

Infinitely many normalized solutions of L^2 -supercritical NLS equations on noncompact metric graphs with localized nonlinearities

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Abstract

We consider the existence of solutions for nonlinear Schrödinger equations on noncompact metric graphs with localized nonlinearities. In an L^2 -supercritical regime, we establish the existence of infinitely many solutions for any prescribed mass.

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1 Introduction and main results

Throughout the paper we assume that \mathcal{G} is a noncompact metric graph which satisfies:

$$\mathcal{G} \text{ has a finite number of edges and vertices, a non trivial compact core } \mathcal{K} \text{ and at least one half-line.} \quad (1.1)$$

The notion of metric graph is detailed in [14]. We recall that if \mathcal{G} is a metric graph with a finite number of edges and vertices, its compact core \mathcal{K} is defined as the metric sub-graph of \mathcal{G} consisting of all the bounded edges of \mathcal{G} (see [4, 34]).

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The paper is devoted to the existence of infinitely many solutions, sometimes called *bound states*, of prescribed mass for the L^2 -supercritical nonlinear Schrödinger (NLS) equation with localized nonlinearities on \mathcal{G}

$$-u'' + \lambda u = \kappa(x)|u|^{p-2}u, \quad (1.2)$$

coupled with the Kirchhoff conditions at the vertices, see (1.5) below. Here $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, $p > 6$, \mathcal{G} satisfies (1.1) and κ is the characteristic function of the compact core \mathcal{K} of \mathcal{G} .

The study of NLS on metric graphs has attracted much attention over the last few decades. Among the reasons coming from physics to consider Schrödinger equations on a metric graph, the most prominent one comes from the study of quantum graphs, see for example [6, 14, 28, 29] and the references therein. In particular, the papers [24, 29] present several interesting motivations for investigating the cases where the nonlinearity affects only a compact part of the graph.

Solutions to (1.2) with prescribed mass, often referred to as *normalized solutions*, correspond to critical points of the NLS energy functional $E(\cdot, \mathcal{G}): H^1(\mathcal{G}) \rightarrow \mathbb{R}$ defined by

$$E(u, \mathcal{G}) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx - \frac{1}{p} \int_{\mathcal{K}} |u|^p dx, \quad (1.3)$$

under the corresponding mass constraint

$$\int_{\mathcal{G}} |u|^2 dx = \mu > 0. \quad (1.4)$$

It is standard to show that $E(\cdot, \mathcal{G})$ is of class C^2 on $H^1(\mathcal{G})$. Note that solutions to (1.2) provide standing waves of the time-dependent focusing NLS on \mathcal{G} ,

$$i \partial_t \psi(t, x) = -\partial_{xx} \psi(t, x) - \kappa(x) |\psi(t, x)|^{p-2} \psi(t, x),$$

via the ansatz $\psi(t, x) = e^{i\lambda t} u(x)$. The constraint (1.4) is meaningful from a dynamics perspective as the mass (or charge), as well as the energy, is conserved by the NLS flow.

Recently, much effort has been devoted to establish the existence of normalized solutions of NLS on metric graphs, in the L^2 -subcritical (i.e., $p \in (2, 6)$) or L^2 -critical regimes (i.e., $p = 6$). In these two regimes, the energy functional $E(\cdot, \mathcal{G})$ is bounded from below and coercive on the mass constraint. A relevant notion is then the one of ground states, namely of solutions which minimize the energy functional on the constraint. For the existence of ground state solutions, the reader can consult [1, 2, 3, 4, 5, 30, 31] for noncompact graphs \mathcal{G} , and [17, 20] for compact ones; some studies are also conducted on the existence of local minimizers, see e.g. [7, 32].

Regarding problems with a localized nonlinearity as in (1.2), existence and non-existence of ground state solutions was discussed in [39] and of bound state solutions in [35] for the L^2 -subcritical case. We refer to [21, 22] for the same problem on the L^2 -critical case. Moreover, in the L^2 -subcritical regime, one may obtain the existence of multiple bound states with negative energy levels by applying genus theory both in the compact case as in [20] and in the noncompact case with localised nonlinearities as in [34].

However, in the L^2 -supercritical regime on general metric graphs, i.e., when $p > 6$, the energy functional $E(\cdot, \mathcal{G})$ is always unbounded from below. Moreover, due to the fact that graphs are not scale invariant, the techniques based on scalings, usually employed in the Euclidean setting and related to the validity of a Pohozaev identity (see [26] or [11, 12, 25, 37, 38]), do not work. These two features make the search for normalized solutions in the L^2 -supercritical regime delicate. Recently, in [19], this issue was considered on compact metric graphs for which the existence of a non-constant solution was proved for small values of $\mu > 0$. In [15], the case of a noncompact graph with a nonlinearity acting only on its compact core was considered. For any mass the existence of at least one positive solution to (1.2) was obtained. Our aim here is to show that, under exactly the same assumptions as in [15], the existence of infinitely many, possibly sign-changing, solutions can be obtained for an arbitrary mass.

Basic notations and main result

For any graph, we write $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, where \mathcal{E} is the set of edges and \mathcal{V} is the set of vertices. Each bounded edge e is identified with a closed bounded interval $I_e = [0, \ell_e]$ (where ℓ_e is the length of e), while each unbounded edge is identified with a closed half-line $I_e = [0, +\infty)$. The length of the shortest path between points provides a natural metric (whence a topology and a Borel structure) on \mathcal{G} . A function $u : \mathcal{G} \rightarrow \mathbb{R}$ is identified with a vector of functions $\{u_e\}_{e \in \mathcal{E}}$, where each u_e is defined on the corresponding interval I_e such that $u|_e = u_e$. Endowing each edge with the Lebesgue measure, one can define $\int_{\mathcal{G}} u(x) dx$ and the space $L^p(\mathcal{G})$ in a natural way, with norm

$$\|u\|_{L^p(\mathcal{G})}^p = \sum_{e \in \mathcal{E}} \|u_e\|_{L^p(e)}^p.$$

The Sobolev space $H^1(\mathcal{G})$ consists of the set of continuous functions $u : \mathcal{G} \rightarrow \mathbb{R}$ such that $u_e \in H^1(\dot{e})$ for every edge e ; the norm in $H^1(\mathcal{G})$ is defined as

$$\|u\|_{H^1(\mathcal{G})}^2 = \sum_{e \in \mathcal{E}} \|u'_e\|_{L^2(e)}^2 + \|u_e\|_{L^2(e)}^2.$$

More details can be found in [3, 4, 14].

We shall study the existence of critical points of the functional $E(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \rightarrow \mathbb{R}$ constrained on the L^2 -sphere

$$H_\mu^1(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) \mid \int_{\mathcal{G}} |u|^2 dx = \mu \right\}.$$

If $u \in H_\mu^1(\mathcal{G})$ is such a critical point, it is standard to show that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u satisfies the following problem:

$$\begin{cases} -u'' + \lambda u = \kappa(x)|u|^{p-2}u & \text{on every edge } e \in \mathcal{E}, \\ \sum_{e \succ v} \frac{du_e}{dx}(v) = 0 & \text{at every vertex } v \in \mathcal{V}, \end{cases} \quad (1.5)$$

where $e \succ v$ means that the edge e is incident at v , and the notation $du_e/dx(v)$ stands for $u'_e(0)$ or $-u'_e(\ell_e)$, according to whether the vertex v is identified with 0 or ℓ_e (namely, the sum involves the derivatives away from the vertex v). The second equation is the so-called *Kirchhoff boundary condition*.

Our main result is the following :

Theorem 1.1. *Let \mathcal{G} be any metric graph satisfying Assumption (1.1) and $p > 6$. Then, for any $\mu > 0$, Problem (1.5) with the mass constraint (1.4) has infinitely many distinct solutions. Moreover, these solutions are associated to positive Lagrange multipliers and correspond to a sequence of critical points of the functional $E(\cdot, \mathcal{G})$ constrained on $H_\mu^1(\mathcal{G})$ whose levels go to $+\infty$.*

In the derivation of the results of [15, 19], a central difficulty was the lack a priori bounds on the Palais-Smale sequences for $E(\cdot, \mathcal{G})$ constrained to $H_\mu^1(\mathcal{G})$. To overcome this difficulty an approach by approximation was developed. It consists in considering the family of functionals $E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \rightarrow \mathbb{R}$ given by

$$E_\rho(u, \mathcal{G}) := \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx - \frac{\rho}{p} \int_{\mathcal{X}} |u|^p dx, \quad \forall u \in H^1(\mathcal{G}), \forall \rho \in \left[\frac{1}{2}, 1 \right]. \quad (1.6)$$

We shall also proceed this way. Clearly a critical point of $E_\rho(\cdot, \mathcal{G})$ constrained to $H_\mu^1(\mathcal{G})$ is a solution to

$$\begin{cases} -u'' + \lambda u = \rho \kappa(x)|u|^{p-2}u & \text{on every edge } e \in \mathcal{E}, \\ \sum_{e \succ v} \frac{du_e}{dx}(v) = 0 & \text{at every vertex } v \in \mathcal{V}, \end{cases} \quad (1.7)$$

where λ is the associated Lagrange multiplier. Denoting by $m(u)$ the Morse index of a solution $u \in H_\mu^1(\mathcal{G})$ to (1.7), we establish

Theorem 1.2. *Let \mathcal{G} satisfy Assumption (1.1) and $p > 6$. For any $\mu > 0$ there exists $N_0 \in \mathbb{N}$ such that for almost every $\rho \in [1/2, 1]$, there exist sequences of Lagrange multipliers $\{\lambda_\rho^N\}_{N=N_0}^\infty \subset \mathbb{R}^+$ and solutions $\{u_\rho^N\}_{N=N_0}^\infty \subset H_\mu^1(\mathcal{G})$ to*

$$\begin{cases} -(u_\rho^N)'' + \lambda_\rho^N u_\rho^N = \rho \kappa(x) |u_\rho^N|^{p-2} u_\rho^N & \text{on every edge } e \in \mathcal{E}, \\ \sum_{e \succ v} \frac{du_\rho^N}{dx}(v) = 0 & \text{at every vertex } v \in \mathcal{V}. \end{cases} \quad (1.8)$$

In addition, $c_\rho^N := E_\rho(u_\rho^N, \mathcal{G}) \xrightarrow{N \rightarrow \infty} +\infty$ uniformly w.r.t. $\rho \in [1/2, 1]$ and $m(u_\rho^N) \leq N + 1$.

To derive Theorem 1.1 from Theorem 1.2, one considers for every fixed $\mu > 0$ and every fixed $N \geq N_0$, a sequence $\{u_{\rho_n}^N\}_{n=1}^\infty$ of solutions to (1.8) where $\rho_n \rightarrow 1^-$ and shows that it converges to some $u^N \in H_\mu^1(\mathcal{G})$. Such $u^N \in H_\mu^1(\mathcal{G})$ will be a solution to (1.4)–(1.5). The point here is to show that the sequence is bounded which in turn is equivalent to showing that the sequence $\{\lambda_{\rho_n}^N\}_{n=1}^\infty \subset \mathbb{R}$ is bounded. In [15, 19] this step was done through a blow-up analysis taking advantage that $u_{\rho_n}^N \in H_\mu^1(\mathcal{G})$ were positive functions. A more general blow-up analysis, in particular for possibly sign-changing solutions, was subsequently performed in [18]. A consequence of this blow-up analysis (see [18, Corollary 1.4]), stated here under our notation in Lemma 6.1, guarantees the boundedness of the sequence of $\{\lambda_{\rho_n}^N\}_{n=1}^\infty \subset \mathbb{R}$ thanks to the boundedness of the Morse index of the solutions $u_{\rho_n}^N \in H_\mu^1(\mathcal{G})$.

Now let us turn to the proof of Theorem 1.2. It relies on an abstract result [16, Theorem 1.12] which we recall here as Theorem 2.5. Used on our family $E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \rightarrow \mathbb{R}$, it will guarantee that, for any $\mu > 0$ and any $N \in \mathbb{N}$, under some geometric conditions, the functional $E_\rho(\cdot, \mathcal{G})$ admits, for almost every $\rho \in [1/2, 1]$, a *bounded* Palais-Smale sequence $\{u_{\rho,n}^N\}_{n=1}^\infty$ at level c_ρ^N which has an ‘‘approximate constrained Morse index at most N .’’

To be more specific, our strategy to prove Theorem 1.2 is the following. First we show that the geometric assumptions on $E_\rho(\cdot, \mathcal{G})$, $\rho \in [1/2, 1]$ are satisfied. Second, we check that the Palais-Smale sequences provided by the application of Theorem 2.5 converge. Finally, we observe that this process guarantees the existence of infinitely many distinct solutions $u_\rho^N \in H_\mu^1(\mathcal{G})$ since $c_\rho^N \rightarrow +\infty$ as $N \rightarrow +\infty$. Let us now provide more information on the first two steps.

The fact that the mentioned geometric assumptions hold is established in Proposition 4.1. Proving this proposition is a central part of the paper and for this we are indebted to ideas from [9, 10, 33]. Our proof of Proposition 4.1 uses the assumption that \mathcal{G} has at least one half-line, and it’s unclear whether a similar result would hold if the graph were compact. As a result the noncompactness of \mathcal{G} appears to be essential in the derivation of Theorem 1.2, see also Remark 1.5.

Regarding the convergence of the bounded Palais-Smale sequences $\{u_{\rho,n}^N\}_{n=1}^\infty$ provided by the application of Theorem 2.5. An essential element of the argument is to establish that the associated sequence of *almost Lagrange multipliers* $\{\lambda_{\rho,n}^N\}_{n=1}^\infty$ (see page 8) converges, up to a subsequence, to a positive $\lambda_\rho^N \in \mathbb{R}$. This is done in two steps. First, making use of the Morse type information carried by the sequence $\{u_{\rho,n}^N\}_{n=1}^\infty$, we show that $\lambda_\rho^N < 0$ is impossible. Here again we use the assumption that our graph contains one half-line, see the proof of Lemma 5.2. Second, to show that $\lambda_\rho^N \neq 0$ requires a specific treatment. In [15, 19] we were dealing with Palais-Smale sequences consisting of non-negative functions and thus their weak limits (which are solutions to (1.7) with possibly a smaller L^2 norm than $\sqrt{\mu}$) were also non-negative. It was then rather direct to show that $\lambda_\rho^N > 0$: see [19, Remark 1.2] in the case of a compact graph, or [15, Proof of Proposition 1.5] in the case of a noncompact graph with a localized nonlinearity. In our problem the weak limits are likely to be sign-changing. In general, there may exist nonzero solutions with a vanishing

Lagrange multiplier, as was already observed in [35, Section 4]. For a simple example (taken from [35, Theorem 4.2 and Remark 4.6]), consider the tadpole graph shown in Figure 1.



Figure 1: A tadpole graph

If the loop has a suitable length, one can put a sign-changing periodic solution of the equation $-u'' = |u|^{p-2}u$ on the loop and extend it by zero on the half-line to obtain a solution of the problem on the whole tadpole graph with a Lagrange multiplier equal to zero. To treat general graphs we make use of EDO techniques in a way which we believe new in this context. Assuming that $\lambda = 0$ in (1.7), we show that the L^2 norm of a solution $u \in H_\mu^1(\mathcal{G})$ goes to infinity as $E(u, \mathcal{G})$ goes to infinity, see Lemma 3.5 for a precise statement. This observation leads to the conclusion that if the suspected energy level, $c_\rho^N \in \mathbb{R}$ is sufficiently high, the case $\lambda_\rho^N = 0$ cannot happen. Having obtained that $\lambda_\rho^N > 0$ and using that the nonlinearity is compactly supported we obtain the convergence of our Palais-Smale sequences and this proves Theorem 1.2.

Remark 1.3. Our multiplicity result Theorem 1.1 is in sharp contrast to what has been observed in [34, 35] in the mass subcritical case $p < 6$. Indeed, [35, Corollary 3.8] shows that for a *graph without cycle* (also called a tree), with at most one *pendant* (see [35] for the terminology), there are no solutions to (1.4)–(1.5) when $p \in [4, 6[$ and $\mu > 0$ is small. Also, in [34, Theorem 1.2], to obtain $k \in \mathbb{N}$ solutions it is necessary to assume that $\mu > \mu(k)$. We have no such limitations in Theorem 1.1.

Remark 1.4. As it was already observed in [34] in the mass subcritical case, the localization of the nonlinearity on the non-trivial compact core is essential to our multiplicity results. Indeed, if the compact core is reduced to a point, \mathcal{G} is a star graph and (1.5) becomes linear. This problem possesses no solution in $H^1(\mathcal{G})$ regardless of the value of $\mu > 0$. On the other hand, if \mathcal{G} is an interval with two half-lines attached to its endpoints and the nonlinearity is not localized, then solutions to (1.4)–(1.5) are the same as those on \mathbb{R} , namely the unique symmetric positive ground state, its opposite, along with their translations (all of which have the same energy level).

Remark 1.5. Let us mention that the issue of multiplicity, even the existence of just two non-trivial solutions, is still open for a general compact graph \mathcal{G} . In [19, Theorem 1.1] only one non-constant solution is obtained (note that there always exists a constant solution to (1.4)–(1.5) on a compact graph).

The paper is organized as follows. In Section 2 we recall with Theorem 2.5 the contents of [16, Theorem 1.6] and explore some of its consequences. In particular, we show that second-order information on Palais-Smale sequences can be used to obtain uniform bounds from below on the associated sequences of almost Lagrange multipliers, see Lemma 2.7. We also derive some results, in Lemma 2.8 and Theorem 2.10, which provide abstract conditions allowing to check that the main assumptions of Theorem 2.5 hold. Most of Section 3 is devoted to show that solutions to (1.7) with $\lambda = 0$ have a L^2 norm going to infinity as their Energy goes to infinity (see Proposition 3.6). In Section 4, we prove Proposition 4.1 which shows that our problem can indeed be treated by an application of Theorem 2.5. In Section 5 we give the proof of Theorem 1.2. Finally, in Section 6 we deduce Theorem 1.1 from Theorem 1.2 making use of the already mentioned blow up analysis result from [18].

2 An Abstract Multiplicity Result

In this section we recall in Theorem 2.5 the contents of [16, Theorem 1.12] and present some of its consequences. We also establish results which permit to check the two main hypotheses the set defined by (2.3)

must satisfy: Lemma 2.8 gives a procedure to prove it is non-void and Theorem 2.10 provides a tool to check the key strict inequality (2.4) appearing in Theorem 2.5.

In order to state [16, Theorem 1.12] we need to recall some definitions.

Let $(E, \langle \cdot, \cdot \rangle)$ and $(H, (\cdot, \cdot))$ be two *infinite-dimensional* Hilbert spaces and assume that $E \hookrightarrow H \hookrightarrow E'$, with continuous injections. For simplicity, assume that the continuous injection $E \hookrightarrow H$ has norm at most 1 and identify E with its image in H . Set

$$\begin{cases} \|u\|^2 = \langle u, u \rangle, & u \in E, \\ |u|^2 = (u, u), & u \in H, \end{cases}$$

and define for $\mu > 0$:

$$S_\mu = \{u \in E \mid |u|^2 = \mu\}.$$

In the context of this paper, we shall have $E = H^1(\mathcal{G})$ and $H = L^2(\mathcal{G})$. Clearly, S_μ is a smooth submanifold of E of codimension 1. Furthermore its tangent space at a given point $u \in S_\mu$ can be considered as the closed subspace of codimension 1 of E given by:

$$T_u S_\mu = \{v \in E \mid (u, v) = 0\}.$$

In the following definition, we denote $\|\cdot\|_*$ and $\|\cdot\|_{**}$ the operator norm of $\mathcal{L}(E, \mathbb{R})$ and of $\mathcal{L}(E, \mathcal{L}(E, \mathbb{R}))$ respectively.

Definition 2.1. Let $\phi : E \rightarrow \mathbb{R}$ be a C^2 -functional on E and $\alpha \in (0, 1]$. We say that ϕ' and ϕ'' are α -Hölder continuous on bounded sets if for any $R > 0$, one can find $M = M(R) > 0$ such that, for any $u_1, u_2 \in B(0, R)$:

$$\|\phi'(u_1) - \phi'(u_2)\|_* \leq M \|u_1 - u_2\|^\alpha, \quad \|\phi''(u_1) - \phi''(u_2)\|_{**} \leq M \|u_1 - u_2\|^\alpha. \quad (2.1)$$

Remark 2.2. Note that, if ϕ'' is α -Hölder continuous on bounded sets, then ϕ' is Lipschitz continuous on bounded sets, whence also α -Hölder continuous on bounded sets.

Definition 2.3. Let ϕ be a C^2 -functional on E . For any $u \in E$, we define the continuous bilinear map:

$$D^2\phi(u) := \phi''(u) - \frac{\phi'(u)[u]}{|u|^2}(\cdot, \cdot).$$

Note that, if u is a critical point of ϕ restricted to the sphere S_μ , then $D^2\phi(u)$, seen as a bilinear form on $T_u S_\mu$, is the second derivative of $\phi|_{S_\mu}$ at u .

Definition 2.4. Let ϕ be a C^2 -functional on E . For any $u \in S_\mu$ and $\theta \geq 0$, we define the *approximate Morse index* by

$$\tilde{m}_\theta(u) = \sup\{\dim L \mid L \text{ is a subspace of } T_u S_\mu \text{ such that } \forall \varphi \in L \setminus \{0\}, D^2\phi(u)[\varphi, \varphi] < -\theta \|\varphi\|^2\}.$$

If u is a critical point for the constrained functional $\phi|_{S_\mu}$ and $\theta = 0$, we say that this is the *Morse index of u as constrained critical point*.

We may now formulate [16, Theorem 1.12]. Its derivation is based on a combination of ideas from [23, 27] implemented in a convenient geometric setting.

Theorem 2.5. Let $I \subset (0, \infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \rightarrow \mathbb{R}$ of the form

$$\Phi_\rho(u) = A(u) - \rho B(u), \quad \rho \in I,$$

where $B(u) \geq 0$ for all $u \in E$ and

$$A(u) \rightarrow +\infty \text{ or } B(u) \rightarrow +\infty \quad \text{as } u \in E \text{ and } \|u\| \rightarrow +\infty. \quad (2.2)$$

Suppose that, for every $\rho \in I$, $\Phi_\rho|_{S_\mu}$ is even and moreover that Φ'_ρ and Φ''_ρ are α -Hölder continuous on bounded sets in the sense of Definition 2.1 for some $\alpha \in (0, 1]$. Finally, suppose that there exists an integer $N \geq 2$ and two odd functions $\gamma_i : \mathbb{S}^{N-2} \rightarrow S_\mu$ where $i = 0, 1$, such that the set

$$\Gamma_N := \{ \gamma \in C([0, 1] \times \mathbb{S}^{N-2}, S_\mu) \mid \forall t \in [0, 1], \gamma(t, \cdot) \text{ is odd, } \gamma(0, \cdot) = \gamma_0, \text{ and } \gamma(1, \cdot) = \gamma_1 \} \quad (2.3)$$

is non void and

$$c_\rho^N := \inf_{\gamma \in \Gamma_N} \max_{(t, a) \in [0, 1] \times \mathbb{S}^{N-2}} \Phi_\rho(\gamma(t, a)) > \max_{a \in \mathbb{S}^{N-2}} \{ \Phi_\rho(\gamma_0(a)), \Phi_\rho(\gamma_1(a)) \}, \quad \forall \rho \in I. \quad (2.4)$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \rightarrow 0^+$ such that, as $n \rightarrow +\infty$,

- (i) $\Phi_\rho(u_n) \rightarrow c_\rho^N$;
- (ii) $\|\Phi'_\rho|_{S_\mu}(u_n)\| \rightarrow 0$;
- (iii) $\{u_n\}$ is bounded in E ;
- (iv) $\tilde{m}_{\zeta_n}(u_n) \leq N$.

Remark 2.6. If the sequence $\{u_n\} \subset S_\mu$ provided by the previous Theorem converges to some $u_\rho \in S_\mu$, then in view of points (i)–(ii), u_ρ is a critical point of $\Phi_\rho|_{S_\mu}$ at level c_ρ^N . Let us show that the Morse index of u_ρ , as a constrained critical point, satisfies $\tilde{m}_0(u_\rho) \leq N$. Assume by contradiction that this is not the case. Then, in view of Definition 2.4, we may assume that there exists a $W_0 \subset T_{u_\rho}S_\mu$ with $\dim W_0 = N + 1$ such that

$$D^2\Phi_\rho(u_\rho)[w, w] < 0 \quad \text{for all } w \in W_0 \setminus \{0\}.$$

Since W_0 is of finite dimension, its unit sphere is compact and there exists $\theta > 0$ such that

$$D^2\Phi_\rho(u_\rho)[w, w] < -\theta\|w\|^2 \quad \text{for all } w \in W_0 \setminus \{0\}.$$

Now, from [16, Corollary 1] or using directly that Φ'_ρ and Φ''_ρ are α -Hölder continuous on bounded sets for some $\alpha \in (0, 1]$, it follows that there exists $\delta > 0$ small enough such that, for any $v \in S_\mu$ satisfying $\|v - u_\rho\| \leq \delta$,

$$D^2\Phi_\rho(v)[w, w] < -\frac{\theta}{2}\|w\|^2 \quad \text{for all } w \in W_0 \setminus \{0\}.$$

In particular, for n large enough, $\|u_n - u_\rho\| \leq \delta$ and $\zeta_n < \theta/2$ (as $\zeta_n \rightarrow 0^+$), so the previous inequality implies

$$D^2\Phi_\rho(u_n)[w, w] < -\frac{\theta}{2}\|w\|^2 < -\zeta_n\|w\|^2 \quad \text{for all } w \in W_0 \setminus \{0\}.$$

Remembering that $\dim W_0 > N$ and observing that Theorem 2.5 (iv) directly implies that if there exists a subspace $W_n \subset T_{u_n}S_\mu$ such that

$$D^2\Phi_\rho(u_n)[w, w] < -\zeta_n\|w\|^2, \quad \text{for all } w \in W_n \setminus \{0\},$$

then necessarily $\dim W_n \leq N$, we have reached a contradiction.

From Theorem 2.5 (ii)–(iii), we deduce in a standard way, see [16, Remarks 1.3] or [13, Lemma 3], that

$$\Phi'_\rho(u_n) + \lambda_n(u_n, \cdot) \rightarrow 0 \quad \text{in } E' \text{ as } n \rightarrow +\infty \quad (2.5)$$

where we have set

$$\lambda_n := -\frac{1}{\mu} \Phi'_\rho(u_n)[u_n]. \quad (2.6)$$

We call the sequence $\{\lambda_n\} \subset \mathbb{R}$ defined in (2.6) the sequence of *almost Lagrange multipliers*.

The following lemma will allow to derive information on such sequences.

Lemma 2.7. *Let $\{u_n\} \subset S_\mu$, $\{\lambda_n\} \subset \mathbb{R}$ and $\{\zeta_n\} \subset \mathbb{R}^+$ with $\zeta_n \rightarrow 0^+$. Assume that, for a given $M \in \mathbb{N}$, the following conditions hold:*

(i) *For large enough $n \in \mathbb{N}$, all subspaces $W_n \subset E$ with the property*

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda_n |\varphi|^2 < -\zeta_n \|\varphi\|^2, \quad \text{for all } \varphi \in W_n \setminus \{0\}, \quad (2.7)$$

satisfy: $\dim(W_n) \leq M$.

(ii) *There exist $\lambda \in \mathbb{R}$, a subspace Y of E with $\dim(Y) \geq M + 1$ and $\zeta > 0$ such that, for large enough $n \in \mathbb{N}$,*

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda |\varphi|^2 \leq -\zeta \|\varphi\|^2, \quad \text{for all } \varphi \in Y. \quad (2.8)$$

Then $\lambda_n > \lambda$ for all large enough $n \in \mathbb{N}$. In particular, if (2.8) holds for any $\lambda < 0$, then $\liminf_{n \rightarrow \infty} \lambda_n \geq 0$.

Proof. Suppose by contradiction that $\lambda_n \leq \lambda$ along a subsequence still denoted $\{\lambda_n\}$. Keep denoting $\{u_n\}$ and $\{\zeta_n\}$ the corresponding subsequences. From (2.8) we have,

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda_n |\varphi|^2 \leq \Phi''_\rho(u_n)[\varphi, \varphi] + \lambda |\varphi|^2 \leq -\zeta \|\varphi\|^2 \quad \text{for all } \varphi \in Y \setminus \{0\}.$$

Now, since $\zeta_n \rightarrow 0^+$, there exists $n_0 \in \mathbb{N}$ such that: $\forall n \geq n_0$, $\zeta_n < \zeta$. Thus, for an arbitrary $n \geq n_0$, we obtain

$$\Phi''_\rho(u_n)[\varphi, \varphi] + \lambda_n |\varphi|^2 \leq -\zeta_n \|\varphi\|^2, \quad \text{for all } \varphi \in Y \setminus \{0\},$$

in contradiction with (2.7) since $\dim(Y) \geq M + 1$. \square

We will now focus on deriving sufficient conditions used to verify the two hypotheses posed on the class of paths Γ_N in Theorem 2.5. The following Lemma, directly inspired by [33, Remark 4.5], deals with the first hypothesis of showing that the set is non void.

Lemma 2.8. *Let $\{u_1, \dots, u_{N-1}\} \subset S_\mu$ and $\{v_1, \dots, v_{N-1}\} \subset S_\mu$ be orthogonal families for the inner product (\cdot, \cdot) . Setting the odd functions*

$$\gamma_0 : \mathbb{S}^{N-2} \rightarrow S_\mu \quad \text{by} \quad \gamma_0(a_1, \dots, a_{N-1}) = \sum_{i=1}^{N-1} a_i u_i$$

and

$$\gamma_1 : \mathbb{S}^{N-2} \rightarrow S_\mu \quad \text{by} \quad \gamma_1(a_1, \dots, a_{N-1}) = \sum_{i=1}^{N-1} a_i v_i,$$

the set Γ_N defined by (2.3) is non void.

Proof. We define the subspace $U = \text{span}\{u_1, \dots, u_{N-1}, v_1, \dots, v_{N-1}\}$ and let $d = \dim(U) \leq 2(N-1)$. Let $R : U \rightarrow U$ be a linear operator such that $Ru_i = v_i$ for $i = 1, 2, \dots, N-1$. Possibly after permutation of the family $\{v_n\}$, we can choose R such that $R \in \text{SO}(d)$ (there may be different choices of R). Now, since $\text{SO}(d)$ is pathwise-connected (see e.g. [36, Section 10.5]), there exists a continuous path $\tilde{\gamma} : [0, 1] \rightarrow \text{SO}(d)$ such that $\tilde{\gamma}(0) = \mathbb{1}$ and $\tilde{\gamma}(1) = R$. Let us define the map,

$$\gamma : [0, 1] \times \mathbb{S}^{N-2} \rightarrow S_\mu : (t, a_1, \dots, a_{N-1}) \mapsto \sum_{i=1}^{N-1} a_i \tilde{\gamma}(t)(u_i).$$

It is clear that γ is continuous, $\gamma(t, \cdot)$ is odd for all t , and $\gamma(0, \cdot) = \gamma_0$, $\gamma(1, \cdot) = \gamma_1$. \square

We now turn to the second hypothesis, which requires finding conditions to insure that the strict inequality (2.4) in Theorem 2.5 is satisfied. At this point, we shall rely on some results from [10]. In particular, the next Lemma is essentially [10, Lemma 3.2].

Lemma 2.9. *Let L_1, L be finite dimensional normed vector spaces such that $\dim(L_1) < \dim(L)$. Let $S = \{u \in L \mid \|u\| = 1\}$, $\alpha \in \mathbb{R}$ and $H = (H_1, H_2) : [0, 1] \times S \rightarrow \mathbb{R} \times L_1$ be a continuous map such that, for all t , $u \mapsto H_1(t, u)$ is even, $u \mapsto H_2(t, u)$ is odd, and*

$$H_1(0, u) < \alpha < H_1(1, u), \quad \text{for } u \in S.$$

Then there exists $(t, u) \in [0, 1] \times S$ such that $H(t, u) = (\alpha, 0)$.

In the proof of our next result we are inspired by [10, Lemma 3.3], see also [9, Lemma 2.3].

Theorem 2.10. *Let $\Phi : E \rightarrow \mathbb{R}$ be a continuous even functional, $d \in \mathbb{N}$, and $\gamma_i : \mathbb{S}^d \rightarrow S_\mu$, $i = 1, 2$, be two odd functions. Assume that the set*

$$\Gamma := \{\gamma \in C([0, 1] \times \mathbb{S}^d, S_\mu) \mid \forall t \in [0, 1], \gamma(t, \cdot) \text{ is odd, } \gamma(0, \cdot) = \gamma_0, \text{ and } \gamma(1, \cdot) = \gamma_1\}$$

is not empty. Assume further that there exists a continuous even functional $J : E \rightarrow \mathbb{R}$, $\beta \in \mathbb{R}$, and $W \subset E$ a subspace with $\dim W \leq d$ such that

$$(H1) \quad J(\gamma_0(s)) < \beta < J(\gamma_1(s)), \text{ for all } s \in \mathbb{S}^d;$$

$$(H2) \quad \max_{s \in \mathbb{S}^d} \max\{\Phi(\gamma_0(s)), \Phi(\gamma_1(s))\} < \inf_{u \in B} \Phi(u) \quad \text{where} \quad B := \{u \in S_\mu \cap W^\perp \mid J(u) = \beta\}.$$

Then

$$c := \inf_{\gamma \in \Gamma} \max_{(t,s) \in [0,1] \times \mathbb{S}^d} \Phi(\gamma(t,s)) \geq \inf_{u \in B} \Phi(u). \quad (2.9)$$

Proof. Let $\gamma \in \Gamma$ be arbitrary and $P : E \rightarrow W$ be the orthogonal projection. Define

$$h : S_\mu \rightarrow \mathbb{R} \times W : u \mapsto (J(u), Pu) \quad \text{and} \quad H = h \circ \gamma : [0, 1] \times \mathbb{S}^d \rightarrow \mathbb{R} \times W.$$

Setting $L = \mathbb{R}^{d+1}$, $S = \mathbb{S}^d$, $L_1 = W$ and $\alpha = \beta$, we see that L, S, L_1, α and H satisfy all the conditions of Lemma 2.9. Therefore there exists $(t_0, s_0) \in [0, 1] \times \mathbb{S}^d$ such that $H(t_0, s_0) = (\beta, 0)$. That is $\gamma(t_0, s_0) \in B$. One deduces that

$$\max_{(t,s) \in [0,1] \times \mathbb{S}^d} \Phi(\gamma(t,s)) \geq \Phi(\gamma(t_0, s_0)) \geq \inf_{u \in B} \Phi(u).$$

This proves that (2.9) holds since $\gamma \in \Gamma$ is arbitrary. \square

3 A Pohožaev type identity and its consequences

In this section we focus on deriving properties of solutions to (1.7) when $\lambda = 0$. Observe that since we are assuming that the compact core is non trivial: \mathcal{G} has at least one bounded edge and thus there is at least one edge where the nonlinearity is acting. Some considerations in this section are slightly more general than what is needed to prove Theorem 1.1.

First let us recall that if u is solution to $-u'' + \lambda u = \rho|u|^{p-2}u$ on some interval $I \subseteq \mathbb{R}$, then the function

$$H_u(x) := \frac{1}{2}(u'(x))^2 + V_\lambda(u(x)) \quad \text{where} \quad V_\lambda(u) := \frac{\rho}{p}|u|^p - \frac{\lambda}{2}|u|^2.$$

is constant on I . Indeed, $H'_u(x) := u'(x) \cdot (u''(x) + V'_\lambda(u(x))) = 0$. We call this constant H_u the *ODE energy*¹ of the solution u on I .

Proposition 3.1 (Pohožaev identity on metric graphs). *Let \mathcal{G} be a metric graph with finitely many edges (bounded or not). Let $p > 2$, $\lambda \in \mathbb{R}$, and $u \in H^1(\mathcal{G})$ be a solution to (1.7). For each bounded edge e of \mathcal{G} , let the ODE energy of the solution u on e be given by*

$$H_u(e) := H_u(x) = \frac{1}{2}|u'(x)|^2 + \frac{\rho}{p}|u(x)|^p - \frac{\lambda}{2}|u(x)|^2, \quad (3.1)$$

where x is an arbitrary point of e . Finally, define

$$P_\rho(u, \mathcal{G}) := \sum_{e \text{ is a bounded edge of } \mathcal{G}} \ell_e H_u(e) \quad (3.2)$$

where ℓ_e is the length of the edge e . Then, one has

$$\frac{1}{2}\|u'\|_{L^2(\mathcal{G})}^2 + \frac{\rho}{p}\|\kappa u\|_{L^p(\mathcal{G})}^p = \frac{\lambda}{2}\|u\|_{L^2(\mathcal{G})}^2 + P_\rho(u, \mathcal{G}).$$

Proof. Let e be a bounded edge of \mathcal{G} . We identify it with the interval $[0, \ell_e]$. Integrating (3.1) on e , we get

$$\frac{1}{2}\|u'\|_{L^2(e)}^2 + \frac{\rho}{p}\|\kappa u\|_{L^p(e)}^p = \frac{\lambda}{2}\|u\|_{L^2(e)}^2 + \ell_e H_u(e). \quad (3.3)$$

Note that (3.3) also holds when e is a half-line if in this case we set $\ell_e H_u(e) := 0$ since $\kappa|_e = 0$ and $u \in H^1(\mathcal{G})$. We end the proof by taking the sum of (3.3) over all edges of \mathcal{G} (whether bounded or not). \square

Lemma 3.2. *Let \mathcal{G} be a metric graph with finitely many edges (bounded or not). Let $p > 2$ and $\lambda \in \mathbb{R}$. Let $u \in H^1(\mathcal{G})$ be a solution to (1.7). Then, one has*

$$E_\rho(u, \mathcal{G}) = \frac{(p-6)\lambda}{2(p+2)}\|u\|_{L^2(\mathcal{G})}^2 + \frac{p-2}{p+2}P_\rho(u, \mathcal{G}),$$

where $P_\rho(u, \mathcal{G})$ is defined by (3.1)–(3.2).

Proof. First note that multiplying $-u'' + \lambda u = \rho\kappa(x)|u|^{p-2}u$ by u and integrating over \mathcal{G} (taking into account the Kirchhoff boundary conditions) we get

$$\|u'\|_{L^2(\mathcal{G})}^2 + \lambda\|u\|_{L^2(\mathcal{G})}^2 = \rho\|\kappa u\|_{L^p(\mathcal{G})}^p. \quad (3.4)$$

¹One needs to be careful not to confuse it with the energy functional E . The quantities E and H will be related through the Pohožaev identity in Proposition 3.1.

From Proposition 3.1 and (3.4), we obtain

$$\begin{aligned}\|u'\|_{L^2(\mathcal{G})}^2 &= \frac{(p-2)\lambda}{p+2}\|u\|_{L^2(\mathcal{G})}^2 + \frac{2p}{p+2}P_\rho(u, \mathcal{G}), \\ \rho\|\kappa u\|_{L^p(\mathcal{G})}^p &= \frac{2p\lambda}{p+2}\|u\|_{L^2(\mathcal{G})}^2 + \frac{2p}{p+2}P_\rho(u, \mathcal{G}).\end{aligned}$$

Thus

$$E_\rho(u, \mathcal{G}) = \frac{1}{2}\|u'\|_{L^2(\mathcal{G})}^2 - \frac{\rho}{p}\|\kappa u\|_{L^p(\mathcal{G})}^p = \frac{(p-6)\lambda}{2(p+2)}\|u\|_{L^2(\mathcal{G})}^2 + \frac{p-2}{p+2}P_\rho(u, \mathcal{G}). \quad \square$$

Let us now establish some relationships between the ODE energy H_u of a solution u on an interval and its L^2 -norm in the case $\lambda = 0$.

Lemma 3.3. *Let $\alpha > 0$ and $p \geq 2$. Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be an τ -periodic² solution of*

$$-u'' = \alpha|u|^{p-2}u.$$

for some $\tau > 0$. Let H_u be the ODE energy of the solution u . Then,

$$\frac{\tau}{8}\left(\frac{pH_u}{\alpha}\right)^{2/p} = \frac{\tau}{8}\|u\|_{L^\infty}^2 \leq \int_0^\tau |u(x)|^2 dx \leq \tau\|u\|_{L^\infty}^2 = \tau\left(\frac{pH_u}{\alpha}\right)^{2/p}. \quad (3.5)$$

Proof. It is advantageous to consider that we are studying periodic solutions of the equation of motion in the potential well defined by

$$V(u) := \frac{\alpha|u|^p}{p},$$

since the equation reads $u'' = -V'(u)$. The ODE energy of the solution u , which is given here by

$$H_u := \frac{1}{2}(u')^2 + V(u),$$

is constant with respect to time. We immediately obtain that

$$\frac{1}{2}(u'(x))^2 \begin{cases} \leq H_u - V(\|u\|_{L^\infty}/2) & \text{for all } x \in [0, \tau] \text{ such that } |u(x)| \geq \|u\|_{L^\infty}/2, \\ \geq H_u - V(\|u\|_{L^\infty}/2) & \text{for all } x \in [0, \tau] \text{ such that } |u(x)| \leq \|u\|_{L^\infty}/2. \end{cases}$$

Therefore, a particle in the potential well always has a smaller speed (in absolute value) when going through the region $[-|u|_\infty, -|u|_\infty/2] \cup [|u|_\infty/2, |u|_\infty]$ than when going through the region $[-|u|_\infty/2, |u|_\infty/2]$. Since both those regions have the same length, we deduce that

$$|A| \geq \frac{1}{2}\tau \quad \text{where } A := \{x \in [0, \tau] \mid |u(x)| \geq \frac{1}{2}\|u\|_{L^\infty}\}$$

as the particle will spend at least half its time in the region where it has a slower speed.

Regarding the inequalities in (3.5), the upper bound is trivial, and the lower bound follows from the inequalities

$$\int_0^\tau |u(x)|^2 dx \geq \int_A |u(x)|^2 dx \geq \frac{1}{4}\|u\|_{L^\infty}^2 |A| \geq \frac{1}{8}\tau\|u\|_{L^\infty}^2.$$

Finally, the equalities in (3.5) follow from the fact that, for periodic solutions, one has

$$H_u = V(\|u\|_{L^\infty}) = \frac{\alpha\|u\|_{L^\infty}^p}{p},$$

since the derivative of the solution vanishes at any maximum or minimum point. □

²I.e. so that $u(x+\tau) = u(x)$ for all $x \in \mathbb{R}$. It is not necessary for τ to be the minimum period.

Lemma 3.4. Let $\alpha > 0$ and $p \geq 2$. The solution of equation

$$-u'' = \alpha|u|^{p-2}u,$$

with initial conditions $u'(0) = 0$ and $u(0) = u_0 > 0$ is $\tau(u_0)$ -periodic, where

$$\tau(u_0) := \frac{C(p)}{\sqrt{\alpha}} u_0^{(2-p)/2}$$

for some constant $C(p) > 0$. Its ODE energy is given by $H_u = V(u_0) = \frac{\alpha}{p} u_0^p$. Moreover, it is (up to time translations) the unique solution of the ODE with this energy, and the unique solution of the ODE with this period.

Proof. It is a standard fact (see e.g. [8, p. 18]) that the period is given by

$$\tau(u_0) = 2 \int_{-u_0}^{u_0} \frac{du}{\sqrt{2(V(u_0) - V(u))}} = \sqrt{\frac{8p}{\alpha}} \int_0^{u_0} \frac{du}{\sqrt{u_0^p - u^p}} = \left(\sqrt{\frac{8p}{\alpha}} \int_0^1 \frac{dt}{\sqrt{1-t^p}} \right) u_0^{1-p/2}.$$

The claim about the energy follows from the definitions. The fact that the set

$$\{(u, v) \in \mathbb{R}^2 \mid \frac{1}{2}v^2 + V(u) = h\}$$

is empty for $h < 0$, is $\{(0, 0)\}$ for $h = 0$ and is a simple closed curve for $h > 0$ implies that no other solutions have this same energy since by a phase plane analysis we obtain that there is a unique orbit of energy h for every $h > 0$. We then deduce from the previous computations that this orbit corresponds to a solution of period $C(\alpha, p) u_0^{1-p/2}$, which ends the proof as the map

$$(0, +\infty) \rightarrow (0, +\infty) : u_0 \mapsto C(\alpha, p) u_0^{1-p/2}$$

is decreasing, so all orbits correspond to solutions with different periods. \square

From here we may deduce that functions with a high ODE energy necessarily have a high L^2 norm.

Corollary 3.5. Let $\ell > 0$, $0 < \underline{\alpha} < \bar{\alpha} < \infty$, and $p > 2$. For every $\underline{\mu} > 0$, there exists $\underline{H} > 0$ such that if $u : [0, \ell] \rightarrow \mathbb{R}$ is a solution to

$$-u'' = \alpha|u|^{p-2}u, \quad \text{with } \alpha \in [\underline{\alpha}, \bar{\alpha}] \text{ and } H_u \geq \underline{H},$$

then

$$\int_0^\ell |u(x)|^2 dx \geq \underline{\mu}.$$

Proof. Lemma 3.4 implies that if the ODE energy $H_u \geq \underline{H}$, then $u_0 \geq (p\underline{H}/\bar{\alpha})^{1/p}$ and so

$$\tau(u_0) \leq \frac{C(p)}{\sqrt{\bar{\alpha}}} \left(\frac{p\underline{H}}{\bar{\alpha}} \right)^{(2-p)/(2p)}.$$

Thus, if \underline{H} is large enough, u is periodic with a period τ less than $\ell/2$. There thus exists some interval $[0, k\tau] \subseteq [0, \ell]$, with $k \in \mathbb{N}$ and $k\tau \geq \ell/2$. Thus u is $k\tau$ -periodic and its ODE energy is at least \underline{H} . Lemma 3.3 implies that the L^2 -norm of u on $[0, k\tau]$ can be made arbitrarily high taking \underline{H} large enough. This ends the proof. \square

The last result of this section will be crucially used to rule out the possibility that the Lagrange multiplier associated to a weak limit of some Palais-Smale sequence is 0.

Proposition 3.6. *Let \mathcal{G} be a metric graph with finitely many edges (bounded or not) and $p > 2$. Let $\{u_n\} \subset H^1(\mathcal{G})$ and $\{\rho_n\} \subset [1/2, 1]$ be sequences such that u_n is a solution to (1.7) with $\rho = \rho_n$ and $\lambda = 0$. If $E_{\rho_n}(u_n, \mathcal{G}) \rightarrow +\infty$, then $\|u_n\|_{L^2(\mathcal{G})} \rightarrow \infty$.*

Proof. First notice that it is sufficient to prove that, up to a subsequence, $\|u_n\|_{L^2(\mathcal{G})} \rightarrow \infty$ because, replaying the argument on an arbitrary subsequence of $\{u_n\}$ will give a sub-subsequence which converges to infinity, which is equivalent to the claim.

Since $E_{\rho_n}(u_n, \mathcal{G}) \rightarrow +\infty$ and $\lambda = 0$, Lemma 3.2 implies that $P_{\rho_n}(u_n, \mathcal{G}) \rightarrow +\infty$. Let $e_0 \in \mathcal{E}$ be a bounded edge such that $\ell_{e_0} H_{u_n}(e_0) \geq \ell_e H_{u_n}(e)$ for all bounded edges $e \in \mathcal{E}$. Given that \mathcal{E} is finite, it is possible to select e_0 independent of n , taking subsequences of $\{u_n\}$ and $\{\rho_n\}$ if necessary. Since

$$P_{\rho_n}(u_n, \mathcal{G}) \leq \text{card}(\mathcal{E}) \ell_{e_0} H_{u_n}(e_0),$$

$H_{u_n}(e_0) \rightarrow +\infty$. At this point, using Corollary 3.5 with $u = u_n$, $\alpha = \rho_n$, and $[0, \ell] = e_0$, we deduce that $\|u_n\|_{L^2(e_0)} \rightarrow \infty$ and thus $\|u_n\|_{L^2(\mathcal{G})} \rightarrow \infty$. \square

4 Infinitely many minimax levels for E_ρ for almost every $\rho \in [\frac{1}{2}, 1]$

This aim of this section is to prove the following result.

Proposition 4.1. *For any $\mu > 0$ and $p > 2$, there exists $N_0 \in \mathbb{N}$ so that if $N \geq N_0$, there exist functions $\gamma_{0,N}$ and $\gamma_{1,N}$ such that the family of functionals*

$$E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\mathcal{G}} |u'|^2 - \frac{\rho}{p} \int_{\mathcal{X}} |u|^p, \quad \rho \in \left[\frac{1}{2}, 1\right]$$

satisfies the assumptions of Theorem 2.5. In particular,

$$\Gamma_N = \{\gamma \in C([0, 1] \times \mathbb{S}^{N-2}, H_\mu^1(\mathcal{G})) \mid \forall t \in [0, 1], \gamma(t, \cdot) \text{ is odd, } \gamma(0, \cdot) = \gamma_{0,N}, \text{ and } \gamma(1, \cdot) = \gamma_{1,N}\} \quad (4.1)$$

is non void and

$$c_\rho^N = \inf_{\gamma \in \Gamma_N} \max_{(t,s) \in [0,1] \times \mathbb{S}^{N-2}} E_\rho(\gamma(t,s), \mathcal{G}) > \max_{s \in \mathbb{S}^{N-2}} \max\{E_\rho(\gamma_0(s), \mathcal{G}), E_\rho(\gamma_1(s), \mathcal{G})\}, \quad \rho \in \left[\frac{1}{2}, 1\right] \quad (4.2)$$

Furthermore, $c_\rho^N \xrightarrow{N \rightarrow +\infty} +\infty$ uniformly w.r.t. $\rho \in [1/2, 1]$. In particular there are infinitely many distinct values of c_ρ^N .

Remark 4.2. Note that the levels c_ρ^N are real numbers for every $N \geq N_0$ and every $\rho \in [1/2, 1]$ since they are defined by infima over nonempty sets (thus $c_\rho^N < +\infty$) and that inequality (4.2) implies that $c_\rho^N > -\infty$.

We consider Theorem 2.5 with the choice of the family $\Phi_\rho = E_\rho(\cdot, \mathcal{G})$. Also $E = H^1(\mathcal{G})$, $H = L^2(\mathcal{G})$, $S_\mu = H_\mu^1(\mathcal{G})$, Setting

$$A(u) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx \quad \text{and} \quad B(u) = \frac{\rho}{p} \int_{\mathcal{X}} |u|^p dx,$$

assumption (2.2) holds, since we have that

$$u \in H_\mu^1(\mathcal{G}), \|u\|_{H^1(\mathcal{G})} \rightarrow +\infty \implies A(u) \rightarrow +\infty.$$

Let E'_ρ and E''_ρ denote respectively the free first and second Fréchet derivatives of E_ρ . Note that B'' , whence E''_ρ , is $\min\{p-2, 1\}$ -Hölder continuous on bounded sets of $H^1(\mathcal{G})$, which, in view of Remark 2.2, implies that assumption (2.1) holds. As such, it only remains to show that the two hypothesis posed on Γ_N hold. This is where Lemma 2.8 and Theorem 2.10 will come into play.

The following two lemmas will provide orthogonal families be to used in Lemma 2.8.

Lemma 4.3. *Let \mathcal{G} be a graph satisfying (1.1), $p > 2$ and $\mu > 0$. For any $\beta > 0$, there exists a sequence of functions $\{\varphi_1, \varphi_2, \dots\}$ such that for any $i, j \in \mathbb{N}^*$ and any $\rho \in [\frac{1}{2}, 1]$:*

- (i) $\varphi_i \in \mathcal{S}_\mu$: $\|\varphi_i'\|_{L^2(\mathcal{G})} = \beta$; $E_\rho(\varphi_i, \mathcal{G}) = \beta^2/2$;
- (ii) φ_i has compact support and $\text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) = \emptyset$ for $i \neq j$;
- (iii) for any $N \geq 2$ and $a \in \mathbb{S}^{N-2}$, $\|(\sum_{i=1}^{N-1} a_i \varphi_i)'\|_{L^2(\mathcal{G})} = \beta$ and $E_\rho(\sum_{i=1}^{N-1} a_i \varphi_i, \mathcal{G}) = \beta^2/2$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R})$ be a function supported on the interval $(0, 1)$ such that $\|\varphi\|_{L^2(\mathbb{R})}^2 = \mu$. Define, for $t \in \mathbb{R}^+$ the function φ^t by

$$\varphi^t(x) := t^{1/2} \varphi(tx). \quad (4.3)$$

If we now view φ as a function in $H^1(\mathcal{G})$ whose support is contained in a half-line which we identify with $[0, \infty)$, we can define

$$\varphi_1 := \varphi^\tau \quad \text{with} \quad \tau := \frac{\beta}{\|\varphi'\|_{L^2(\mathcal{G})}}.$$

The function φ_1 satisfies (i). Indeed, for any $t > 0$, $\|\varphi^t\|_{L^2(\mathcal{G})} = \|\varphi\|_{L^2(\mathcal{G})} = \mu$, and a direct calculation yields

$$\|\varphi_1'\|_{L^2(\mathcal{G})}^2 = \tau^2 \|\varphi'\|_{L^2(\mathcal{G})}^2 = \beta^2. \quad (4.4)$$

Finally, since φ_1 is supported in the half-line, we have

$$E_\rho(\varphi_1, \mathcal{G}) = \frac{1}{2} \|\varphi_1'\|_{L^2(\mathcal{G})}^2 = \frac{\beta^2}{2}.$$

Define now, for $i \geq 2$,

$$\varphi_i(x) := \varphi_1\left(x - \frac{i-1}{\tau}\right).$$

Since the φ_i are translations of φ_1 they still satisfy (i). Also, observe that by definition $\text{supp}(\varphi_i) \subset (\frac{i-1}{\tau}, \frac{i}{\tau})$ and so they all have disjoint compact supports. This is (ii). Finally, observe that for $a \in \mathbb{S}^{N-2}$

$$\left\| \left(\sum_{i=1}^{N-1} a_i \varphi_i \right)' \right\|_{L^2(\mathcal{G})}^2 = \sum_{i=1}^{N-1} a_i^2 \|\varphi_i'\|_{L^2(\mathcal{G})}^2 = \beta^2,$$

from which (iii) follows, ending the proof. \square

Lemma 4.4. *Let \mathcal{G} be a graph satisfying (1.1), $p > 6$ and $\mu > 0$. For any fixed integer $N \geq 2$ and any given values of $\bar{\beta} > 0$, $\bar{b} > 0$, there exist functions $\bar{\varphi}_1, \dots, \bar{\varphi}_N$, compactly supported in \mathcal{K} , such that for all $i, j \in \{1, \dots, N\}$ and all $\rho \in [1/2, 1]$,*

- (i) $\bar{\varphi}_i \in \mathcal{S}_\mu$: $\|\bar{\varphi}_i'\|_{L^2(\mathcal{G})} \geq \bar{\beta}$;
- (ii) $\text{supp}(\bar{\varphi}_i) \cap \text{supp}(\bar{\varphi}_j) = \emptyset$ for $i \neq j$;
- (iii) if $a \in \mathbb{S}^{N-2}$ then $\|(\sum_{i=1}^{N-1} a_i \bar{\varphi}_i)'\|_{L^2(\mathcal{G})} \geq \bar{\beta}$ and $E_\rho(\sum_{i=1}^{N-1} a_i \bar{\varphi}_i, \mathcal{G}) \leq \bar{b}$.

Proof. Let $e = [0, \ell_e]$ be any bounded edge of \mathcal{G} . Let $\varphi \in C_c^\infty((0, \ell_e/N))$ be any function such that $\|\varphi\|_{L^2(\mathbb{R})} = \mu$. Using the notation (4.3), we notice that $\text{supp}(\varphi^t) \subset (0, \ell_e/N)$ whenever $t \geq 1$. Define the functions

$$\bar{\varphi}_i := \varphi^t \left(x - \frac{(i-1)\ell_e}{N} \right), \quad i = 1, \dots, N,$$

where $t \geq 1$ will be chosen later. Note that

$$\text{supp}(\bar{\varphi}_i) \subset \left(\frac{(i-1)\ell_e}{N}, \frac{i\ell_e}{N} \right)$$

so the functions $\bar{\varphi}_i$ have disjoint supports and (ii) is satisfied. Viewing now $\bar{\varphi}_i$ as functions in $H^1(\mathcal{G})$ supported in e , we may compute the energy of the function $\sum_{i=1}^{N-1} a_i \bar{\varphi}_i$ with $a \in \mathbb{S}^{N-2}$ as follows

$$\begin{aligned} E_p \left(\sum_{i=1}^{N-1} a_i \bar{\varphi}_i, \mathcal{G} \right) &= \frac{1}{2} \int_e \left| \sum_{i=1}^N a_i \bar{\varphi}_i' \right|^2 dx - \frac{p}{p} \int_e \left| \sum_{i=1}^{N-1} a_i \bar{\varphi}_i \right|^p dx \\ &= \frac{t^2}{2} \sum_{i=1}^{N-1} a_i^2 \int_0^{\ell_e/N} |\varphi'|^2 dx - \frac{p t^{(p-2)/2}}{p} \sum_{i=1}^{N-1} |a_i|^p \int_0^{\ell_e/N} |\varphi|^p \\ &\leq \frac{t^2 \|\varphi'\|_{L^2(\mathcal{G})}^2}{2} - \frac{C t^{(p-2)/2} \|\varphi\|_{L^p(\mathcal{X})}^p}{2p} \xrightarrow{t \rightarrow +\infty} -\infty \end{aligned}$$

where $C := \min_{a \in \mathbb{S}^{N-2}} \sum_{i=1}^{N-1} |a_i|^p$. Thus, for all $\bar{b} \in \mathbb{R}$, there exists $T_0 > 0$ such that for all $t > T_0$ we have $E_p(\sum_{i=1}^N a_i \bar{\varphi}_i, \mathcal{G}) < \bar{b}$. As a result, if we choose

$$t := \max \left\{ 1, \frac{\bar{\beta}}{\|\varphi'\|_{L^2(\mathcal{G})}}, T_0 \right\},$$

the functions $\bar{\varphi}_i$ satisfy all of the desired properties. Indeed, $\bar{\varphi}_i \in H_\mu^1(\mathcal{G})$ and from (4.4) we have

$$\|\bar{\varphi}_i'\|_{L^2(\mathcal{G})} = t \|\varphi'\|_{L^2(\mathcal{G})} \geq \bar{\beta}, \quad (4.5)$$

which implies (i). Finally, the choice of t , (4.5) and $\|(\sum_{i=1}^{N-1} a_i \bar{\varphi}_i)'\|_{L^2(\mathcal{G})}^2 = \sum_{i=1}^{N-1} a_i^2 \|\bar{\varphi}_i'\|_{L^2(\mathcal{G})}^2$ show (iii), ending the proof. \square

Now let $\{V_N\}$ be a sequence of linear subspaces of $H^1(\mathcal{G})$ with $\dim(V_N) = N$ which is exhausting $H^1(\mathcal{G})$ in the sense that

$$\bigcup_{N \geq 1} V_N$$

is dense in $H^1(\mathcal{G})$. We recall that for separable Hilbert spaces, such as $H^1(\mathcal{G})$, such sequence always exists.

Our next lemma is an adaptation of [9, Lemma 2.1].

Lemma 4.5. *For any $p > 2$ there holds:*

$$S_N := \inf_{u \in V_{N-2}^\perp} \frac{\int_{\mathcal{G}} |u'|^2 + |u|^2}{\left(\int_{\mathcal{X}} |u|^p \right)^{2/p}} \rightarrow \infty, \quad \text{as } N \rightarrow \infty.$$

Proof. Suppose by contradiction that there exists a sequence $\{u_N\} \subset V_{N-2}^\perp$ such that $\|u_N\|_{L^p(\mathcal{X})} = 1$ and $\|u_N\|_{H^1(\mathcal{G})}$ is bounded. In particular, up to a subsequence, there exists $u \in H^1(\mathcal{G})$ such that $u_N \rightharpoonup u$ in $H^1(\mathcal{G})$ (and thus in $H^1(\mathcal{X})$) and therefore $u_N \rightarrow u$ in $L^p(\mathcal{X})$. Let $v \in H^1(\mathcal{G})$. Because $\{V_N\}$ exhausts $H^1(\mathcal{G})$, there exists a sequence $\{v_N\} \subset H_\mu^1(\mathcal{G})$ such that, for all $N \in \mathbb{N}$, $v_N \in V_{N-2}$ and $v_N \rightarrow v$ in $H^1(\mathcal{G})$. Taking the scalar product in $H^1(\mathcal{G})$ we have

$$|\langle u_N, v \rangle| \leq |\langle u_N, v - v_N \rangle| + |\langle u_N, v_N \rangle| = |\langle u_N, v - v_N \rangle| \leq \|u_N\|_{H^1(\mathcal{G})} \|v - v_N\|_{H^1(\mathcal{G})} \xrightarrow{N \rightarrow \infty} 0.$$

It follows that $u_N \rightharpoonup 0 = u$ in contradiction with $\|u_N\|_{L^p(\mathcal{X})} = 1$. \square

We now define

$$\beta_N := \left(\frac{S_N^{p/2}}{L} \right)^{1/(p-2)} \quad \text{where} \quad L = L(p) := \frac{3}{p} \max_{x>0} \frac{(\mu + x^2)^{p/2}}{\mu + x^p}. \quad (4.6)$$

As an immediate consequence of Lemma 4.5, we have that $\beta_N \rightarrow \infty$. Thus if we define

$$b_\rho^N := \inf_{u \in B_N} E_\rho(u, \mathcal{G}) \quad \text{where} \quad B_N := \{u \in V_{N-2}^\perp \cap H_\mu^1(\mathcal{G}) \mid \|u'\|_{L^2(\mathcal{G})} = \beta_N\} \quad (4.7)$$

we obtain that

Lemma 4.6. $b_\rho^N \rightarrow +\infty$ as $N \rightarrow +\infty$, uniformly in $\rho \in [1/2, 1]$.

Proof. For every $u \in B_N$ we have

$$\begin{aligned} E_\rho(u, \mathcal{G}) &= \frac{1}{2} \int_{\mathcal{G}} |u'|^2 - \frac{\rho}{p} \left(\int_{\mathcal{X}} |u|^p \right)^{\frac{2}{p} \cdot \frac{p}{2}} \geq \frac{1}{2} \int_{\mathcal{G}} |u'|^2 - \frac{1}{p} \left(\frac{\mu + \int_{\mathcal{G}} |u'|^2}{S_N} \right)^{p/2} \\ &\geq \frac{1}{2} \|u'\|_{L^2(\mathcal{G})}^2 - \frac{L}{3S_N^{p/2}} (\mu + \|u'\|_{L^2(\mathcal{G})}^p) \\ &= \frac{1}{2} \beta_N^2 - \frac{1}{3} \beta_N^{2-p} (\mu + \beta_N^p) \\ &= \frac{1}{6} \beta_N^2 + o(1). \end{aligned}$$

The proof is completed by taking the infimum over B_N . □

We are finally in position to give the

Proof of Proposition 4.1. We have already proved that (2.1) and (2.2) hold. Let $\{V_N\}$ with $\dim(V_N) = N$ be an exhausting sequence of $H^1(\mathcal{G})$ and, for each $N \geq 2$, define the values β_N and b_ρ^N respectively by (4.6) and (4.7). By Lemma 4.5 and Lemma 4.6 both sequences $\{\beta_N\}$ and $\{b_\rho^N\}$ diverge.

Consider now a sequence of functions $\{\varphi_i\}_{i=1}^\infty$ as given by Lemma 4.3 taking $\beta = 1$ and a set of N functions $\{\bar{\varphi}_i\}_{i=1}^N$ given by Lemma 4.4 taking $\bar{\beta} = 2\beta_N$ and $\bar{b} = 1$. Moreover, define the functions

$$\gamma_{0,N}: \mathbb{S}^{N-2} \rightarrow H_\mu^1(\mathcal{G}) : (a_1, \dots, a_{N-1}) \mapsto \sum_{i=1}^{N-1} a_i \varphi_i, \quad \gamma_{1,N}: \mathbb{S}^{N-2} \rightarrow H_\mu^1(\mathcal{G}) : (a_1, \dots, a_{N-1}) \mapsto \sum_{i=1}^{N-1} a_i \bar{\varphi}_i,$$

which satisfy, for every $N \geq 2$ and $a \in \mathbb{S}^{N-2}$,

$$\begin{cases} \|\gamma_{0,N}(a)'\|_{L^2(\mathcal{G})} = 1, \\ E_\rho(\gamma_{0,N}(a), \mathcal{G}) = \frac{1}{2}, \end{cases} \quad \text{and} \quad \begin{cases} \|\gamma_{1,N}(a)'\|_{L^2(\mathcal{G})} \geq 2\beta_N, \\ E_\rho(\gamma_{1,N}(a), \mathcal{G}) \leq 1. \end{cases}$$

From Lemma 2.8 we know that the set

$$\Gamma_N = \{\gamma \in C([0, 1] \times \mathbb{S}^{N-2}, H_\mu^1(\mathcal{G})) \mid \forall t \in [0, 1], \gamma(t, \cdot) \text{ is odd, } \gamma(0, \cdot) = \gamma_{0,N}, \text{ and } \gamma(1, \cdot) = \gamma_{1,N}\}$$

is not empty.

Now, we want to use Theorem 2.10 with the choice $\Phi = E_\rho(\cdot, \mathcal{G})$, $d = N - 2$, $J(u) = \|u'\|_{L^2(\mathcal{G})}$, $\beta = \beta_N$, and $W = V_{N-2}$. We easily check that its assumptions (H1) and (H2) are satisfied for any N sufficiently large (uniformly in ρ) and thus (4.2) also holds. Finally, using $b_\rho^N \rightarrow +\infty$ as $N \rightarrow \infty$ and (2.9), we get that $c_\rho^N \rightarrow +\infty$ as $N \rightarrow \infty$. □

5 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. As a consequence of Proposition 4.1, we may apply Theorem 2.5 to the family of functionals given by (1.6). From Theorem 2.5 and the considerations just after it (see in particular (2.5)–(2.6)), for all $N \in \mathbb{N}$ large enough and for almost every $\rho \in [1/2, 1]$, we deduce the existence of a bounded sequence $\{u_{\rho,n}^N\}_{n=1}^\infty \subset H_\mu^1(\mathcal{G})$, that we shall simply denote $\{u_n\}$, such that

$$E_\rho(u_n, \mathcal{G}) \rightarrow c_\rho^N \quad (5.1)$$

and

$$E'_\rho(u_n, \mathcal{G}) + \lambda_n(u_n, \cdot) \rightarrow 0 \quad \text{in the dual of } H_\mu^1(\mathcal{G}), \quad (5.2)$$

where

$$\lambda_n := -\frac{1}{\mu} E'_\rho(u_n, \mathcal{G})[u_n]. \quad (5.3)$$

Finally, there exists a sequence $\{\zeta_n\} \subset \mathbb{R}^+$ with $\zeta_n \rightarrow 0^+$ such that, if the inequality

$$\int_{\mathcal{G}} |\varphi'|^2 + (\lambda_n - (p-1)\rho\kappa(x)|u_n|^{p-2}) \varphi^2 dx = E''_\rho(u_n, \mathcal{G})[\varphi, \varphi] + \lambda_n \|\varphi\|_{L^2(\mathcal{G})}^2 < -\zeta_n \|\varphi\|_{H^1(\mathcal{G})}^2 \quad (5.4)$$

holds for any $\varphi \in W_n \setminus \{0\}$ in a subspace W_n of $T_{u_n} H_\mu^1(\mathcal{G})$, then the dimension of W_n is at most N .

Since $\{u_n\} \subset H^1(\mathcal{G})$ is bounded, passing to a subsequence we may assume that there exists $u_\rho^N \in H^1(\mathcal{G})$ such that

$$u_n \rightharpoonup u_\rho^N \quad \text{in } H^1(\mathcal{G}), \quad (5.5)$$

$$u_n \rightarrow u_\rho^N \quad \text{in } L^r_{\text{loc}}(\mathcal{G}) \text{ for all } r \geq 2. \quad (5.6)$$

Observe also that, since $\{u_n\} \subset H^1(\mathcal{G})$ is a bounded sequence, it follows from (5.3) that $\{\lambda_n\} \subset \mathbb{R}$ is bounded. As before, passing to a subsequence, there exists $\lambda_\rho^N \in \mathbb{R}$ such that $\lim_{n \rightarrow +\infty} \lambda_n = \lambda_\rho^N$.

The sequences $\{\lambda_\rho^N\}_{N=1}^\infty \subset \mathbb{R}$ and $\{u_\rho^N\}_{N=1}^\infty \subset H_\mu^1(\mathcal{G})$ are the candidates to prove Theorem 1.2. We begin by verifying that the limit $u_\rho^N \in H^1(\mathcal{G})$ solves (1.8). Indeed, using (5.2) and the fact that $\lim_{n \rightarrow +\infty} \lambda_n = \lambda_\rho^N$, we get

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} (E'_\rho(u_n, \mathcal{G}) + \lambda_n(u_n, \cdot))[\eta] \\ &= \lim_{n \rightarrow \infty} \left[\int_{\mathcal{G}} u'_n \eta' + \lambda_n \int_{\mathcal{G}} u_n \eta - \rho \int_{\mathcal{X}} |u_n|^{p-2} u_n \eta \right] \\ &= \int_{\mathcal{G}} (u_\rho^N)' \eta' + \lambda_\rho^N \int_{\mathcal{G}} u_\rho^N \eta - \rho \int_{\mathcal{X}} |u_\rho^N|^{p-2} u_\rho^N \eta \end{aligned} \quad (5.7)$$

for every $\eta \in H^1(\mathcal{G})$. We have thus proved the claim.

We now focus on proving the strong convergence of the sequences $\{u_n\} \subset H^1(\mathcal{G})$ to insure that the limits u_ρ^N belong to the mass constraint $H_\mu^1(\mathcal{G})$.

Proposition 5.1. *The following convergence holds:*

$$\int_{\mathcal{G}} |(u_n - u_\rho^N)'|^2 + \lambda_\rho^N \int_{\mathcal{G}} |u_n - u_\rho^N|^2 \xrightarrow{n \rightarrow \infty} 0.$$

In particular, if $\lambda_\rho^N > 0$, the sequence $\{u_n\}$ converges strongly in $H^1(\mathcal{G})$.

Proof. First, rewriting (5.2) as follows:

$$\begin{aligned} o(1)\|\eta\|_{H^1(\mathcal{G})} &= \int_{\mathcal{G}} u'_n \eta' - \rho \int_{\mathcal{X}} |u_n|^{p-2} u_n \eta + \lambda_n \int_{\mathcal{G}} u_n \eta \\ &= \int_{\mathcal{G}} u'_n \eta' - \rho \int_{\mathcal{X}} |u_n|^{p-2} u_n \eta + \lambda_p^N \int_{\mathcal{G}} u_n \eta + (\lambda_n - \lambda_p^N) \int_{\mathcal{G}} u_n \eta, \end{aligned}$$

we get

$$\int_{\mathcal{G}} u'_n \eta' - \rho \int_{\mathcal{X}} |u_n|^{p-2} u_n \eta + \lambda_p^N \int_{\mathcal{G}} u_n \eta = o(1)\|\eta\|_{H^1(\mathcal{G})}. \quad (5.8)$$

Now, taking the difference between (5.8) and (5.7), choosing $\eta = \eta_n := u_n - u_p^N$ and taking into account (5.6) and that $\{\eta_n\}$ is bounded, we obtain

$$\begin{aligned} o(1) &= o(1)\|\eta_n\|_{H^1(\mathcal{G})} = \int_{\mathcal{G}} (u'_n - (u_p^N)') \eta'_n - \rho \int_{\mathcal{X}} (|u_n|^{p-2} u_n - |u_p^N|^{p-2} u_p^N) \eta_n + \lambda_p^N \int_{\mathcal{G}} (u_n - u_p^N) \eta_n \\ &= \int_{\mathcal{G}} (u'_n - (u_p^N)') \eta'_n + \lambda_p^N \int_{\mathcal{G}} (u_n - u_p^N) \eta_n + o(1)\|\eta_n\|_{H^1(\mathcal{G})} \\ &= \int_{\mathcal{G}} |(u_n - u_p^N)'|^2 + \lambda_p^N \int_{\mathcal{G}} |u_n - u_p^N|^2 + o(1) \end{aligned}$$

This proves the claim. \square

In order to apply Proposition 5.1 we need to show that the assumption $\lambda_p^N > 0$ holds. We will do this in two steps. In first place, we will show that $\lambda_p^N < 0$ is not possible by making use of Lemma 2.7. The following result will aid to check that its assumptions hold.

Lemma 5.2. *For any $\lambda < 0$ and $d \in \mathbb{N}$, there exists a subspace Y of $H^1(\mathcal{G})$ with $\dim(Y) = d$ such that*

$$E''_{\rho}(u_n, \mathcal{G})[w, w] + \lambda \|w\|_{L^2(\mathcal{G})}^2 = \int_{\mathcal{G}} |w'|^2 dx + \lambda \int_{\mathcal{G}} |w|^2 dx \leq \frac{\lambda}{2} \|w\|_{H^1(\mathcal{G})}^2, \quad \forall w \in Y.$$

Proof. We proceed similarly to the proof of Lemma 4.3. Take $\varphi \in C_c^\infty(\mathbb{R})$ with $\text{supp } \varphi \subset (0, 1)$ and such that $\int_0^{+\infty} |\varphi|^2 dx = 1$. Viewing φ as a function in $H^1(\mathcal{G})$ whose support is contained in a half-line which we identify with $[0, \infty)$, we define (using the notation of (4.3))

$$\varphi_1 := \varphi^\tau,$$

where $\tau > 0$ is taken small enough so that

$$\tau^2 \|\varphi'\|_{L^2(\mathbb{R})}^2 + \lambda \leq \frac{\lambda}{2} (\tau^2 \|\varphi'\|_{L^2(\mathbb{R})}^2 + 1). \quad (5.9)$$

One has that

$$\|\varphi_1\|_{L^2(\mathcal{G})} = 1, \quad \|\varphi_1'\|_{L^2(\mathcal{G})} = \tau^2 \|\varphi'\|_{L^2(\mathcal{G})}.$$

Define now, for $i \geq 2$,

$$\varphi_i(x) := \varphi_1 \left(x - \frac{i-1}{\tau} \right).$$

Since $\text{supp}(\varphi_i) \subset \left(\frac{i-1}{\tau}, \frac{i}{\tau} \right)$, all the φ_i have disjoint supports. Let $Y \subset H^1(\mathcal{G})$ be the subspace generated by $\varphi_1, \dots, \varphi_d$. Any element $w \in Y$ writes as

$$w := \sum_{i=1}^d \theta_i \varphi_i.$$

where $\theta_1, \dots, \theta_d \in \mathbb{R}$. By direct calculations we have

$$\begin{aligned} \int_{\mathcal{G}} |w'|^2 dx + \lambda \int_{\mathcal{G}} |w|^2 dx &= \tau^2 \left(\sum_{i=1}^d \theta_i^2 \|\varphi'\|_{L^2(\mathbb{R})}^2 \right) + \lambda \left(\sum_{i=1}^d \theta_i^2 \right) \\ &= (\tau^2 \|\varphi'\|_{L^2(\mathbb{R})}^2 + \lambda) \sum_{i=1}^d \theta_i^2. \end{aligned}$$

Similarly, $\|w\|_{H^1(\mathcal{G})}^2 = (\tau^2 \|\varphi'\|_{L^2(\mathbb{R})}^2 + 1) \sum_{i=1}^d \theta_i^2$. Therefore, (5.9) implies that

$$\int_{\mathcal{G}} |w'|^2 dx + \lambda \int_{\mathcal{G}} |w|^2 dx = (\tau^2 \|\varphi'\|_{L^2(\mathbb{R})}^2 + \lambda) \sum_{i=1}^d \theta_i^2 \leq \frac{\lambda}{2} (\tau^2 \|\varphi'\|_{L^2(\mathbb{R})}^2 + 1) \sum_{i=1}^d \theta_i^2 = \frac{\lambda}{2} \|w\|_{H^1(\mathcal{G})}^2.$$

The fact that w vanishes outside the half-line justifies the equality in the claim, ending the proof. \square

Observe that the codimension of $T_{u_n} H_\mu^1(\mathcal{G})$ in $H^1(\mathcal{G})$ is one. Thus, if inequality (5.4) holds for every $\varphi \in W_n \setminus \{0\}$ for a subspace W_n of $H^1(\mathcal{G})$, then the dimension of W_n is at most $N + 1$. Let $\lambda < 0$. Let Y be the space of dimension $d = N + 2$ provided by Lemma 5.2. We may thus apply Lemma 2.7 to obtain that

$$\lambda_\rho^N \geq 0. \quad (5.10)$$

Combining Proposition 5.1 and (5.10), we get that

$$\int_{\mathcal{G}} |(u_n - u_\rho^N)'|^2 \rightarrow 0.$$

Using in addition (5.6) and recalling that the nonlinearity acts only on the compact core \mathcal{K} , we obtain that $E_\rho(u_n, \mathcal{G}) \rightarrow E_\rho(u_\rho^N, \mathcal{G})$. In particular, in view of (5.1), it follows that

$$E_\rho(u_\rho^N, \mathcal{G}) = c_\rho^N. \quad (5.11)$$

We will now prove that $\lambda_\rho^N = 0$ is not possible either, assuming that $N \in \mathbb{N}$ is large enough uniformly in $\rho \in [1/2, 1]$. It is here that we will use what has been developed in Section 3: assume by contradiction that there exists a subsequence $\{u_{\rho_k}^{N_k}\}_{k=1}^\infty$, with $N_k \rightarrow +\infty$ and $\rho_k \in [1/2, 1]$ for all k , such that the weak limits $u_{\rho_k}^{N_k} \in H^1(\mathcal{G})$ have an associated $\lambda_{\rho_k}^{N_k}$ which is 0. By Proposition 4.1, $c_\rho^N \xrightarrow{N \rightarrow \infty} +\infty$ uniformly w.r.t. ρ , and thus we have from (5.11) that $E_{\rho_k}(u_{\rho_k}^{N_k}, \mathcal{G}) \rightarrow +\infty$ as $k \rightarrow \infty$. This is in contradiction with Proposition 3.6 since $\{u_{\rho_k}^{N_k}\}_{k=1}^\infty \subset H_\mu^1(\mathcal{G})$. In conclusion, we have $\lambda_\rho^N > 0$.

Finally let us show that the Morse index $m(u_\rho^N)$ of u_ρ^N as a solution to (1.8) satisfies $m(u_\rho^N) \leq N + 1$. We recall that the Morse index of a solution $u \in H^1(\mathcal{G})$ of (1.7) is defined as the maximal dimension of a subspace $W \subset H^1(\mathcal{G})$ such that $Q(\varphi; u, \mathcal{G}) < 0$ for all $\varphi \in W \setminus \{0\}$, where

$$Q(\varphi; u, \mathcal{G}) := \int_{\mathcal{G}} |\varphi'|^2 + (\lambda - \kappa(x)(p-1)\rho|u|^{p-2})\varphi^2 dx.$$

We also note the relationship between the Morse index of a solution to (1.7) and the Morse index as a constrained critical point (refer to Definition 2.4) via the equality

$$\begin{aligned} D^2 E_\rho(u_\rho^N, \mathcal{G})[w, w] &:= E_\rho''(u_\rho^N, \mathcal{G})[w, w] + \lambda_\rho^N(w, w) \\ &= \int_{\mathcal{G}} \left[|w'|^2 + (\lambda_\rho^N - (p-1)\kappa(x)|u_\rho^N|^{p-2})w^2 \right] dx, \quad \text{for all } w \in H^1(\mathcal{G}). \end{aligned} \quad (5.12)$$

Since $u_{\rho,n}^N \rightarrow u_{\rho}^N$ as $n \rightarrow \infty$, we know from Remark 2.6 that the Morse index of $u_{\rho}^N \in H_{\mu}^1(\mathcal{G})$ as a constrained critical point is less than N . In view of (5.12) and of the fact that $H_{\mu}^1(\mathcal{G})$ is of codimension 1 in $H^1(\mathcal{G})$ we deduce

$$m(u_{\rho}^N) \leq N + 1. \quad (5.13)$$

Summarizing what has been observed so far we can give the

Proof of Theorem 1.2. For any $\mu > 0$ and any $N \in \mathbb{N}$ sufficiently large, we have shown that the particular bounded Palais-Smale sequence, satisfying (5.1)–(5.4), provided for almost every $\rho \in [1/2, 1]$ by the application of Theorem 2.5 is converging. This leads to the existence of sequence of couples $\{(\lambda_{\rho}^N, u_{\rho}^N)\} \subset (0, +\infty) \times H_{\mu}^1(\mathcal{G})$ which are solutions to (1.8). We also have by (5.11) that $E(u_{\rho}^N, \mathcal{G}) = c_{\rho}^N \rightarrow +\infty$. The estimate (5.13) completes the proof. \square

6 Proof of Theorem 1.1

Let $\mu > 0$ and $N \in \mathbb{N}$ be sufficiently large. By Theorem 1.2, it is possible to choose a sequence $\rho_n \rightarrow 1^-$, and a corresponding sequence of critical points $u_{\rho_n}^N \in H_{\mu}^1(\mathcal{G})$ of $E_{\rho_n}(\cdot, \mathcal{G})$ constrained to $H_{\mu}^1(\mathcal{G})$, at the level $c_{\rho_n}^N$ and having a Morse index $m(u_{\rho_n}^N) \leq N + 1$. Additionally, the Lagrange multipliers satisfy $\lambda_{\rho_n}^N > 0$.

To prove Theorem 1.1, it clearly suffices to show that $\{u_{\rho_n}^N\} \subset H_{\mu}^1(\mathcal{G})$ converges. For this the key point is to show that $\{u_{\rho_n}^N\} \subset H^1(\mathcal{G})$ is bounded. The monotonicity of c_{ρ}^N , as a function of $\rho \in [1/2, 1]$ implies that $\{c_{\rho_n}^N\}$ is bounded as it belongs to $[c_1^N, c_{1/2}^N]$ with $c_1^N, c_{1/2}^N \in \mathbb{R}$ (see Remark 4.2). In addition, since, thanks to the Kirchhoff boundary condition

$$\int_{\mathcal{G}} |(u_{\rho_n}^N)'|^2 + \lambda_{\rho_n}^N (u_{\rho_n}^N)^2 dx = \rho_n \int_{\mathcal{X}} |u_{\rho_n}^N|^p dx,$$

it follows that

$$c_{\rho_n}^N = E_{\rho_n}(u_{\rho_n}^N, \mathcal{G}) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |(u_{\rho_n}^N)'|^2 dx - \frac{\lambda_{\rho_n}^N \mu}{p}.$$

Therefore

$$\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |(u_{\rho_n}^N)'|^2 dx = c_{\rho_n}^N + \frac{\lambda_{\rho_n}^N \mu}{p}$$

and thus, if $\{\lambda_{\rho_n}^N\} \subset (0, +\infty)$ is bounded, then $\{u_{\rho_n}^N\} \subset H^1(\mathcal{G})$ is bounded as well. At this point to conclude the proof of Theorem 1.1 we just need to make use of the following result which is [18, Corollary 1.4] adapted to our notation.

Lemma 6.1. *Let \mathcal{G} be a metric graph satisfying Assumption (1.1), and $p > 6$. Assume that $(\rho_n) \subseteq [1/2, 1]$ is a sequence converging to 1. Let $\{(\lambda_n, u_n)\} \subseteq \mathbb{R} \times H^1(\mathcal{G})$ be a sequence of solutions to*

$$\begin{cases} -u'' + \lambda u = \rho \kappa(x) |u|^{p-2} u & \text{on every edge } e \in \mathcal{E}, \\ \sum_{e \succ v} u'_e(v) = 0 & \text{at every vertex } v \in \mathcal{V}, \end{cases}$$

and satisfy additionally, for some $\mu > 0$,

$$\int_{\mathcal{G}} |u_n|^2 dx = \mu, \quad \text{for all } n \in \mathbb{N}$$

and whose Morse indices $m(u_n)$ are bounded. Then, the sequence $\{\lambda_n\} \subset \mathbb{R}$ is bounded from above.

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