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Freudenthal duality in conformal field theory

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Abstract: Rotational Freudenthal duality (RFD) relates two extremal Kerr-Newman (KN) black holes (BHs) with different angular momenta and electric-magnetic charges, but with the same Bekenstein-Hawking entropy. Through the Kerr/CFT correspondence (and its KN extension), a four-dimensional, asymptotically flat extremal KN BH is endowed with a dual thermal, two-dimensional conformal field theory (CFT) such that the Cardy entropy of the CFT is the same as the Bekenstein-Hawking entropy of the KN BH itself. Using this connection, we study the effect of the RFD on the thermal CFT dual to the KN extremal (or doubly-extremal) BH. We find that the RFD maps two different thermal, two-dimensional CFTs with different temperatures and central charges, but with the same asymptotic density of states, thereby matching the Cardy entropy. We also discuss the action of the RFD on doubly-extremal rotating BHs, finding a spurious branch in the non-rotating limit, and determining that for this class of BH solutions the image of the RFD necessarily over-rotates.

Keywords: Black Holes, Black Holes in String Theory, Scale and Conformal Symmetries

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Contents

1 Introduction

Freudenthal duality (FD), as originally introduced, is a non-linear, anti-involutive map between the electric-magnetic (e.m.) charge configurations supporting two static, spherically symmetric extremal black holes (BHs) as solutions in four-dimensional $\mathcal{N} = 2$ supergravity in asymptotically flat space, and defined as the gradient of the entropy in the e.m. charge symplectic space [1–3]. The study of this duality was further extended for BHs in gauged supergravity [4] and in many other contexts [5–10]. A remarkable feature of FD is that the BHs have the same Bekenstein-Hawking entropy, despite being supported by different, Freudenthal-dual, e.m. charges; moreover, they also share the same attractor values of the moduli fields at the event horizon. This can be traced back to the fact that the Bekenstein-Hawking entropy of the aforementioned class of BHs is a homogeneous, degree-two function of its e.m. charges. Hence, extending this duality to more generic cases, such as non-extremal and/or rotating (stationary) BHs is not straightforward, as their entropy loses the above-mentioned homogeneity [11]. Recently, the present authors have extended this idea to near-extremal BHs, stating that two different near-extremal BHs can share the same entropy when their charges are related by a suitable non-linear relation, which necessarily yields that they have different temperatures [12]. This (non-anti-involutive) duality has been named the generalised FD (GFD). Later on, a further extension to over- and under-rotating stationary extremal BHs has been formulated in [13], necessarily yielding different angular momenta for the Freudenthal-dual BHs. This (non-anti-involutive) duality has been named rotational FD (RFD). It is interesting to note that entropy stays invariant also for a class of Kerr-Newman-deSitter BHs with the same charge and angular momentum but with different masses [14, 15].

So far, the understanding and the application of GFD/RFD has been limited to the macroscopic regime of BHs in Maxwell-Einstein-scalar theories of gravity (which are not necessarily supersymmetric, but could be considered as bosonic sectors of four-dimensional supergravity[1]). As a BH behaves as a thermal object with a finite temperature and entropy, one might wonder to count the degeneracy of its microstates. It is indeed possible to count the microstates for supersymmetric, extremal BHs with large charges [16–18]. In this paper, we attempt to understand the effect of FD on the microscopic entropy of an extremal Kerr-Newman BH in asymptotically flat space in four dimensions. Through the Kerr/CFT correspondence [19], there exists a two-dimensional, thermal conformal field theory (CFT) that is dual to the near horizon geometry of the extremal Kerr BH in asymptotically flat space. The Cardy entropy, corresponding to the density of states of such a CFT, exactly produces the Bekenstein-Hawking entropy of the Kerr BH. After the seminal paper [19], the Kerr/CFT correspondence has been generalized to the presence of Maxwell fields [20] and thus to the Kerr-Newman extremal BH solutions (which can be regarded as solution to the bosonic sector of $\mathcal{N} = 2$, $D = 4$ "pure" supergravity). A consistent formulation within (ungauged and gauged) supergravity was already put forward in [21], within the so-called extremal black hole/CFT correspondence (see also [22], and further [23] for a comprehensive review). Here, we will analyze the effect of RFD on the density of states of a two-dimensional CFT that is dual to an asymptotically flat, generally dyonic Kerr-Newman extremal BH, regarded as a solution to a Maxwell-Einstein-scalar system, which can (but does not necessarily have to) be conceived as the purely bosonic sector of a ($\mathcal{N} \geqslant 2$-extended, ungauged) supergravity theory in four space-time dimensions.

Our plan for the paper is as follows.

We begin with a brief discussion of RFD in section 2, specifically review RFD for extremal black hole in section 2.1. In section 2.2 we give a general proof of the uniqueness of RFD (as well as a treatment of the "spurious" branch of solutions in the non-rotating limit) for doubly-extremal Kerr-Newman BHs, thus completing the analysis given in [13]; interestingly, we find that any RFD-transformed doubly-extremal KN BH is over-rotating. After that, in section 3 we elaborate the details of the Kerr/CFT correspondence (and its aforementioned extensions and generalizations to Maxwell-Einstein-scalar systems), focusing our attention specially on the duality in presence of (running) scalar fields. Then in section 4 we discuss the action of the RFD on the Cardy entropy in the CFT dual, both for the extremal and doubly-extremal rotating BHs, providing a novel microscopic picture of RFD. We finish with a outlook and implications of our findings in section 5. Finally, for pedagogic completeness, in appendix A we provide a concise review of the Cardy formula for CFT.

2 Rotational Freudenthal duality (RFD)

In this section, we briefly discuss the effect of Rotational Freudenthal duality on extremal black holes [13] and also analyze in detail the same for double-extremal rotating black holes.

[1]For the interplay between BPS supersymmetry-preserving properties of (extremal) BHs and FD, see section 8.4 of [10].

2.1 RFD for extremal black holes

The Bekenstein-Hawking entropy of an asymptotically flat, extremal, under-/over- rotating stationary (Kerr-Newman, generally dyonic) charged BH (regarded as a solution to Einstein gravity, coupled to Maxwell fields and running scalar fields, within the so-called 'Maxwell-Einstein-scalar system', which can further be regarded as the purely bosonic sector of four-dimensional supergravity theories), with e.m. charges collected in the symplectic vector Q and angular momentum J, is respectively given by [24, 25],

$$S_{\text{under}}(Q,J) = \sqrt{S_0^2(Q) - J^2} \tag{2.1}$$

$$S_{\text{over}}(Q,J) = \sqrt{J^2 - S_0^2(Q)} \tag{2.2}$$

where $S_0(Q)$ is the extremal entropy in the absence of rotation. Though extremal, the entropies of such extremal BHs lose the homogeneity property on Q due to the presence of angular momentum. Hence, a canonical method of establishing FD does not hold for them. As mentioned, in [13] the present authors have formulated a generalised version of FD, named RFD, for this class of BHs: two rotating BHs with angular momenta J and $J+\delta J$ have the same entropy i.e.

$$S_{\text{under(over)}}(Q,J) = S_{\text{under(over)}}\left(\Omega\frac{\partial S_{\text{under(over)}}(Q,J+\delta J)}{\partial Q}, J+\delta J\right), \tag{2.3}$$

when their e.m. charges are RFD-dual, ie. they are related by

$$Q \to \hat{Q}(Q,J+\delta J) = \Omega\frac{\partial S_{\text{under(over)}}(Q,J+\delta J)}{\partial Q}. \tag{2.4}$$

Consistently, the $J \to 0^{(+)}$ limit of the extremal, under-rotating BH entropy is[2] $S_0(Q)$. By analysing the solutions to a sextic equation of δJ, it has been shown both analytically and numerically that this mapping is unique.

Introducing a real and positive parameter α for the mother black hole as $\alpha = \frac{J^2}{S_0^2}$, one can find that the angular momenta $\hat{J} = J + \delta J$ of the RFD dual black hole behaves as follows [13]

$$\hat{J}^2 = (J+\delta J)^2 = f(\alpha) S_0^2, \tag{2.5}$$

$$f(\alpha) := \frac{1}{2^{2/3}}\left(1+\sqrt{1+\frac{4}{27}(\alpha-2)^3}\right)^{2/3} + \frac{1}{2^{2/3}}\left(1-\sqrt{1+\frac{4}{27}(\alpha-2)^3}\right)^{2/3} + \frac{1}{3}(\alpha+1). \tag{2.6}$$

From (2.6) and also from the plot shown in figure 1, we see the following properties

$$f(\alpha) = \alpha \Leftrightarrow \alpha = 2; \tag{2.7}$$

$$0 \leqslant \alpha < 2 : f(\alpha) > \alpha; \tag{2.8}$$

$$\alpha > 2 : f(\alpha) < \alpha, \tag{2.9}$$

[2]When the U-duality group is a 'generalized group of type E_7' of non-degenerate resp. degenerate type, S_0 equals (in units of π) $\sqrt{|I_4(Q)|}$ or $|I_2(Q)|$, where $I_4(Q)$ and $I_2(Q)$ are the (unique, primitive) polynomial of e.m. charges (homogeneous of degree four resp. two), invariant under the (non-transitive, symplectic) action of the U-duality group itself.

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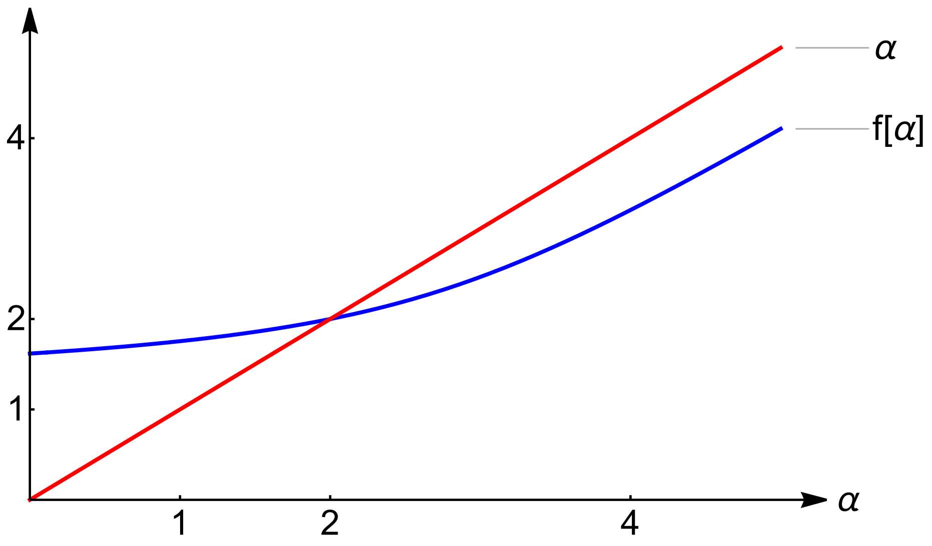

Figure 1. $f(\alpha)$ vs. α.

which will be useful later. $\alpha > 1$ (resp. $\alpha < 1$) corresponds to an over-rotating (resp. under-rotating) BHs, whereas $\alpha = 0$ corresponds to a non-rotating (i.e., static, RN) BH.

Moreover, in [13] it has been further emphasized that RFD cannot induce a change in the BH rotation regime; the story goes differently for *doubly-extremal* rotating solutions: in section 2.2 we will show that the (suitably defined) RFD *necessarily* maps to over-rotating solutions.

2.2 RFD on *doubly-extremal* BHs

After the treatment of [24], a rotating BH with angular momentum $\tilde{J}$ and non-rotating (i.e., static) entropy $S_0 \equiv S_0(Q)$, which is *doubly-extremal*, i.e. it is coupled to *constant* scalar fields, has a Bekenstein-Hawking entropy

$$S_{BH}^{\text{doubly-extr.}} = \sqrt{S_0^2\,(Q) + 4\pi^2 \tilde{J}^2}. \tag{2.10}$$

For convenience, we absorb 2π in the definition of the angular momentum as $2\pi\tilde{J} =: J$, hence the entropy can be written as

$$S_{BH}^{\text{doubly-extr.}} = \sqrt{S_0^2\,(Q) + J^2}, \tag{2.11}$$

where $S_0\,(Q)$ is the (purely Q-dependent) non-rotating (i.e., static) Bekenstein-Hawking entropy (i.e., the entropy of the RN extremal BH obtained by letting $J \to 0^{(+)}$). The functional form of the BH entropy (2.11), which contains the sum of two squares under the square root, has not been treated in the analysis given in [13]; here, we fill this gap and, along the same lines of reasoning of [13], we briefly discuss the possibility to define a RFD for rotating, doubly-extremal BHs (which we will define RFD$^{\text{doubly-extr.}}$).

First of all, we observe that, as for the analysis given in [13], a naïve definition of RFD with unvaried J is not possible. Indeed, one would obtain the following transformation of the e.m. charge symplectic vector:

$$\mathrm{RFD}^{\text{doubly-extr.}}: Q \longmapsto \hat{Q}(Q,J) := \Omega \frac{\partial S_{BH}^{\text{doubly-extr.}}}{\partial Q} = \frac{S_0(Q)}{\sqrt{S_0^2(Q)+J^2}} \Omega \frac{\partial S_0(Q)}{\partial Q}. \tag{2.12}$$

Thus, the condition of invariance of the BH entropy under $\mathrm{RFD}^{\text{doubly-extr.}}$ reads

$$S_{BH}^{\text{doubly-extr.}}(Q,J) = S_{BH}^{\text{doubly-extr.}}\left(\hat{Q}(Q,J),J\right) \Leftrightarrow S_0^2 J^2 (J^2 + 2S_0^2) = 0, \tag{2.13}$$

whose only possible solution is $J = 0$, thus trivializing the $\mathrm{RFD}^{\text{doubly-extr.}}$ to FD itself.

Therefore, we proceed to consider a transformation of the angular momentum J under the RFD, as well. The corresponding transformation rule of Q reads

$$\mathrm{RFD}^{\text{doubly-extr.}}: Q \longmapsto \hat{Q}(Q,J+\delta J) := \Omega \frac{\partial S_{BH}^{\text{doubly-extr.}}(Q,J+\delta J)}{\partial Q} = \frac{S_0}{\sqrt{S_0^2+(J+\delta J)^2}} \Omega \frac{\partial S_0(Q)}{\partial Q}. \tag{2.14}$$

This implies that the condition of invariance of the BH entropy under $\mathrm{RFD}^{\text{doubly-extr.}}$, i.e.

$$S_{BH}^{\text{doubly-extr.}}(Q,J) = S_{BH}^{\text{doubly-extr.}}\left(\hat{Q}(Q,J+\delta J),J+\delta J\right), \tag{2.15}$$

holds *iff* δJ satisfies the following inhomogeneous algebraic equation of degree six:

$$b_1(\delta J)^6 + b_2(\delta J)^5 + b_3(\delta J)^4 + b_4(\delta J)^3 + b_5(\delta J)^2 + b_6(\delta J) + b_7 = 0, \tag{2.16}$$

with

$$\begin{aligned} b_1 &= -1, \\ b_2 &= -6J, \\ b_3 &= -14J^2 - S_0^2, \\ b_4 &= -4J(4J^2 + S_0^2), \\ b_5 &= -9J^4 - 4J^2 S_0^2 + S_0^4, \\ b_6 &= -2J(J^4 + S_0^4), \\ b_7 &= J^2 S_0^2 (J^2 + 2S_0^2). \end{aligned} \tag{2.17}$$

In order to find out two rotating doubly-extremal BHs with different angular momenta but the same entropy, having their dyonic charges related by RFD (2.14), we need to look for a real solution of (2.16), namely for $\delta J = \delta J(J, S_0) \in \mathbb{R}$ such that the transformed angular momentum is (cf. the start of section 4)

$$\hat{J} := J + \delta J \in \mathbb{R}^+. \tag{2.18}$$

Non-rotating limit: the spurious, "golden" branch of doubly-extremal BHs. Before dealing with a detailed analysis of the roots of the algebraic equation (2.16), let us investigate the limit $J \to 0^{(+)}$ of the set of solutions to (2.16). In this limit, $\hat{J} \to \delta J^{(+)} \in \mathbb{R}^+$, and eq. (2.16) reduces to

$$(\delta J)^6 + S_0^2(\delta J)^4 - S_0^4(\delta J)^2 = 0, \tag{2.19}$$

which consistently admits the solution $\delta J = 0$. This is no surprise, since in the $J \to 0^{(+)}$ limit, $\mathrm{RFD}^{\text{doubly-extr.}}$ simplifies down to its usual, non-rotating definition (namely, to FD, [2]).

Interestingly, other two real solutions to the cubic algebraic homogeneous equation (in $(\delta J)^2$) (2.19) exist. In fact, eq. (2.19) can be solved by two more real roots[3]

$$\delta J_{\pm} := \pm\sqrt{\phi - 1}\, S_0\left(Q\right) = \pm\frac{1}{\sqrt{\phi}} S_0\left(Q\right), \tag{2.20}$$

with $\phi := \frac{1+\sqrt{5}}{2}$ being the so-called *golden ratio* (see e.g. [26]). Since $\hat{J} \to \delta J^{(+)} \in \mathbb{R}^+$, only δJ_+ in (2.20) has a sensible physical meaning. Correspondingly, within the $J \to 0^+$ (i.e., non-rotating) limit, the unique non-vanishing, physically sensible solution for the angular momentum transformation reads

$$\hat{J}^{\text{doubly-extr.}}_{\text{golden}} := \delta J_+ = \sqrt{\phi - 1} S_0\left(Q\right) = \sqrt{\frac{-1+\sqrt{5}}{2}} S_0(Q). \tag{2.21}$$

This analysis shows the existence, in the limit $J \to 0^+$, of a spurious, "golden" branch $\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\,\text{golden}}$ of $\mathrm{RFD}^{\text{doubly-extr.}}$, which in the non-rotating limit $J \to 0^{(+)}$ does *not* reduce to the usual FD, but rather it allows to map a non-rotating doubly-extremal BH to an under-rotating (stationary) double-extremal one; this can be depicted as

$$\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\,\text{golden}} : \underbrace{\begin{cases} S = S_0\left(Q\right) \\ J = 0 \end{cases}}_{\text{static doubly-extremal BH}} \longrightarrow \underbrace{\begin{cases} S = S^{\text{doubly-extr.}}_{BH}\left(\hat{Q}^{\text{doubly-extr.}}_{\text{golden}}, \hat{J}^{\text{doubly-extr.}}_{\text{golden}}\right) \\ J = \hat{J}^{\text{doubly-extr.}}_{\text{golden}} \end{cases}}_{\text{under-rotating doubly-extremal BH}}, \tag{2.22}$$

where $\hat{J}^{\text{doubly-extr.}}_{\text{golden}}$ is defined by (2.21), $\hat{Q}^{\text{doubly-extr.}}_{\text{golden}}$ is defined by

$$\begin{aligned}
\hat{Q}^{\text{doubly-extr.}}_{\text{golden}} &:= \Omega \frac{\partial S^{\text{doubly-extr.}}_{BH}\left(Q, \hat{J}^{\text{doubly-extr.}}_{\text{golden}}\right)}{\partial Q} = \frac{S_0\left(Q\right)}{\sqrt{\left(\hat{J}^{\text{doubly-extr.}}_{\text{golden}}\right)^2 + S_0^2\left(Q\right)}} \Omega \frac{\partial S_0\left(Q\right)}{\partial Q} \\
&= \frac{S_0\left(Q\right)}{\sqrt{\left(\phi - 1\right) S_0^2(Q) + S_0^2\left(Q\right)}} \Omega \frac{\partial S_0\left(Q\right)}{\partial Q} = \frac{1}{\sqrt{\phi}} \Omega \frac{\partial S_0\left(Q\right)}{\partial Q} \\
&= \sqrt{\phi - 1}\,\Omega \frac{\partial S_0\left(Q\right)}{\partial Q},
\end{aligned} \tag{2.23}$$

and $\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\,\text{golden}}$ denotes such a spurious, "golden" branch of the non-rotating limit of RFD applied to doubly-extremal BHs, defined by (2.14):

$$\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\,\text{golden}} : \begin{cases} Q \to \hat{Q}^{\text{doubly-extr.}}_{\text{golden}} = \sqrt{\phi - 1}\,\Omega \frac{\partial S_0(Q)}{\partial Q}; \\ \\ J = 0 \to \hat{J}^{\text{doubly-extr.}}_{\text{golden}}; \end{cases} \tag{2.24}$$

Note that in the last step of (2.23) we used the crucial property of the golden ratio ϕ, namely

$$\phi - 1 = \frac{1}{\phi}. \tag{2.25}$$

[3] There exist also two purely imaginary roots with $\delta J^2 = -S_0^2\left(\frac{1+\sqrt{5}}{2}\right)$, which we ignore.

Consistently, the total doubly-extremal BH entropy $S_{BH}^{\text{doubly-extr.}}$ is preserved by the map $\text{RFD}_{J\to 0^+,\,\text{golden}}^{\text{doubly-extr.}}$ (2.24), because

$$\begin{aligned}
S_{BH}^{\text{doubly-extr.}}\left(\hat{Q}_{\text{golden}}^{\text{doubly-extr.}},\hat{J}_{\text{golden}}^{\text{doubly-extr.}}\right) &= \sqrt{S_0^2\left(\hat{Q}_{\text{golden}}^{\text{doubly-extr.}}\right)+\left(\hat{J}_{\text{golden}}^{\text{doubly-extr.}}\right)^2} \\
&= \sqrt{S_0^2\left(\sqrt{\phi-1}\Omega\frac{\partial S_0(Q)}{\partial Q}\right)+(\phi-1)\,S_0^2(Q)} \\
&= \sqrt{(\phi-1)^2\,S_0^2\left(\Omega\frac{\partial S_0(Q)}{\partial Q}\right)+(\phi-1)\,S_0^2(Q)} \\
&= \sqrt{\phi^2-\phi}S_0(Q)=S_0(Q)\,,
\end{aligned} \tag{2.26}$$

where in the last step the crucial property (2.25) has been used again. Moreover, in achieving (2.26), we have also exploited two crucial properties of the static (doubly-)extremal BH entropy $S_0(Q)$, namely its homogeneity of degree two in the e.m. charges,

$$S_0(\lambda Q)=\lambda^2 S_0(Q),\ \forall\lambda\in\mathbb{R}, \tag{2.27}$$

and its invariance under the usual FD [2],

$$S_0(Q)=S_0\left(\Omega\frac{\partial S_0(Q)}{\partial Q}\right). \tag{2.28}$$

Thus, the two doubly-extremal BHs in the l.h.s. and r.h.s. of (2.22) have the same Bekenstein-Hawking entropy, preserved by the map $\text{RFD}_{J\to 0^+,\,\text{golden}}^{\text{doubly-extr.}}$ (2.24).

Remark 1. It is worth remarking that the stationary doubly-extremal BH in the r.h.s. of (2.22), namely the image of a doubly-extremal, static BH under the map $\text{RFD}_{J\to 0^+,\,\text{golden}}^{\text{doubly-extr.}}$ (2.24), is necessarily *under-rotating*. In fact,

$$0<\phi-1<1\Rightarrow\hat{J}_{\text{golden}}^{\text{doubly-extr.}}=\sqrt{\phi-1}S_0(Q)<S_0(Q)\Leftrightarrow\hat{J}_{\text{golden}}^{\text{doubly-extr.}}-S_0^2(Q)<0, \tag{2.29}$$

which pertains to the under-rotating case. This is to be contrasted with what holds for extremal but not doubly-extremal rotating BHs; cf. remark 1 in section 3.1 of [13].

Remark 2. The comparison with the spurious, "golden" branch of the RFD applied to (under- or over-)rotating, extremal (but not doubly-extremal) BHs, discovered in [13], is particularly relevant. From eqs. (2.21) and (2.23), the following relations hold:

$$\hat{J}_{\text{golden}}^{\text{doubly-extr.}}=\sqrt{\frac{\phi-1}{\phi}}\hat{J}_{\text{golden}}=\frac{\hat{J}_{\text{golden}}}{\phi}; \tag{2.30}$$

$$\hat{Q}_{\text{golden}}^{\text{doubly-extr.}}=-\sqrt{\frac{\phi-1}{\phi}}\hat{Q}_{\text{golden}}=-\frac{\hat{Q}_{\text{golden}}}{\phi}, \tag{2.31}$$

where $\hat{J}_{\text{golden}}$ and $\hat{Q}_{\text{golden}}$ are respectively given by eqs. (3.8) and (3.10) of [13].

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Remark 3. In a sense, the use of the spurious, "golden" branch $\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\ \text{golden}}$ (2.24) of the map RFD (2.14) can be regarded as a kind of solution-generating technique, which generates an under-rotating, stationary doubly-extremal BH from a non-rotating, static BH, while keeping the Bekenstein-Hawking entropy fixed. Indeed, while both such BHs are asymptotically flat, their near-horizon geometry changes under $\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\ \text{golden}}$:

$$\underset{\text{static doubly-extremal BH [27]}}{AdS_2 \otimes S^2} \overset{\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\ \text{golden}}}{\longrightarrow} \underset{\text{under-rotating doubly-extremal BH [28]}}{AdS_2 \otimes S^1}. \tag{2.32}$$

This is totally analogous to what holds for extremal (but not doubly-extremal) rotating BHs; see remark 2 in section 3.1 of [13].

Remark 4. The expression of the Bekenstein-Hawking entropy $S_{BH}^{\text{doubly-extr.}}$ of doubly-extremal (stationary) rotating BHs, given by (3.17) or equivalently by (2.11), is suggestive of a representation of the two fundamental quantities S_0 and J characterizing such BHs in terms of elements of the *complex* numbers $\mathbb{C}$, i.e. respectively as

$$Z := S_0 + \mathbf{i}J \in \mathbb{C} \Rightarrow S_{BH}^{\text{doubly-extr.}} = |Z| := \sqrt{Z\bar{Z}} = \sqrt{S_0^2 + J^2}, \tag{2.33}$$

where the bar stands for the complex conjugation.

Thus, any map acting on S_0 and J of a doubly-extremal rotating (stationary) BH, as the RFD map (2.14) itself, can be represented as acting on the Argand-Gauss plane $\mathbb{C}[S_0, J]$ (or, equivalently, $\mathbb{C}[J, S_0]$) itself. Since complex numbers are a field (because they do not contain divisors of 0), there may exist "wild" automorphisms [29] of $\mathbb{C}$, depending on the so-called 'axiom of choice'. The four discrete fundamental automorphisms are $\mathbb{I}$, $-\mathbb{I}$, $\mathbf{C}$ and $-\mathbf{C}$, where $\mathbb{I}$ and $\mathbf{C}$ respectively denote the identity map and the aforementioned conjugation map; interestingly, since $S_0 \in \mathbb{R}_0^+$ and $J \in \mathbb{R}^+$, none (but, trivially, $\mathbb{I}$) of such discrete automorphisms are physically allowed, and the physically sensible quadrant of $\mathbb{C}[S_0, J]$ is only the first one.

In particular, the RFD non-linear map (2.14) map acts on any $Z \in \mathbb{C}[S_0, J]$ as a norm-preserving transformation, because, by definition, it preserves the Bekenstein-Hawking entropy $S_{BH}^{\text{doubly-extr.}}$, as given by the defining condition (2.11). Thus, considering a (stationary, asymptotically flat) doubly-extremal rotating BH with Bekenstein-Hawking entropy $S_{BH}^{\text{doubly-extr.}} = \mathcal{S} \in \mathbb{R}^+$, its RFD-dual extremal BH will belong to the arc of the circle of squared radius $|Z|^2 = \mathcal{S}^2$ within the first quadrant of $\mathbb{C}[S_0, J]$.

In the non-rotating limit, in light of the treatment above, the action of the RFD map (2.14) on a non-rotating doubly-extremal BH (represented as a point of the S_0 axis — with its origin excluded — in $\mathbb{C}[S_0, J]$, thus with coordinates $(S_0, 0)$) may be nothing but the identity $\mathbb{I}$ (in the usual branch of $\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+}$) or a point with coordinates $\left((\phi - 1)\, S_0, \sqrt{\phi - 1} S_0\right) = \left(\frac{S_0}{\phi}, \frac{S_0}{\sqrt{\phi}}\right)$ along the arc of the circle defined as $\mathcal{C} := \left\{Z \in \mathbb{C}[S_0, J] : |Z|^2 = S_0^2\right\}$ (in the spurious, "golden" branch $\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+,\ \text{golden}}$ discussed above). Again, this is to be contrasted with remark 3 in section 3.1 of [13].

Analytical solution. By recalling the definition (2.18), the sextic equation (2.16) can be rewritten as

$$-\hat{J}^6 + \hat{J}^4(J^2 - S_0^2) + \hat{J}^2 S_0^2(S_0^2 + 2J^2) + J^2 S_0^4 = 0, \tag{2.34}$$

and by setting $x := \hat{J}^2$, one obtains the following cubic inhomogeneous equation in x:

$$-x^3 + x^2(J^2 - S_0^2) + xS_0^2(S_0^2 + 2J^2) + J^2 S_0^4 = 0. \tag{2.35}$$

As discussed e.g. in [13] (see also refs. therein), the turning points of the parametric curve

$$g(x) := -x^3 + x^2(J^2 - S_0^2) + xS_0^2(S_0^2 + 2J^2) + J^2 S_0^4 \tag{2.36}$$

tell us about the root of the equation (2.35). By solving $\frac{dg(x)}{dx} = 0$, we find the two turning points $T_i = (x_i, g(x_i)), i = 1, 2$ with coordinates

$$T_1 = (-S_0^2, -S_0^6), \tag{2.37}$$

$$T_2 = \left(\frac{1}{3}(2J^2 + S_0^2), \frac{\mathcal{B}}{27}\right), \tag{2.38}$$

$$\mathcal{B} := \left(4(J^2 + 2S_0^2)^3 - 27S_0^6\right) = (4J^6 + 24J^4 S_0^2 + 48J^2 S_0^4 + 5S_0^6) > 0. \tag{2.39}$$

Hence, the discriminant of $g(x)$ is $\Delta = S_0^6 \mathcal{B} > 0$. Thus, the equation $g(x) = 0$ always has three real roots. Note that $g(0) = J^2 S^0$ is always on the positive $y \equiv g(x)$ axis. The two turning points T_1 and T_2 lie in the third quadrant and first quadrant respectively in the x *vs.* $y = g(x)$ plot. This implies that $g(x)$ has only one positive real root which is physically sensible, namely $x = \hat{J}^2$, always a positive and real quantity. Thus, acting with $\mathrm{RFD}^{\text{doubly-extr.}}_{J\to 0^+}$ on a doubly-extremal KN BH with charge Q and angular momentum J, one obtains another doubly-extremal rotating unique BH with charge $\hat{Q}(Q, \hat{J})$ and angular momentum $\hat{J}$. A typical plot of $g(x)$ *vs.* x (for $J = 2, S_0 = 7$) is shown in figure 2.

Numerical solution. We can also solve the equation (2.16) numerically, and see that there exist only two real solutions for δJ, namely δJ_1 and δJ_2. Then, these two real solutions can be numerically analyzed by choosing some positive and real values for J and S_0: it is easy to realize that there is only one solution for δJ that makes $\hat{J} := J + \delta J$ always positive. For example, with $S_0 = 2$, we plot the $J + \delta J_i$ *vs.* J_i $(i = 1, 2)$ curves as follows

Exact solution. A convenient way of solving (2.35) is to write it in the *depressed form*

$$\mathbf{t}^3 + \mathbf{p}\mathbf{t} + \mathbf{q} = 0, \tag{2.40}$$

with $\mathbf{t} := x - \frac{1}{3}(J^2 - S_0^2)$, $\mathbf{p} := -\frac{1}{3}(J^2 + 2S_0^2)^2$, $\mathbf{q} := (\frac{S_0^6}{2} - \frac{\mathcal{B}}{54})$. Therefore, the solution to (2.40) is given by

$$\mathbf{t} = \sqrt[3]{-\frac{\mathbf{q}}{2} + \sqrt{\frac{\mathbf{q}^2}{4} + \frac{\mathbf{p}^3}{27}}} + \sqrt[3]{-\frac{\mathbf{q}}{2} - \sqrt{\frac{\mathbf{q}^2}{4} + \frac{\mathbf{p}^3}{27}}}. \tag{2.41}$$

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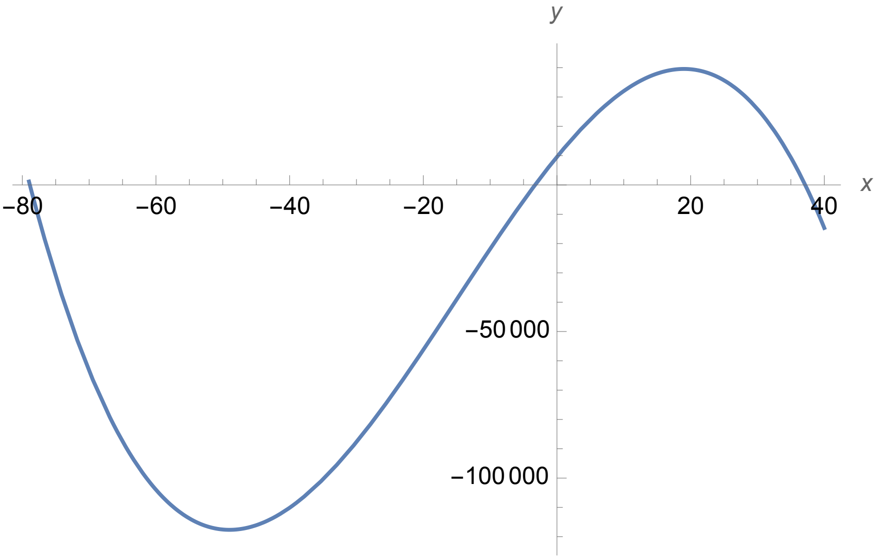
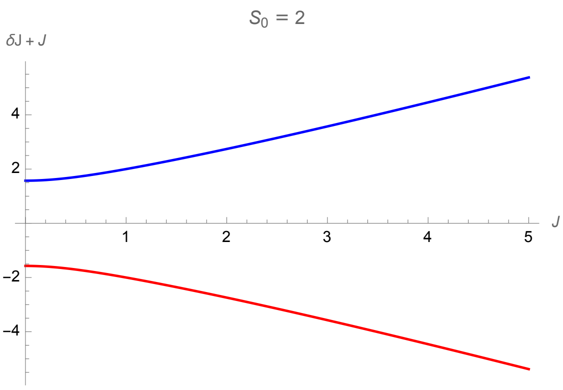
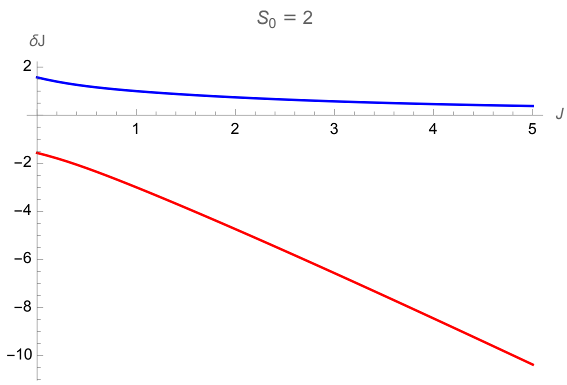


Figure 2. Typical plot of $g(x)$ for $J \neq 0$.



(a) Fixed Entropy.



(b) Fixed entropy.

Figure 3. Two real roots of (2.16).

One should note that in the above expression the operations "$\sqrt{\ }$" and "$\sqrt[3]{\ }$" represents the principle value of the expression making (2.41) always real. Plugging back the definitions of $\mathbf{t}, \mathbf{p}$ and $\mathbf{q}$, one finds that the positive real root can be written as

$$\hat{J}^2 = x = \frac{1}{3}(J^2 - S_0^2) - \omega\left(\frac{S_0^3}{2} + \sqrt{\frac{-\mathcal{B}}{108}}\right)^{\frac{2}{3}} - \omega^2\left(\frac{S_0^3}{2} - \sqrt{\frac{-\mathcal{B}}{108}}\right)^{\frac{2}{3}} \tag{2.42}$$

where $\omega(= -\frac{1}{2} - i\frac{\sqrt{3}}{2})$ and ω^2 are the two non trivial cubic roots of unity with the relation that $\omega^2 = 1/\omega$. Similar to the case of extremal black hole, defining $\alpha := J^2/S_0^2$, this formula can be recast as follows:

$$\begin{aligned}\hat{J}^2 &= \frac{S_0^2}{3}(\alpha - 1) - \omega\frac{S_0^2}{2^{2/3}}\left(1 + \sqrt{1 - \frac{4}{27}(\alpha+2)^3}\right)^{\frac{2}{3}} - \omega^2\frac{S_0^2}{2^{2/3}}\left(1 - \sqrt{1 - \frac{4}{27}(\alpha+2)^3}\right)^{\frac{2}{3}} \\ &=: S_0^2 h(\alpha)\end{aligned} \tag{2.43}$$

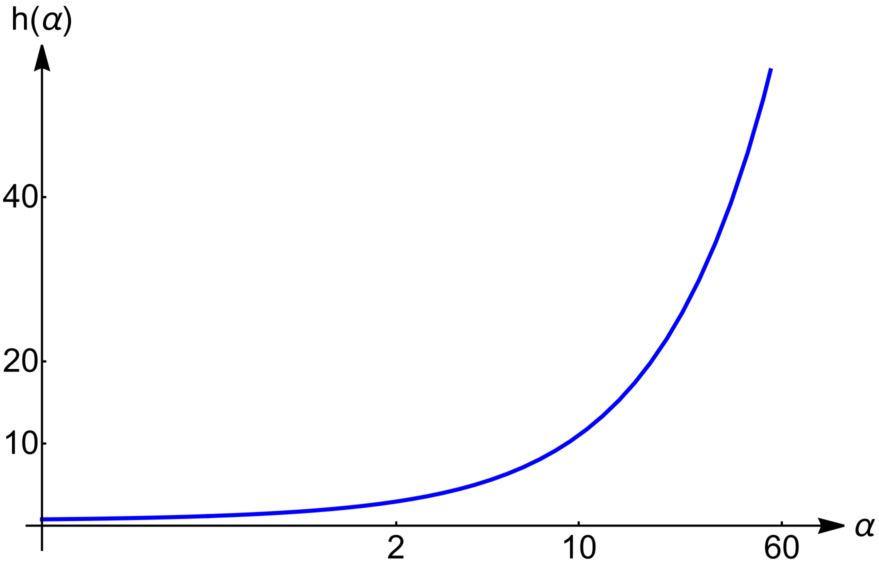


Figure 4. Semi-log plot of $h(\alpha)$ *vs.* α.

where

$$h(\alpha)=\frac{1}{3}(\alpha-1)-\frac{\omega}{2^{2/3}}\left(1+\sqrt{1-\frac{4}{27}(\alpha+2)^3}\right)^{\frac{2}{3}}-\frac{\omega^2}{2^{2/3}}\left(1-\sqrt{1-\frac{4}{27}(\alpha+2)^3}\right)^{\frac{2}{3}} \tag{2.44}$$

to be compared with $f(\alpha)$, defined in (2.6).

We can express the RFD$^{\text{doubly-extr.}}$-transformed entropy as

$$\begin{aligned}S_{BH}^{\text{doubly-extr.}}(\hat{Q}(Q,\hat{J}),\hat{J}) &:= \sqrt{S_0^2\left(\hat{Q}(Q,\hat{J})\right)+\hat{J}^2} = \sqrt{S_0^2\left(\frac{S_0}{\sqrt{S_0^2+\hat{J}^2}}\Omega\frac{\partial S_0(Q)}{\partial Q}\right)+\hat{J}^2}\\ &= \sqrt{\frac{S_0^4}{\left(S_0^2+\hat{J}^2\right)^2}S_0^2\left(\Omega\frac{\partial S_0(Q)}{\partial Q}\right)+\hat{J}^2} = \sqrt{\frac{S_0^6}{\left(S_0^2+\hat{J}^2\right)^2}+\hat{J}^2}\\ &= \sqrt{\tilde{S}_0{}^2(Q,\hat{J})+\hat{J}^2},\end{aligned} \tag{2.45}$$

where

$$\tilde{S}_0(Q,\hat{J}) := \frac{S_0^3(Q)}{S_0^2(Q)+\hat{J}^2} \overset{(2.44)}{=} \frac{S_0(Q)}{1+h(\alpha)}. \tag{2.46}$$

Note that,consistently, $\tilde{S}_0(Q,\hat{J})$ is nothing but the RFD$^{\text{doubly-extr.}}$-transformed non-rotating BH entropy

$$\begin{aligned}RFD^{\text{doubly-extr.}}: S_0(Q)\mapsto S_0\left(\hat{Q}\right) &= S_0\left(\frac{S_0}{\sqrt{S_0^2+\hat{J}^2}}\Omega\frac{\partial S_0(Q)}{\partial Q}\right)\\ &= \frac{S_0^2}{S_0^2(Q)+\hat{J}^2}S_0\left(\Omega\frac{\partial S_0(Q)}{\partial Q}\right) = \frac{S_0^3}{S_0^2(Q)+\hat{J}^2} = \tilde{S}_0(Q,\hat{J}),\end{aligned} \tag{2.47}$$

and thus the real positive parameter

$$\hat{\alpha} := \frac{\hat{J}^2}{\tilde{S}_0^2} \tag{2.48}$$

has the physically sensible interpretation of RFD$^{\text{doubly-extr.}}$-transformed α. Of course, the RFD$^{\text{doubly-extr.}}$-invariance of the doubly-extremal KN BH entropy (2.15) implies that

$$S_{BH}^{\text{doubly-extr.}}(\hat{Q}(Q,\hat{J}),\hat{J}) = S_{BH}^{\text{doubly-extr.}}(Q,J) = \sqrt{S_0^2 + J^2} = S_0\sqrt{1+\alpha}. \tag{2.49}$$

Finally, eqs. (2.46) and (2.46), together with definition (2.48), yield that

$$\hat{\alpha} = h(\alpha)(1+h(\alpha))^2, \tag{2.50}$$

whereas (2.48) and (2.50) imply

$$\tilde{S}_0\sqrt{1+\hat{\alpha}} = S_0\sqrt{1+\alpha} \overset{(2.44)}{\Leftrightarrow} \frac{\sqrt{1+\hat{\alpha}}}{1+h(\alpha)} = \sqrt{1+\alpha}, \tag{2.51}$$

which, by virtue of (2.50), allows one to determine an equation satisfies by $h(\alpha)$ itself:[4]

$$\begin{aligned} 1+h(\alpha) &= \sqrt{\tfrac{1+\hat{\alpha}}{1+\alpha}} \overset{(2.50)}{=} \sqrt{\frac{1+h(\alpha)(1+h(\alpha))^2}{1+\alpha}}; \\ &\Updownarrow \\ (1+\alpha-h(\alpha))\,(1+h(\alpha))^2 &= 1, \end{aligned} \tag{2.53}$$

where the *iff* (i.e., "$\Updownarrow$") implication holds because $h(\alpha) > 0\ \forall\alpha \geqslant 0$, as evident from the corresponding plot shown in figure 4.

$\hat{\alpha} > 1$ always. At $\alpha = 1 \Leftrightarrow J^2 = S_0^2$, it holds that $\hat{\alpha} \approx 15.58 \Rightarrow \hat{J}^2 > \tilde{S}_0^2$. On the other hand, $\hat{\alpha} = 1 \Leftrightarrow \hat{J}^2 = \tilde{S}_0^2 \Leftrightarrow h(\alpha)(1+h(\alpha))^2 = 1$, implying that $h(\alpha) \approx 0.4656$. Therefore, if $h(\alpha) < 0.4656$, then $\hat{\alpha} < 1$, and if $h(\alpha) > 0.4656$, then $\hat{\alpha} > 1$. One can check that $h(0) \approx 0.62$, and also that $h(\alpha)$ is a monotonically increasing function (from graph 4). Hence, for any value of $\alpha \geqslant 0$ it holds that $\hat{\alpha} > 1$. This means that, regardless the under-rotating ($0 \leqslant \alpha < 1$) or over-rotating ($\alpha > 1$) regime of (stationary) rotation of the starting doubly-extremal KN BH, the RFD-transformed doubly-extremal KN BH is always over-rotating: $\hat{\alpha} > 1$. Again, this is clarified in figure 5.

3 Kerr/CFT correspondence and its extensions

The Kerr/CFT correspondence [19] establishes a duality between a four-dimensional, asymptotically flat, extremal Kerr BH and a two-dimensional (thermal) CFT. This duality arises

[4] An analogous equation for $f(\alpha)$ defined in (2.6) can be derived from the treatment given in section 7 of [13]:

$$(1-f(\alpha))^2(1-\alpha+f(\alpha)) = 1. \tag{2.52}$$

Actually, $f(\alpha)$ resp. $h(\alpha)$ given by equations resp. (2.6) & (2.44), are the unique physically sound solution of the functional equation resp. (2.53) & (2.52). (ie, they are the only solutions implying $\hat{J}$ real and positive.)

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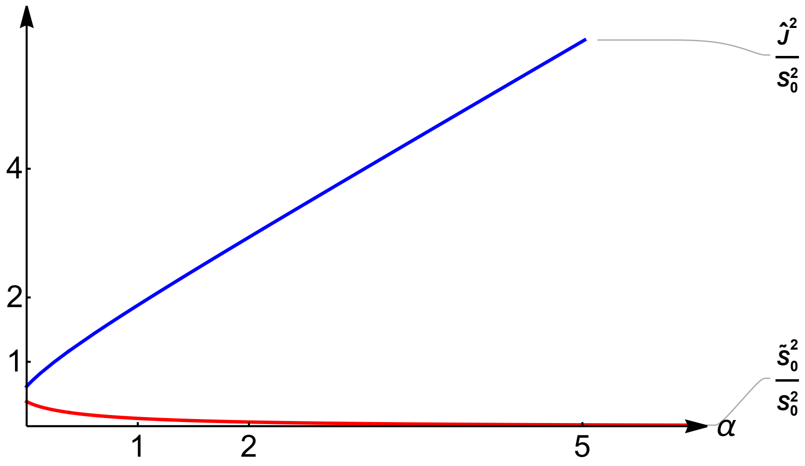

Figure 5. Any RFD-tranformed doubly-extremal KN BH is always over-rotating.

from the realization that the near-horizon extremal Kerr (NHEK) metric is essentially equivalent to a warped AdS_3 space. In contrast to the $\mathrm{SL}(2,\mathbb{R})_R \times \mathrm{SL}(2,\mathbb{R})_L$ isometry group of AdS_3, the warped AdS_3 has an isometry group of $\mathrm{SL}(2,\mathbb{R})_R \times \mathrm{U}(1)$. Following a similar analysis by Brown and Henneaux [30], the asymptotic symmetry group of the extremal Kerr BH has been studied in [19]. With a suitable choice of fall off of the metric at the boundary, the U(1) symmetry of $\mathrm{SL}(2,\mathbb{R})_R \times \mathrm{U}(1)$ gets enhanced to the infinite-dimensional Virasoro algebra with a central charge $c_L = 12\tilde{J}$, where $\tilde{J}$ is the angular momentum of the dual BH. To compute the statistical entropy of this CFT, we need the temperature T_L which cannot be proportional to the Hawking temperature of the dual extremal Kerr BH, as that is vanishing. In order to find out T_L, it becomes necessary to define a proper quantum vacuum. As the Kerr geometry lacks a global time-like Killing vector, there is no Hartle-Hawking vacuum. We need to consider Frolov and Thorne vacuum [31–33] which is analogous to the Hartle-Hawking vacuum in the near-horizon region. This leads to the finding the temperature $T_L = \frac{1}{2\pi}$, while $T_R = 0$. Hence following the Cardy formula, we find the entropy of the extremal Kerr BH as

$$S^{\mathrm{Kerr}}_{\mathrm{Cardy}} = \frac{\pi^2}{3} c_L \, T_L = 2\pi \tilde{J} = S^{\mathrm{Kerr}}_{BH} \,, \tag{3.1}$$

which thus agrees with the Bekenstein-Hawking entropy for the same.

3.1 Maxwell fields

The Kerr/CFT correspondence can be extended to the so-called Kerr-Newman/CFT correspondence, to include Maxwell fields (and thus, extremal KN BH solutions), generally also in presence of scalar fields [20–22] (see also [23] for a comprehensive review, and the treatment below). An extremal KN BH is characterized by its charge q and the angular momentum $\tilde{J}$. The near-horizon isometry is given by $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$. The asymptotic symmetry group for the near-horizon of the extremal KN BH contains two sets of Virasoro algebras with central charges $c_L = c_R = 12\tilde{J}$. Introducing the Frolov-Thorne vacuum as in the case of Kerr BH, we find the temperatures of the two CFTs as $T_L = \frac{r_+}{2\pi a} - \frac{q^2}{4\pi M a}$ and $T_R = 0$, where M and r_+ are the mass and the outer event horizon of the BH with $a = \tilde{J}/M$. Hence, the Cardy entropy is

$$S^{\mathrm{KN}}_{\mathrm{Cardy}} = \frac{\pi^2}{3}(c_L T_L + c_R T_R) \tag{3.2}$$

$$= \frac{\pi^2}{3} c_L T_L \tag{3.3}$$

$$= \pi(2Mr_+ - q^2) = S^{\mathrm{KN}}_{BH}, \tag{3.4}$$

with S^{KN}_{BH} being the Bekenstein-Hawking entropy of the extremal KN BH. To summarize, after the treatment given in [24] and [22, 23], the CFT dual to an extremal KN BH with angular momentum $\tilde{J}$ and (electric) charge Q is endowed with the following central charges and temperatures:

$$c = 12\tilde{J}; \tag{3.5}$$

$$T_L = \frac{\sqrt{q^4 + 4\tilde{J}^2}}{4\pi\tilde{J}}; \tag{3.6}$$

$$T_R = 0. \tag{3.7}$$

Hence the Bekenstein-Hawking entropy S^{KN}_{BH} and the Cardy entropy $S^{\mathrm{KN}}_{\mathrm{Cardy}} \equiv S_{CFT}$ of the dual CFT are

$$S^{\mathrm{KN}}_{BH} = S_{CFT} = \pi\sqrt{q^4 + 4\tilde{J}^2}. \tag{3.8}$$

As to be expected, the $q \to 0$ and $\tilde{J} \to 0^{(+)}$ limits of this formula respectively yield the extremal Kerr BH entropy (3.1) and the extremal (static) RN entropy πq^2.

3.2 Scalar fields

It has been conjectured and proved that the central charge of a CFT dual to an extremal, rotating (Kerr or KN) BH which solves Einstein equations of a theory containing scalars and other fields depends only on the gravitational field [22, 23]. Nevertheless, the Bekenstein-Hawking entropy is still the same as the Cardy entropy of the dual CFT.

Here, we discuss briefly this further generalization of the Kerr/CFT correspondence. The gravitational action of our concerned theory is of the form [22–24]

$$\begin{aligned} \mathcal{S} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\Big(&R - \frac{1}{2}f_{AB}(\Phi)\partial_\mu\Phi^A\partial^\mu\Phi^B \\ &-V(\Phi) - k_{IJ}(\Phi)F^I_{\mu\nu}F^{J\mu\nu} + h_{IJ}(\Phi)\epsilon^{\mu\nu\delta\tau}F^I_{\mu\nu}F^J_{\delta\tau}\Big). \end{aligned} \tag{3.9}$$

Other than gravity, the theory involves scalars Φ and gauge fields A^I_μ with field strength $F^I_{\mu\nu}$. Here μ, ν, run over the space-time indices, A, B denotes the scalar indices and I, J denotes the gauge field indices. Since we are going to consider asymptotically flat extremal BH solutions, in the rest of the treatment we will assume $V = 0$, so we may be considering (the purely bosonic sector of) ungauged supergravity, or gauged supergravity with flat Abelian gaugings,[5] as well. We assume that $f_{AB}(\Phi)$ and $k_{IJ}(\Phi)$ are positive definite and maximal

[5]This choice is immaterial, since the solutions are exactly the same, as far as the supersymmetry (BPS-) preserving properties, and in general fermions, are not concerned [34].

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rank, of course. The near-horizon region of any stationary, rotating extremal BH of this theory has $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)$ isometry. Hence, the near-horizon geometry takes the form [22–24]

$$\begin{aligned} ds^2 &= \Gamma(\theta)\left[-r^2dt^2+\frac{dr^2}{r^2}+\beta(\theta)^2d\theta^2+\gamma(\theta)^2\left(d\phi+\rho r dt\right)^2\right] \\ &= \Omega(\theta)^2e^{2\Psi(\theta)}\left[-r^2dt^2+\frac{dr^2}{r^2}+\beta(\theta)^2d\theta^2\right. \\ &\quad\left.+\Omega(\theta)^{-2}e^{-4\Psi(\theta)}\left(d\phi+\rho r dt\right)^2\right]; \end{aligned} \tag{3.10}$$

$$\Phi^A = \Phi^A(\theta), \quad A^I = f^I(\theta)\left(d\phi+\rho r dt\right)+\frac{e^I}{\rho}d\phi. \tag{3.11}$$

The BH entropy is then given by

$$S_{BH} = \frac{1}{4G_N\hbar}\int d\theta d\phi\sqrt{g_{\theta\theta}g_{\phi\phi}} = \frac{\pi}{2G_N\hbar}\int d\theta\Omega(\theta)\beta(\theta), \tag{3.12}$$

with $\Gamma(\theta)>0, \gamma(\theta)\geq 0$. $\Phi^A(\theta), f^I(\theta), k$ and e_I are real parameters who get fixed by solving the equations of motion.

On the other hand, the Cardy entropy of the dual $2D$ CFT is

$$S_{CFT} = \frac{\pi^2}{3}cT_L, \tag{3.13}$$

with [22, 23]

$$c = \frac{3\rho}{G_N\hbar}\int d\theta\sqrt{g_{\theta\theta}g_{\phi\phi}}; \tag{3.14}$$

$$T_L = \frac{1}{2\pi\rho}. \tag{3.15}$$

Hence, the Cardy entropy is

$$S_{CFT} = \frac{\pi}{2G_N\hbar}\int d\theta\sqrt{g_{\theta\theta}g_{\phi\phi}}, \tag{3.16}$$

and it matches with the Bekenstein-Hawking entropy (3.12).

3.2.1 Frozen scalars

As discussed in 2.2, the Bekenstein-Hawking entropy of rotating, doubly-extremal BHs with fixed scalars at the asymptote is as follows,

$$S_{BH}^{\text{doubly-extr.}} = \sqrt{S_0^2\left(Q\right)+4\pi^2\tilde{J}^2}. \tag{3.17}$$

For doubly-extremal solutions, this expression generalizes the purely electric, KN formula (3.8), because πq^2 gets generalized to a generic, dyonic expression $S_0\left(Q\right)$ of the extremal (static) RN entropy. By the aforementioned KN/CFT correspondence, the dual CFT is equipped with

$$c = 12\tilde{J}; \tag{3.18}$$

$$T_L = \frac{\sqrt{S_0^2\left(Q\right)+4\pi^2\tilde{J}^2}}{4\pi^2\tilde{J}}; \tag{3.19}$$

$$T_R = 0, \tag{3.20}$$

thus yielding

$$S_{CFT} = \sqrt{(S_0)^2 + 4\pi^2 \tilde{J}^2} = S_{BH}^{\text{doubly-extr.}}. \tag{3.21}$$

Completing the analysis done in [13], in section 2.2 we have proved that there exists a unique, Freudenthal dual BH for doubly-extremal rotating BHs, and we have also found a 'spurious' (but 'golden'!) branch of solutions in the non-rotating limit; moreover, we have also pointed out that the RFD always boosts a rotating doubly-extremal BH (with any ratio of $S_0(Q)$ and $2\pi\tilde{J}$) to an over-rotating regime.

3.2.2 Running scalars

After the treatment of [24], for rotating BHs which are *extremal but not doubly-extremal* (i.e., which are coupled to running scalar fields), the Bekenstein-Hawking entropy is given by (2.1)–(2.2) (which we repeat here for convenience,with $2\pi\tilde{J} =: J$),

$$S_{\text{under}} = \sqrt{S_0^2(Q) - 4\pi^2\tilde{J}^2} = S_0(Q)\sqrt{1-\alpha}; \tag{3.22}$$

$$S_{\text{over}} = \sqrt{4\pi^2\tilde{J}^2 - S_0^2(Q)} = S_0(Q)\sqrt{\alpha-1}; \tag{3.23}$$

$$\alpha := \frac{4\pi^2\tilde{J}^2}{S_0^2(Q)} = \alpha\left(Q, \tilde{J}\right) \geqslant 0, \tag{3.24}$$

depending whether the BH is (stationarily) under- or over- rotating, respectively. Conveniently, these forms of extremal Bekenstein-Hawking entropy can be written as

$$S_{\text{under(over)}} = S_0\sqrt{|1-\alpha|}, \tag{3.25}$$

where $\alpha > 1$ resp. $\alpha < 1$ for over- resp. under- rotating extremal BHs. By the aforementioned KN/CFT correspondence, the central charge (3.14) and the temperature (3.15) of the corresponding, dual CFT can be computed to read

$$c = 12\tilde{J}; \tag{3.26}$$

$$T_L = \frac{1}{2\pi}\sqrt{\frac{|1-\alpha|}{\alpha}}, \tag{3.27}$$

and thus the Cardy entropy reads

$$S_{CFT} = S_0\sqrt{|1-\alpha|} = S_{\text{under(over)}}, \tag{3.28}$$

therefore again matching the Bekenstein-Hawking BH entropy (3.25).

4 RFD in CFT

In the previous sections, we have separately discussed the RFD on both extremal and double-extremal black holes and the KN/CFT correspondence. As proved in [13], it is here worth recalling that two asymptotically flat, extremal (stationary) rotating BHs possessing different e.m. charges Q resp. $\hat{Q}$, as well as different angular momenta J resp. $\hat{J} := J + \delta J$, have the same Bekenstein-Hawking entropy when their e.m. charges are related by RFD $\hat{Q} := \Omega\frac{\partial S(Q,\hat{J})}{\partial Q}$.

A similar duality also holds for the double-extremal black holes. As reviewed in section 3 and section 3.2, the Bekenstein-Hawking entropy of these BHs precisely matches the Cardy entropy of their dual CFTs (according to the Kerr/CFT correspondence and its various extensions reviewed above). This naturally leads to the question of how RFD operates on these dual CFTs, which is the main scope of this paper. In this section, we are going to analyse that in detail for both extremal and doubly-extremal BHs.

4.1 Central charge and temperature

We now consider applying the RFD to a rotating, extremal (or doubly-extremal) BH with e.m. charge Q and angular momentum J. From the KN/CFT correspondence, the dual CFT possesses a temperature T_L and central charge c_L; therefore, the Cardy entropy is generally given by (3.3), with c_L given by (3.5). Let us denote the central charge and temperature of the CFT that is dual to the RFD-transformed BH as c_L' and T_L', with $c_L' = 12\widetilde{\hat{J}}$, where $\widetilde{\hat{J}} := 2\pi\hat{J} = 2\pi\left(J + \delta J\right)$. For convenience, we introduce a running function $w(\alpha)$ such that

$$w(\alpha) = \begin{cases} f(\alpha) & \text{for an extremal BH} \\ h(\alpha) & \text{for a doubly-extremal BH} \end{cases} \tag{4.1}$$

Thus, it holds that

$$c_L' \overset{(2.5),(2.43)}{=} 2\pi\cdot 12\sqrt{w(\alpha)}S_0 = 2\pi\cdot 12\sqrt{\frac{w(\alpha)}{\alpha}}J \overset{(3.5)}{=} \sqrt{\frac{w(\alpha)}{\alpha}}c_L. \tag{4.2}$$

Since the RFD preserves the Bekenstein-Hawking BH entropy, the KN/CFT correspondence and (3.3) thus imply that

$$\frac{\pi^2}{3}c_L'T_L' = \frac{\pi^2}{3}c_L T_L \Leftrightarrow T_L' = \frac{c_L T_L}{c_L'} \overset{(3.5)}{=} \frac{J T_L}{\hat{J}} \overset{(2.5),(2.43)}{=} \sqrt{\frac{\alpha}{w(\alpha)}}T_L. \tag{4.3}$$

Hence, we find that, for both extremal and doubly-extremal BHs, two different (thermal) CFTs with different temperatures and central charges have the same Cardy entropy when their dual extremal resp. doubly-extremal rotating BHs (through the Kerr/CFT correspondence and its generalizations) have their e.m. charges and angular momenta related by the RFD.

4.2 Remark

By recalling (3.3), it is clear that $S^{\mathrm{KN}}_{\mathrm{Cardy}}$ is invariant under a simultaneous dilatation on c_L and T_L with opposite weights:

$$\left.\begin{aligned} c_L &\to e^{\xi}c_L, \\ T_L &\to e^{\varphi}T_L; \end{aligned}\right\} \Rightarrow S^{\mathrm{KN}}_{\mathrm{Cardy}} \to S^{\mathrm{KN}}_{\mathrm{Cardy}} \Leftrightarrow \xi = -\varphi. \tag{4.4}$$

Therefore, it is clear that the action (4.2)–(4.3) of the RFD on the CFT side of the KN/CFT correspondence, namely on c and $T_{(L)}$ of the dual CFT, is a particular case of the dilatational symmetry (4.4) of $S^{\mathrm{KN}}_{\mathrm{Cardy}}$, with

$$\xi = -\varphi = \frac{1}{2}\ln\left(\frac{w\left(\alpha\right)}{\alpha}\right). \tag{4.5}$$

By definition (cf. (4.1)), $w(\alpha) = f(\alpha)$ defined in (2.6) for extremal, and $w(\alpha) = h(\alpha)$ as in (2.44) for doubly-extremal BH.

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4.3 (Asymptotic) Density of states

We further delve into the action of the RFD on the density of states of the thermal CFTs which are dual, through the KN/CFT correspondence, to two RFD-dual rotating extremal (or doubly-extremal) BHs. With the help of the thermodynamic relation eq. (A.14) between temperature T and the energy Δ, we can deduce the expression of the RFD-transformed energy,

$$\Delta' = \sqrt{\frac{\alpha}{w(\alpha)}}\Delta. \tag{4.6}$$

The holomorphic part of the RFD-transformed density of states can be written as[6]

$$\rho'(\Delta') = \int e^{2\pi \mathbf{i} g'(\tau)} Z(-1/\tau) d\tau, \tag{4.7}$$

where

$$g'(\tau) := -\tau\Delta' + \frac{c'}{24\tau} + \frac{c'\tau}{24}. \tag{4.8}$$

As in [35, 36], the indefinite integral appearing in (4.7) can be evaluated by using the steepest descent method; for a large value of Δ', the exponent is extremised at the point

$$\tau_0' = \sqrt{\frac{c'}{-24\Delta' + c'}} \approx \mathbf{i}\sqrt{\frac{w(\alpha)}{\alpha}}\sqrt{\frac{c}{24\Delta}}, \tag{4.9}$$

to be contrasted with $\tau_0 \approx \mathbf{i}\sqrt{\frac{c}{24\Delta}}$. Therefore, one obtains the relation

$$\tau_0' = \tau_0\sqrt{\frac{w(\alpha)}{\alpha}}. \tag{4.10}$$

With all this together, eq. (4.8) gets simplified as follows:

$$g'(\tau) \approx g'(\tau_0') = -\tau_0'\Delta' + \frac{c'}{24\tau_0'} + \frac{c'\tau_0'}{24} \overset{(4.6),\,(4.10)}{=} -\tau_0\Delta + \frac{c}{24\tau_0} + \frac{w(\alpha)}{\alpha}\frac{c\tau_0}{24}. \tag{4.11}$$

Therefore, the only modification at the saddle point value of the exponent comes through the last term.

Finally, we can write the density of states of the thermal CFT which is dual to the RFD-transformed extremal (or doubly-extremal) rotating BH, for large[7] Δ', as

$$\begin{aligned}\rho'(\Delta') &\approx \exp\left(4\pi\sqrt{\frac{c\Delta}{24}}\right)\exp\left(-\frac{2\pi c}{24}\sqrt{\frac{c}{24\Delta}}\frac{w(\alpha)}{\alpha}\right) Z\left(-1/\left(\mathbf{i}\sqrt{w(\alpha)/\alpha}\sqrt{c/24\Delta}\right)\right)\\ &\approx \exp\left(2\pi\sqrt{\frac{c\Delta}{6}}\right) = \rho(\Delta).\end{aligned} \tag{4.12}$$

Thus, one can conclude that *the asymptotic density of states remains invariant under RFD*, which is expected due to the invariance of the Cardy entropy itself. Note that, in order to derive (4.12), the following approximations have been used:[8]

$$\exp\left(-\frac{2\pi c}{24}\sqrt{\frac{c}{24\Delta}}\frac{w(\alpha)}{\alpha}\right) \overset{\Delta\gg 1}{\approx} 1; \tag{4.13}$$

[6]In the present paper, we will denote the imaginary unit by $\mathbf{i}$.

[7]Large value of Δ' $\implies$ large value of Δ as they are related through eq. (4.6), since the ratio $\alpha/w(\alpha)$ is always finite and non-negative.

[8]In (4.14) the modular symmetry of Z has been used (see the *résumé* in appendix A and refs. therein).

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$$Z\left(-1/\left(\mathbf{i}\sqrt{w(\alpha)/\alpha}\sqrt{c/24\Delta}\right)\right)\overset{\Delta\gg 1}{\approx} 1. \tag{4.14}$$

This is exactly the line of thought that one uses in order to derive the Cardy formula for CFT; see appendix A, in which the treatment of [35, 36] is sketchily reviewed.

5 Discussion and outlook

Rotational Freudenthal duality (RFD) is a generalization of Freudenthal duality for asymptotically flat, rotating, stationary, generally dyonic extremal BHs, introduced in [13]. It yields that two (under- or over-) rotating extremal BHs have the same Bekenstein-Hawking entropy in spite of having their e.m. charges related by a non-linear relation when they have certain distinct (uniquely and analytically determined) angular momenta. In this paper, we begin by completing the analysis done in [13]. In section 2.2 we have proved that there exists a unique, Freudenthal dual BH for *doubly-extremal* rotating BHs, and we have also found a 'spurious' (but 'golden'!) branch of solutions in the non-rotating limit. Intriguingly, we have also obtained that the RFD will *necessarily* boost a rotating doubly-extremal BH (with any ratio of $S_0(Q)$ and J) to an over-rotating regime. All these results surely warrant some further future investigation.

In the subsequent part of the paper, we have primarily aimed at understanding the action of the RFD on the microscopic entropy of extremal (or doubly-extremal) Kerr-Newman (KN) BHs. Using the Kerr/CFT correspondence (and its further extensions to Maxwell and scalar fields, thus applicable to Maxwell-Einstein-scalar systems, as the ones occurring in the purely bosonic sector of four-dimensional supergravities), we have investigated how the RFD influences the density of states of a thermal CFT, which is dual to the extremal (or doubly-extremal) rotating BH under consideration.

On the CFT side of the KN/CFT correspondence, we found that the RFD relates two two-dimensional thermal CFTs, whose density of states is the same, though they have different temperatures and central charges (but the same Cardy entropy!). More explicitly, by recalling (4.2) and (4.3), we have determined that the action of the RFD on the (left) central charge and on the (left) temperature reads as follows:

$$c_L \to c_L' := \sqrt{\frac{w(\alpha)}{\alpha}}c_L; \tag{5.1}$$

$$T_L \to T_L' := \sqrt{\frac{\alpha}{w(\alpha)}}T_L \overset{(3.27)}{=} \frac{1}{2\pi}\sqrt{\frac{\alpha}{w(\alpha)}}\sqrt{\frac{|1-\alpha|}{\alpha}} = \frac{1}{2\pi}\sqrt{\frac{|1-\alpha|}{w(\alpha)}}. \tag{5.2}$$

It is amusing to observe that the net effect of the RFD on the temperature is the replacement of α with $w(\alpha)(=f(\alpha)$ defined in (2.6) or $h(\alpha)$ defined in (2.44)) in the denominator of T_L (3.27):

$$T_L = \frac{1}{2\pi}\sqrt{\frac{|1-\alpha|}{\alpha}} \to T_L' = \frac{1}{2\pi}\sqrt{\frac{|1-\alpha|}{w(\alpha)}}. \tag{5.3}$$

One should note two interesting domains of α.

1. **Extremal Black hole:** the RFD-self-dual value $\alpha = 2$, at which the RFD trivializes to the identity (because $w(2) \equiv f(2) = 2$; cf. (2.7)): it corresponds to a peculiar over-rotating extremal BH, with $J^2 = 2S_0^2(Q)$, discussed in section 2.2 of [13].

   Doubly-extremal Black hole: on the contrary to its extremal counterpart, nothing similar happens for $w(\alpha) = h(\alpha)$, which nicely ties in to the fact that *RFD transformed doubly-extremal KN BH is always over-rotating* cf. section 2.2.

2. **Extremal Black hole:** the second one is the interval $\alpha \in [0, 2)$, in which $\frac{f(\alpha)}{\alpha} < 1$. Therefore, one could have naïvely made the observation that for this range of α the RFD might map a unitary CFT (with $c > 1$) to a non-unitary one (with $c < 1$) by virtue of the formula (4.2); this would have resulted in a conceptual issue, since one of the core assumptions of the Kerr/CFT correspondence (and its extensions) is that the dual CFT is unitary. But a careful look at the values of $\sqrt{\frac{f(\alpha)}{\alpha}}$ reveals that this function has a (global) minimum at $\alpha \approx 3.75$, at which $\sqrt{\frac{f(\alpha)}{\alpha}} \approx 0.894$, as shown in figure 6. Since the Kerr/CFT correspondence works in the large c limit (i.e., for $c \gg 1$), the central charge will still remain large under the action of the RFD. This observation raises the interesting question whether one could analytically extend the definition of the RFD for all values of c of a CFT and analyze the effect of RFD itself on the unitarity of the theory. We leave this to further future investigation.

   Double-extremal Black hole: in this case $\frac{h(\alpha)}{\alpha} > 1 \; \forall \, \alpha \in (0, \infty)$ implying no such unitarity issue as above. This can easily be observed from (2.53) and the aforementioned observation that $h(\alpha) > 0 \, \forall \, \alpha \in (0, \infty)$, which implies $(1 + h(\alpha))^2 > 1$. Therefore $\forall \, \alpha \in (0, \infty)$ we have

$$0 < 1 + \alpha - h(\alpha) < 1 \implies \alpha - h(\alpha) < 0 \implies \frac{h(\alpha)}{\alpha} > 1. \tag{5.4}$$

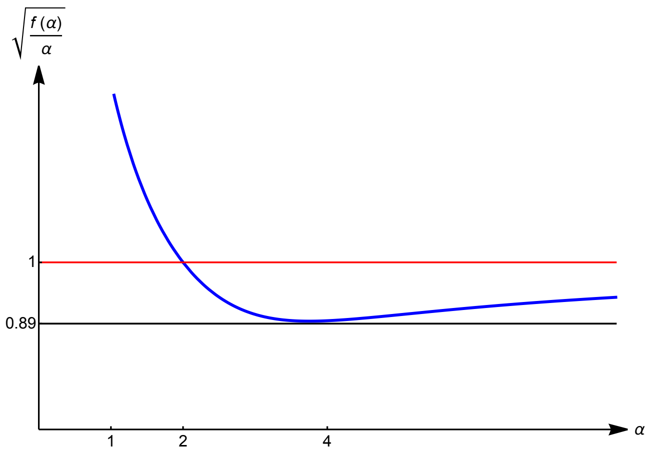

Figure 6. Plot of $\sqrt{\frac{f(\alpha)}{\alpha}}$ shown in blue with a minima at $\alpha \approx 3.75$.

Though our analysis did not address the effect of the RFD on the degeneracy of BH microstates, still it employed the KN/CFT correspondence. Explicitly, we exploit the fact that the microscopic/statistical (Cardy) entropy of a two-dimensional thermal CFT (on the CFT side of the correspondence) matches the macroscopic/thermodynamical (Bekenstein-Hawking) entropy of the asymptotically flat KN extremal (or doubly-extremal) BH under consideration (on the BH side of the correspondence). An intriguing further development may concern the study of the action of the RFD on the bound states of D-branes [16] or M-branes [17] that account for the Bekenstein-Hawking BH entropy itself.

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It will also be interesting to extend the present investigation to asymptotically AdS, KN extremal (or doubly-extremal) BH solutions, in which one would be able to apply the AdS/CFT correspondence and the KN/CFT correspondence to the (spacial) asymptotical and near-horizon limits of the scalar flow, respectively. This research venue appears particularly promising, and we keep it as a future endeavour.

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A Cardy formula: a quick *résumé*

The Cardy formula plays a crucial role in our analysis. In this appendix, we provide a concise overview of the derivation of the Cardy formula [35, 36], which is employed to calculate the density of states in a two-dimensional CFT and subsequently determine its statistical entropy.

Thus, we start with a two-dimensional CFT endowed with the usual pair of Virasoro algebras governing the left- and right- moving modes, each characterized by a central charge denoted as c. Exploiting the conformal symmetry, it becomes possible to map the theory from a plane to a cylinder without loss of generality. Moreover, one can compactify the cylinder into a torus by using a complex modulus $\tau = \tau_1 + \mathbf{i}\tau_2$. The torus partition function can then be written as follows

$$Z(\tau,\bar{\tau}) = \operatorname{Tr} e^{2\pi\mathbf{i}\tau L_0} e^{-2\pi\mathbf{i}\bar{\tau}\bar{L}_0}, \tag{A.1}$$

with L_0 and $\bar{L}_0$ being the global generators of the holomorphic and the anti-holomorphic part. Since the trace is over all states in the Hilbert space, one can further write the partition function as

$$Z(\tau,\bar{\tau}) = \operatorname{Tr} e^{2\pi\mathbf{i}\tau L_0} e^{-2\pi\mathbf{i}\bar{\tau}\bar{L}_0} = \sum_{\{\Delta,\bar{\Delta}\}} \rho(\Delta,\bar{\Delta}) e^{2\pi\mathbf{i}\Delta\tau} e^{-2\pi\mathbf{i}\bar{\Delta}\bar{\tau}}, \tag{A.2}$$

with $\rho(\Delta,\bar{\Delta})$ being the density of states and Δ denoting the conformal scaling dimension. Using the *residue trick*, one can then express the density of states in terms of the partition function as

$$\rho(\Delta,\bar{\Delta}) = \frac{1}{(2\pi\mathbf{i})^2} \int \frac{dq}{q^{\Delta+1}} \frac{d\bar{q}}{\bar{q}^{\bar{\Delta}+1}} Z(q,\bar{q}), \tag{A.3}$$

where $q = e^{2\pi\mathbf{i}\tau}$ and $\bar{q} = e^{2\pi\mathbf{i}\bar{\tau}}$ and the contour encloses the point $(q,\bar{q}) \equiv (0,0)$.

A crucial step in deriving the Cardy formula is the definition of the function

$$Z_0(\tau, \bar{\tau}) := \mathrm{Tr}\, e^{2\pi \mathbf{i}\left(L_o - \frac{c}{24}\right)\tau} e^{-2\pi \mathbf{i}\left(\bar{L}_o - \frac{c}{24}\right)\bar{\tau}}, \tag{A.4}$$

which enjoys modular invariance under the transformation $\tau \to -1/\tau$. Therefore, one can recast the density of states given by eq. (A.3) as

$$\rho(\Delta, \bar{\Delta}) = \frac{1}{(2\pi \mathbf{i})^2} \int \frac{dq}{q^{\Delta+1}} \frac{d\bar{q}}{\bar{q}^{\bar{\Delta}+1}} Z_0(q, \bar{q}) q^{\frac{c}{24}} \bar{q}^{-\frac{c}{24}}. \tag{A.5}$$

From now onwards, we will focus on the holomorphic parts only assuming a similar treatment with the anti-holomorphic sector. Using the modular symmetry of $Z_0(\tau)$, one obtains

$$Z_0(\tau) = Z_0\left(-\frac{1}{\tau}\right) = e^{\frac{2\pi \mathbf{i} c}{24\tau}} Z\left(-\frac{1}{\tau}\right). \tag{A.6}$$

Using all the above relations together, we have the holomorphic part of the density of states rewritten as

$$\rho(\Delta) = \int e^{2\pi \mathbf{i} g(\tau)} Z(-1/\tau) d\tau, \tag{A.7}$$

with the definition

$$g(\tau) := -\tau\Delta + \frac{c}{24\tau} + \frac{c\tau}{24}. \tag{A.8}$$

The whole point of using the modular symmetry is to claim the fact that $Z(-1/\tau)$ approaches a constant value for large values of τ_2, which is evident by the construction. Therefore, we can now use the *steepest descent method* to calculate the integral in eq. (A.7). In order to proceed with the steepest descent, one has to look for the value of $\tau = \tau_0$ such that the power of the exponent in eq. (A.7) is extremised. This implies that for $dg/d\tau \to 0$ one obtains

$$\tau_0 = \sqrt{\frac{c}{-24\Delta + c}} \approx \mathbf{i}\sqrt{\frac{c}{24\Delta}}, \tag{A.9}$$

where we have assumed Δ to be large ($\Delta \gg c$) or, in other words, we are looking for asymptotic density of states through eq. (A.7). Therefore, at the saddle point for large Δ values, we have

$$\begin{aligned} \rho(\Delta) &\approx \exp\left(4\pi\sqrt{\frac{c\Delta}{24}}\right) \exp\left(-\frac{2\pi c}{24}\sqrt{\frac{c}{24\Delta}}\right) Z\left(-\frac{1}{i\sqrt{c/24\Delta}}\right) \\ &\approx \exp\left(2\pi\sqrt{\frac{c\Delta}{6}}\right) Z(i\infty) \approx \exp\left(2\pi\sqrt{\frac{c\Delta}{6}}\right), \end{aligned} \tag{A.10}$$

where

$$\exp\left(-\frac{2\pi c}{24}\sqrt{\frac{c}{24\Delta}}\right) \overset{\Delta \gg 1}{\approx} 1, \tag{A.11}$$

and modular symmetry implies that

$$Z(i\infty) \overset{\Delta \gg 1}{\approx} 1. \tag{A.12}$$

Thus, without even knowing many details of the theory, one can arrive at the definition of entropy as

$$S_{\text{Cardy}} = \log\left(\rho(\Delta)\right) = 2\pi\sqrt{\frac{c\Delta}{6}}. \tag{A.13}$$

Since in this formula c is the central charge and Δ is the analogue of energy, the temperature of the CFT can be defined through the relation

$$\frac{1}{T} = \frac{dS_{\text{Cardy}}}{d\Delta} = \pi\sqrt{\frac{c}{6\Delta}}. \tag{A.14}$$

This brings us to the famous Cardy formula

$$S_{\text{Cardy}} = \frac{\pi^2}{3}cT. \tag{A.15}$$

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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