ - (\partial^{i+1} \lambda^1)_+ L, \end{aligned} \quad (\text{D.56})$$

which coincides with λ_{N-i-1} given by the proposed solution (3.106).

Solution 28. We insert the solution (3.106) for λ_i into the expression (3.107) for the transformation of L and obtain

$$\begin{aligned} \delta_\lambda L &= \partial (\partial^{N-1} (\lambda^1 L)_+ - (\partial^{N-1} \lambda^1)_+ L) - \lambda_{11} L \\ &\quad + \sum_{k=2}^N u_k (\partial^{N-k} (\lambda^1 L)_+ - (\partial^{N-k} \lambda^1)_+ L) \\ &= L (\lambda^1 L)_+ - (\partial^N \lambda^1)_+ L - \sum_{k=2}^N u_k (\partial^{N-k} \lambda^1)_+ L \\ &= L (\lambda^1 L)_+ - (L \lambda^1)_+ L, \end{aligned} \quad (\text{D.57})$$

which gives the desired result (3.108).

Solution 29.

$$\begin{aligned}
i\{\hat{\mathcal{L}}_m, \Lambda_n\} &= \sum_{q \in \mathbb{Z}} i\{\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_{n+q} \hat{\mathcal{L}}_{-q}\} \\
&= \sum_{q \in \mathbb{Z}} \left((m-n-q) \hat{\mathcal{L}}_{m+n+q} + \delta_{m,-n-q} \frac{c}{12} m(m^2-1) \right) \hat{\mathcal{L}}_{-q} \\
&\quad + \sum_{q \in \mathbb{Z}} \hat{\mathcal{L}}_{n+q} \left((m+q) \hat{\mathcal{L}}_{m-q} + \delta_{m,q} \frac{c}{12} m(m^2-1) \right) \\
&= \sum_{q \in \mathbb{Z}} (m-n-q) \hat{\mathcal{L}}_{m+n+q} \hat{\mathcal{L}}_{-q} + \sum_{q \in \mathbb{Z}} (2m+q) \hat{\mathcal{L}}_{n+m+q} \hat{\mathcal{L}}_{-q} \\
&\quad + \frac{c}{6} m(m^2-1) \hat{\mathcal{L}}_{m+n} \\
&= (3m-n) \Lambda_{m+n} + \frac{c}{6} m(m^2-1) \hat{\mathcal{L}}_{m+n}, \tag{D.58}
\end{aligned}$$

where in the third equation we shifted the summation variable in the second sum, $q \rightarrow q+m$.

Solution 30. We rewrite the commutator of the modes L_m and L_n of $T(z)$ as a contour integral (see (4.11)),

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^{m+1} w^{n+1} \mathcal{R}(T(z)T(w)) \\
&= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \right), \tag{D.59}
\end{aligned}$$

where we inserted the singular part of the OPE of T with itself (4.14). Now we first evaluate the residue integral in z , succeeded by the residue integral in w :

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2\pi i} \oint_0 dw w^{n+1} \left(\frac{c}{2} \cdot \frac{1}{6} (m+1)m(m-1)w^{m-2} \right. \\
&\quad \left. + 2T(w)(m+1)w^m + \partial_w T(w)w^{m+1} \right) \\
&= \frac{c}{12} m(m^2-1) \delta_{m+n,0} + 2(m+1)L_{m+n} \\
&\quad + \frac{1}{2\pi i} \oint_0 dw \sum_{p \in \mathbb{Z}} L_p (-p-2) w^{m+n-p-1} \\
&= \frac{c}{12} m(m^2-1) \delta_{m+n,0} + (m-n)L_{m+n}. \tag{D.60}
\end{aligned}$$

Solution 31. We want to find the state corresponding to the current $W^{(s)}(z)$. For that purpose we act with $W^{(s)}(z)$ on the vacuum and take $z \rightarrow 0$. Notice that the modes $W_m^{(s)}$ annihilate the vacuum for $m > -s$, so that we find

$$W^{(s)}(z)\mathcal{Q} = \sum_{m \leq -s} z^{-m-s} W_m^{(s)} \mathcal{Q} \xrightarrow{z \rightarrow 0} W_{-s}^{(s)} \mathcal{Q}. \quad (\text{D.61})$$

The state corresponding to $W^{(s)}$ is therefore $W_{-s}^{(s)} \mathcal{Q}$.

Solution 32. The OPE of two currents of spin s and t , respectively, is given by

$$W^{(s)}(z)W^{(t)}(w) = V(W^{(s)}(z-w)W_{-t}^{(t)}\mathcal{Q}; w) = \sum_{n \leq t} (z-w)^{-n-s} V(W_n^{(s)}W_{-t}^{(t)}\mathcal{Q}; w). \quad (\text{D.62})$$

The normal ordered product $(W^{(s)}W^{(t)})$ of the fields occurs in the term where the exponent of $(z-w)$ is zero, therefore

$$(W^{(s)}W^{(t)})(w) = V(W_{-s}^{(s)}W_{-t}^{(t)}\mathcal{Q}; w). \quad (\text{D.63})$$

Solution 33. If the commutator of L_m and $W_n^{(s)}$ is given by (4.28), then we find

$$L_1 W_{-s}^{(s)} \mathcal{Q} = [L_1, W_{-s}^{(s)}] \mathcal{Q} = (2s-1)W_{-s+1}^{(s)} \mathcal{Q} = 0, \quad (\text{D.64})$$

hence $W^{(s)}$ is quasi-primary.

On the other hand, if $W^{(s)}$ is quasi-primary, it satisfies the OPE (4.23) (with weight $h=s$). The OPE determines the commutation relation of the modes by (4.11), so we find

$$\begin{aligned} [L_m, W_n^{(s)}] &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^{m+1} w^{n+s-1} T(z)W^{(s)}(w) \\ &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^{m+1} w^{n+s-1} \\ &\quad \times \left(\text{higher-order poles} + \frac{0}{(z-w)^3} + \frac{sW^{(s)}(w)}{(z-w)^2} + \frac{\partial_w W^{(s)}(w)}{z-w} \right). \end{aligned} \quad (\text{D.65})$$

For $-1 \leq m \leq 1$, the term z^{m+1} is at most quadratic, so the higher-order poles in $(z-w)$ (fourth and higher) do not contribute to the residue integral. Specialising on this case ($m \in \{-1, 0, 1\}$) we get

$$\begin{aligned} [L_m, W_n^{(s)}] &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^{m+1} w^{n+s-1} \left(\frac{sW^{(s)}(w)}{(z-w)^2} + \frac{\partial_w W^{(s)}(w)}{z-w} \right) \\ &= \frac{1}{2\pi i} \oint_0 dw w^{n+s-1} \left(s(m+1)w^m W^{(s)}(w) + w^{m+1} \partial_w W^{(s)}(w) \right) \\ &= \sum_{p \in \mathbb{Z}} \frac{1}{2\pi i} \oint_0 dw w^{m+n-p-1} (s(m+1) + (-p-s)) W_p^{(s)} \\ &= ((s-1)m-n) W_{m+n}^{(s)}. \end{aligned} \quad (\text{D.66})$$

Solution 34. The modes of $\partial^2 T$ are

$$(\partial^2 T)_m = (m+2)(m+3)L_m, \quad (\text{D.67})$$

the modes of (TT) are (see (4.22))

$$(TT)_m = \sum_{p \geq 2} L_{-p} L_{m+p} + \sum_{p \geq 1} L_{m+p} L_{-p}, \quad (\text{D.68})$$

so that the modes of Λ are

$$\Lambda_m = \sum_{p \geq 2} L_{-p} L_{m+p} + \sum_{p \geq 1} L_{m+p} L_{-p} - \frac{3}{10}(m+2)(m+3)L_m. \quad (\text{D.69})$$

We now want to rewrite this such that the quadratic terms are mode normal ordered (meaning that the L_m with greater mode number appears on the right). Let us consider the first sum. The operators are mode normal ordered for $2p \geq -m$, therefore all terms that appear are mode normal ordered for $m \geq -4$. (Similarly the terms in the second sum are always mode normal ordered for $m \leq -2$.)

Assume now that $m < -4$. Then the second sum is already mode normal ordered, and for the first sum we can write

$$\sum_{p \geq 2} L_{-p} L_{m+p} = \sum_{p \geq 2} : L_{-p} L_{m+p} : + \sum_{p=2}^{N_m} [L_{-p}, L_{m+p}], \quad (\text{D.70})$$

where

$$N_m = \begin{cases} -\frac{m+2}{2} & \text{for } m \text{ even} \\ -\frac{m+1}{2} & \text{for } m \text{ odd.} \end{cases} \quad (\text{D.71})$$

Using the commutation relations (notice that by assumption $m \neq 0$) we find

$$\begin{aligned} \sum_{p \geq 2} L_{-p} L_{m+p} &= \sum_{p \geq 2} : L_{-p} L_{m+p} : + \sum_{p=2}^{N_m} (-m-2p)L_m \\ &= \sum_{p \geq 2} : L_{-p} L_{m+p} : + (-m(N_m-1) - N_m(N_m+1) + 2)L_m \\ &= \sum_{p \geq 2} : L_{-p} L_{m+p} : + g(m)L_m, \end{aligned} \quad (\text{D.72})$$

with

$$g(m) = \begin{cases} \frac{1}{4}(m+2)(m+4) & \text{for } m \text{ even} \\ \frac{1}{4}(m+3)^2 & \text{for } m \text{ odd.} \end{cases} \quad (\text{D.73})$$

Similarly one can show that for all $m \in \mathbb{Z}$ we have

$$(TT)_m = \sum_{p \geq 2} : L_{-p} L_{m+p} : + \sum_{p \geq 1} : L_{m+p} L_{-p} : + g(m)L_m, \quad (\text{D.74})$$

and therefore

$$\Lambda_m = \sum_{p \geq 2} : L_{-p} L_{m+p} : + \sum_{p \geq 1} : L_{m+p} L_{-p} : + f(m) L_m \quad (\text{D.75})$$

with

$$f(m) = \begin{cases} -\frac{1}{20}(m^2 - 4) & \text{for } m \text{ even} \\ -\frac{1}{20}(m^2 - 9) & \text{for } m \text{ odd.} \end{cases} \quad (\text{D.76})$$

Solution 37. When we apply the differential d to (5.6), we arrive at the condition for consistency,

$$\bar{h}_b \wedge dC^{a_1 \dots a_n b} + \frac{n}{3} M^2 \bar{h}^{(a_1} \wedge dC^{a_2 \dots a_n)} = 0. \quad (\text{D.77})$$

Inserting (5.6), we indeed find

$$\begin{aligned} & \bar{h}_b \wedge \left(\bar{h}_c C^{a_1 \dots a_n b c} + \frac{n+1}{3} M^2 \bar{h}^{(a_1} C^{a_2 \dots a_n b)} \right) \\ & + \frac{n}{3} M^2 \bar{h}^{(a_1} \wedge \left(\bar{h}_c C^{a_2 \dots a_n) c} + \frac{n-1}{3} M^2 \bar{h}^{a_2} C^{a_3 \dots a_n)} \right) = 0, \end{aligned} \quad (\text{D.78})$$

where one uses that $\bar{h}_b \wedge \bar{h}_c$ is antisymmetric in b and c .

Solution 38. We apply the Lorentz covariant differential ∇ to (5.9), and we obtain after using (2.38)

$$\frac{n}{\ell^2} \bar{e}_b \wedge \bar{e}^{(a_1} C^{a_2 \dots a_n) b} = -\bar{e}_b \wedge \nabla C^{a_1 \dots a_n b} - \left(\frac{n}{3} M^2 - \frac{n(n-1)}{\ell^2} \right) \bar{e}^{(a_1} \wedge \nabla C^{a_2 \dots a_n)}. \quad (\text{D.79})$$

On the right-hand side, we insert the expression (5.9) for ∇C^{\dots} , and check that both sides agree. We only have to check this for the terms proportional to $\frac{1}{\ell^2}$, the rest of the calculation is similar to the consistency of (5.6) (see exercise 37). The $\frac{1}{\ell^2}$ -terms on the right-hand side of (D.79) are

$$\begin{aligned} & \frac{n(n+1)}{\ell^2} \bar{e}_b \wedge \bar{e}^{(a_1} C^{a_2 \dots a_n) b} + \frac{n(n-1)}{\ell^2} \bar{e}^{(a_1} \wedge \bar{e}_b C^{a_2 \dots a_n) b} \\ & = \left(\frac{n^2}{\ell^2} - \frac{n(n-1)}{\ell^2} \right) \bar{e}_b \wedge \bar{e}^{(a_1} C^{a_2 \dots a_n) b}, \end{aligned} \quad (\text{D.80})$$

which indeed agrees with the left-hand side of (D.79).

Solution 40. Applying the differential to the right-hand side of (5.31a) and inserting back (5.31a) one obtains

$$\begin{aligned} d(-AC + C\tilde{A}) &= -dA C + A \wedge dC + dC \wedge \tilde{A} + C d\tilde{A} \\ &= -dA C + A \wedge (-AC + C\tilde{A}) + (-AC + C\tilde{A}) \wedge \tilde{A} + C d\tilde{A} \\ &= -(dA + A \wedge A) C + C (d\tilde{A} + \tilde{A} \wedge \tilde{A}), \end{aligned} \quad (\text{D.81})$$

which indeed vanishes when A and \tilde{A} are solutions of the Chern–Simons equation of motion (2.14). The argument is similar for (5.31b).

Solution 41. Using (5.35) and the fact that the epsilon tensor $\epsilon_{\alpha\beta}$ is antisymmetric and normalized such that $\epsilon_{01} = 1$, we obtain

$$[L_{11}, L_{00}]_{\star} = 4\epsilon_{10}L_{10} = -4L_{10}, \quad (\text{D.82a})$$

$$[L_{01}, L_{00}]_{\star} = 2\epsilon_{10}L_{00} = -2L_{00}, \quad (\text{D.82b})$$

$$[L_{01}, L_{11}]_{\star} = 2\epsilon_{01}L_{11} = 2L_{11}. \quad (\text{D.82c})$$

Therefore by identifying $L_0 = \frac{1}{2}L_{10}$, $L_{+1} = \frac{1}{2}L_{00}$ and $L_{-1} = \frac{1}{2}L_{11}$ we recover the $\mathfrak{sl}(2, \mathbb{R})$ commutation relations.

Solution 43. The differential version of the star product can be obtained by straightforward computation:

$$\begin{aligned} (f \star g)(y) &= \frac{1}{(2\pi)^2} \int d^2u d^2v e^{ivu} e^{u\partial^{y'}} f(y') e^{v\partial^{y''}} g(y'') \Big|_{y'=y''=y} \\ &= \frac{1}{(2\pi)^2} \int d^2u d^2v e^{iv(u-i\partial^{y'})} e^{u\partial^{y'}} f(y') g(y'') \Big|_{y'=y''=y} \\ &= f(y) e^{-i\overleftarrow{\partial}_y \overrightarrow{\partial}_y} g(y), \end{aligned} \quad (\text{D.83})$$

where we have used Taylor's theorem, $f(y+u) = e^{u\partial^y} f(y)$, in the first line and the identity $\delta^{(2)}(u) = \frac{1}{(2\pi)^2} \int d^2v e^{ivu}$ in the last step. The arrow on the derivative indicates on which function the derivative acts (in the above formula $\overleftarrow{\partial}$ acts on f) without introducing additional signs.

Solution 44. The most convenient way to check (5.44) is using (5.43) from which we obtain

$$\begin{aligned} y_{\alpha} \star f(y) &= y_{\alpha} (1 - i\overleftarrow{\partial}_y^{\sigma} \overrightarrow{\partial}_{\sigma}^y) f(y) \\ &= (y_{\alpha} - i(\partial_y^{\sigma} y_{\alpha}) \partial_{\sigma}^y) f(y) \\ &= (y_{\alpha} + i\partial_{\alpha}^y) f(y), \end{aligned} \quad (\text{D.84})$$

where we have used $\partial_y^{\sigma} y_{\alpha} = \epsilon^{\sigma\delta} \partial_{\delta}^y y_{\alpha} = \epsilon^{\sigma\delta} \epsilon_{\delta\alpha} = -\delta_{\alpha}^{\sigma}$. In complete analogy one derives $f(y) \star y_{\alpha} = (y_{\alpha} - i\partial_{\alpha}^y) f(y)$. From these results (5.45) follows immediately,

$$y_{\alpha} \star f(y) - f(y) \star y_{\alpha} = 2i\partial_{\alpha}^y f(y), \quad (\text{D.85})$$

and analogously for the anticommutator.

Solution 45. To prove these identities, it is important to remember that $L_{\alpha\beta} = -\frac{i}{2}y_{(\alpha} \star y_{\beta)}$ (see (5.34)). Using (5.45) we then obtain

$$\begin{aligned}
[L_{\alpha\beta}, f(y)]_{\star} &= -\frac{i}{2} [y_{(\alpha} \star y_{\beta)}, f(y)]_{\star} \\
&= -\frac{i}{2} \left(y_{(\alpha} \star [y_{\beta)}, f(y)]_{\star} + [y_{(\alpha}, f(y)]_{\star} \star y_{\beta)} \right) \\
&= \{y_{(\alpha}, \partial_{\beta)}^y f(y)\}_{\star} \\
&= 2y_{(\alpha} \partial_{\beta)}^y f(y), \tag{D.86}
\end{aligned}$$

which confirms (5.46a). To prove the result (5.46b) most efficiently we employ (5.44) and compute

$$\begin{aligned}
\{L_{\alpha\beta}, f(y)\}_{\star} &= -\frac{i}{2} \{y_{(\alpha} \star y_{\beta)}, f(y)\}_{\star} \\
&= -\frac{i}{2} \left((y + i\partial^y)_{(\alpha} (y + i\partial^y)_{\beta)} f(y) + (y - i\partial^y)_{(\alpha} (y - i\partial^y)_{\beta)} f(y) \right) \\
&= -i \left(y_{\alpha} y_{\beta} - \partial_{\alpha}^y \partial_{\beta}^y \right) f(y). \tag{D.87}
\end{aligned}$$

In the last step we have used $\partial_{(\alpha}^y y_{\beta)} = 0$. Applying the result for the anticommutator we can straightforwardly check (5.36):

$$-\frac{1}{4} \{L_{\alpha\beta}, L^{\alpha\beta}\}_{\star} = \frac{1}{4} i (y_{\alpha} y_{\beta} - \partial_{\alpha}^y \partial_{\beta}^y) \left(-\frac{i}{2}\right) y^{\alpha} y^{\beta} = -\frac{3}{4}. \tag{D.88}$$

To obtain the last equation we have used the fact that $y_{\alpha} y^{\alpha} = 0$ and $\partial_{\alpha}^y \partial_{\beta}^y y^{\alpha} y^{\beta} = \delta_{\beta}^{\alpha} \partial_{\alpha}^y y^{\beta} + 2\partial_{\alpha}^y y^{\alpha} = 6$.

Solution 46. First we check the intermediate result given in the exercise

$$\hat{y}^{\alpha} \star \hat{y}_{\alpha} = \frac{1}{2} \epsilon^{\alpha\beta} [\hat{y}^{\alpha}, \hat{y}^{\beta}]_{\star} = i\epsilon^{\alpha\beta} \epsilon_{\alpha\beta} (1 + \nu k) = -2i(1 + \nu k). \tag{D.89}$$

Similarly one derives

$$L_{\alpha\beta} = -\frac{i}{4} (\hat{y}_{\alpha} \star \hat{y}_{\beta} + \hat{y}_{\beta} \star \hat{y}_{\alpha}) = -\frac{i}{2} \hat{y}_{\alpha} \star \hat{y}_{\beta} - \frac{1}{2} \epsilon_{\alpha\beta} (1 + \nu k). \tag{D.90}$$

These results can be used to check the value of the quadratic Casimir

$$\begin{aligned}
C_2 &= -\frac{1}{32} \left(-i\hat{y}_{\alpha} \star \hat{y}_{\beta} - \epsilon_{\alpha\beta} (1 + \nu k) \right) \left(-i\hat{y}^{\alpha} \star \hat{y}^{\beta} - \epsilon^{\alpha\beta} (1 + \nu k) \right) \\
&= -\frac{1}{32} \left(-\hat{y}^{\alpha} \star \hat{y}^{\beta} \star \hat{y}_{\beta} \star \hat{y}_{\alpha} + 2i\hat{y}^{\alpha} \star \hat{y}^{\beta} (1 + \nu k) - 2(1 + \nu k)^2 \right) \\
&= -\frac{1}{16} (3 + 2\nu k - \nu^2), \tag{D.91}
\end{aligned}$$

where we have used $\hat{y}_{\alpha} k = -k\hat{y}_{\alpha}$ and $k^2 = 1$ to obtain the last equation.

Solution 47. Combining (5.58b) and the fact that $f^{\mu_1 \dots \mu_s}$ is completely symmetric it follows that

$$f_{\nu}^{\nu\mu_3\cdots\mu_s} \sim f^{\alpha_1\cdots\alpha_{2s}} \epsilon_{\alpha_1\alpha_3} \epsilon_{\alpha_2\alpha_4} \bar{e}_{\alpha_5\alpha_6}^{\mu_3} \cdots \bar{e}_{\alpha_{2s-1}\alpha_{2s}}^{\mu_s} = 0. \quad (\text{D.92})$$

Solution 48. Explicit computation gives

$$\begin{aligned} D^{AdS} D^{AdS} F &= (d \cdot -\mathcal{A}^{AdS} \wedge \star \cdot + (-1)^{|F|+1} \cdot \wedge \star \mathcal{A}^{AdS}) \\ &\quad (dF - \mathcal{A}^{AdS} \wedge \star F + (-1)^{|F|} F \wedge \star \mathcal{A}^{AdS}) \\ &= F \wedge \star (d\mathcal{A}^{AdS} - \mathcal{A}^{AdS} \wedge \star \mathcal{A}^{AdS}) \\ &\quad - (d\mathcal{A}^{AdS} - \mathcal{A}^{AdS} \wedge \star \mathcal{A}^{AdS}) \wedge \star F \\ &\quad + (\mathcal{A}^{AdS} - \mathcal{A}^{AdS}) \wedge \star dF \\ &\quad + dF \wedge \star \left((-1)^{|F|} \mathcal{A}^{AdS} + (-1)^{|F|+1} \mathcal{A}^{AdS} \right) \\ &\quad + (-1)^{|F|} (\mathcal{A}^{AdS} \wedge \star F \wedge \star \mathcal{A}^{AdS} - \mathcal{A}^{AdS} \wedge \star F \wedge \star \mathcal{A}^{AdS}). \end{aligned} \quad (\text{D.93})$$

The first two terms vanish upon imposing the equations of motion for \mathcal{A}^{AdS} while the other terms are identically zero.

Solution 49. In order to simplify notation, we define

$$\hat{y}_{\alpha(s)} = \hat{y}_{\alpha_1} \star \cdots \star \hat{y}_{\alpha_s}. \quad (\text{D.94})$$

We first show (5.73). We start with the fully symmetrised expression

$$\hat{y}_{(\alpha} \star \hat{y}_{\beta(s))} = \frac{1}{s+1} \left(\hat{y}_{\alpha} \star \hat{y}_{\beta(s)} + \sum_{j=1}^s \hat{y}_{\beta_1} \star \cdots \star \hat{y}_{\beta_j} \star \hat{y}_{|\alpha|} \star \hat{y}_{\beta_{j+1}} \star \cdots \star \hat{y}_{\beta_s} \right), \quad (\text{D.95})$$

and in all terms on the right-hand side, we bring \hat{y}_{α} to the left by using the commutation relations (5.47). We arrive at

$$\hat{y}_{(\alpha} \star \hat{y}_{\beta(s))} = \hat{y}_{\alpha} \star \hat{y}_{\beta(s)} - \sum_{j=1}^s \sum_{m=0}^{j-1} 2i(1 + \nu k (-1)^{j-1-m}) \epsilon_{\alpha(\beta_1} \hat{y}_{\beta_2} \star \cdots \star \hat{y}_{\beta_s}), \quad (\text{D.96})$$

where we also used that k anticommutes with the \hat{y}_{β} . After evaluating the sum, we obtain

$$\hat{y}_{(\alpha} \star \hat{y}_{\beta(s))} = \hat{y}_{\alpha} \star \hat{y}_{\beta(s)} - i \left(s + \frac{\frac{1}{2}(1 - (-1)^s) + s}{s+1} \nu k \right) \epsilon_{\alpha(\beta_1} \hat{y}_{\beta_2} \star \cdots \star \hat{y}_{\beta_s}), \quad (\text{D.97})$$

which confirms (5.73). The identity (5.74) can be derived along very similar lines.

Using this formula, it follows that

$$\hat{y}_{\alpha_1} \star \hat{y}_{\alpha_2} \star \hat{y}_{\beta(s)} = \hat{y}_{\alpha_1} \star \left(\hat{y}_{\alpha_2} \star \hat{y}_{\beta(s)} + i \left(s + \frac{\frac{1}{2}(1 - (-1)^s) + s}{s+1} \nu k \right) \epsilon_{\alpha_2(\beta_1} \hat{y}_{\beta_2} \star \cdots \star \hat{y}_{\beta_s} \right). \quad (\text{D.98})$$

Applying the same formula again, one obtains that

$$\begin{aligned}
& \hat{y}_{\alpha_1} \star \hat{y}_{\alpha_2} \star \hat{y}_{\beta(s)} \\
&= \hat{y}_{(\alpha_1 \star \hat{y}_{\alpha_2} \star \hat{y}_{\beta(s)})} + i \left(s + 1 + \frac{\frac{1}{2}(1-(-)^{s+1})+s+1}{s+2} \nu k \right) \epsilon_{\alpha_1(\alpha_2 \hat{y}_{\beta(s)})} \\
&\quad + i \left(s - \frac{\frac{1}{2}(1-(-)^s)+s}{s+1} \nu k \right) \left\{ \frac{s+1}{s} \epsilon_{\alpha_2(\alpha_1 \hat{y}_{\beta(s)})} - \frac{1}{s} \epsilon_{\alpha_2 \alpha_1 \hat{y}_{\beta(s)}} \right. \\
&\quad \left. + i \left(s - 1 + \frac{\frac{1}{2}(1-(-)^{s-1})+s-1}{s} \nu k \right) \epsilon_{\alpha_1(\beta_1 \epsilon_{|\alpha_2| \beta_2} \hat{y}_{\beta_3} \star \cdots \star \hat{y}_{\beta_s})} \right\}.
\end{aligned} \tag{D.99}$$

Following completely analogous steps, one can show that $\hat{y}_{\beta(s)} \star \hat{y}_{\alpha_1} \star \hat{y}_{\alpha_2}$ is equal to

$$\begin{aligned}
& \hat{y}_{\beta(s)} \star \hat{y}_{\alpha_1} \star \hat{y}_{\alpha_2} \\
&= \hat{y}_{(\beta(s) \star \hat{y}_{\alpha_1} \star \hat{y}_{\alpha_2})} - i \left(s + 1 + \frac{\frac{1}{2}(1-(-)^{s+1})+(-)^{s+2}(s+1)}{s+2} \nu k \right) \epsilon_{\alpha_2(\alpha_1 \hat{y}_{\beta(s)})} \\
&\quad - i \left(s + \frac{\frac{1}{2}(1-(-)^s)+(-)^{s+1}s}{s+1} \nu k \right) \left\{ \frac{s+1}{s} \epsilon_{\alpha_1(\alpha_2 \hat{y}_{\beta(s)})} - \frac{1}{s} \epsilon_{\alpha_1 \alpha_2 \hat{y}_{\beta(s)}} \right. \\
&\quad \left. - i \left(s - 1 + \frac{\frac{1}{2}(1-(-)^{s-1})+(-)^s(s-1)}{s} \nu k \right) \epsilon_{\alpha_1(\beta_1 \epsilon_{|\alpha_2| \beta_2} \hat{y}_{\beta_3} \star \cdots \star \hat{y}_{\beta_s})} \right\}.
\end{aligned} \tag{D.100}$$

Using these two results (and simplifying them for even s), it is straightforward to check that (5.72) indeed holds when s is even.

Solution 51. Two identities are convenient to use

$$[f(y) \star z_\alpha g(y, z)]_{z=0} = [i \partial_\alpha^y f(y) \star g(y, z)]_{z=0}, \tag{D.101a}$$

$$z^\alpha (f(y) \star z_\alpha g(y, z)) = i z^\alpha (\partial_\alpha^y f(y) \star g(y, z)). \tag{D.101b}$$

Both identities can be easily checked using the integral representation of the star product. For example, the first identity can be derived as follows

$$\begin{aligned}
& \frac{1}{(2\pi)^2} \int d^2u d^2v e^{ivu} f(y+u) (z-v)_\alpha g(y+v, z-v) \Big|_{z=0} \\
&= \frac{1}{(2\pi)^2} \int d^2u d^2v e^{ivu} f(y+u) (-v_\alpha) g(y+v, -v) \\
&= \frac{1}{(2\pi)^2} \int d^2u d^2v (-i\partial_\alpha^u) e^{ivu} f(y+u) g(y+v, -v) \\
&= \frac{1}{(2\pi)^2} \int d^2u d^2v e^{ivu} (i\partial_\alpha^u) f(y+u) g(y+v, -v) \\
&= \frac{1}{(2\pi)^2} \int d^2u d^2v e^{ivu} (i\partial_\alpha^y f(y+u)) g(y+v, -v) \\
&= (i\partial_\alpha^y f(y) \star g(y, z)) \Big|_{z=0}. \tag{D.102}
\end{aligned}$$

The other identity follows along similar lines. Now, consider the equation of motion for the linear perturbations in the one-form,

$$\begin{aligned}
D^{AdS} \mathbf{A}^{(1)} &= -\frac{i}{2} D^{AdS} z^\alpha \Gamma_0 \langle D^{AdS} \mathcal{S}_\alpha^{(1)} \rangle \Big|_{z=0} \\
&= -\frac{i}{2} \left\{ \mathcal{A}^{AdS}, z^\alpha \Gamma_0 \left\langle [\mathcal{A}^{AdS}, z_\alpha t^{(1)}]_\star \right\rangle \right\}_\star \Big|_{z=0}, \tag{D.103}
\end{aligned}$$

where we have used the fact that $[d, z_\alpha] = 0$ and \mathcal{A}^{AdS} denotes the AdS background. Furthermore, we have defined $\mathcal{S}_\alpha^{(1)} = z_\alpha s^{(1)}$ where

$$s^{(1)} = \Gamma_1 \langle \mathbf{C}^{(1)}(y) \star \varkappa \rangle = \Gamma_1 \left\langle (\mathcal{C}^{(1)}(y)\psi + \tilde{\mathcal{C}}^{(1)}(y)) \star \varkappa \right\rangle. \tag{D.104}$$

We shall now apply the identities (D.101) to the equation of motion,

$$D^{AdS} \mathbf{A}^{(1)} = \frac{i}{2} \left\{ \partial_y^\alpha \mathcal{A}^{AdS}, \Gamma_0 \left\langle [\partial_\alpha^y \mathcal{A}^{AdS}, s^{(1)}]_\star \right\rangle \right\}_\star \Big|_{z=0}. \tag{D.105}$$

The derivative of \mathcal{A}^{AdS} is given by

$$\partial_y^\alpha \mathcal{A} = -\frac{i}{2} (\bar{e}^{\alpha\beta} \phi + \bar{\omega}^{\alpha\beta}) y_\beta, \tag{D.106}$$

and hence we find

$$D^{AdS} \mathbf{A}^{(1)} = -\frac{i}{8} \left\{ (\bar{e}^{\alpha\beta} \phi + \bar{\omega}^{\alpha\beta}) y_\beta, \Gamma_0 \left\langle [(\bar{e}_\alpha^\gamma \phi + \bar{\omega}_\alpha^\gamma) y_\gamma, s^{(1)}]_\star \right\rangle \right\}_\star \Big|_{z=0}. \tag{D.107}$$

We shall only focus on the terms proportional to $\bar{e} \wedge \bar{e}$ and $\mathcal{C}^{(1)}$ (the other terms vanish which can be shown using the same methods that we use in the following). Let us denote the projection to this part by $|_{\bar{e}\tilde{\mathcal{C}}}$, and we get

$$D^{AdS} \mathbf{A}^{(1)} \Big|_{\bar{e}\tilde{\mathcal{C}}} = -\frac{i}{8} \bar{e}^{\alpha\beta} \wedge \bar{e}_\alpha^\gamma \left\{ y_\beta, \Gamma_0 \left\langle \{y_\gamma, \Gamma_1 \langle \mathcal{C}^{(1)} \psi \star \varkappa \rangle\}_\star \right\rangle \right\}_\star \Big|_{z=0}. \tag{D.108}$$

Next we use

$$\Gamma_1 \langle \mathcal{C}^{(1)}(y) \star \mathcal{X} \rangle = \Gamma_1 \langle \mathcal{C}^{(1)}(-z) e^{iyz} \rangle = \int_0^1 d\tau \tau e^{i\tau y z} \mathcal{C}^{(1)}(-\tau z) \quad (\text{D.109})$$

and the identity

$$\{y_\alpha, f(y, z)\}_\star = 2(y - i\partial^z)_\alpha f(y, z) \quad (\text{D.110})$$

that can be straightforwardly verified using the integral representation of the star product. We obtain (using the two-form E defined in (5.127))

$$\begin{aligned} D^{AdS} \mathbf{A}^{(1)}|_{\bar{z}\bar{e}\bar{\mathcal{C}}} &= \frac{i}{2} E^{\alpha\beta} \int_0^1 \int_0^1 dt d\tau (y - i\partial^z)_\alpha (y - i\frac{1}{t}\partial^z)_\beta \tau \mathcal{C}^{(1)}(-t\tau z) e^{it\tau y z} \psi|_{z=0} \\ &= \frac{i}{2} E^{\alpha\beta} \int_0^1 \int_0^1 dt d\tau ((1-t)y - i\partial^z)_\alpha ((1-t)y - i\frac{1}{t}\partial^z)_\beta \tau \mathcal{C}^{(1)}(-t\tau z) \psi|_{z=0} \\ &= \frac{i}{2} E^{\alpha\beta} \int_0^1 \int_0^1 dt d\tau ((1-t)y + it\tau\partial^u)_\alpha ((1-t)y + it\tau\partial^u)_\beta \tau \mathcal{C}^{(1)}(u) \psi|_{u=0}. \end{aligned} \quad (\text{D.111})$$

It remains to evaluate the integrals:

$$yy: \int_0^1 \int_0^1 dt d\tau (1-t\tau)(1-\tau)\tau = \frac{1}{8}, \quad (\text{D.112a})$$

$$\partial\partial: \int_0^1 \int_0^1 dt d\tau t\tau^3 = \frac{1}{8}, \quad (\text{D.112b})$$

$$y\partial: \int_0^1 \int_0^1 dt d\tau (1-t\tau)\tau^2 = \frac{5}{24}, \quad (\text{D.112c})$$

$$\partial y: \int_0^1 \int_0^1 dt d\tau (1-\tau)t\tau^2 = \frac{1}{24}. \quad (\text{D.112d})$$

The $y\partial$ and ∂y contributions can be combined because $E^{\alpha\beta}$ is symmetric. We therefore obtain the final result

$$D^{AdS} \mathbf{A}^{(1)}|_{\bar{z}\bar{e}\bar{\mathcal{C}}} = \frac{i}{16} E^{\alpha\beta} (y_\alpha + i\partial_\alpha^u)(y_\beta + i\partial_\beta^u) \mathcal{C}^{(1)}(u) \psi|_{u=0}. \quad (\text{D.113})$$

Using the same methods, one can show that all contributions involving the twisted zero-form $\mathcal{C}^{(1)}$ cancel out.

Solution 52. Our aim is to show that for M defined in (5.128) we have

$$D^{AdS} (M\psi) = -\frac{i}{16} E^{\alpha\beta} (y_\alpha + i\partial_\alpha^u)(y_\beta + i\partial_\beta^u) \mathcal{C}^{(1)}(u) \psi|_{u=0}. \quad (\text{D.114})$$

We introduce a 0-form

$$F_{\alpha\beta} = \int_0^1 dt (t^2 - 1) (y_\alpha + it^{-1}\partial_\alpha^y)(y_\beta + it^{-1}\partial_\beta^y) \mathcal{C}^{(1)}(ty), \quad (\text{D.115})$$

such that $M = -\frac{i}{8}\phi \bar{e}^{\alpha\beta} F_{\alpha\beta}$. Then

$$D^{AdS}(M\psi) = -\frac{i}{8}\phi \left(d\bar{e}^{\alpha\beta} F_{\alpha\beta} - \bar{e}^{\alpha\beta} \wedge (dF_{\alpha\beta} - [\bar{\omega}, F_{\alpha\beta}]_* - \{\bar{e}, F_{\alpha\beta}\}_*) \right) \psi. \quad (\text{D.116})$$

We compute

$$[\bar{\omega}, F_{\alpha\beta}]_* = \bar{\omega}^{\gamma\delta} y_{(\gamma} \partial_{\delta)} F_{\alpha\beta} \quad , \quad \{\bar{e}, F_{\alpha\beta}\}_* = -\frac{i}{2} \bar{e}^{\gamma\delta} (y_\gamma y_\delta - \partial_\gamma^y \partial_\delta^y) F_{\alpha\beta} \phi. \quad (\text{D.117})$$

To evaluate $dF_{\alpha\beta}$ we use the equation of motion for $\mathcal{C}^{(1)}$ which implies

$$d\mathcal{C}^{(1)}(ty) = -\frac{i}{2} \bar{e}^{\gamma\delta} (t^2 y_\gamma y_\delta - t^{-2} \partial_\gamma^y \partial_\delta^y) \mathcal{C}^{(1)}(ty) \phi + \bar{\omega}^{\gamma\delta} y_{(\gamma} \partial_{\delta)}^y \mathcal{C}^{(1)}(ty). \quad (\text{D.118})$$

The last ingredient we need is

$$d\bar{e}^{\alpha\beta} = -2\bar{\omega}^{\gamma(\alpha} \wedge \bar{e}_{\gamma}^{\beta)} \quad (\text{D.119})$$

which can be inferred from $d\mathcal{A}^{AdS} = \mathcal{A}^{AdS} \wedge *_{\mathcal{A}^{AdS}}$. Combining everything we observe that in B all contributions containing $\bar{\omega}$ drop out, and we arrive at

$$\begin{aligned} D^{AdS}(M\psi) &= \frac{1}{16} \bar{e}^{\alpha\beta} \wedge \bar{e}^{\gamma\delta} \int_0^1 dt (t^2 - 1) \\ &\quad \left\{ \left(yy + \frac{2i}{t} y\partial - \frac{1}{t^2} \partial\partial \right)_{(\alpha\beta)} \left(t^2 yy - \frac{1}{t^2} \partial\partial \right)_{\gamma\delta} \right. \\ &\quad \left. - (yy - \partial\partial)_{\gamma\delta} \left(yy + \frac{2i}{t} y\partial - \frac{1}{t^2} \partial\partial \right)_{(\alpha\beta)} \right\} \mathcal{C}^{(1)}(ty) \psi. \end{aligned} \quad (\text{D.120})$$

Here we introduced the notation

$$\begin{aligned} \left(yy + \frac{2i}{t} y\partial - \frac{1}{t^2} \partial\partial \right)_{(\alpha\beta)} &= y_\alpha y_\beta + \frac{i}{t} (y_\alpha \partial_\beta^y + y_\beta \partial_\alpha^y) - \frac{1}{t^2} \partial_\alpha^y \partial_\beta^y \\ &= \left(y_{(\alpha} + \frac{i}{t} \partial_{(\alpha}^y \right) \left(y_{\beta)} + \frac{i}{t} \partial_{\beta)}^y \right). \end{aligned} \quad (\text{D.121})$$

The two-form built from the vielbein can be written as

$$\bar{e}^{\alpha\beta} \wedge \bar{e}^{\gamma\delta} = E^{(\gamma|(\alpha} e^{\beta)|\delta)}. \quad (\text{D.122})$$

When the epsilon tensor hits two adjacent y 's or two adjacent derivatives, there is no contribution, otherwise it produces the y -counting operator

$$y \cdot \partial^y = e^{\beta\delta} y_\delta \partial_\beta^y. \quad (\text{D.123})$$

We now commute all y 's to the left of the counting operator, and all derivatives to the right. After some algebra, and defining $u = ty$ such that $\partial^y = t \partial^u$ we arrive at

$$\begin{aligned} D^{AdS}(M\psi) &= \frac{1}{16} E^{\alpha\gamma} \int_0^1 dt (t^2 - 1) \\ &\quad \left\{ i y_\alpha y_\gamma \left(4t + (t^2 - 1) \frac{1}{t} y \cdot \partial^y \right) - y_\alpha \left(4t + (t^2 - 1) \frac{1}{t} y \cdot \partial^y \right) \partial_\gamma^u \right. \\ &\quad \left. - y_\gamma \left(4t + (t^2 - 1) \frac{1}{t} y \cdot \partial^y \right) \partial_\alpha^u - i \left(4t + (t^2 - 1) \frac{1}{t} y \cdot \partial^y \right) \partial_\alpha^u \partial_\gamma^u \right\} \mathcal{C}^{(1)}(ty) \psi. \end{aligned} \quad (\text{D.124})$$

Now we use $(y \cdot \partial^y) f(ty) = t \frac{d}{dt} f(ty)$, and we finally obtain

$$\begin{aligned} D^{AdS}(M\psi) &= \frac{1}{16} E^{\alpha\gamma} \int_0^1 dt (t^2 - 1) \\ &\quad \left((t^2 - 1) \frac{d}{dt} + 4t \right) (i y_\alpha y_\gamma - y_\alpha \partial_\gamma^u - y_\gamma \partial_\alpha^u - i \partial_\alpha^u \partial_\gamma^u) \mathcal{C}^{(1)}(ty) \psi \\ &= \frac{1}{16} E^{\alpha\gamma} \int_0^1 dt \frac{d}{dt} \left((t^2 - 1)^2 (i y_\alpha y_\gamma - y_\alpha \partial_\gamma^u - y_\gamma \partial_\alpha^u - i \partial_\alpha^u \partial_\gamma^u) \mathcal{C}^{(1)}(ty) \psi \right) \\ &= -\frac{i}{16} E^{\alpha\gamma} (y_\alpha + i \partial_\alpha^u) (y_\gamma + i \partial_\gamma^u) \mathcal{C}^{(1)}(u) \Big|_{u=0} \psi. \end{aligned} \quad (\text{D.125})$$

References

1. X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, *Rev. Mod. Phys.* **84** (2012) 987 [[1007.0435](#)].
2. D. Ponomarev, *Basic Introduction to Higher-Spin Theories*, *Int. J. Theor. Phys.* **62** (2023) 146 [[2206.15385](#)].
3. A. Bengtsson, *Higher Spin Field Theory*, Texts and Monographs in Theoretical Physics, De Gruyter (2020), [10.1515/9783110451771](#).
4. E.S. Fradkin, “The problem of unification of all interactions and self-consistency.” Talk presented for the reception of the Dirac medal (Trieste, 1989).
5. D.J. Gross, *High-Energy Symmetries of String Theory*, *Phys. Rev. Lett.* **60** (1988) 1229.
6. A. Sagnotti, *Notes on Strings and Higher Spins*, *J. Phys. A* **46** (2013) 214006 [[1112.4285](#)].
7. R. Rahman and M. Taronna, *From Higher Spins to Strings: A Primer*, *Lect. Notes Phys.* **1028** (2024) 1 [[1512.07932](#)].
8. S. Giombi, *Higher Spin — CFT Duality*, in *Proceedings of the 2015 Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, pp. 137–214, 2017, DOI [[1607.02967](#)].
9. M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett. B* **243** (1990) 378.
10. M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys. Lett. B* **567** (2003) 139 [[hep-th/0304049](#)].

11. X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, in *1st Solvay Workshop on Higher Spin Gauge Theories*, pp. 132–197, 2004 [[hep-th/0503128](#)].
12. V.E. Didenko and E.D. Skvortsov, *Elements of Vasiliev Theory*, *Lect. Notes Phys.* **1028** (2024) 269 [[1401.2975](#)].
13. D. Ponomarev and E.D. Skvortsov, *Light-Front Higher-Spin Theories in Flat Space*, *J. Phys. A* **50** (2017) 095401 [[1609.04655](#)].
14. K. Krasnov, E. Skvortsov and T. Tran, *Actions for self-dual Higher Spin Gravities*, *JHEP* **08** (2021) 076 [[2105.12782](#)].
15. N. Boulanger, A. Campoleoni and S. Pekar, *New higher-spin curvatures in flat space*, *Phys. Rev. D* **108** (2023) L101904 [[2306.05367](#)].
16. A.Y. Segal, *Conformal higher spin theory*, *Nucl. Phys. B* **664** (2003) 59 [[hep-th/0207212](#)].
17. X. Bekaert, N. Boulanger, A. Campoleoni, M. Chiodaroli, D. Francia, M. Grigoriev et al., *Snowmass White Paper: Higher Spin Gravity and Higher Spin Symmetry*, in *Proceedings of the 2021 US Community Study on the Future of Particle Physics: Snowmass 2021, 2022* [[2205.01567](#)].
18. B. Binengar, *Relativistic Field Theories in Three-dimensions*, *J. Math. Phys.* **23** (1982) 1511.
19. S. Deser, R. Jackiw and G. 't Hooft, *Three-Dimensional Einstein Gravity: Dynamics of Flat Space*, *Annals Phys.* **152** (1984) 220.
20. S. Deser and R. Jackiw, *Three-Dimensional Cosmological Gravity: Dynamics of Constant Curvature*, *Annals Phys.* **153** (1984) 405.
21. M. Bañados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)].
22. M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys.Rev.* **D48** (1993) 1506 [[gr-qc/9302012](#)].
23. S. Carlip, *Quantum gravity in 2+1 dimensions*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2003), [10.1017/CBO9780511564192](#).
24. P. Kraus, *Lectures on black holes and the AdS(3) / CFT(2) correspondence*, *Lect. Notes Phys.* **755** (2008) 193 [[hep-th/0609074](#)].
25. G. Compère, *The Kerr/CFT correspondence and its extensions*, *Living Rev. Rel.* **15** (2012) 11 [[1203.3561](#)].
26. M. Blencowe, *A Consistent Interacting Massless Higher Spin Field Theory in $D = (2+1)$* , *Class.Quant.Grav.* **6** (1989) 443.
27. E. Bergshoeff, M.P. Blencowe and K.S. Stelle, *Area Preserving Diffeomorphisms and Higher Spin Algebra*, *Commun. Math. Phys.* **128** (1990) 213.
28. A. Achúcarro and P.K. Townsend, *A Chern–Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories*, *Phys. Lett. B* **180** (1986) 89.
29. E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, *Nucl.Phys.* **B311** (1988) 46.
30. C. Aragone and S. Deser, *Hypersymmetry in $D = 3$ of Coupled Gravity Massless Spin 5/2 System*, *Class. Quant. Grav.* **1** (1984) L9.
31. M.R. Gaberdiel and R. Gopakumar, *An AdS₃ Dual for Minimal Model CFTs*, *Phys.Rev.* **D83** (2011) 066007 [[1011.2986](#)].
32. M.R. Gaberdiel and R. Gopakumar, *Minimal Model Holography*, *J.Phys.* **A46** (2013) 214002 [[1207.6697](#)].
33. M. Henneaux and S.-J. Rey, *Nonlinear W_∞ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, *JHEP* **12** (2010) 007 [[1008.4579](#)].
34. A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [[1008.4744](#)].
35. I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical $O(N)$ vector model*, *Phys. Lett. B* **550** (2002) 213 [[hep-th/0210114](#)].
36. B. Sundborg, *Stringy gravity, interacting tensionless strings and massless higher spins*, *Nucl. Phys. B Proc. Suppl.* **102** (2001) 113 [[hep-th/0103247](#)].

37. A. Mikhailov, *Notes on higher spin symmetries*, [hep-th/0201019](#).
38. E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl. Phys. B* **644** (2002) 303 [[hep-th/0205131](#)], [Erratum: *Nucl. Phys. B* **660** (2003) 403].
39. E. Sezgin and P. Sundell, *Holography in 4D (super) higher spin theories and a test via cubic scalar couplings*, *JHEP* **07** (2005) 044 [[hep-th/0305040](#)].
40. S. Giombi and X. Yin, *The Higher Spin/Vector Model Duality*, *J.Phys. A* **46** (2013) 214003 [[1208.4036](#)].
41. M. Gutperle and P. Kraus, *Higher Spin Black Holes*, *JHEP* **05** (2011) 022 [[1103.4304](#)].
42. J. de Boer and J.I. Jottar, *Thermodynamics of higher spin black holes in AdS₃*, *JHEP* **01** (2014) 023 [[1302.0816](#)].
43. C. Bunster, M. Henneaux, A. Pérez, D. Tempo and R. Troncoso, *Generalized Black Holes in Three-dimensional Spacetime*, *JHEP* **05** (2014) 031 [[1404.3305](#)].
44. M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Black holes in three dimensional higher spin gravity: A review*, *J. Phys. A* **46** (2013) 214001 [[1208.5182](#)].
45. A. Pérez, D. Tempo and R. Troncoso, *Higher Spin Black Holes*, *Lect. Notes Phys.* **892** (2015) 265 [[1402.1465](#)].
46. A. Castro, *Lectures on Higher Spin Black Holes in AdS₃ Gravity*, *Acta Phys. Polon. B* **47** (2016) 2479.
47. S. Prokushkin and M.A. Vasiliev, *Higher spin gauge interactions for massive matter fields in 3-D AdS space-time*, *Nucl.Phys.* **B545** (1999) 385 [[hep-th/9806236](#)].
48. C. Fronsdal, *Massless Fields with Integer Spin*, *Phys.Rev.* **D18** (1978) 3624.
49. C. Fronsdal, *Singletons and Massless, Integral Spin Fields on de Sitter Space (Elementary Particles in a Curved Space. 7.*, *Phys. Rev.* **D20** (1979) 848.
50. S. Fredenhagen, O. Krüger and K. Mkrtchyan, *Vertex-Constraints in 3D Higher Spin Theories*, *Phys. Rev. Lett.* **123** (2019) 131601 [[1905.00093](#)].
51. M. Grigoriev, K. Mkrtchyan and E. Skvortsov, *Matter-free higher spin gravities in 3D: Partially-massless fields and general structure*, *Phys. Rev. D* **102** (2020) 066003 [[2005.05931](#)].
52. J. Hoppe, *Quantum theory of a massless relativistic surface and a two-dimensional bound state problem*, Ph.D. thesis, Massachusetts Institute of Technology, 1982. <https://dspace.mit.edu/handle/1721.1/15717>.
53. J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations: A graduate course for physicists*, Cambridge University Press (10, 2003).
54. B.L. Feigin, *The lie algebras and cohomologies of lie algebras of differential operators*, *Russian Mathematical Surveys* **43** (1988) 169.
55. C.N. Pope, L.J. Romans and X. Shen, *W(infinity) and the Racah-Wigner Algebra*, *Nucl. Phys. B* **339** (1990) 191.
56. E.S. Fradkin and V.Y. Linetsky, *Infinite dimensional generalizations of simple Lie algebras*, *Mod. Phys. Lett. A* **5** (1990) 1967.
57. M. Bordemann, J. Hoppe and P. Schaller, *INFINITE DIMENSIONAL MATRIX ALGEBRAS*, *Phys. Lett. B* **232** (1989) 199.
58. B. Khesin and F. Malikov, *Universal Drinfeld-Sokolov reduction and matrices of complex size*, *Commun. Math. Phys.* **175** (1996) 113 [[hep-th/9405116](#)].
59. M.A. Vasiliev, *Higher Spin Algebras and Quantization on the Sphere and Hyperboloid*, *Int.J.Mod.Phys.* **A6** (1991) 1115.
60. S. Monnier, *Finite higher spin transformations from exponentiation*, *Commun. Math. Phys.* **336** (2015) 1 [[1402.4486](#)].
61. C. Ahn, *The Large N 't Hooft Limit of Coset Minimal Models*, *JHEP* **10** (2011) 125 [[1106.0351](#)].
62. M.R. Gaberdiel and C. Vollenweider, *Minimal Model Holography for SO(2N)*, *JHEP* **08** (2011) 104 [[1106.2634](#)].
63. A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W-symmetries in three-dimensional higher-spin gauge theories*, *JHEP* **09** (2011) 113 [[1107.0290](#)].

64. C. Vollenweider, *Infinitely Generated Quantum W Algebras in Minimal Model Holography*, Ph.D. thesis, ETH Zurich, 2015. [10.3929/ethz-a-010528211](#).
65. M. Kelm, *Holographic Minimal Models and Orbifold Conformal Field Theories*, Ph.D. thesis, Zurich, ETH, 2016. [10.3929/ethz-a-010735840](#).
66. T. Basile, N. Boulanger and F. Buisseret, *Structure constants of $shs[\lambda]$: the deformed-oscillator point of view*, *J. Phys. A* **51** (2018) 025201 [[1604.04510](#)].
67. M.G. Eastwood, *Higher symmetries of the Laplacian*, *Annals Math.* **161** (2005) 1645 [[hep-th/0206233](#)].
68. C. Iazeolla and P. Sundell, *A Fiber Approach to Harmonic Analysis of Unfolded Higher-Spin Field Equations*, *JHEP* **10** (2008) 022 [[0806.1942](#)].
69. N. Boulanger and E.D. Skvortsov, *Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime*, *JHEP* **09** (2011) 063 [[1107.5028](#)].
70. E. Joung and K. Mkrtchyan, *Notes on higher-spin algebras: minimal representations and structure constants*, *JHEP* **05** (2014) 103 [[1401.7977](#)].
71. A. Campoleoni and S. Pekar, *Carrollian and Galilean conformal higher-spin algebras in any dimensions*, *JHEP* **02** (2022) 150 [[2110.07794](#)].
72. A. Castro, E. Hijano and A. Lepage-Jutier, *Unitarity Bounds in AdS₃ Higher Spin Gravity*, *JHEP* **06** (2012) 001 [[1202.4467](#)].
73. J.D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun.Math.Phys.* **104** (1986) 207.
74. M. Bañados, *Three-dimensional quantum geometry and black holes*, *AIP Conf.Proc.* **484** (1999) 147 [[hep-th/9901148](#)].
75. M. Henneaux, A. Pérez, D. Tempo and R. Troncoso, *Chemical potentials in three-dimensional higher spin anti-de Sitter gravity*, *JHEP* **12** (2013) 048 [[1309.4362](#)].
76. M. Bañados, *Global charges in Chern–Simons field theory and the (2+1) black hole*, *Phys. Rev. D* **52** (1996) 5816 [[hep-th/9405171](#)].
77. A. Castro, N. Iqbal and E. Llabrés, *Eternal Higher Spin Black Holes: a Thermofield Interpretation*, *JHEP* **08** (2016) 022 [[1602.09057](#)].
78. M. Bañados, R. Canto and S. Theisen, *Higher spin black holes in three dimensions: Remarks on asymptotics and regularity*, *Phys. Rev. D* **94** (2016) 024002 [[1601.05827](#)].
79. R. Benguria, P. Cordero and C. Teitelboim, *Aspects of the Hamiltonian Dynamics of Interacting Gravitational Gauge and Higgs Fields with Applications to Spherical Symmetry*, *Nucl.Phys.* **B122** (1977) 61.
80. M. Blagojevic, *Gravitation and gauge symmetries*, Institute of Physics Publishing, London (2002).
81. D. Grumiller and M. Riegler, *Most general AdS₃ boundary conditions*, *JHEP* **10** (2016) 023 [[1608.01308](#)].
82. F. Alessio, G. Barnich, L. Ciambelli, P. Mao and R. Ruzziiconi, *Weyl charges in asymptotically locally AdS₃ spacetimes*, *Phys. Rev. D* **103** (2021) 046003 [[2010.15452](#)].
83. R. Ruzziiconi and C. Zwickel, *Conservation and Integrability in Lower-Dimensional Gravity*, *JHEP* **04** (2021) 034 [[2012.03961](#)].
84. M. Geiller, C. Goeller and C. Zwickel, *3d gravity in Bondi-Weyl gauge: charges, corners, and integrability*, *JHEP* **09** (2021) 029 [[2107.01073](#)].
85. A. Campoleoni, L. Ciambelli, A. Delfante, C. Marteau, P.M. Petropoulos and R. Ruzziiconi, *Holographic Lorentz and Carroll frames*, *JHEP* **12** (2022) 007 [[2208.07575](#)].
86. C. Troessaert, *Enhanced asymptotic symmetry algebra of AdS₃*, *JHEP* **08** (2013) 044 [[1303.3296](#)].
87. A. Campoleoni and M. Henneaux, *Asymptotic symmetries of three-dimensional higher-spin gravity: the metric approach*, *JHEP* **03** (2015) 143 [[1412.6774](#)].
88. O. Coussaert, M. Henneaux and P. van Driel, *The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant*, *Class.Quant.Grav.* **12** (1995) 2961 [[gr-qc/9506019](#)].

89. M. Bañados, K. Bautier, O. Coussaert, M. Henneaux and M. Ortiz, *Anti-de Sitter / CFT correspondence in three-dimensional supergravity*, *Phys.Rev.* **D58** (1998) 085020 [[hep-th/9805165](#)].
90. M. Henneaux, L. Maoz and A. Schwimmer, *Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity*, *Annals Phys.* **282** (2000) 31 [[hep-th/9910013](#)].
91. J. Balog, L. Feher, L. O’Raifeartaigh, P. Forgacs and A. Wipf, *Toda Theory and W Algebra From a Gauged WZNW Point of View*, *Annals Phys.* **203** (1990) 76.
92. P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys. Rept.* **223** (1993) 183 [[hep-th/9210010](#)].
93. B. Khesin and R. Wendt, *The Geometry of Infinite-Dimensional Groups*, Springer Berlin, Heidelberg (2009).
94. A. Campoleoni, S. Fredenhagen and J. Raeymaekers, *Quantizing higher-spin gravity in free-field variables*, *JHEP* **02** (2018) 126 [[1712.08078](#)].
95. M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Spacetime Geometry in Higher Spin Gravity*, *JHEP* **10** (2011) 053 [[1106.4788](#)].
96. J.M. Figueroa-O’Farrill, J. Mas and E. Ramos, *A One parameter family of Hamiltonian structures for the KP hierarchy and a continuous deformation of the nonlinear W(KP) algebra*, *Commun. Math. Phys.* **158** (1993) 17 [[hep-th/9207092](#)].
97. L.A. Dickey, *Lectures on Classical W-Algebras*, *Acta Appl. Math.* **47** (1997) 243.
98. M. Bershadsky and H. Ooguri, *Hidden SL(n) Symmetry in Conformal Field Theories*, *Commun. Math. Phys.* **126** (1989) 49.
99. S.L. Lukyanov and V.A. Fateev, *Additional symmetries and exactly soluble models in two-dimensional conformal field theory*, *Soviet Scientific Reviews A, Physics: 15.2* (1990) .
100. M. Ammon and J. Erdmenger, *Gauge/gravity duality: Foundations and applications*, Cambridge University Press, Cambridge (4, 2015).
101. H. Nastase, *Introduction to the ADS/CFT Correspondence*, Cambridge University Press (9, 2015).
102. J. Penedones, *TASI lectures on AdS/CFT*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, pp. 75–136, 2017, DOI [[1608.04948](#)].
103. P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York (1997), [10.1007/978-1-4612-2256-9](#).
104. M.R. Gaberdiel, *An Introduction to conformal field theory*, *Rept. Prog. Phys.* **63** (2000) 607 [[hep-th/9910156](#)].
105. R. Blumenhagen and E. Plauschinn, *Introduction to conformal field theory: with applications to String theory*, vol. 779, Springer Berlin, Heidelberg (2009), [10.1007/978-3-642-00450-6](#).
106. T. Arakawa, *Introduction to w-algebras and their representation theory*, in *Perspectives in Lie Theory*, F. Callegaro, G. Carnovale, F. Caselli, C. De Concini and A. De Sole, eds., (Cham), pp. 179–250, Springer International Publishing (2017), DOI [[1605.00138](#)].
107. T. Procházka, *W-algebras and integrability*, [2402.10016](#).
108. A.B. Zamolodchikov, *Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory*, *Theor. Math. Phys.* **65** (1985) 1205.
109. K. Thielemans, *A Mathematica package for computing operator product expansions*, *Int. J. Mod. Phys. C* **2** (1991) 787.
110. M.R. Gaberdiel and R. Gopakumar, *Triality in Minimal Model Holography*, *JHEP* **07** (2012) 127 [[1205.2472](#)].
111. T. Procházka, *Exploring \mathcal{W}_∞ in the quadratic basis*, *JHEP* **09** (2015) 116 [[1411.7697](#)].
112. A.R. Linshaw, *Universal two-parameter \mathcal{W}_∞ -algebra and vertex algebras of type $\mathcal{W}(2,3,\dots,N)$* , *Compos. Math.* **157** (2021) 12 [[1710.02275](#)].
113. O. Tsybaliuk, *The affine Yangian of gl_1 , and the infinitesimal Cherednik algebras*, Ph.D. thesis, MIT, 2014. <http://hdl.handle.net/1721.1/90192>.
114. T. Procházka, *\mathcal{W} -symmetry, topological vertex and affine Yangian*, *JHEP* **10** (2016) 077 [[1512.07178](#)].

115. M.R. Gaberdiel, R. Gopakumar, W. Li and C. Peng, *Higher Spins and Yangian Symmetries*, *JHEP* **04** (2017) 152 [[1702.05100](#)].
116. F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, *Coset Construction for Extended Virasoro Algebras*, *Nucl. Phys. B* **304** (1988) 371.
117. V.A. Fateev and A.V. Litvinov, *Correlation functions in conformal Toda field theory. I.*, *JHEP* **11** (2007) 002 [[0709.3806](#)].
118. I. Bakas and E. Kiritsis, *Bosonic Realization of a Universal W-Algebra and Z_∞ Parafermions*, *Nucl. Phys. B* **343** (1990) 185 [Erratum: *Nucl. Phys. B* **350** (1991) 512].
119. M.R. Gaberdiel, K. Jin and W. Li, *Perturbations of $W(\infty)$ CFTs*, *JHEP* **10** (2013) 162 [[1307.4087](#)].
120. P. Goddard, A. Kent and D.I. Olive, *Unitary Representations of the Virasoro and super-Virasoro Algebras*, *Commun. Math. Phys.* **103** (1986) 105.
121. V.A. Fateev and S.L. Lukyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with $Z(n)$ Symmetry*, *Int. J. Mod. Phys. A* **3** (1988) 507.
122. M.R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, *Partition Functions of Holographic Minimal Models*, *JHEP* **08** (2011) 077 [[1106.1897](#)].
123. C.-M. Chang and X. Yin, *A semilocal holographic minimal model*, *Phys. Rev. D* **88** (2013) 106002 [[1302.4420](#)].
124. A. Jevicki and J. Yoon, *Field Theory of Primaries in W_N Minimal Models*, *JHEP* **11** (2013) 060 [[1302.3851](#)].
125. D. Roggenkamp and K. Wendland, *Limits and degenerations of unitary conformal field theories*, *Commun. Math. Phys.* **251** (2004) 589 [[hep-th/0308143](#)].
126. A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, *Conical Defects in Higher Spin Theories*, *JHEP* **02** (2012) 096 [[1111.3381](#)].
127. E. Perlmutter, T. Procházka and J. Raeymaekers, *The semiclassical limit of W_N CFTs and Vasiliev theory*, *JHEP* **05** (2013) 007 [[1210.8452](#)].
128. A. Campoleoni and S. Fredenhagen, *On the higher-spin charges of conical defects*, *Phys.Lett.* **B726** (2013) 387 [[1307.3745](#)].
129. J. Raeymaekers, *Quantization of conical spaces in 3D gravity*, *JHEP* **03** (2015) 060 [[1412.0278](#)].
130. J. Raeymaekers, *Conical spaces, modular invariance and $c_{p,1}$ holography*, *JHEP* **03** (2021) 189 [[2012.07934](#)].
131. M. Ammon, P. Kraus and E. Perlmutter, *Scalar fields and three-point functions in $D=3$ higher spin gravity*, *JHEP* **07** (2012) 113 [[1111.3926](#)].
132. C.-M. Chang and X. Yin, *Correlators in W_N Minimal Model Revisited*, *JHEP* **10** (2012) 050 [[1112.5459](#)].
133. T. Creutzig, Y. Hikida and P.B. Rønne, *$N=1$ supersymmetric higher spin holography on AdS_3* , *JHEP* **02** (2013) 019 [[1209.5404](#)].
134. M.R. Gaberdiel and R. Gopakumar, *Large $N=4$ Holography*, *JHEP* **09** (2013) 036 [[1305.4181](#)].
135. T. Creutzig, Y. Hikida and P.B. Rønne, *Extended higher spin holography and Grassmannian models*, *JHEP* **11** (2013) 038 [[1306.0466](#)].
136. M.R. Gaberdiel and R. Gopakumar, *Higher Spins & Strings*, *JHEP* **11** (2014) 044 [[1406.6103](#)].
137. P. Kessel, *The physics of higher-spin theories*, Ph.D. thesis, Humboldt University Berlin, 2016. [10.18452/17700](#).
138. L. Castellani, R. D'Auria and P. Fré, *Supergravity*, Supergravity and Superstrings: A Geometric Perspective, World Scientific (1991).
139. I. Fujisawa and R. Nakayama, *Metric-Like Formalism for Matter Fields Coupled to 3D Higher Spin Gravity*, *Class.Quant.Grav.* **31** (2014) 015003 [[1304.7941](#)].
140. I. Fujisawa, K. Nakagawa and R. Nakayama, *AdS/CFT for 3D Higher-Spin Gravity Coupled to Matter Fields*, *Class. Quant. Grav.* **31** (2014) 065006 [[1311.4714](#)].
141. M.A. Vasiliev, *Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions*, *Int.J.Mod.Phys.* **D5** (1996) 763 [[hep-th/9611024](#)].

142. S. Prokushkin and M.A. Vasiliev, *3-d higher spin gauge theories with matter*, in *Theory of Elementary Particles: Proceedings of the 31st International Symposium Ahrenshoop, September 2-6 1997, Buckow, Germany*, 12, 1998 [[hep-th/9812242](#)].
143. R. Bonezzi, N. Boulanger, E. Sezgin and P. Sundell, *An Action for Matter Coupled Higher Spin Gravity in Three Dimensions*, *JHEP* **05** (2016) 003 [[1512.02209](#)].
144. M.A. Vasiliev, *Equations of motion for $d = 3$ massless fields interacting through Chern–Simons higher spin gauge fields*, *Mod. Phys. Lett.* **A7** (1992) 3689.
145. P. Kessel, G. Lucena Gómez, E. Skvortsov and M. Taronna, *Higher Spins and Matter Interacting in Dimension Three*, *JHEP* **11** (2015) 104 [[1505.05887](#)].
146. A.V. Korybut, A.A. Sevostyanova, M.A. Vasiliev and V.A. Veretin, *Disentanglement of topological and dynamical fields in 3d higher-spin theory within shifted homotopy approach*, *Phys. Lett. B* **838** (2023) 137718 [[2211.15778](#)].
147. N. Boulanger, P. Kessel, E.D. Skvortsov and M. Taronna, *Higher spin interactions in four-dimensions: Vasiliev versus Fronsdal*, *J. Phys. A* **49** (2016) 095402 [[1508.04139](#)].
148. E.D. Skvortsov and M. Taronna, *On Locality, Holography and Unfolding*, *JHEP* **11** (2015) 044 [[1508.04764](#)].
149. C. Sleight and M. Taronna, *Higher-Spin Gauge Theories and Bulk Locality*, *Phys. Rev. Lett.* **121** (2018) 171604 [[1704.07859](#)].
150. D. Ponomarev, *A Note on (Non)-Locality in Holographic Higher Spin Theories*, *Universe* **4** (2018) 2 [[1710.00403](#)].
151. V.E. Didenko, O.A. Gelfond, A.V. Korybut and M.A. Vasiliev, *Homotopy Properties and Lower-Order Vertices in Higher-Spin Equations*, *J. Phys. A* **51** (2018) 465202 [[1807.00001](#)].
152. A. Castro, E. Hijano, A. Lepage-Jutier and A. Maloney, *Black Holes and Singularity Resolution in Higher Spin Gravity*, *JHEP* **01** (2012) 031 [[1110.4117](#)].
153. P. Kraus and E. Perlmutter, *Partition functions of higher spin black holes and their CFT duals*, *JHEP* **11** (2011) 061 [[1108.2567](#)].
154. M.R. Gaberdiel, K. Jin and E. Perlmutter, *Probing higher spin black holes from CFT*, *JHEP* **10** (2013) 045 [[1307.2221](#)].
155. H.S. Tan, *On scalar propagators of three-dimensional higher-spin black holes*, *JHEP* **09** (2016) 137 [[1608.00226](#)].
156. M. Bañados, R. Canto and S. Theisen, *The Action for higher spin black holes in three dimensions*, *JHEP* **07** (2012) 147 [[1204.5105](#)].
157. A. Pérez, D. Tempo and R. Troncoso, *Higher spin gravity in 3D: Black holes, global charges and thermodynamics*, *Phys.Lett.* **B726** (2013) 444 [[1207.2844](#)].
158. A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Towards metric-like higher-spin gauge theories in three dimensions*, *J.Phys.* **A46** (2013) 214017 [[1208.1851](#)].
159. A. Pérez, D. Tempo and R. Troncoso, *Higher spin black hole entropy in three dimensions*, *JHEP* **04** (2013) 143 [[1301.0847](#)].
160. J. de Boer and J.I. Jottar, *Boundary conditions and partition functions in higher spin AdS_3/CFT_2* , *JHEP* **04** (2016) 107 [[1407.3844](#)].
161. P. Kraus and E. Perlmutter, *Probing higher spin black holes*, *JHEP* **02** (2013) 096 [[1209.4937](#)].
162. M. Ammon, A. Castro and N. Iqbal, *Wilson Lines and Entanglement Entropy in Higher Spin Gravity*, *JHEP* **10** (2013) 110 [[1306.4338](#)].
163. J. de Boer and J.I. Jottar, *Entanglement Entropy and Higher Spin Holography in AdS_3* , *JHEP* **04** (2014) 089 [[1306.4347](#)].
164. E. Perlmutter, *Comments on Renyi entropy in AdS_3/CFT_2* , *JHEP* **05** (2014) 052 [[1312.5740](#)].
165. J. de Boer, A. Castro, E. Hijano, J.I. Jottar and P. Kraus, *Higher spin entanglement and \mathcal{W}_N conformal blocks*, *JHEP* **07** (2015) 168 [[1412.7520](#)].
166. E. Perlmutter, *Bounding the Space of Holographic CFTs with Chaos*, *JHEP* **10** (2016) 069 [[1602.08272](#)].
167. P. Narayan and J. Yoon, *Chaos in Three-dimensional Higher Spin Gravity*, *JHEP* **07** (2019) 046 [[1903.08761](#)].

168. L.F. Alday, J.-B. Bae, N. Benjamin and C. Jorge-Diaz, *On the Spectrum of Pure Higher Spin Gravity*, *JHEP* **12** (2020) 001 [2009.01830].
169. S. Zhao, C. Northe, K. Weisenberger and R. Meyer, *Charged moments in W_3 higher spin holography*, *JHEP* **05** (2022) 166 [2202.11111].
170. M.R. Gaberdiel and R. Gopakumar, *Stringy Symmetries and the Higher Spin Square*, *J. Phys. A* **48** (2015) 185402 [1501.07236].
171. L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *The Worldsheet Dual of the Symmetric Product CFT*, *JHEP* **04** (2019) 103 [1812.01007].
172. J. Raeymaekers, *On tensionless string field theory in AdS_3* , *JHEP* **07** (2019) 019 [1903.09647].
173. S. Prohazka and M. Riegler, *Higher Spins Without (Anti-)de Sitter*, *Universe* **4** (2018) 20 [1710.11105].
174. H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller and J. Rosseel, *Spin-3 Gravity in Three-Dimensional Flat Space*, *Phys. Rev. Lett.* **111** (2013) 121603 [1307.4768].
175. H.A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, *Asymptotically flat spacetimes in three-dimensional higher spin gravity*, *JHEP* **09** (2013) 016 [1307.5651].
176. M. Ammon, D. Grumiller, S. Prohazka, M. Riegler and R. Wutte, *Higher-Spin Flat Space Cosmologies with Soft Hair*, *JHEP* **05** (2017) 031 [1703.02594].
177. M. Ammon, M. Pannier and M. Riegler, *Scalar Fields in 3D Asymptotically Flat Higher-Spin Gravity*, *J. Phys. A* **54** (2021) 105401 [2009.14210].
178. A. Campoleoni, H.A. Gonzalez, B. Oblak and M. Riegler, *BMS Modules in Three Dimensions*, *Int. J. Mod. Phys. A* **31** (2016) 1650068 [1603.03812].
179. S. Fredenhagen and P. Kessel, *Metric- and frame-like higher-spin gauge theories in three dimensions*, *J.Phys. A* **48** (2015) 035402 [1408.2712].
180. I. Fujisawa and R. Nakayama, *Second-Order Formalism for 3D Spin-3 Gravity*, *Class.Quant.Grav.* **30** (2013) 035003 [1209.0894].
181. K. Mkrtchyan, *Cubic interactions of massless bosonic fields in three dimensions*, *Phys. Rev. Lett.* **120** (2018) 221601 [1712.10003].
182. P. Kessel and K. Mkrtchyan, *Cubic interactions of massless bosonic fields in three dimensions II: Parity-odd and Chern–Simons vertices*, *Phys. Rev. D* **97** (2018) 106021 [1803.02737].
183. S. Fredenhagen, O. Krüger and K. Mkrtchyan, *Constraints for Three-Dimensional Higher-Spin Interactions and Conformal Correlators*, *Phys. Rev. D* **100** (2019) 066019 [1812.10462].
184. M. Henneaux, G. Lucena Gómez, J. Park and S.-J. Rey, *Super- $W(\infty)$ Asymptotic Symmetry of Higher-Spin AdS_3 Supergravity*, *JHEP* **06** (2012) 037 [1203.5152].
185. K. Hanaki and C. Peng, *Symmetries of Holographic Super-Minimal Models*, *JHEP* **08** (2013) 030 [1203.5768].
186. H.S. Tan, *Exploring Three-dimensional Higher-Spin Supergravity based on $sl(N|N-1)$ Chern–Simons theories*, *JHEP* **11** (2012) 063 [1208.2277].
187. C. Candu, C. Peng and C. Vollenweider, *Extended supersymmetry in AdS_3 higher spin theories*, *JHEP* **12** (2014) 113 [1408.5144].
188. T. Creutzig, Y. Hikida and P.B. Rønne, *Higher spin AdS_3 supergravity and its dual CFT*, *JHEP* **02** (2012) 109 [1111.2139].
189. C. Candu and M.R. Gaberdiel, *Supersymmetric holography on AdS_3* , *JHEP* **09** (2013) 071 [1203.1939].
190. M. Beccaria, C. Candu, M.R. Gaberdiel and M. Groher, *$N=1$ extension of minimal model holography*, *JHEP* **07** (2013) 174 [1305.1048].
191. M. Grigoriev, I. Lovrekovic and E. Skvortsov, *New Conformal Higher Spin Gravities in 3d*, *JHEP* **01** (2020) 059 [1909.13305].
192. N. Boulanger, A. Campoleoni, V. Lekeu and E. Skvortsov, *Strange higher-spin topological systems in 3D*, *JHEP* **05** (2024) 109 [2312.03382].
193. Y.M. Zinoviev, *On higher spin cubic interactions in $d = 3$* , *JHEP* **11** (2021) 022 [2109.08480].

194. Y.M. Zinoviev, *On massive higher spin supermultiplets in $d=3$* , *Nucl. Phys. B* **996** (2023) 116351 [2307.16464].
195. T. Damour and S. Deser, 'Geometry' of Spin 3 Gauge Theories, *Ann. Inst. H. Poincaré Phys. Theor.* **47** (1987) 277.
196. B. Chen, J. Long and J.-B. Wu, *Spin-3 topologically massive gravity*, *Phys. Lett. B* **705** (2011) 513 [1106.5141].
197. A. Bagchi, S. Lal, A. Saha and B. Sahoo, *Topologically Massive Higher Spin Gravity*, *JHEP* **10** (2011) 150 [1107.0915].
198. E.A. Bergshoeff, M. Kovacevic, J. Rosseel, P.K. Townsend and Y. Yin, *A spin-4 analog of 3D massive gravity*, *Class. Quant. Grav.* **28** (2011) 245007 [1109.0382].
199. B. Chen and J. Long, *High Spin Topologically Massive Gravity*, *JHEP* **12** (2011) 114 [1110.5113].
200. N. Boulanger, D. Ponomarev, E. Sezgin and P. Sundell, *New unfolded higher spin systems in AdS_3* , *Class. Quant. Grav.* **32** (2015) 155002 [1412.8209].
201. D. Dalmazi and A.L.R.d. Santos, *Higher spin analogs of linearized topologically massive gravity and linearized new massive gravity*, *Phys. Rev. D* **104** (2021) 085023 [2107.08879].
202. E.A. Bergshoeff, O. Hohm and P.K. Townsend, *On Higher Derivatives in 3D Gravity and Higher Spin Gauge Theories*, *Annals Phys.* **325** (2010) 1118 [0911.3061].
203. I.L. Buchbinder, T.V. Snegirev and Y.M. Zinoviev, *Lagrangian formulation of the massive higher spin supermultiplets in three dimensional space-time*, *JHEP* **10** (2015) 148 [1508.02829].
204. S.M. Kuzenko and M. Tsulaia, *Off-shell massive $N=1$ supermultiplets in three dimensions*, *Nucl. Phys. B* **914** (2017) 160 [1609.06910].
205. N. Boulanger, P. Sundell and M. Valenzuela, *Three-dimensional fractional-spin gravity*, *JHEP* **02** (2014) 052 [1312.5700], [Erratum: *JHEP* **03** (2016) 076].
206. N. Boulanger, P. Sundell and M. Valenzuela, *Gravitational and gauge couplings in Chern–Simons fractional spin gravity*, *JHEP* **01** (2016) 173 [1504.04286], [Erratum: *JHEP* **03** (2016) 075].
207. P. Schuster and N. Toro, *A new class of particle in $2 + 1$ dimensions*, *Phys. Lett. B* **743** (2015) 224 [1404.1076].
208. M. Gary, D. Grumiller and R. Rashkov, *Towards non-AdS holography in 3-dimensional higher spin gravity*, *JHEP* **03** (2012) 022 [1201.0013].
209. H. Afshar, M. Gary, D. Grumiller, R. Rashkov and M. Riegler, *Non-AdS holography in 3-dimensional higher spin gravity - General recipe and example*, *JHEP* **11** (2012) 099 [1209.2860].
210. M. Gary, D. Grumiller, S. Prohazka and S.-J. Rey, *Lifshitz Holography with Isotropic Scale Invariance*, *JHEP* **08** (2014) 001 [1406.1468].
211. E. Bergshoeff, D. Grumiller, S. Prohazka and J. Rosseel, *Three-dimensional Spin-3 Theories Based on General Kinematical Algebras*, *JHEP* **01** (2017) 114 [1612.02277].
212. D. Chernyavsky and D. Sorokin, *Three-dimensional (higher-spin) gravities with extended Schrödinger and l -conformal Galilean symmetries*, *JHEP* **07** (2019) 156 [1905.13154].
213. R. Caroca, D.M. Peñafiel and P. Salgado-Rebolledo, *Nonrelativistic spin-3 symmetries in $2+1$ dimensions from expanded and extended Nappi-Witten algebras*, *Phys. Rev. D* **107** (2023) 064034 [2208.00602].
214. P. Concha, C. Henríquez-Báez and E. Rodríguez, *Non-relativistic and ultra-relativistic expansions of three-dimensional spin-3 gravity theories*, *JHEP* **10** (2022) 155 [2208.01013].
215. S. Golkar, D.X. Nguyen, M.M. Roberts and D.T. Son, *Higher-Spin Theory of the Magnetorotons*, *Phys. Rev. Lett.* **117** (2016) 216403 [1602.08499].
216. A. Cappelli and E. Randellini, *Multipole Expansion in the Quantum Hall Effect*, *JHEP* **03** (2016) 105 [1512.02147].
217. Z. Liu, A. Gromov and Z. Papić, *Geometric quench and nonequilibrium dynamics of fractional quantum Hall states*, *Phys. Rev. B* **98** (2018) 155140 [1803.00030].

218. M.A. Vasiliev, 'Gauge' form description of massless fields with arbitrary spin. (in Russian), *Yad.Fiz.* **32** (1980) 855 [English translation: *Sov. J. Nucl. Phys.* **32** (1980) 439].
219. V. Lopatin and M.A. Vasiliev, *Free Massless Bosonic Fields of Arbitrary Spin in d-dimensional De Sitter Space*, *Mod.Phys.Lett.* **A3** (1988) 257.
220. Y.M. Zinoviev, *Frame-like gauge invariant formulation for massive high spin particles*, *Nucl. Phys. B* **808** (2009) 185 [[0808.1778](#)].
221. E.S. Fradkin and M.A. Vasiliev, *Candidate to the Role of Higher Spin Symmetry*, *Annals Phys.* **177** (1987) 63.
222. D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints*, Springer Berlin, Heidelberg (1990), [10.1007/978-3-642-83938-2](#).
223. A.W. Wipf, *Hamilton's formalism for systems with constraints*, *Lect. Notes Phys.* **434** (1994) 22 [[hep-th/9312078](#)].
224. M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press (1994).